These 16 types form a basis into which any 4x4 matrix can be expanded.

I. Photons – need an equation for them, analogous to Dirac Eq

(1) Show $\Box^2 A = j^\mu$ if $\partial_\mu A^\mu = 0$

D'Alembertian $\quad\quad$ gauge choice

Recall $j^\mu = (\rho, \vec{J})$

Define
$$F^{\mu\nu} = \begin{array}{c|cccc} & (t) & (x) & (y) & (z) \\ \hline (t) & 0 & -E_x & -E_y & -E_z \\ (x) & E_x & 0 & -B_z & B_y \\ (y) & E_y & B_z & 0 & -B_x \\ (z) & E_z & -B_y & B_x & 0 \end{array}$$

Recall Maxwell's Eq

i) $\nabla \cdot E = \rho$

ii) $\nabla \times E + \frac{dB}{dt} = 0$

iii) $\nabla \cdot B = 0$

iv) $\nabla \times B - \frac{dE}{dt} = \vec{J}$

(i) and (iv) can be written as $\partial_\mu F^{\mu\nu} = j^\nu$ $\quad\quad$ Eq 1

Ex $\frac{d}{dt} F^{tt} + \frac{d}{dx} F^{xt} + \frac{d}{dy} F^{yt} + \frac{d}{dz} F^{zt} = j^t$

$$\underbrace{\frac{d}{dt} 0 + \frac{d}{dx} E_x + \frac{d}{dy} E_y + \frac{d}{dz} E_z = \rho}_{\nabla \cdot E = \rho}$$

Because $\nabla \cdot (\nabla \times \text{anything}) = 0$,
(iii) can be rewritten by defining A such that

$$B = \nabla \times A \qquad \text{"Eq 2"}$$

Plug this into (ii) to get

$$\nabla \times E + \frac{d}{dt}(\nabla \times A) = 0$$

$$\downarrow$$

$$\nabla \times \left(E + \frac{dA}{dt}\right) = 0$$

Because $\nabla \times (\nabla \text{ anything}) = 0$, this can be written as

$$E + \frac{dA}{dt} = -\nabla V$$

$$\downarrow$$

$$E = -\nabla V - \frac{dA}{dt} \qquad \text{"Eq 3"}$$

If we define $A^\mu \equiv (V, \vec{A})$, then
Eq 2 and Eq 3 can be written compactly as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Plug this into Eq 1

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu$$

$$\downarrow$$

$$\underbrace{\partial_\mu \partial^\mu}_{\Box^2} A^\nu - \partial^\nu \overbrace{(\partial_\mu A^\mu)}^{\to 0} = j^\nu$$

In the Lorentz gauge, $\partial_\mu A^\mu = 0$

$$\boxed{\Box^2 A^\nu = j^\nu}$$ Maxwell's Equations

$\Box^2 A^\mu = j^\mu$ if we allow $\partial_\mu A^\mu = 0$ (Lorentz condition)

EM field (photon)

current of spinor that interacts with it.

Additional gauge freedom:

We can choose a gauge (ex, Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$, If photon is free, $j^{\mu}=0$

$$\downarrow$$

$$D^2 A^{\mu} = 0$$

Solutions look like $A^{\mu} = \varepsilon(\vec{q}) e^{-iq\cdot x}$

$$= \varepsilon(\vec{q})\, e^{+iEt} e^{-i\vec{q}\cdot\vec{x}}$$

Check: $D^2 A^{\mu} = \left[\frac{\partial^2}{\partial t^2} - \nabla^2\right] \varepsilon(\vec{q}) e^{iEt} e^{-i\vec{q}\cdot\vec{x}}$

$$= \left[-E^2 - (-\vec{q})^2\right] A^{\mu}$$

$$= \left[-E^2 + q^2\right] A^{\mu}$$

$$= 0$$

for photon $m=0$, so $E^2 = q^2$

Facts about $\varepsilon(\vec{q})$

1) Called the photon polarization vector

2) index μ tells us it has 4 components, but the $\partial_\mu A^\mu$ requirement implies $\partial_\mu \varepsilon^\mu = 0$, so subtract 1 degree of freedom (component)

