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PROPERTIES OF YANG-MILLS FIELD WITH AXIALLY SYMMETRIC EXTERNAL COLOR CHARGE SOURCES

NGUYEN QUOC HOAN
Hanoi University of Science and Technology -
1 Dai Co Viet Road, Hanoi 10000, Vietnam

Abstract. *The physical characteristics of the solutions of Yang-Mills equations with two point color charge sources and for topological indices $n \geq 2$ are studied. The space distributions of corresponding non-Abelian field strengths and energy density are simulated and plotted. The dependence of the field strengths and the energy density on topological indices is discussed. By using the same algorithm of solving, new solutions for the Yang-Mills field due to a series of color point charges which lie in a straight line are found and plotted explicitly. It is shown that the solutions for gauge potentials and field energy have the form of vortex strings.*

I. INTRODUCTION

For field equations of gauge theories there exist solutions, characterized by nontrivial topological properties. They are called "topological defects", which play an important role in particle physics and cosmology. Some examples are monopoles (zero-dimensional defects), vortex strings (one-dimensional defects), domain walls (two-dimensional defects). Since the systems of gauge field equations are coupled nonlinear differential equations of several field variables, the solutions that have been found by analytical methods are only for some specific cases of high symmetry properties. For example, the Wu-Yang monopole [1], 't Hooft-Polyakov monopole [2-4], instanton [5] solutions to the systems of Yang-Mills and Yang-Mills-Higgs equations were found by virtue of using a spherically symmetric ansatz. In general cases, for systems with lower symmetries, the integration of the corresponding field equations must be done numerically. Nowadays different numerical methods are widely applied in order to discover more new solutions, and a lot of important solutions have been presented in the literature (see, for example, [6]-[9]).

Much attention have been paid also to the solutions to the systems of the Yang-Mills fields coupled to external sources. Notable are the charge screening solutions and the magnetic dipole solutions by Sikivie , Weiss [10], the type-I and type-II solutions by Jackiw, Jacobs, Rebbi [11]. The solutions carry an gauge-invariant topological index, identified with the magnetic charge. The solutions to the Yang-Mills fields with axial-symmetric external sources and with higher topological indices were constructed in Refs. [12],[13].

The problem of the Yang-Mills field equations with two color point charges and higher topological indices ($n \geq 2$) has been investigated in Ref. [14], where the solutions to the field equations were found numerically by the relaxation method [15]. In the present paper, some physical characteristics of the solutions obtained in [14] are studied in more details. Namely, the space distributions of the non-Abelian "electric", "magnetic" field and field energy density are simulated and plotted for the cases $n = 2$ and $n = 3$. The results show explicitly the dependence of the field and energy on the topological index (Sec. II). Furthermore, by the same algorithm for solving, as in Ref. [14], new solutions for the case of a series of color point charges located in a straight line are exhibited (Sec. III). In Section IV, Conclusion, a discussion is given about the obtained results.

II. PROPERTIES OF THE YANG-MILLS FIELDS WITH TWO POINT COLOR CHARGE SOURCES

The $SU(2)$ Yang-Mills equations in the presence of an external static source is

$$D_\mu F^{\mu\nu a} = j^{\nu a}, \tag{1}$$

where $j^{\nu a}$ is the non-Abelian color charge external static source current, and

$$F^{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2}$$

is the non-Abelian field strength tensor of electrodynamics. In Eq. (2) g is the gauge field coupling constant. Here we shall set $g = 1$, and the restoration can be done by a rescaling with the coefficient $1/g$ for potentials and field strengths and $1/g^2$ for Lagrangian and energy.

