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ALGEBRAIC LAGRANGIAN GEOMETRY: FROM GEOMETRIC QUANTIZATION TO MIRROR SYMMETRY

NIKOLAY TYURIN
BLTPh JINR (Dubna) and HSE (Moscow)

ALAG — **Abelian Lagrangian Algebraic Geometry**, proposed by A. Tyurin and A. Gorodentsev, [1], — is a programme indeed. As *input* one has (M, ω) — compact 1 - connected symplectic manifold s.t. $[\omega] \in H^2(M, \mathbb{Z})$; as *output* one gets $\mathcal{B}_S^{hw,r}$ — an infinite dimensional Kahler manifold which is called *moduli space of half weighted Bohr - Sommerfeld lagrangian cycles of fixed topological type and volume.*

Recall, for such a case (L, a) — **prequantization data** on M — are line bundle L with hermitian connection a s.t. $c_1(L) = [\omega], F_a = 2\pi i \omega$. Submanifold $S \subset M$ is **lagrangian** iff $\omega|_S \equiv 0$; **Bohr - Sommerfeld** if $a|_S$ has trivial periods. Then the moduli space $\mathcal{B}_S^{hw,r} = \{(S, \theta) | \int_S \theta^2 = r\}$ where S is a Bohr - Sommerfeld (with respect to prequantization data) oriented lagrangian submanifold and θ is a half weight. The parameters are: (integer) S has a fixed smooth type and orientation, $[S] \in H_n(M, \mathbb{Z})$ is fixed; (real) r - volume. Local theory — based on the Darboux - Weinstein theorem:

$$T_{(S,\theta)} \mathcal{B}_S^{hw,r} = C^\infty(S, \mathbb{R})/const \oplus C^\infty(S, \mathbb{R})/const.$$

Moreover pairs of smooth functions (ϕ, ψ) give canonical coordinate system in which

$$\Omega_{(S,\theta)} = \int_S (\psi_2 \phi_1 - \psi_1 \phi_2) \theta^2$$
$$G_{(S,\theta)} = \int_S (\phi_1 \phi_2 + \psi_1 \psi_2) \theta^2$$

— the ingredients of canonical Kahler structure. One can prove, [2]
Existence theorem: *If $S \subset M$ is a smooth orientable lagrangian submanifold representing homology class $[S] \in H_n(M, \mathbb{Z})$. Then there exists such integer k that for any level $k' > k$ the moduli space $\mathcal{B}_{S,k'}^{hw,r}$ is non empty.*

The main aim of A. Tyurin and A. Gorodentsev: factorize $\mathcal{B}_S^{hw,r}$ with respect to Sym$M$ which should lead to a finite dimensional Kahler manifold. Thus it should be a new construction of Mirror Symmetry! But $\dim \mathcal{B}_S^{hw,r} = \dim \mathrm{Sym} M$, thus one would get just a 0-dimensional set which removes the advantage of Kahler structure.

First application of ALAG is a new method of **quantization**. Quantization in the broadest context is a procedure:

$$\begin{array}{ccc} (M, \omega) - \text{sympl. man.} & \mapsto & \mathcal{H} - \text{Hilbert sp.} \\ \text{(class. phase space)} & \mapsto & \text{(quantum phase space)} \\ C^\infty(M, \mathbb{R}) & \stackrel{q}{\to} & \mathrm{Op}(\mathcal{H}) \\ \text{(class.observ.)} & \mapsto & \text{(quantum observ.)} \end{array}$$

where q have to satisfy conditions from the Dirac list:
1. $q(a+bf) = a\mathrm{Id} + \mathrm{bq(f)}$ (linearity),
2. $q(\{f_1, f_2\}_\omega) = i[\hat{F}_1, \hat{F}_2]$ (*the correspondence principle*),
3. irreducibility.
So in short: q must be an irreducible representation of the Poisson algebra.

Example: geometric quantization. If (M, ω) admits an integrable complex structure,then

$$\mathcal{H}_k = H^0(M_I, L^k)$$

and we have a number of methods how to construct such a map q.

