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SOLUTIONS FOR YANG-MILLS FIELD WITH SINGULAR SOURCE TERMS AND HIGHER TOPOLOGICAL INDICES

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Abstract. *We find solutions for Yang-Mills field with two point sources in the sector of higher topological indices. Mathematically, this is a nonlinear problem of elliptic partial differential equations with $\delta-$function singular source terms and axial symmetry. The obtained numerical solutions by relaxation methods for field functions are presented. We show that the screening of the source charges is actually present and caused by the vector component of the gauge potential. The total energies of the system are calculated for a range of charge magnitudes, and the plot of energy versus charge magnitude is presented.*

I. INTRODUCTION

Lattice analogues of continuum field systems help theorists in finding answers to several problems of quantum field theory, for which purely analytical methods are inadequate. An interesting example is the lattice formulation of QCD, from which a variety of results have been established (see, for example, a recent review [1]). In this approach, the functional integrals defining QCD correlation functions [2] were evaluated numerically by Monte Carlo technique. Another problem-solving strategy in field theory is to first make analytic approximations, which replace the field theoretical problem by a classical variational problem, leading to a system of partial differential equations which are then solved numerically [3].

For non-Abelian gauge theories some interesting solutions, like the Wu-Yang monopole [4], non-Abelian plane wave [5], 't Hooft-Polyakov monopole [6], [7], instanton [8], meron [9] solutions have been discovered analytically. However, such solutions concern to ansatz that has such high symmetries, as spherically symmetrical or one-dimensional. For solutions with lower symmetries, namely, axially symmetrical multimonopole [10], monopole-antimonopole [11], vortex ring [12] solutions of the Yang-Mills-Higgs equations, field equations are reduced to more complicated systems of partial differential equations, and the solutions have been found numerically.

Apart from sourceless Yang-Mills equations and the Yang-Mills-Higgs system, the interest has been paid also to the system of an Yang-Mills gauge field coupled to an external source, knowledge about which can be useful in several respects [13], [14], [15],

[16]. [17], [18]. First, it has important Abelian counterpart in electrodynamics to bring into comparison. Secondly, one may think it as a classical model of a heavy color quark and its color field. One of interesting phenomena for this system is the instability which is related to color screening and may be regarded as a model for color confinement in non-Abelian gauge theories [13].

The aim of the present work is to set up a lattice version for the system of Yang-Mills field in the presence of two point sources carrying non-Abelian color charge. The features of the problem are (i) the system possesses axial symmetry; (ii) the source term in field equation is (δ-function) point-like singular; (iii) we find the solution that belong to the sectors of higher topological indices. It has been known that for the system of Yang-Mills field with external sources, solutions have non-trivial topological structure and can be classified by the index of mapping from coordinate space to the unit sphere in the SU(2) isospace of the external charge [19], [20], [21], [22]. In Sec. II we resume the equations, the boundary conditions and the ansatz for the gauge potentials. Then the computational lattice is described in Sec. III. The obtained solutions are plotted and shown in Sec. IV, and some discussion about our results is given in Sec. V, Conclusion.

II. FIELD EQUATIONS AND AXISYMMETRICAL ANSATZ

The $SU(2)$ Yang-Mills equations in the presence of an external current j_a^ν are

$$\left(D^\mu F^a_{\mu\nu}\right) = j^a_\nu, \tag{1}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu, \tag{2}$$

where g is the gauge field coupling constant. Latin and Greek indices run from 1 to 3 and from 0 to 3, respectively. Making the conventional rescaling of the gauge potentials by the coupling constant g, in Eq. (2) one can set $g = 1$. The space-time is flat, and we use the metric which is determined by $g_{00} = -g_{ii} = 1$.