3) Choose Coulomb gauge:

$\vec{\nabla}\cdot\vec{A} = 0$

$\downarrow$ recall $\nabla = -i\vec{p} = -i\vec{q}$

$\vec{q}\cdot\vec{A} = 0$

$\downarrow$ But $\vec{A} \propto \vec{\varepsilon}$

$\vec{q}\cdot\vec{\varepsilon} = 0$

Lose 1 more dof/component of $\varepsilon(q)$

Now only (2 left)

4) From $\vec{\varepsilon}\cdot\vec{q}=0$ we infer that the 2 possible polarization ε_1 and ε_2 are both $\perp$ to $\vec{q}$ (i.e., $\perp$ to direction of propagation of photon)

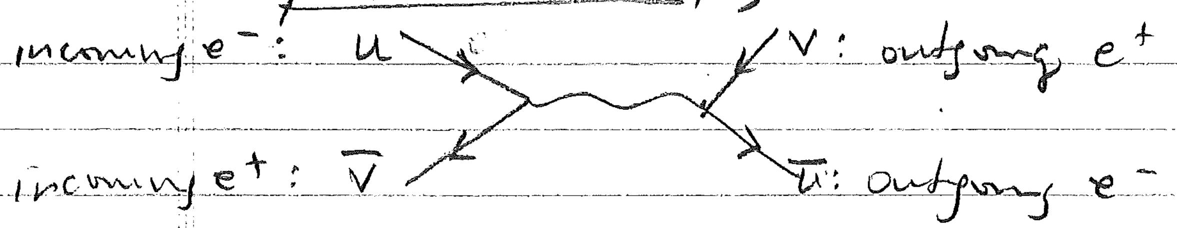
Let $\hat{q} = \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

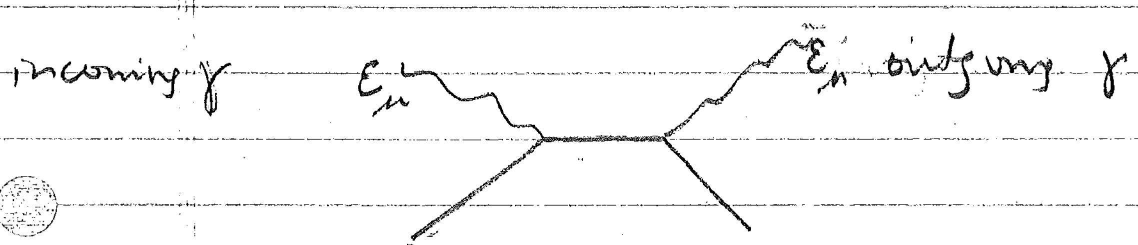
Then $\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

I. Feynman rules for QED.

Amend the "Toy Theory Rules" to include

1) Identify the particle species as it propagates <u>forward in time</u>, e.g. use these labels:





Vertex factor: $ie\gamma^\mu$. Note $e = \sqrt{\alpha}$ when $\hbar = c = 1$

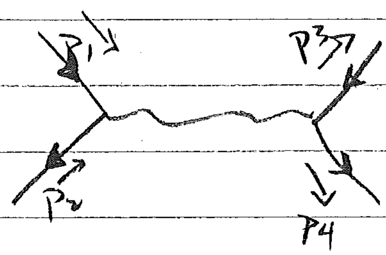
Propagator $e^\pm$: $\dfrac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$

Propagator photon: $\dfrac{-ig_{\mu\nu}}{q^2}$

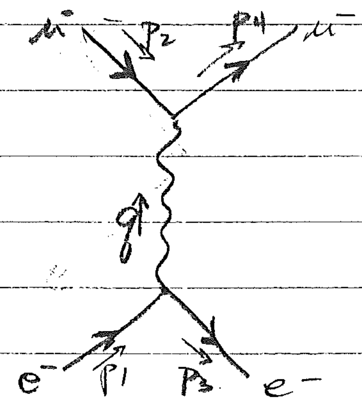
Otherwise same rules re factors of $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$ at vertices and $\int \frac{d^4 q}{(2\pi)^4}$ over internal lines.

because matrices are involved, must assemble the matrix elt. in the right order.