On the other hand, **Geometric Formulation** of Quantum Mechanics, proposed by A. Ashtekar and T. Schilling, [3], gives a vocabulry:

$$\begin{array}{ccc}
\mathcal{H}, <;> & \mapsto & \mathbb{P}\mathcal{H}, (\Omega, G, I) \\
\psi \in \mathcal{H} & \mapsto & p \in \mathbb{P}\mathcal{H} \\
\hat{F} \in \mathrm{Op}(\mathcal{H}) & \mapsto & f \in C^\infty_q(\mathbb{P}, \mathbb{R}) \subset C^\infty \\
 & & \mathrm{Lie}_{X_f} G \equiv 0 \\
i[\hat{F}, \hat{K}] & \mapsto & \{f, k\}_\Omega \\
\frac{\partial \psi}{\partial t} = -i\hat{H}\psi & \mapsto & \dot{p} = X_h \\
\psi - \text{ eigenvector } \hat{H} & \mapsto & p = \mathbb{P}(\psi) - \text{crit.point} \\
\lambda - \text{eigenvalue} & \mapsto & \lambda - \text{crit. value} \\
\text{projections} & \mapsto & \text{geodesic distances}
\end{array}$$

Difference between CM and QM in the presence of riemannian metric, which responds for probabilistic aspects. It distinguishes a subspace $C^\infty_q(\mathbb{P}, \mathbb{R}) \subset C^\infty(\mathbb{P}, \mathbb{R})$ — space of *symbols* (in F. Berezin terminology). Ashtekar and Schilling ask are there other Kahler manifolds which can play the role of quantum phase space? F.e., in Geometric Quantization it is the given symplectic manifold *itself* (with some compatible integrable complex structure).

Geometric formulation inspires the introduction of new notion — **Algebro-geometric quantization**, where one finds for a symplectic manifold (M, ω) certain **Kahler (algebraic)** manifold $\mathcal{K}$ together with a map

$$q : C^\infty(M, \mathbb{R}) \to C^\infty_q(\mathcal{K}, \mathbb{R}),$$

respects the modified conditions from the Dirac list:
1. $q(a + bf) = a + bq(f), \forall a, b \in \mathbb{R}, f \in C^\infty(M, \mathbb{R})$
2. $q(\{f_1, f_2\}_\omega) = \{q(f_1), q(f_2)\}_\Omega$
3. irreducibility (= $\forall p \in \mathcal{K}, v \in T_p\mathcal{K}$ there exists $f \in C^\infty(M, \mathbb{R})$ s.t. $X_{q(f)}(p) = v$ and $\ker q = 0$).

If $\mathcal{K} = \mathbb{P}\mathcal{H}$ then **AG - quantization** $\equiv$ **quantization**.

It was proved in [4], [5] that $\mathcal{K} = \mathcal{B}^{hw,r}_S$ is a solution of AGQ, so ALAG solves the problem of Algebro - geometric quantization. The correspondence is given explicitly

$$q(f) = F_\tau(f)(S, \theta) = \tau \int_S f|_S \theta^2$$

where $\tau \in \mathbb{R}$ is a parameter. Then
1. $F_\tau(a + bf) = a\tau r + bF_\tau(f)$;
2. $\{F_\tau(f_1), F_\tau(f_2)\}_\Omega = 2\tau F_\tau(\{f_1, f_2\}_\omega)$;
3. F_τ is irreducible.
Thus the Dirac conditions are satisfied if
$r = \tau = \frac{1}{2}$.

Thus one can say that every classical mechanical system, represented by (M, ω), **contains** some quantum mechanical system, represented by $\mathcal{B}_S^{hw,r}$, and the dynamics (classical and quantum) are compatible — and it follows the Copenhagen programme:
"... the physical predictions of a quantum theory must be formulated in terms of classical concepts... in addition to the usual structures any sensitive quantum theory it has to admit an appropriate passage to a classical limit ... but the correspondence between quantum theory and classical theory has to be based not only on numerical coincidences but on an analogy between their mathematical structures. Classical theory does approximate the quantum theory but it does do even more - it supplies a frame to some interpretation of the quantum theory..."

But let us come back to the background idea of A. Tyurin and A. Gorodentsev: construct from SG- object some AG-object, namely, from a finite dimensional symplectic manifold some finite dimensional Kahler (algebraic) manifold. This would give a new approach in **Mirror Symmetry**, understood as a duality between Algebraic Geometry and Symplectic Geometry.

Example: in Homological Mirror Symmetry (M. Kontsevich) "duality" means that some category derived from AG of M (the derived category of coherent sheaves) is equivalent to some category derived from SG of W (Fukaya category). But what about "real" geometry?

Example (A. Tyurin, C. Vafa):

Let M, W — two CY_3- manifolds,

$AG(M)$	|	$SG(W)$
$m \in H^{2*}(M, \mathbb{Z})$	|	$w \in H^3(W, \mathbb{Z})$
realization of m	|	realization of w
by stable vect. bundles	|	by special lag. cycles
$\mathcal{M}_{st}(m)$	$\equiv$	$\mathcal{M}_{SpLAG}(w)$

Problem: on the LHS there are geometrical objects and constructions indeed while on the RHS the theory of SpLAG is still not completed yet.

