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ON A PHASE TRANSITION OF BOSE GAS

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Abstract. *The Cornwall-Jackiw-Tomboulis (CJT) effective action at finite temperature is applied to study the phase transition in Bose gas. The effective potential, which preserves the Goldstone theorem, is found in the Hartree-Fock (HF) approximation. This quantity is then used to consider the equation of state (EOS) and phase transition of the system.*

I. INTRODUCTION

Nowadays, the research of phase transition has become one of the most topical fields in both theoretics and experiment since it is closely related to quantum field theory, fundamental particle physics, condensed matter physics, and cosmology. However, around the critical points of phase transition, many properties of physical systems have an anomalous alteration, that is difficult for observation in perturbation series. Accordingly, interest in finding and developing an adequate formalism, which provides a reliable description of critical phenomena have been growing in several recent years. As was pointed out in [1], the CJT effective action is most suited for this purpose.

In this paper, basing on the CJT effective action approach, we reconsider the phase transition at high temperature of Bose Gas. The paper is organized as follows. In section II, the CJT effective action at finite temperature is calculated and renormalized. Section III is devoted to determining several important physical properties of system. The conclusion and discussion are given in section IV.

II. EFFECTIVE POTENTIAL IN HF APPROXIMATION

Let us begin with the Bose gas given by the Lagrangian

$$\mathcal{L} = \phi^* \left(-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} \right) \phi - \mu\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \tag{1}$$

where μ represents the chemical potential of the field ϕ, m the mass of ϕ atom, and λ the coupling constant. In the tree approximation the condensate density ϕ_0^2 corresponds to local minimum of the potential. It fulfills

$$-\mu\phi_0 + \frac{\lambda}{2}\phi_0^3 = 0, \tag{2}$$

yielding (for $\phi \neq 0$)

$$\frac{\phi_0^2}{2} = \frac{\mu}{\lambda}. \tag{3}$$

Now let us focus on the calculation of effective potential in HF approximation. At first the field operator ϕ is decomposed

$$\phi = \frac{1}{\sqrt{2}}(\phi_0 + \phi_1 + i\phi_2). \tag{4}$$

Inserting (4) into (1) we get, among others, the interaction Lagrangian

$$\mathcal{L}_{int} = \frac{\lambda}{2}\phi_0\phi_1(\phi_1^2 + \phi_2^2) + \frac{\lambda}{8}(\phi_1^2 + \phi_2^2)^2,$$

and the inverse propagator in the tree approximation

$$D_0^{-1}(k) = \begin{pmatrix} \frac{\vec{k}^2}{2m} - \mu + \frac{3\lambda}{2}\phi_0^2 & -\omega \\ \omega & \frac{\vec{k}^2}{2m} - \mu + \frac{\lambda}{2}\phi_0^2 \end{pmatrix}. \tag{5}$$

From (3) and (5) it follows that

$$E = +\sqrt{\left(\frac{\vec{k}^2}{2m} + \lambda\phi_0^2\right)\frac{\vec{k}^2}{2m}}\ , \tag{6}$$

which is the Bogoliubov dispersion relation for Bose gas in the broken phase.
For small momenta equation (6) reduces to

$$E \approx +k\sqrt{\frac{\lambda\phi_0^2}{2m}}\ , \tag{7}$$

associating with Goldstone boson due to $U(1)$ breaking.

Next the CJT effective potential is calculated in the HF approximation [2]. The propagator is expressed in the form [3],

$$D^{-1} = \begin{pmatrix} \frac{\vec{k}^2}{2m} + M_1 & -\omega \\ \omega & \frac{\vec{k}^2}{2m} + M_2 \end{pmatrix}.$$

Following closely [4] we arrive at the CJT effective potential $V_\beta^{CJT}(\phi_0, D)$ at finite temperature in the HF approximation

$$\begin{aligned} V_\beta^{CJT}(\phi_0, D) = &-\frac{\mu}{2}\phi_0^2 + \frac{\lambda}{8}\phi_0^4 + \frac{1}{2}\int_\beta tr\Big\{\ln D^{-1}(k) + D_0^{-1}(k;\phi_0)D - 1\!\!1\Big\} \\ &+\frac{3\lambda}{8}\Big[\int_\beta D_{11}(k)\Big]^2 + \frac{3\lambda}{8}\Big[\int_\beta D_{22}(k)\Big]^2 + \frac{\lambda}{4}\Big[\int_\beta D_{11}(k)\Big]\Big[\int_\beta D_{22}(k)\Big]. \end{aligned} \tag{8}$$