We consider the static external source that has the following form

$$j^a_\nu = \delta^0_\nu e^a Q[\delta(\overrightarrow{x} - \overrightarrow{a_1}) - \delta(\overrightarrow{x} - \overrightarrow{a_2})], \tag{3}$$

where $\overrightarrow{a_1}, \overrightarrow{a_2}$ are the position vectors of positive and negative color charges, respectively. Working in cylindrical coordinates ρ, z, ϕ, in which two point charges are located symmetrically on the z axis, we have $\overrightarrow{a_1} = \{\rho = 0, \phi = 0, z = a\}; \overrightarrow{a_2} = \{\rho = 0, \phi = 0, z = -a\}$, then the system of gauge field and external source possesses axial symmetry about z axis.

An interesting axial-symmetric ansatz for the Yang-Mills potentials has been found in Refs. [15], [21], and has also been used in Ref. [22]. It is given by

$$A^a_0 = e^a \Phi_n(\rho, z), \quad A^a_i = \delta^a_3 \phi_i A_n(\rho, z), \tag{4a}$$

where

$$e^a = \delta^a_1 cos(n\phi) + \delta^a_1 sin(n\phi), \quad \phi_i = \varepsilon_{ij3}\frac{x_i}{\rho}. \tag{4b}$$

Here e^a are the unit vectors giving the orientations of sources in the isospin (target) space of gauge fields, and the number n in Eq. (4b) is the winding number, i. e. the number of times the circle spanned by the vector e^a is covered, when the azimuth ϕ sweeps one round from 0 to 2π. One uses n as an index (topological index) which labels classes of solutions.

The Lagrange functional, from which equations of motion (1) are derived, after the substitution of ansatz (4), takes the following form

$$\begin{aligned} L = \pi \int_0^\infty \rho d\rho \int_{-\infty}^{+\infty} dz \times \\ \times \left\{ \left[\left(\frac{\partial \Phi_n}{\partial \rho} \right)^2 + \left(\frac{\partial \Phi_n}{\partial z} \right)^2 \right] - \left[\left(\frac{\partial A_n}{\partial \rho} \right)^2 + \left(\frac{\partial A_n}{\partial z} \right)^2 \right] \right. \\ \left. + \left[\Phi_n^2 \left(A_n - \frac{n}{\rho} \right)^2 - \left(\frac{A_n}{\rho} \right)^2 \right] \right\} \\ -Q\Phi_n(\rho = 0, z = a) + Q\Phi_n(\rho = 0, z = -a), \end{aligned} \tag{5}$$

where we have integrated out the δ-function in the source terms [two last terms in Eq. (5)]. The problem is to find the profile functions $\Phi_n(\rho, z)$ and $A_n(\rho, z)$ that minimize functional (5) and belong to the class with the topological index n. The boundary conditions for both $\Phi_n(\rho, z)$ and $A_n(\rho, z)$ are 0 at infinity.

III. DISCRETIZATION OF THE CONTINUUM PROBLEM

Introducing the computational lattice

$$\rho_i = i.\Delta\rho, \quad i = 0, 1, 2, ...,$$

$$z_j = j.\Delta z, \quad j = ... - 2, -1, 0, 1, 2, ...,$$

the discrete form of L from (5) is written as follows

$$\begin{aligned} L = \pi \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_i \left\{ \left[\frac{(\Phi_n)_{i+1,j} - (\Phi_n)_{i,j}}{\Delta\rho} \right]^2 + \left[\frac{(\Phi_n)_{i,j+1} - (\Phi_n)_{i,j}}{\Delta z} \right]^2 \right. \\ - \left[\frac{(A_n)_{i+1,j} - (A_n)_{i,j}}{\Delta\rho} \right]^2 - \left[\frac{(A_n)_{i,j+1} - (A_n)_{i,j}}{\Delta z} \right]^2 \\ \left. + (\Phi_n)_{ij}^2 \left[(A_n)_{ij} - \frac{n}{\rho_i} \right]^2 - \left[\frac{(A_n)_{ij}}{\rho_i} \right]^2 \right\} \Delta\rho\Delta z \\ -Q\left(\Phi_n\right)_{0,a} + Q\left(\Phi_n\right)_{0,-a}, \end{aligned} \tag{6}$$

(in our setting the sources lie at nodes of the computational lattice $a = N.\Delta z$, N is integer).

