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ENTANGLEMENT IN FERMI GAS AND BCS SYSTEMS

PHAN MINH TRUONG, LE THI TRUC PHUONG, HO TRUNG DUNG
Ho Chi Minh city Institute of Physics, 1 Mac Dinh Chi Str., District 1, Ho Chi Minh city

Abstract. *Besides being of fundamental interest, entanglement is an important resource in quantum communication and information processing. The properties of the entanglement in many-body systems have attracted much attention recently. Using separability conditions in the form of inequalities derived by Hillery and coworkers* [1, 2]*, we explore the bipartite entanglement in noninteracting Fermi gas and Bardeen-Cooper-Schrieffer (BCS) superconducting systems at zero temperature. It is shown that these inequalities can detect entanglement between two particles within an electron pair in the ground state of a BCS system. In the case of Fermi gases, entanglement may be found below some upper bound on the interparticle distance.*

I. INTRODUCTION

One of the most striking aspects of quantum mechanics is the superposition principle which has no counterpart in classical physics. A direct consequence of this principle is the entanglement of physical states, which has proven to be a valuable resource in quantum information processing. Detection, classification, and quantification of entanglement have been subjects of intense research in the recent years. However, determining whether or not a state is entangled remains a formidable task. Methods such as the Peres-Horodecki positive partial transpose condition, entanglement witnesses, and hierarchies of entanglement conditions exist (see, for example, [3] for a review), but are not always straightforward to apply. Quantum entanglement may lead to further insight into the physics of many-body systems [3]. Tripartite entanglement of a noninteracting Fermi gas has been investigated using parameterized entanglement witnesses [4] and many-body entanglement in one-dimensional noninteracting ultracold atomic Fermi gases has been explored using entanglement entropy [5]. In the BCS system, bipartite entanglement between a pair of modes with two opposite wave vectors $\pm\mathbf{k}$ has been studied using the concurrence [6] and the partial transpose condition [7].

In recent work [1, 2], Hillery and coworkers have proposed two inequalities which can be used to detect the presence of entanglement in a bipartite system

$$\left|\langle A^\dagger B\rangle\right|^2 > \langle A^\dagger A B^\dagger B\rangle, \tag{1}$$

$$\left|\langle AB\rangle\right|^2 > \langle A^\dagger A\rangle\langle B^\dagger B\rangle, \tag{2}$$

where $A(B)$ is an operator on the Hilbert space of the first(second) subsystem. These (sufficient) conditions have been applied to spin systems in [8] and extended to multipartite systems in [9]. It has been shown in particular that the multipartite version can detect entanglement in generalized Greenberger-Horne-Zeilinger states while the existing spin squeezing inequalities cannot [9].

In this communication, we report some preliminary results of our investigation on the bipartite entanglement in the noninteracting Fermi gas and the BCS superconducting system at zero temperature. The task is to find suitable operators A and B such that the conditions (1) and (2) are satisfied. For Fermi gases, we establish an upper bound for the interparticle distances within which entanglement may be detected by (1). For the BCS system, we show that the inequalities (1) and (2) provide a means of entanglement detection more simple and straightforward than the entanglement witnesses and entropy used in [4, 5].

II. ENTANGLEMENT IN THE FERMI GAS

Consider a system of N noninteracting spin-1/2 fermions in a box of volume Ω. By the Pauli exclusion principle, at most two particles, one with spin up and one with spin down, can occupy the same momentum state $\mathbf{k}$. In the ground state at absolute zero temperature, the energy levels are filled from the bottom up until all N particles are accommodated. The occupied orbitals may be represented as points inside a sphere in $\mathbf{k}$ space. The energy (momentum) at the surface of the sphere is the Fermi energy ε_F (Fermi momentum $\mathbf{k}_F$). The ground state of the system can be written as [10]

$$\begin{aligned} |\Phi_0\rangle &= c_1^\dagger c_2^\dagger c_3^\dagger ... c_{\mathbf{k}}^\dagger ... c_{\mathbf{k}_F}^\dagger |\Phi_{\text{vac}}\rangle \\ &= \left|1_1 1_2 1_3 ... 1_N 0_{N+1} ... 0 ...\right\rangle \end{aligned} \tag{3}$$

where $|\Phi_{\text{vac}}\rangle$ is the vacuum state in which no particles are present, $c^\dagger$ and c are fermion operators, and the states are numbered in order of increasing energy. The annihilation of a particle at position $\mathbf{x}$ is represented by the field operator

$$\Psi(\mathbf{x}) = \sum_i c_i \varphi_i(\mathbf{x}), \qquad \varphi_i(\mathbf{x}) = \frac{1}{\sqrt{\Omega}} e^{-i\mathbf{k}_i \mathbf{x}}, \tag{4}$$

where $\varphi_i(\mathbf{x})$ is a one-particle eigenfunction. We shall explore the entanglement between two particles at different positions in the state (3) by using the field operators (4).