Here

$$\int_\beta f(k) = T\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}f(\omega_n,\vec{k}).$$

Starting from (8) we obtain, respectively,

a - The gap equation

$$\mu - \frac{\lambda}{2}\phi_0^2 - \Sigma_1 = 0. \tag{9}$$

b- The Schwinger-Dyson (SD) equation

$$D^{-1} = D_0^{-1}(k;\phi_0) + \Sigma, \tag{10}$$

where

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}, \tag{11}$$

and

$$\Sigma_1 = \frac{3\lambda}{2}\int_\beta D_{11}(k) + \frac{\lambda}{2}\int_\beta D_{22}(k), \quad \Sigma_2 = \frac{\lambda}{2}\int_\beta D_{11}(k) + \frac{3\lambda}{2}\int_\beta D_{22}(k)$$

$$M_1 = -\mu + \frac{3\lambda}{2}\phi_0^2 + \Sigma_1, \quad M_2 = -\mu + \frac{\lambda}{2}\phi_0^2 + \Sigma_2. \tag{12}$$

The explicit form for propagator comes out from combining (9) and (10),

$$D^{-1} = \begin{pmatrix} \frac{\vec{k}^2}{2m} - \mu + \frac{3\lambda}{2}\phi_0^2 + \Sigma_1 & -\omega \\ \omega & \frac{\vec{k}^2}{2m} - \mu + \frac{\lambda}{2}\phi_0^2 + \Sigma_2 \end{pmatrix}, \tag{13}$$

which clearly show that the Goldstone theorem fails in the HF approximation. In order to restore it, we use the method developed in [5], adding a correction ΔV to V_β^{CJT}

$$\tilde{V}_\beta^{CJT} = V_\beta^{CJT} + \Delta V_\beta^{CJT}, \tag{14}$$

with

$$\Delta V_\beta^{CJT} = \frac{x\lambda}{2}[P_{11}^2 + P_{22}^2 - 2P_{11}P_{22}]$$

$$P_{aa} = \int_\beta D_{aa}, \quad (a = 1 \text{ or } 2). \tag{15}$$

It is easily checked that choosing $x = -1/2$ we are led to effective potential $\tilde{V}_\beta^{CJT}$

$$\begin{aligned}\tilde{V}_\beta^{CJT}(\phi_0, D) &= -\frac{\mu}{2}\phi_0^2 + \frac{\lambda}{8}\phi_0^4 + \frac{1}{2}\int_\beta tr\Big\{\ln D^{-1}(k) + [D_0^{-1}(k;\phi_0)D] - 1\!\!1\Big\} \\ &+ \frac{\lambda}{8}P_{11}^2 + \frac{\lambda}{8}P_{22}^2 + \frac{3\lambda}{4}P_{11}P_{22},\end{aligned} \tag{16}$$

which obeys three requirements imposed in [5]: (i) it restores the Goldstone theorem in the broken symmetry phase, (ii) it does not change the HF equations for the mean fields, and (iii) it does not change results in the phase of restored symmetry.

From (16), instead of (9), (12) and (13), we get:

a- The gap equation

$$-\mu + \frac{\lambda}{2}\phi_0^2 + \Sigma_2^* = 0. \tag{17}$$

At critical temperature we have $\phi_0 = 0$, and Eq. (17) give $\mu = \Sigma_2^*$, which manifest exactly the Hugenholz - Pines theorem [6].

b- The SD equation

$$D^{-1} = D_0^{-1}(k;\phi_0) + \Sigma^*, \tag{18}$$

in which

$$\Sigma^* = \begin{pmatrix} \Sigma_1^* & 0 \\ 0 & \Sigma_2^* \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{2}P_{11} + \frac{3\lambda}{2}P_{22} & 0 \\ 0 & \frac{3\lambda}{2}P_{11} + \frac{\lambda}{2}P_{22} \end{pmatrix}$$

Combining (17) and (18) we get the form for inverse propagator

$$D^{-1} = \begin{pmatrix} \frac{\vec{k}^2}{2m} + M_1^* & -\omega \\ \omega & \frac{\vec{k}^2}{2m} + M_2^*, \end{pmatrix}.$$

in which

$$M_1^* = -\mu + \frac{3\lambda}{2}\phi_0^2 + \Sigma_1^*, \quad M_2^* = -\mu + \frac{\lambda}{2}\phi_0^2 + \Sigma_2^*. \tag{19}$$