Differentiating Eq. (6) with respect to $(\Phi_n)_{i,j}$ and $(A_n)_{i,j}$ gives the discrete form of field equations,

$$\begin{aligned} 2\pi\rho_i \left(\frac{\Delta z}{\Delta\rho} \right) \left\{ \left[(\Phi_n)_{i,j} - (\Phi_n)_{i-1,j} \right] - \left[(\Phi_n)_{i+1,j} - (\Phi_n)_{i,j} \right] \right\} \\ +2\pi\rho_i \left(\frac{\Delta\rho}{\Delta z} \right) \left\{ \left[(\Phi_n)_{i,j} - (\Phi_n)_{i,j-1} \right] - \left[(\Phi_n)_{i,j+1} - (\Phi_n)_{i,j} \right] \right\} \end{aligned}$$

$$+2\pi\rho_i\Delta\rho\Delta z\left(\Phi_n\right)_{i,j}\left[\left(A_n\right)_{i,j}-\frac{n}{\rho_i}\right]^2=Q\delta_{i,0}\left(\delta_{0,N}-\delta_{0,-N}\right), \tag{7a}$$

$$\left(\frac{\Delta z}{\Delta\rho}\right)\left[\left(A_n\right)_{i+1,j}-2\left(A_n\right)_{i,j}+\left(A_n\right)_{i-1,j}\right]+$$
$$\left(\frac{\Delta\rho}{\Delta z}\right)\left[\left(A_n\right)_{i,j+1}-2\left(A_n\right)_{i,j}+\left(A_n\right)_{i,j-1}\right]+$$
$$\Delta\rho\Delta z\left(\Phi_n\right)^2_{i,j}\left[\left(A_n\right)_{i,j}-\frac{n}{\rho_i}\right]-\Delta\rho\Delta z\frac{A_{i,j}}{\rho_i^2}=0. \tag{7b}$$

Eqs. (7a) and (7b) then give us a set of linear equations in unknown variables $(\Phi_n)_{i,j}$, $(A_n)_{i,j}$ which are solved by iterative method. The results of solving are given in the next Section.

IV. SOLUTION FOR FIELD FUNCTIONS AND ENERGY

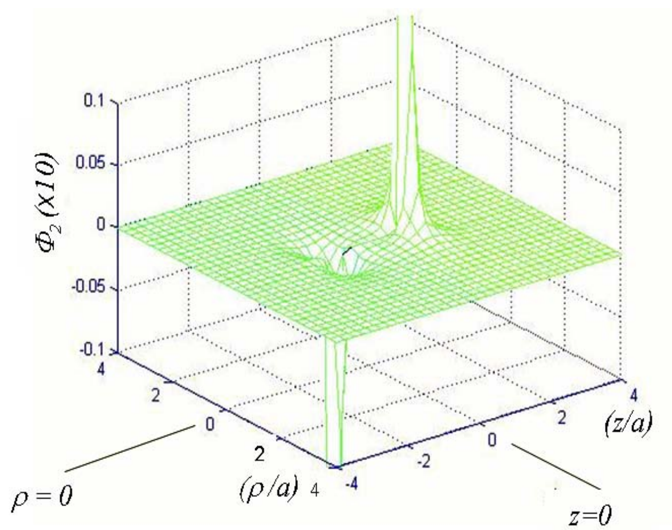

Fig. 1. Graph for $\Phi_2(\rho, z)$ magnified vertically by a factor of 10

We show several obtained numerical solutions for the profile functions of Yang-Mills field $\Phi_n(\rho, z)$ and $A_n(\rho, z)$ belonging to the solution classes with topological index $n = 2$. Figures 1 and 2 give values of Φ_2 and A_2 (plotted vertically) on a plan passing through the axis of rotation (about ρ axis) with the chosen value $Q = 5$. The lengths on ρ and z axes are in units of a.