Distinguish between momentum arrows (off the lines) and fermion arrows (on the lines)



Concentrate on fermion lines. Track each one backward through the diagram while writing factors left to right. Ex:



"p_3"

ν not μ, different vertex

$$\mathcal{M} = \int \bar{u}(3)(ie\gamma^{\mu})u(1)\,\frac{(-ig_{\mu\nu})}{q^2}\,\bar{u}(4)(ie\gamma^{\nu})u(2)$$

$$\cdot \underbrace{(2\pi)^4\delta^4(p_2+q-p_4)}_{\text{upper vertex}} \cdot \underbrace{(2\pi)^4\delta^4(p_1-q-p_3)}_{\text{lower vertex}} \cdot \frac{d^4q}{(2\pi)^4}$$

one internal line

Apply $\int d^4q \; \delta(p_1 - q - p_3)$ $\quad q \to p_1 - p_3$

Mult $-i \cdot i \cdot i \to i$

Apply $g_{\mu\nu}\gamma^{\nu} \to \gamma_{\mu}$

What remains:

$$\frac{i\,(2\pi)^4 e^2\, \bar{u}(3)\gamma^{\mu}u(1)\,\bar{u}(4)\gamma_{\mu}u(2)\,\delta^4(p_1 + p_2 - p_3 - p_4)}{(p_1 - p_3)^2}$$

To get M, mult by i and cancel $(2\pi)^4\,\delta^4(p\ldots)$

$$M = \frac{-e^2[\bar{u}(3)\gamma^{\mu}u(1)][\bar{u}(4)\gamma_{\mu}u(2)]}{(p_1 - p_3)^2}$$

To get a testable prediction, need to plug this into Fermi's Golden Rule for scattering.

Recall $\frac{d\sigma}{d\Omega} \sim |M|^2$

Note cc

$$\text{So } |M|^2 = \frac{e^4}{(p_1 - p_3)^4}[\bar{u}(3)\gamma^{\mu}u(1)][\bar{u}(4)\gamma_{\mu}u(2)][\bar{u}(3)\gamma^{\nu}u(1)]^*[\bar{u}(4)\gamma_{\nu}u(2)]^*$$

dummy index reflects which terms are contracted (multiplied)

Notice this includes terms of the form

$$[\bar{u}(a)\,\Gamma_1\,u(b)][\bar{u}(a)\,\Gamma_2\,u(b)]^*$$

e.g. γ^μ and γ_μ or $\gamma^\mu\gamma^5$ etc (depending on the bilinear covariant)

If the colliding beams have not been intentionally polarized, we have to average over all possible spin combinations of the colliding fermions.

So we need: $\frac{1}{4}\sum_{\text{spin } a=1}^{2}\;\sum_{\text{spin } b=1}^{2}$ in front.

It turns out that ("Casimir's Trick", Griffiths p. 251

$$\sum\sum_{\text{all spins}}[\bar{u}(a)\,\Gamma_1\,u(b)][\bar{u}(a)\,\Gamma_2\,u(b)]^* =$$

$$Tr[\Gamma_1(\not{p}_b+m_b)\bar{\Gamma}_2(\not{p}_a+m_a)]$$

$\bar{\Gamma}_2 \equiv \gamma^0\Gamma_2^\dagger\gamma^0$

$p_b^\mu\gamma_\mu$ (implied sum over μ)

standard matrix multiplication

$Trace(A) = \sum_i A_{ii}$

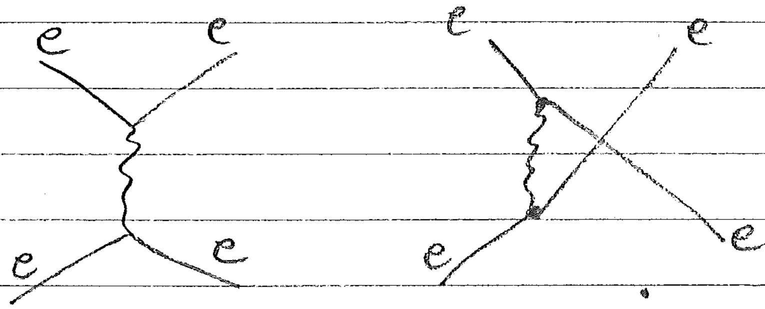
This "$\not{p}+m$" form comes from directly computing $\sum u\bar{u}$ including the $\sqrt{E+m}$ normalization.