Owing to (17) M_2^* vanishes in broken phase and

$$D^{-1} = \begin{pmatrix} \frac{\vec{k}^2}{2m} + M_1^* & -\omega \\ \omega & \frac{\vec{k}^2}{2m} \end{pmatrix}. \tag{20}$$

It is obvious that the dispersion relation related to (20) reads

$$E = \sqrt{\frac{\vec{k}^2}{2m}\left(\frac{\vec{k}^2}{2m} + M_1^*\right)} \longrightarrow \sqrt{\frac{M_1^*}{2m}}\ k \quad \text{as} \quad k \to 0,$$

which express the Goldstone theorem. Due to the Landau criteria for superfluidity [7] the idealized Bose gas turns out to be superfluid in broken phase and speed of sound in condensate is given by

$$C = \sqrt{\frac{M_1^*}{2m}}.$$

Ultimately the one-particle-irreducible effective potential $\tilde{V}_\beta(\phi_0)$ is read off from (16) with D fulfilling (18),

$$\begin{aligned}\tilde{V}_\beta(\phi_0) &= -\frac{\mu}{2}\phi_0^2+\frac{\lambda}{8}\phi_0^4+\frac{1}{2}\int_\beta tr\ln D^{-1}(k)+\frac{1}{2}\Big(-M_1^*-\mu+\frac{3\lambda}{2}\phi_0^2\Big)P_{11}\\ &+ \frac{1}{2}\Big(-\mu+\frac{\lambda}{2}\phi_0^2\Big)P_{22}+\frac{\lambda}{8}P_{11}^2+\frac{\lambda}{8}P_{22}^2+\frac{3\lambda}{4}P_{11}P_{22}. \qquad (21)\end{aligned}$$

Since $\tilde{V}_\beta^{CJT}(\phi_0, D)$ and $\tilde{V}_\beta(\phi_0)$ contain divergent integrals, corresponding to zero temperature contributions, we must proceed to the regularization. To this end, we make use of the dimensional regularization by performing momentum integration in $d = 3-\epsilon$ dimensions and then taking $\epsilon \to 0$. The regularized integrals then turn out to be finite [8]. We therefore find the effective potentials consisting of only finite terms.

III. PHYSICAL PROPERTIES

III.1. Equations of state

Let us now consider EOS starting from the effective potential. To this end, we begin with the pressure defined by

$$P = -\,\tilde{V}_\beta^{CJT}(\phi_0, D)\Big|_{\text{at minimum}}, \qquad (22)$$

from which the total particle density is determined

$$\rho = \frac{\partial P}{\partial \mu}.$$

Taking into account the fact that derivative of $\tilde{V}_\beta^{CJT}(\phi_0, D)$ with respect to its argument vanishes at minimum we get

$$\rho = -\frac{\partial V_\beta^{CJT}}{\partial \mu} = \frac{\phi_0^2}{2}+\frac{P_{11}}{2}+\frac{P_{22}}{2}. \qquad (23)$$

Hence, the gap equation (17) becomes

$$\mu = \lambda\rho + \lambda P_{11}, \qquad (24)$$

Combining Eqs. (19), (22) and (23) together produces the following expression for the pressure

$$P = \frac{\lambda}{2}\,\rho^2 - \frac{1}{2}\int_\beta tr\ln D^{-1}k) - \frac{\lambda}{2}P_{11}^2 + \lambda\,\rho\,P_{11}. \qquad (25)$$

The free energy follows from the Legendre transform

$$E = \mu\rho - P,$$

and reads

$$E = \frac{\lambda}{2}\rho^2 + \frac{1}{2}\int_\beta tr\ln D^{-1}(k) + \frac{\lambda}{2}P_{11}^2. \qquad (26)$$

Eqs. (25) and (26) constitute the EOS governing all thermodynamical processes, in particular, phase transitions of the system.