Let us begin with the inequality (1). A possible choice of A and B is

$$A = \Psi(\mathbf{x}) = \sum_i c_i \varphi_i(\mathbf{x}), \quad B = \Psi(\mathbf{x}') = \sum_j c_j \varphi_j(\mathbf{x}'). \tag{5}$$

Then

$$\begin{aligned} \langle\Phi_0|A^\dagger B|\Phi_0\rangle &= \langle\Phi_0| \sum_{i,j} c_i^\dagger c_j \varphi_i^*(\mathbf{x}) \varphi_j(\mathbf{x}') |\Phi_0\rangle \\ &= \sum_{i=1}^{N} \varphi_i^*(\mathbf{x}) \varphi_i(\mathbf{x}') \\ &= \frac{1}{\Omega} \sum_{i=1}^{N} e^{i\mathbf{k}_i(\mathbf{x}-\mathbf{x}')}. \end{aligned} \tag{6}$$

To calculate the sum in Eq. (6), we convert it into an integral

$$\sum_{\mathbf{k}} \to \frac{\Omega}{(2\pi)^3} \int d^3k. \tag{7}$$

Going over to the spherical coordinate system and fixing the z-axis along the line connecting the two positions $(\mathbf{x} - \mathbf{x}')$, we obtain

$$\begin{aligned} \langle \Phi_0 | A^\dagger B | \Phi_0 \rangle &= \frac{1}{(2\pi)^2} \int d\theta \sin\theta \int_0^{k_F} dk k^2 \exp[-ik|\mathbf{x} - \mathbf{x}'| \cos\theta] \\ &= -\frac{k_F^3}{2\pi^2} \frac{1}{(\Delta x)^2} \left\{ \cos\Delta x - \frac{1}{\Delta x} \sin\Delta x \right\}, \end{aligned} \tag{8}$$

where the dimensionless distance $\Delta x = k_F |\mathbf{x} - \mathbf{x}'|$ has been introduced. Note that the upper limit of the k-integral is k_F because the system is in the ground state. Next we calculate the right hand side of (1)

$$\begin{aligned} &\langle \Phi_0 | A^\dagger A B^\dagger B | \Phi_0 \rangle = \langle \Phi_0 | \sum_{i,j,i'j'} c_i^\dagger c_j c_{i'}^\dagger c_{j'} \varphi_i^*(\mathbf{x}) \varphi_j(\mathbf{x}) \varphi_{i'}^*(\mathbf{x}') \varphi_{j'}(\mathbf{x}') | \Phi_0 \rangle \\ &= \langle \Phi_0 | \sum_{i,i'} c_i^\dagger c_i c_{i'}^\dagger c_{i'} \varphi_i^*(\mathbf{x}) \varphi_i(\mathbf{x}) \varphi_{i'}^*(\mathbf{x}') \varphi_{i'}(\mathbf{x}') | \Phi_0 \rangle \\ &\quad + \langle \Phi_0 | \sum_{i,i'} c_i^\dagger c_{i'} c_{i'}^\dagger c_i \varphi_i^*(\mathbf{x}) \varphi_{i'}(\mathbf{x}) \varphi_{i'}^*(\mathbf{x}') \varphi_i(\mathbf{x}') | \Phi_0 \rangle \\ &= \sum_{i,i'=1}^{N} \varphi_i^*(\mathbf{x}) \varphi_i(\mathbf{x}) \varphi_{i'}^*(\mathbf{x}') \varphi_{i'}(\mathbf{x}') + \sum_{i=1}^{N} \sum_{i'=N+1}^{\infty} \varphi_i^*(\mathbf{x}) \varphi_{i'}(\mathbf{x}) \varphi_{i'}^*(\mathbf{x}') \varphi_i(\mathbf{x}') \\ &= \left(\sum_{i=1}^{N} \frac{1}{\Omega} \right)^2 - \left| \sum_{i=1}^{N} \varphi_i^*(\mathbf{x}) \varphi_i(\mathbf{x}') \right|^2, \end{aligned} \tag{9}$$