The calculation of a Tr of a product is simplified by 16 trace identities (Griffiths 252-3), for example $Tr(\gamma^5)=0$ and $Tr(\not{a}\not{b}) = 4a\cdot b$

- Related amplitudes

Recall "M" = Σ(amplitudes of all processes that begin + end with the states of interest)

Could be more than 1, e.g.



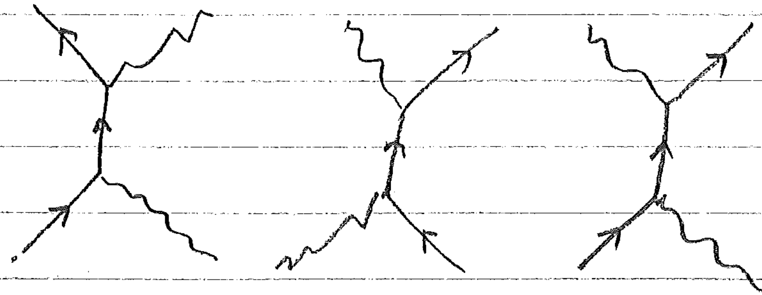
Facts

(1) If 2 competing process differ only by exchange of identical fermions, subtract them instead of adding

(2) If a process is known to be possible, then all other processes that can be obtained by twisting the lines are also possible

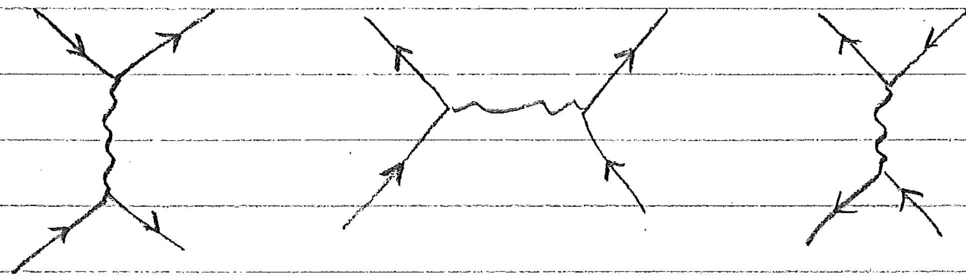
e.g.

(i)



— or —

(ii)



This relationship is called Crossing Symmetry (Griffiths p. 22) and can be expressed as:

If: $A + B \rightarrow C + D$

Then also:

antiparticle

$A \rightarrow \bar{B} + C + D$

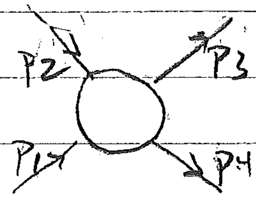
$A + \bar{C} \rightarrow \bar{B} + D$

$\bar{C} + \bar{D} \rightarrow \bar{A} + \bar{B}$

This QFT result follows directly from construction of matrix elts. using Feynman rules

(3) Mandelstam variables.

Consider



Define $s \equiv (p_1+p_2)^2 = (p_3+p_4)^2$

$$t \equiv (p_3-p_1)^2 = (p_4'-p_2')^2$$

$$u \equiv (p_4-p_1)^2 = (p_3'-p_2')^2$$

The Mandelstam variables. They are Lorentz invariants, so useful for theoretical calculations but less transparent than energy and angle.
Notice s = the total initial 4-moment~. $(E_1+E_2)^2 - (\vec{p}_1+\vec{p}_2)^2$. In a symmetric collider, $\vec{p}_1 = -\vec{p}_2$, so $s = (E_{TOT})^2$