To proceed further it is interesting to consider the high temperature regime, $T/\mu \gg 1$. Introducing the effective chemical potential

$$\overline{\mu} = \mu - \Sigma_2^*,$$

the gap equation (17) can be rewritten as

$$\frac{\lambda}{2}\phi_0^2 = \overline{\mu}_1,$$

which yield

$$\frac{\phi_0^2}{2} = \frac{\overline{\mu}}{\lambda}. \tag{27}$$

Eq. (27) resemble (3) with μ replaced by $\overline{\mu}$.
It is evident that the symmetry breaking at $T = 0$ is restored at $T = T_c$ if

$$\phi_0^2 = 0.$$

Using the high temperature expansions of all integrals appearing in V_β and related quantities, we find the critical temperature T_c

$$T_c = 2\pi\left[\frac{\mu}{2m^{3/2}\lambda\zeta(3/2)}\right]^{2/3}. \tag{28}$$

and the pressure to first order in λ for temperature just below the critical temperature

$$P = \frac{\lambda}{2}\rho^2 + \frac{m^{3/2}\zeta(5/2)}{2\sqrt{2}\pi^{3/2}}T^{5/2} + \frac{m^3\lambda[\zeta(3/2)]^2}{16\pi^3}T^3,$$

which is the well-known result of Lee and Yang for Bose gas [9] without invoking the double counting subtraction as was done in Ref. [10].
Based on the formula

$$E = -\frac{\partial}{\partial\beta}[\beta P(\mu)]_\mu, \quad \beta = 1/T,$$

the high temperature behaviour of the free energy density is also derived in the same approximation

$$E = -\frac{1}{2}\lambda\rho^2 - \frac{3m^{3/2}\lambda\rho\zeta(3/2)}{4\sqrt{2}\pi^{3/2}}T^{3/2} + \frac{3m^{3/2}\zeta(5/2)}{4\sqrt{2}\pi^{3/2}}T^{5/2} + \frac{m^3\lambda[\zeta(3/2)]^2}{8\pi^3}T^3.$$

Next the low temperature regime, $T/\mu \ll 1$, is concerned. Basing on the low temperature expansions of all quantities we are able to write the low temperature behaviour of the equations for M_1^* as follows

$$M_1^* = 2\lambda\rho - \frac{2\sqrt{2}M_1^{*3/2}m^{3/2}\lambda}{3\pi^2} - \frac{2\sqrt{2}m^3\lambda\pi^2}{15M_1^{*5/2}}T^4$$

which require a self-consistent solution for M_1^* as function of density and temperature. The first approximation we can choose is

$$M_1^* \simeq 2\lambda\rho. \tag{29}$$

and we arrive at the low temperature dependence of chemical potential

$$\mu = \lambda\rho + \frac{4m^{3/2}\lambda^{5/2}\rho^{3/2}}{3\pi^2} + \frac{m^{3/2}\pi^2}{60\lambda^{3/2}\rho^{5/2}}T^4,$$

and pressure

$$P = \frac{\lambda\rho^2}{2} + \frac{4m^{3/2}\lambda^{5/2}\rho^{5/2}}{5\pi^2} + \frac{\pi^2 m^{3/2}T^4}{36\lambda^{3/2}\rho^{3/2}} - \frac{m^3T^4}{45\rho} - \frac{8m^3\lambda^4\rho^3}{9\pi^2} - \frac{\pi^4 m^3 T^8}{7200\lambda^4\rho^5}. \tag{30}$$

It is worth to mention that Eq. (30) does not coincide with [10] because several T-dependent terms were missed in that work. Accordingly we get the equation for free energy

$$E = \mu\rho - P = \frac{\lambda\rho^2}{2} + \frac{8m^{3/2}\lambda^{5/2}\rho^{5/2}}{15\pi^2} - \frac{\pi^2 m^{3/2}T^4}{90\lambda^{3/2}\rho^{3/2}} + \frac{m^3T^4}{45\rho} + \frac{8m^3\lambda^4\rho^3}{9\pi^4} + \frac{\pi^2 m^3 T^8}{7200\lambda^4\rho^5}.$$

III.2. Numerical study

In order to get some insight to the phase transition of the Bose gas, let us choose the model parameters, which are close to the experimental settings, namely

$$\lambda = 10^{-11}eV^{-2}, \quad \mu = 10^{-11}\ eV, \quad = 80\ GeV.$$

Solving self-consistently the gap and the SD equations (17), (18) and (19) we obtain the T dependence of M_1^* given in Fig. 1 and ϕ_0 shown in Fig. 2. As is seen from these figures the symmetry restoration takes place at $T_c \simeq 300$ nK and phase transition is second order. This statement is confirmed again in Fig. 3, providing the evolution of $V_\beta[\phi_0, T]$ with respect to ϕ_0.