For the external sources with two point color charges the system possesses axial symmetry. By applying the axial-symmetric ansatz for the Yang-Mills potentials in Refs. [12], [13], the Lagrangian functional for the Yang-Mills equations with two point color charges is reduced to the following form in cylindrical coordinates [14]

$$L = \pi \int_0^\infty \rho d\rho \int_{-\infty}^{+\infty} dz \times$$

$$\left\{ \left[\left(\frac{\Phi_n}{\partial \rho} \right)^2 + \left(\frac{\Phi_n}{\partial z} \right)^2 \right] - \left[\left(\frac{A_n}{\partial \rho} \right)^2 + \left(\frac{A_n}{\partial z} \right)^2 \right] + \left[\Phi_n^2 \left(A_n - \frac{n}{\rho} \right)^2 - \left(\frac{A_n}{\rho} \right)^2 \right] \right\}$$

$$-Q\Phi_n(\rho = 0, z = a) + Q\Phi_n(\rho = 0, z = -a), \tag{3}$$

In Eqs. (3) $(\rho = 0, z = \pm a)$ are the coordinates of two point sources, $A_n(\rho, z)$, $\Phi_n(\rho, z)$ are the field profiles, n is the topological index of the solution, which defines which homotopy class the solution belongs to. The numerical solutions are equilibrium gauge configurations for Lagrangian (3) and can be found by the relaxation method [15]. The schema of discretization and minimization for Lagrangian (3), as well as the results for the case $n = 2$ were presented in Ref. [14]. Here we find the solutions for the case of $n = 3, 4$, and then calculate and plot explicitly the corresponding electric, magnetic non-Abelian field strengths and the energy density.

The expression for the non-Abelian electric field vector is

$$E^{ai}_n = F^{a0i} = -\left[\delta^{a1} cos(n\varphi) + \delta^{a2} sin(n\varphi) \right] \partial^i \Phi_n$$

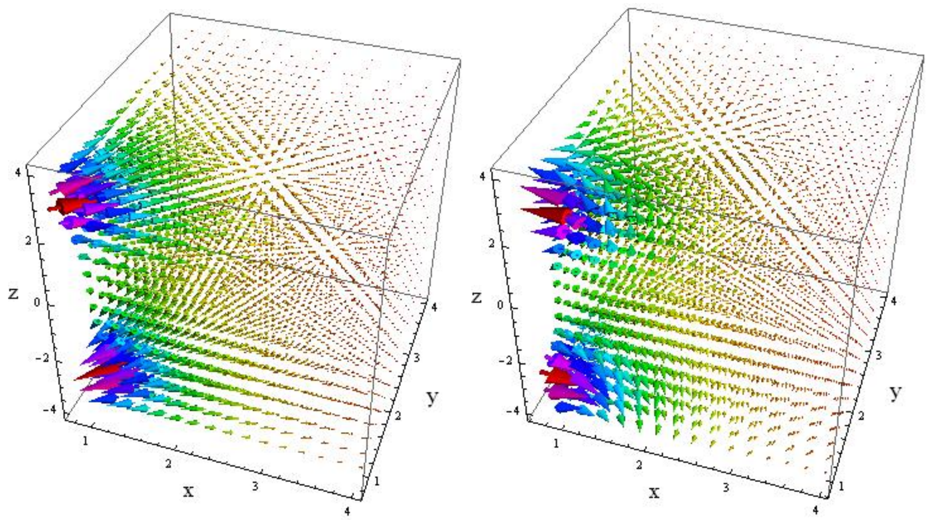

Fig. 1. The 3D-plot of the non-Abelian electric field vector $\vec{E}^1_n$ for $n = 2$ (left) and for $n = 3$ (right)

$$-\left[\delta^{a2} cos(n\varphi) - \delta^{a1} sin(n\varphi) e^{\phi} \Phi_n\right]\left[A_n - \frac{n}{\rho}\right]. \quad (4)$$

Substituting the obtained numerical solutions for the field profiles $A_n(\rho, z)$,