So a collider's energy is often quoted as,

$$(\text{example Tevatron}) \quad \sqrt{s} = 1.96\,\text{TeV}$$

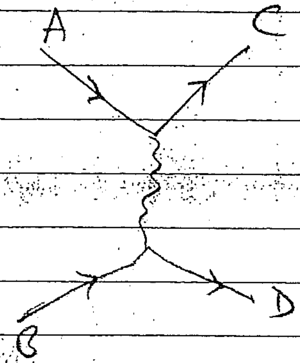
Also $t = (p_3-p_1)^2 = (E_3-E_1)^2 - (\vec{p}_3-\vec{p}_1)^2$

In an elastic collision, $E_1 = E_3$, then

$$t = -(\vec{p}_3-\vec{p}_1)^2 = -(\text{momentum transfer})^2$$

1) Jargon:

Consider $AB \to CD$

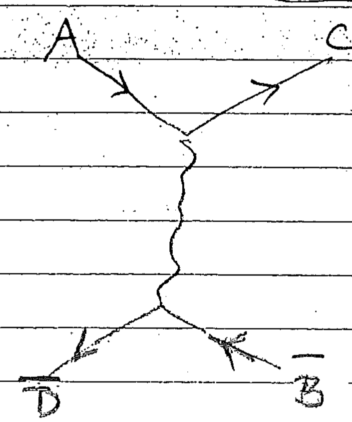


Because $s = (p_A + p_B)^2$ and we named the incoming particles A and B, this is the "s-channel process" among the family of crossed diagrams

(so scattering particles on particle = s channel)

Cross it to become:

$A\bar{D} \to \bar{B}C$



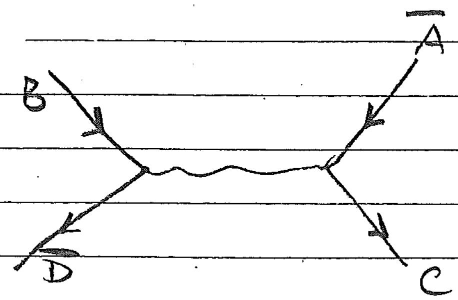
Now A and $\bar{D}$ are incoming

Note $u = (p_A - p_D)^2 = (p_A + (-p_D))^2$

So $\sqrt{u}$ is the center of mass energy for this process, and it is called the "u-channel diagram."

(so scattering particles on antiparticles is u-channel)

Cross again:
(rotate)



B and $\bar{D}$ are incoming, $\bar{A}$ and C are outgoing

$$(\text{c-o-m energy})^2 \text{ is } (p_B - p_D)^2 \text{ or } \underbrace{(p_C - p_A)^2}_{\text{t-channel}}$$

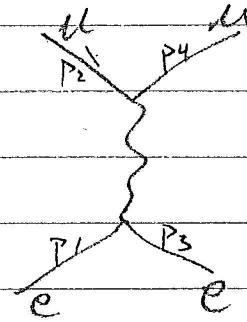
This is the "t-channel process"

So annihilating particle on antiparticle is t-channel

Note also $s+t+u = \sum_{i=1}^{4} m_i^2$

Calculating a QED cross section

Recall we had



Apply Casimir's Trick to $|M|^2$ to get

$$\langle |M| \rangle^2 = \frac{1}{4} \frac{e^2}{(p_1 - p_3)^2} \mathrm{Tr}\left[\gamma^\mu (\not{p}_e^{(1)} + m_e)\gamma^\nu (\not{p}_e^{(3)} + m_e)\right] \cdot$$

$$\mathrm{Tr}\left[\gamma_\mu (\not{p}_M^{(2)} + m_M)\gamma_\nu (\not{p}_M^{(4)} + m_M)\right]$$

average over spins

"Trace 1"

"Trace 2"

capital M means "muon" here.