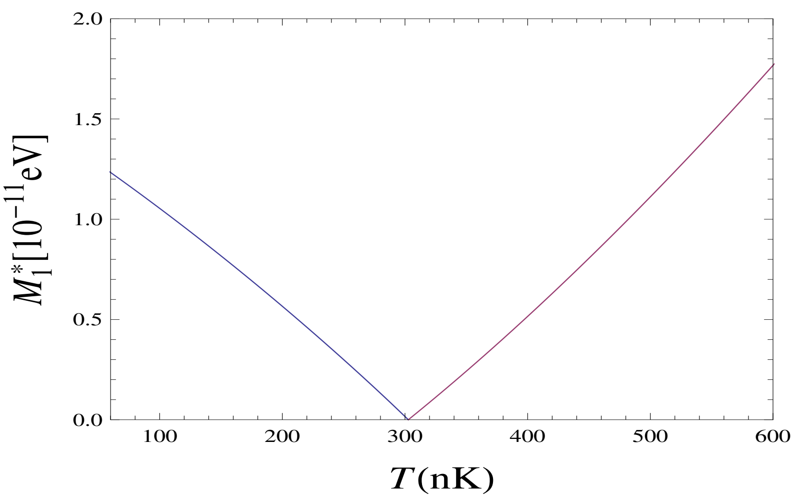

Fig. 1. The T dependence of M_1^*.

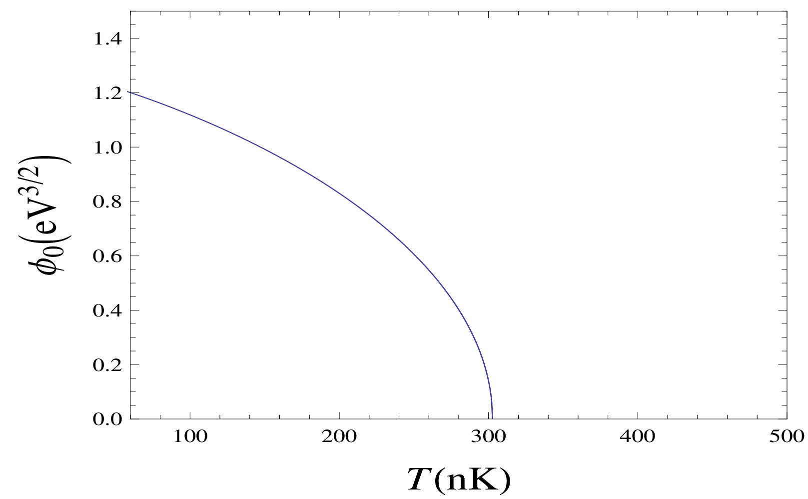

Fig. 2. The T dependence of ϕ_0^*.

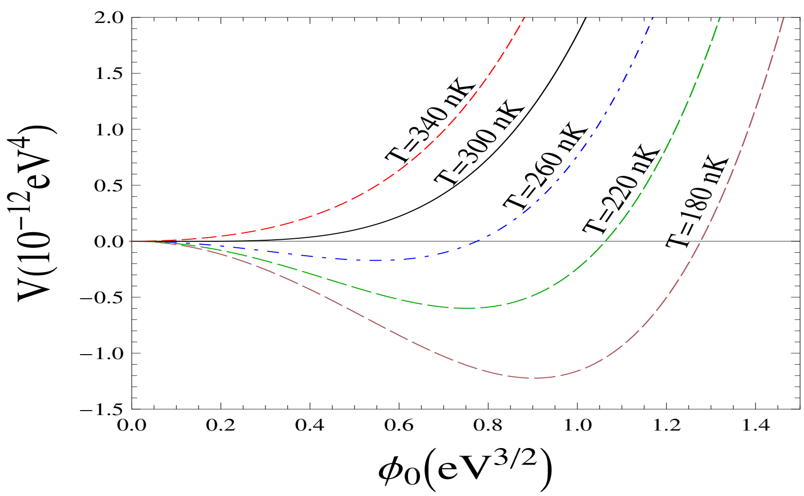

Fig. 3. The ϕ_0 dependence of $V_\beta[\phi_0, T]$ at several values of T around T_c.

IV. CONCLUSION AND OUTLOOK

Due to growing interest of phase transition we considered a non-relativistic model of idealized Bose gas. We have obtained the effective potential in the HF approximation, which is renormalized and respects Goldstone theorem.The expression for pressure, which depends on particles densities, was derived together with the free energy. The EOS 's at low and high temperatures were considered in detail, giving rise to the well-known formula of Lee and Yang and other results for single Bose gas. It was indicated that the symmetry restoration takes place at $T_c \simeq 300$ nK and phase transition is second order.

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