where going from the third equation to the fourth, we have rewritten the sum over i' in the second term as $\sum_{i'=N+1}^{\infty} \ldots = \sum_{i'=1}^{\infty} \ldots - \sum_{i'=1}^{N} \ldots$ and made use of the relationship $\sum_{i=1}^{\infty} \varphi_i(\mathbf{x}) \varphi_i^*(\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$. The delta function vanishes $\delta(\mathbf{x} - \mathbf{x}') = 0$ because the particles under consideration are located at different positions. Next we convert the sums in Eq. (9) into integrals in accordance with Eq. (7). The first (self-energy) term can be found to be equal to $\left(\frac{k_F^3}{2\pi^2} \frac{1}{3} \right)^2$, whereas the second (correlation) term is nothing else rather than $|\langle \Phi_0 | A^\dagger B | \Phi_0 \rangle|^2$ [cf. Eq. (6)]. Then

$$\langle \Phi_0 | A^\dagger A B^\dagger B | \Phi_0 \rangle = \frac{k_F^6}{4\pi^4} \left\{ \frac{1}{9} - \frac{1}{(\Delta x)^4} \left(\cos\Delta x - \frac{1}{\Delta x} \sin\Delta x \right)^2 \right\}. \tag{10}$$

Combining (1), (8), and (10) yields the following condition on the interparticle distance

$$\frac{1}{(\Delta x)^4} \left(\cos\Delta x - \frac{1}{\Delta x} \sin\Delta x \right)^2 > \frac{1}{18}. \tag{11}$$

The above inequality contains oscillating functions of Δx and it is impossible to extract from it a transparent analytical relationship for Δx. We can, however, examine the limits. For very large distances $\Delta x \to \infty$, the left hand side tends to zero and the inequality cannot be satisfied. That is, no entanglement can be detected for well separated particles. When $\Delta x \to 0$, making Taylor expansions of the trigonometric functions, it can be found that the left hand side tends to 1/9, which is larger than the value of 1/18 on the right hand side. This means there may be a range of moderate distances where the entanglement is detectable. Exact numerical evaluation of (11) (not shown) yields a maximal interparticle distance $\Delta r \lesssim 1.8$ for which an entanglement detection may be possible.

It is noteworthy that the choice of $A = \Psi^\dagger(\mathbf{x}),\ B = \Psi^\dagger(\mathbf{x}')$ gives rise to divergences of the type $\sum_{i=N+1}^{\infty} \frac{1}{\Omega} = \frac{1}{2\pi^2}\int_{k_F}^{\infty} dk k^2$ in the self-energy term. The choices of $A = \Psi^\dagger(\mathbf{x}),\ B = \Psi(\mathbf{x}')$, and $A = \Psi(\mathbf{x}),\ B = \Psi^\dagger(\mathbf{x}')$ are also not interesting because for these $\langle\Phi_0|A^\dagger B|\Phi_0\rangle = 0$. With respect to the second inequality (2), if one picks $A = \Psi(\mathbf{x}),\ B = \Psi(\mathbf{x}')$ or $A = \Psi^\dagger(\mathbf{x}),\ B = \Psi^\dagger(\mathbf{x}')$, the left hand side vanishes $\langle\Phi_0|AB|\Phi_0\rangle = 0$, whereas if one picks $A = \Psi^\dagger(\mathbf{x}),\ B = \Psi(\mathbf{x}')$, or $A = \Psi(\mathbf{x}),\ B = \Psi^\dagger(\mathbf{x}')$, divergences arise.

III. ENTANGLEMENT IN THE BCS SYSTEM

When conduction electrons interact with the lattice vibrations, they are scattered from one state $\mathbf{k}$ to another $\mathbf{k}'$. The coupling between electrons and virtual phonons causes a slight attraction between two electrons, leading to the creation of Cooper pairs. The electron pairs have an energy slightly lower than the Fermi level and leave an energy gap above them. For temperatures such that the thermal energy is less than the band gap, the material exhibits zero resistivity. By eliminating the phonon operators and dropping the repulsive interaction, one obtains the effective electron-electron Hamiltonian [10]

$$H_{\text{red}} = \sum \varepsilon_{\mathbf{k}}(c^\dagger_{\mathbf{k}}c_{\mathbf{k}} + c^\dagger_{-\mathbf{k}}c_{-\mathbf{k}}) - V\sum c^\dagger_{\mathbf{k}'}c^\dagger_{-\mathbf{k}'}c_{-\mathbf{k}}c_{\mathbf{k}}. \quad (12)$$

This so-called BCS reduced Hamiltonian operates within the subspace of Cooper pairs $\mathbf{k} + \mathbf{k}' = 0$, which here are assumed to have antiparallel spins. The ground state of the system is

$$|\Phi_g\rangle = \prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}}c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}})|\Phi_{\text{vac}}\rangle, \quad (13)$$

where for convenience the filled Fermi sea is redefined as the vacuum state $|\Phi_{\text{vac}}\rangle$ and $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ are real constants satisfying $u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1$.