Expand Trace 1:

$$\underbrace{Tr[\gamma^\mu \not{p}_e(1) \gamma^\nu \not{p}_e(3)]}_{} + m_e Tr[\gamma^\mu \not{p}_e(1)\gamma^\nu] +$$

$$\underset{\displaystyle \overset{\|}{0}}{}$$

$$m_e Tr[\gamma^\mu \gamma^\nu \not{p}_e(3)] + m_e^2 \underbrace{Tr[\gamma^\mu \gamma^\nu]}_{4g^{\mu\nu}}$$

$$\overset{\|}{0}$$

Note $\gamma^\mu \gamma^\nu \not{p} = \gamma^\mu \gamma^\nu p^\alpha \gamma_\alpha$ ← odd # of γ matrices

It can be shown that $Tr[\text{odd \# } \gamma\text{'s}] = 0$

Note $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

Note $Tr(\gamma^\mu \not{p}_e(1) \gamma^\nu \not{p}_e(3)) =$

$$= Tr(\gamma^\mu \gamma^\lambda p_{e\lambda}(1) \gamma^\nu \gamma^\sigma p_{e\sigma}(3))$$

$$= p_{e\lambda}(1) p_{e\sigma}(3) \underbrace{Tr(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma)}_{}$$

$$4(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\nu\lambda})$$

$$= 4[p^\mu p^\nu - g^{\mu\nu}(p_e(1) \cdot p_e(3)) + p_e^\mu(3) p_e^\nu(1)]$$

$$\text{Trace } 1 = 4[p_e^\mu(1) p_e^\nu(3) + p_e^\mu(3) p_e^\nu(1) + g^{\mu\nu}(m^2 - \{p_e(1) \cdot p_e(3)\})]$$

Trace 2 is same with $e \to M$

$1 \to 2$

$3 \to 4$

$\mu, \nu \to \mu, \nu$

Plug in Trace 1 and Trace 2 to get

$$\langle |\mathcal{M}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^2}\Big[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)$$

"$p_1 \cdot p_2$" comes from $p_1^\mu p_{2\mu}$

$$-(p_1\cdot p_3)m_M^2 - (p_2\cdot p_4)m_e^2 + 2(m_e m_M)^2\Big]$$

Now find the differential cross section.
Note if the "muon" were a fixed target and the "electron" is non-relativistic, we expect this should be the Rutherford formula. Check to see:

Fermi's Golden Rule:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_M}\right)^2 |\mathcal{M}|^2$$

electron before: $p_1 = (E, \vec{p}_1)$
muon before: $p_2 = (m_M, 0)$ — fixed target

electron after: $p_3 = (E, \vec{p}_3)$
muon after: $p_4 = (m_M, 0)$

point particles have no internal structure so scatter must be elastic

Elastic: $|\vec{p}_1| = |\vec{p}_3| \equiv |\vec{p}|$

$$(p_1-p_3)^2 = (E-E)^2 - (\vec{p}_1 - \vec{p}_3)^2 = -\vec{p}_1^{\,2} - \vec{p}_3^{\,2} + 2\vec{p}_1\cdot\vec{p}_3$$
$$= -2p^2 + 2p^2\cos\theta$$
$$= -2p^2(1-\cos\theta)$$
$$= -2p^2\left(2\sin^2\frac{\theta}{2}\right)$$

$$(p_1 \cdot p_3) = p_1^0 p_3^0 - \vec{p}_1 \cdot \vec{p}_3$$

$$= E^2 - \vec{p}_1 \cdot \vec{p}_3$$

$E^2 = m^2 + p^2$

$$= m^2 + p^2 - (p^2 \cos\theta)$$
$$= m^2 + p^2(1 - \cos\theta)$$
$$= m^2 + p^2\left(2\sin^2\frac{\theta}{2}\right)$$

Similarly $(p_1 \cdot p_2)(p_3 \cdot p_4) = (m_M E)^2$

$(p_1 \cdot p_4)(p_2 \cdot p_3) = (m_M E)^2$

$(p_2 \cdot p_4) = m_M^2$

Plug in:

$$\langle |M|^2 \rangle = \left[\frac{e^2 m_M}{p^2 \sin^2\frac{\theta}{2}}\right]^2 \left(m_e^2 + p^2\cos^2\frac{\theta}{2}\right)$$

Note $e^2 = \alpha \cdot 4\pi$

$$\frac{d\sigma}{d\Omega} = \left[\frac{\alpha\hbar}{2p^2 \sin^2\frac{\theta}{2}}\right]^2 \left(m_e^2 + p^2\cos^2\frac{\theta}{2}\right)$$