The graph of field energy H versus non-Abelian charge Q for the mention solutions is given in Fig.5. The field energy is calculated by the following equation, where the static potential energy V_{static} is determined by extremizing the Lagrangian functional with respect to the function $\Phi_n(\rho, z), A_n(\rho, z)$.

$$H = V_{static} = -ext_{\Phi_n,A_n}L\left(\Phi_n, A_n\right), \tag{8}$$

For comparison, we plot $H(Q)$ for the Coulomb solution (the solution $\Phi_c(\rho, z)$ of the

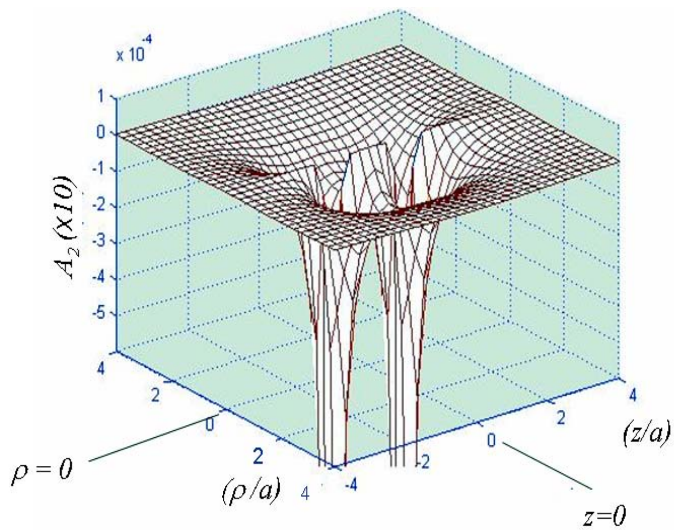

Fig. 2. Graph for $A_2(\rho, z)$ magnified vertically by a factor of 10

equation $\triangle\Phi = -j^0$). It is seen in Fig. 3 that at all Q values the energy for our solution is lower than that for the Coulomb solution, and as Q increases, the difference is more significant. In the graph, energy H is measured in units of H_0, the energy of the Coulomb solution ($H_0 = Q^2/8\pi a$).

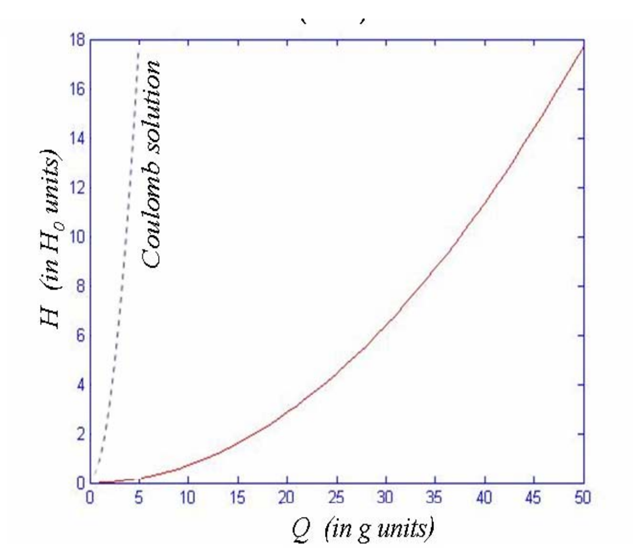

Fig. 3. Graph of field energy H vs Q, ($n = 2$)

V. CONCLUSION

We have successfully set up a lattice version of the Yang-Mills field coupled to an external static source density. The source term has singularity of delta-function form and non-trivial topological structure. The obtained solutions manifest explicitly screening characterization: The spreading area of the field profiles is very small. Apart from the solutions for topological index $n = 2$, as shown in the text, we have also obtained the solutions for $n = 3, 4$ (not shown here), and have observed that for higher n, the effect of screening is stronger.

In summary, by direct calculation we have found the explicit solutions for the Yang-Mills field coupled to a pair of particle-antiparticle carrying color charge. The solutions may be useful for several problems of non-Abelian gauge field theories. We have also demonstrated that there exists actually the screening of color charges in the considered system, a way to understand the confinement of color charges.

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