To explore the entanglement between particles in the ground state (13), let us begin with particles within a Cooper pair. Apparent choices of A and B are

$$A = c_{\mathbf{q}}, \quad B = c^\dagger_{-\mathbf{q}} \quad (14)$$

$$A = c^\dagger_{\mathbf{q}}, \quad B = c_{-\mathbf{q}}, \quad (15)$$

$$A = c_{\mathbf{q}}, \quad B = c_{-\mathbf{q}}, \quad (16)$$

$$A = c^\dagger_{\mathbf{q}}, \quad B = c^\dagger_{-\mathbf{q}}. \quad (17)$$

For the first inequality (1), the choice (14) is equivalent to (15) and the choice (16) is equivalent to (17). Consider first (14). Using the state (13) and (14) in (1), we derive for the left hand side of (1)

$$\begin{aligned}\left\langle\Phi_g|A^\dagger B|\Phi_g\right\rangle &= \langle\Phi_g|c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}}|\Phi_g\rangle\\ &= \langle\Phi'_g|(u_{\mathbf{q}}+v_{\mathbf{q}}c_{\mathbf{q}}c_{-\mathbf{q}})c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}}(u_{\mathbf{q}}+v_{\mathbf{q}}c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}})|\Phi'_g\rangle\\ &= \langle\Phi'_g|v_{\mathbf{q}}c_{\mathbf{q}}c_{-\mathbf{q}}c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}}u_{\mathbf{q}}|\Phi'_g\rangle\\ &= v_{\mathbf{q}}u_{\mathbf{q}},\end{aligned}\tag{18}$$

where the notation $|\Phi'_g\rangle=\prod_{\mathbf{k}\neq\mathbf{q}}(u_{\mathbf{k}}+v_{\mathbf{k}}c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}})|\Phi_{\rm vac}\rangle$ has been introduced. Similarly for the right hand side

$$\begin{aligned}\left\langle\Phi_g|A^\dagger AB^\dagger B|\Phi_g\right\rangle &= \langle\Phi'_g|(u_{\mathbf{q}}+v_{\mathbf{q}}c_{\mathbf{q}}c_{-\mathbf{q}})c^\dagger_{\mathbf{q}}c_{\mathbf{q}}c_{-\mathbf{q}}c^\dagger_{-\mathbf{q}}(u_{\mathbf{q}}+v_{\mathbf{q}}c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}})|\Phi'_g\rangle\\ &= 0,\end{aligned}\tag{19}$$

where we have used the Pauli exclusion principle to put $c^\dagger_{-\mathbf{q}}c^\dagger_{-\mathbf{q}}|\Phi'_g\rangle=0$. Eqs. (18) and (19) indicate that the inequality (1) is satisfied, that is, the state is entangled, for arbitrary nonvanishing values of $v_{\mathbf{q}}$ and $u_{\mathbf{q}}$. It is remarkable that the presence of entanglement can be detected after just a few simple steps. For the choice (16) [or equivalently (17)], it is not difficult to verify that

$$\left\langle\Phi_g|c^\dagger_{\mathbf{q}}c_{-\mathbf{q}}|\Phi_g\right\rangle=0.\tag{20}$$

Therefore $\left\langle\Phi_g|A^\dagger B|\Phi_g\right\rangle=0$ and the inequality (1) cannot detect the presence of entanglement.

Next we go over to the inequality (2). It is not satisfied by choosing (14) since $\left\langle\Phi_g|AB|\Phi_g\right\rangle=\left\langle\Phi_g|c_{\mathbf{q}}c^\dagger_{-\mathbf{q}}|\Phi_g\right\rangle=0$ [cf. Eq. (20)]. It is also not satisfied by choosing A and B as in Eq. (15) since $\left\langle\Phi_g|AB|\Phi_g\right\rangle=\left\langle\Phi_g|c^\dagger_{\mathbf{q}}c_{-\mathbf{q}}|\Phi_g\right\rangle=0$. For the choice (16), we have that

$$\begin{aligned}\left\langle\Phi_g|AB|\Phi_g\right\rangle &= \langle\Phi_g|c_{\mathbf{q}}c_{-\mathbf{q}}|\Phi_g\rangle\\ &= v_{\mathbf{q}}u_{\mathbf{q}}\end{aligned}\tag{21}$$