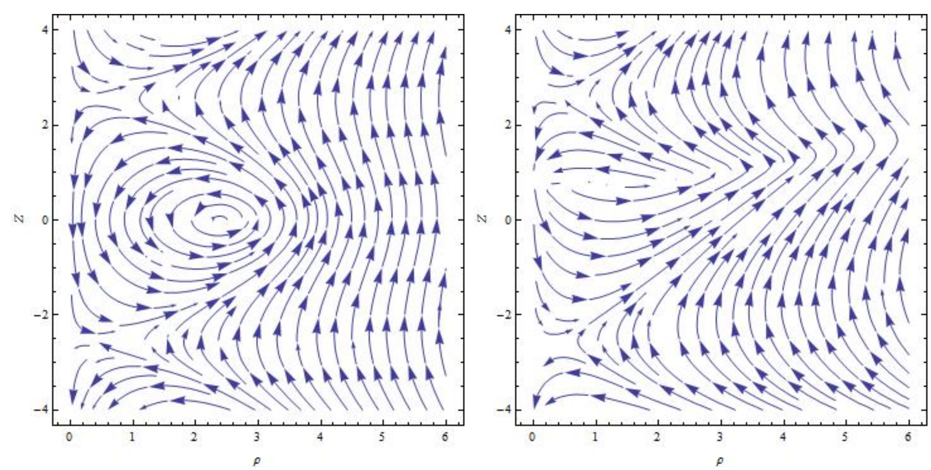

Fig. 2. The field lines of the non-Abelian magnetic field vector component $B^3_n(\rho, z)$ for $n = 2$ (left) and for $n = 3$ (right)

$\Phi_n(\rho, z)$, one has the numerical values for E^{ai}_n. The 3D-plots of the electric field vector (transformed to Cartesian coordinates) for the component with $a = 1$, and two different indices $n = 2$ and $n = 3$ are given in Fig. 1, for illustration. Hereafter the numerical value of the charge parameter is taken $Q = 3$.

The non-Abelian magnetic field is given by

$$B_n^{ai} = \frac{1}{2}\varepsilon^{ijk}F_{jk}^a = \delta_3^a\varepsilon^{ijk}\partial_j\left(A_n\hat{\varphi}_k\right), \tag{5}$$

where $\hat{\varphi}$ is the unit vector of coordinate φ. Note that the magnetic field vectors lie in the (ρ, z)-plane, and it is better to observe the magnetic field lines in this plane (see Fig. 2).

The plots of the field energy density

$$H = 2\pi\rho\left[\frac{1}{2}\left(\nabla\Phi_n\right)^2 + \Phi_n^2\right]\left[A_n - \frac{n}{\rho}\right]^2, \tag{6}$$

for the cases $n = 2$ and $n = 3$ are presented in Fig. 3.

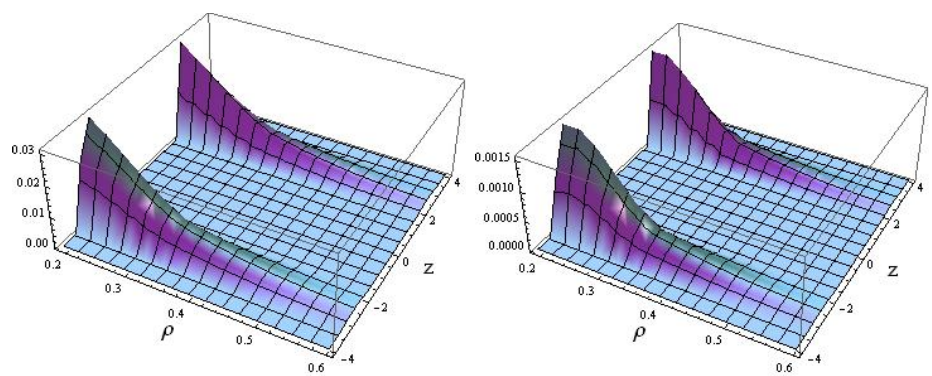

Fig. 3. The distributions of non-abelian fields energy density $H_n(\rho, z)$ in plane $(\rho,\, z)$, for $n = 2$ (left) and for $n = 3$ (right)

Figures 1, 2, 3 give a visual concept about the Yang-Mills field solutions, and their dependence on topological index n. One can see that when the picture of non-Abelian electric and magnetic field vectors changes slowly as increasing n, the fields, and in particular, the field energy density have a remarkable change in magnitude. For example, the magnitude of the energy maximum in the case of $n = 3$ is smaller than that for $n = 2$ in about 20 times.