Main observations for today:
1. some standard gauge theory construction can be generalized for ALAG;
2. there are some natural bundles (which we call **Floer bundles**) over the moduli space $\mathcal{B}_S$ of Bohr - Sommerfeld lagrangian cycles;
- thus one hopes that ALAG is usefull in MS.

Can we reproduce certain constructions from the Donaldson gauge theory ([6]) in ALAG?

We can construct a universal object — incidence cycle

$$\begin{array}{ccccc} M & \stackrel{p}{\leftarrow} & \mathcal{U} & \stackrel{q}{\rightarrow} & \mathcal{B}_S \\ & & \cap & & \\ & & M \times \mathcal{B}_S & & \end{array}$$

$\mathcal{U} = \{(x,S)\}$ where x — point of M, $S \subset M$ — Bohr - Sommerfeld cycle such that $x \in S$. It is a problem: orientability of $\mathcal{B}$, but suppose we can fix an orientation. Then $[\mathcal{U}]^{PD} \in H^n(M \times \mathcal{B}_S, \mathbb{Z})$ defines

$$\begin{array}{rcl} \mu_{\mathcal{U}} : H_i(M, \mathbb{Z}) & \rightarrow & H^{n-i}(\mathcal{B}_S, \mathbb{Z}) \\ \mu_{\mathcal{U}}(\sigma) & = & [\mathcal{U}]^{PD}/\sigma \end{array}$$

— generalized μ - classes in ALAG, analogeous to standard μ - classes in the Donaldson theory.

Toy example: consider $\mathbb{CP}^1 = S^2$ with standard symplectic form. Then the moduli space $\mathcal{B}_S$ is given by smooth loops $\gamma \subset S^2$ s.t. $\int_{int\gamma} \omega = \int_{out\gamma} \omega\}$. Cosnider minimal B. - S. cycles for F. - S. metric:

$$\mathcal{B}_S \supset \mathcal{M}_{min} = \{\text{big circles}\}, dim_{\mathbb{R}} \mathcal{M}_{min} = 2$$

But for this case $n = 1$ and $\mu_{\mathcal{U}}([pt]) \in H^1(\mathcal{B}_S, \mathbb{R})$.

At the same time the universal object $\mathcal{U}$ can be used to transport bundles and sheaves from M to $\mathcal{B}_S$.For any $S_1 \subset M$ one has

$$\mathcal{F}|_{S \in \mathcal{B}_S} = FH(S, S_1, \mathbb{C})$$

with Floer cohomology of the pair (S, S_1) as a fiber. But the Floer cohomology is stable with respect to local Hamiltonian deformations therefore it is a complex bundle on $\mathcal{B}_S$. It's natural to call it **Floer bundle** $\mathcal{F}_{S_1}$. It depends not on S_1, but on the class of Hamiltonian deformation of S_1. The Floer bundle $\mathcal{F}_{S_1}$ carries some **canonical singular connection** A_{S_1}:

$$\mathcal{B}_S \supset \mathcal{B}(S_1) = \{S | S \text{ transversal to } S_1\},$$

then the intersection points $S \cap S_1 = \{p_1, ..., p_m\}$ induce framings of $\mathcal{F}|_{\mathcal{B}(S_1)}$ over each small neighborhood and it gives a smooth connection over $\mathcal{B}_S \backslash \mathcal{B}(S_1)$. The singular set

$$\text{Sing}\, A_{S_1} = \mathcal{B}_S \backslash \mathcal{B}(S_1).$$

This connection **depends** strictly on the cycle S_1.

Why we are interested in vector bundles over $\mathcal{B}_S$? May be it is possible to revise the idea of A. Tyurin and A. Gorodentsev: if one finds some natural holomorphic vector bundle $\mathcal{E} \rightarrow \mathcal{B}_S^{hw,r}$ of finite rank k which is equivariant with respect to Sym$M$ - action then it were k - dimensional Kahler manifold

$$\mathcal{E}/\text{SymM} = \text{W},$$

which could be understand as a mirror partner of the given symplectic manifold M. And one sees that the Floer bundles are equivariant with respect to the action of $\text{Sym}^0 M$. So our main strategy: extend the Floer bundles in a natural way to $\mathcal{B}_S^{hw,r}$. We hope to reach some result in this direction in a future.

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