[cf. Eq. (18)], and

$$\begin{aligned}\left\langle\Phi_g|A^\dagger A|\Phi_g\right\rangle &= \langle\Phi_g|c^\dagger_{\mathbf{q}}c_{\mathbf{q}}|\Phi_g\rangle\\ &= \langle\Phi'_g|(u_{\mathbf{q}}+v_{\mathbf{q}}c_{\mathbf{q}}c_{-\mathbf{q}})c^\dagger_{\mathbf{q}}c_{\mathbf{q}}(u_{\mathbf{q}}+v_{\mathbf{q}}c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}})|\Phi'_g\rangle\\ &= \langle\Phi'_g|v_{\mathbf{q}}c_{\mathbf{q}}c_{-\mathbf{q}}c^\dagger_{\mathbf{q}}c_{\mathbf{q}}v_{\mathbf{q}}c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}}|\Phi'_g\rangle\\ &= v^2_{\mathbf{q}},\end{aligned}\tag{22}$$

$$\begin{aligned}\left\langle\Phi_g|B^\dagger B|\Phi_g\right\rangle &= \langle\Phi_g|c^\dagger_{-\mathbf{q}}c_{-\mathbf{q}}|\Phi_g\rangle\\ &= v^2_{\mathbf{q}}.\end{aligned}\tag{23}$$

Then the inequality (2) is satisfied if $u^2_{\mathbf{q}}>v^2_{\mathbf{q}}$. For the choice (17), it can be found that $\left\langle\Phi_g|AB|\Phi_g\right\rangle=\langle\Phi_g|c^\dagger_{\mathbf{q}}c^\dagger_{-\mathbf{q}}|\Phi_g\rangle=v_{\mathbf{q}}u_{\mathbf{q}}$, while $\left\langle\Phi_g|A^\dagger A|\Phi_g\right\rangle=\langle\Phi_g|c_{\mathbf{q}}c^\dagger_{\mathbf{q}}|\Phi_g\rangle=u^2_{\mathbf{q}}$ and $\left\langle\Phi_g|B^\dagger B|\Phi_g\right\rangle=\langle\Phi_g|c_{-\mathbf{q}}c^\dagger_{-\mathbf{q}}|\Phi_g\rangle=u^2_{\mathbf{q}}$, so that the inequality (2) is satisfied if $v^2_{\mathbf{q}}>u^2_{\mathbf{q}}$. Thus with respect to this inequality, the two choices (16) and (17) complement each other.

It is tempting to consider the entanglement between particles from different Cooper pairs and the entanglement between different Cooper pairs. In the first case, we employ the following single electron operators

$$A = c_{\mathbf{q}}(c^{\dagger}_{\mathbf{q}}), \quad B = c^{\dagger}_{\mathbf{q'}}(c_{\mathbf{q'}}), \tag{24}$$

$$A = c_{\mathbf{q}}(c^{\dagger}_{\mathbf{q}}), \quad B = c_{\mathbf{q'}}(c^{\dagger}_{\mathbf{q'}}), \tag{25}$$

while in the second case, we use the pair operators

$$A = b_{\mathbf{q}}(b^{\dagger}_{\mathbf{q}}), \quad B = b^{\dagger}_{\mathbf{q'}}(b_{\mathbf{q'}}), \tag{26}$$

$$A = b_{\mathbf{q}}(b^{\dagger}_{\mathbf{q}}), \quad B = b_{\mathbf{q'}}(b^{\dagger}_{\mathbf{q'}}), \tag{27}$$

where $\mathbf{q} \neq \mathbf{q'}$ and $b_{\mathbf{q}}$ is the pair annihilation operator $b_{\mathbf{q}} = c_{\mathbf{q}}c_{-\mathbf{q}}$. Unfortunately, for all these choices, it has been found that neither the condition (1) nor the condition (2) can detect entanglement.

In summary, we have tested different combinations of the operators A and B in the inequalities (1) and (2) to find out which one can best reveal entanglement in the ground state of the noninteracting Fermi gas and the BCS superconducting system. In the Fermi gas system, it has been shown that entanglement may exist between two spatially separated particles, with the interparticle distance being bounded from above. This bound has been established. In the BCS system, the above mentioned inequalities have been shown to be capable of detecting entanglement between particles in a Cooper pair with much less effort as compared to the concurrence and the partial transpose condition used in earlier work. Future work may include an investigation of the multipartite entanglement in Fermi gases, which would allow for direct comparison with the existing literature. The case of nonzero temperature may also be of interest.

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