III. YANG-MILLS FIELD SOLUTIONS WITH SERIES OF POINT COLOR CHARGES LOCATED IN A STRAIGHT LINE

By using the same algorithm of solving, the Yang-Mills field solutions for the case of the sources which have the form of a series of point color charge are obtained. We have changed the last two term in Lagrangian functional (3) into

$$-\sum_n Q\Phi_n(\rho = 0, z = na), n = 0, \pm 1, \pm 2, ... \tag{7}$$

and performed the calculation in FOTRAN programming. The obtained results for the gauge potentials and the corresponding energy density become z-independent. The plots of the profile functions for gauge potentials Φ_n and A_n, as functions of only coordinate ρ, are given in Figures 4, where parameter Q is taken, as before, to be $Q = 3$. The distributions of non-abelian fields energy density $H_n(\rho, z)$ in plane $(\rho,\, z)$ are shown in Fig. 5.

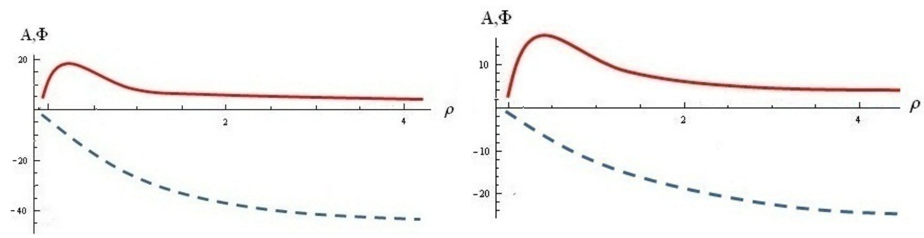

Fig. 4. The profile functions $\Phi_n(\rho)$ - red solid curve, $A_n(\rho)$ - dashed curve for $n = 2$ (left) and for $n = 3$ (right)

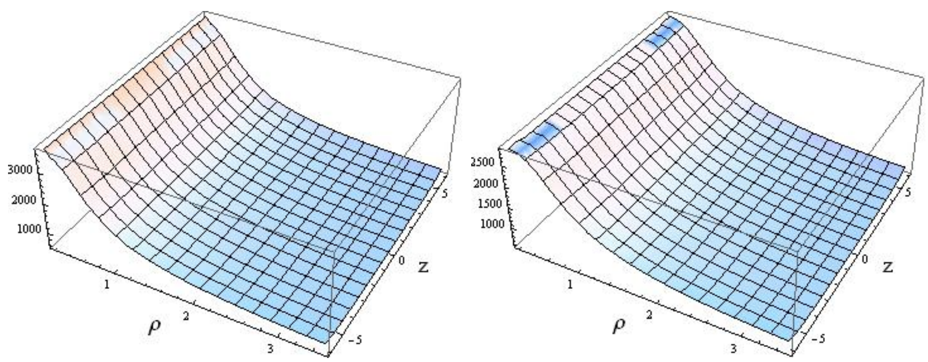

Fig. 5. The distributions of the non-abelian fields energy density $H_n(\rho, z)$ $H_n(\rho, z)$ in plane $(\rho,\, z)$, for $n = 2$ (left) and for $n = 3$ (right)

It is seen that the solutions for the gauge potentials and the energy density have the form of a vortex. The results of calculation for different values of parameter Q and of index n show that the global extrema of the gauge potentials and the energy density increase fast with Q and decrease with increasing n.

IV. CONCLUSION

We have investigated static solutions of the $SU(2)$ Yang-Mills theory with external sources which have the form of two point color charges and then a series of point color charges. The numerical calculations are obtained by solving the system of two coupled non-linear partial differential equations for profile functions of the Yang-Mills potentials. For the case of two point color sources, the dependence of non-Abelian gauge field strengths and energy density on topological indices and the parameter Q of color point charges have been discussed. Some main conclusions are as follows: (i) The space distribution of of the non-Abelian gauge electric and magnetic changes when topological index n varies. (ii) The magnitude of maxima of field strengths and energy increases with varying parameter Q fast, and decrease with increasing topological index n.

By numerical solutions for the Yang-Mills solutions in the case of external sources that have the form of a series of point color charges, the results show the vortex form of Yang-Mills potentials and energy density. These solutions provide an example of vortex string configuration, the extended object which plays an important role in particle and cosmology (see, for example,[16]).

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