"Mott x-sectn"

$\downarrow$

Non-rel. approx $p^2 \ll m^2$

$$\left(\frac{e^2}{2mv^2 \sin^2\frac{\theta}{2}}\right)^2$$

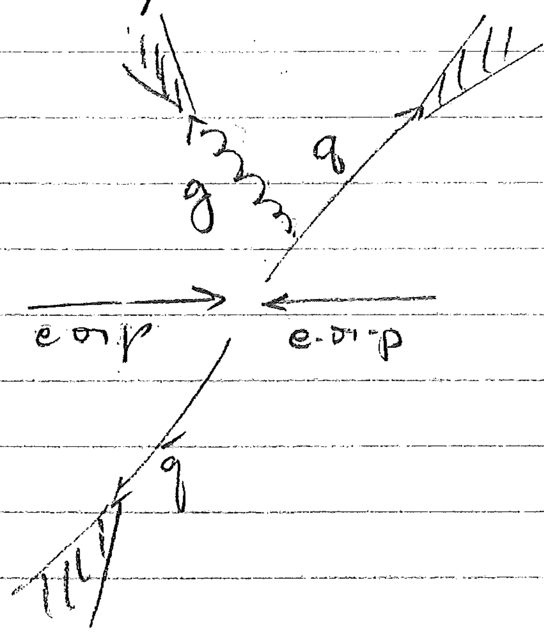
Rutherford

I QCD

The mediator: gluons
Because of confinement, isolated quarks and gluons are not observed.
The bound states that they form preserve the initial parton energy + momentum. These <u>collimated</u> sprays of particles are called <u>jets</u>.

Quarks constantly emit gluons: "gluon splitting"
Evidence for the existence of gluons: 3-jet events

Sau Lan Wu (TASSO Expt)
1979



$$S \equiv \frac{3\left(\sum_i p_{\perp i}^2\right)_{min}}{2\sum_i \vec{p}_i^{\,2}}$$

early method for extracting jets from background: "sphericity"
with development of technology: precision tracking.

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Evidence for Jet Structure in Hadron Production by e^+e^- Annihilation*

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(Received 8 October 1975)

We have found evidence for jet structure in $e^+e^- \to$ hadrons at center-of-mass energies of 6.2 and 7.4 GeV. At 7.4 GeV the jet-axis angular distribution integrated over azimuthal angle was determined to be proportional to $1+(0.78\pm0.12)\cos^2\theta$.

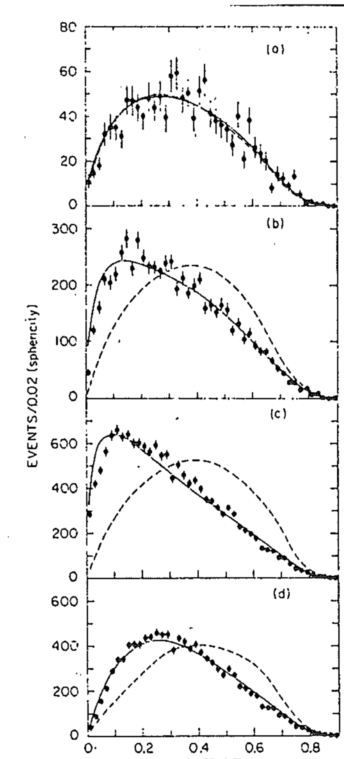

FIG. 2. Observed sphericity distributions for data, jet model with $\langle p_\perp \rangle = 315$ MeV/c (solid curves), and phase-space model (dashed curves) for (a) $E_{c.m.} = 3.0$ GeV; (b) $E_{c.m.} = 6.2$ GeV; (c) $E_{c.m.} = 7.4$ GeV; and (d) $E_{c.m.} = 7.4$ GeV, events with largest $x < 0.4$. The distributions for the Monte Carlo models are normalized to the number of events in the data.

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EVIDENCE FOR PLANAR EVENTS IN e^+e^- ANNIHILATION AT HIGH ENERGIES

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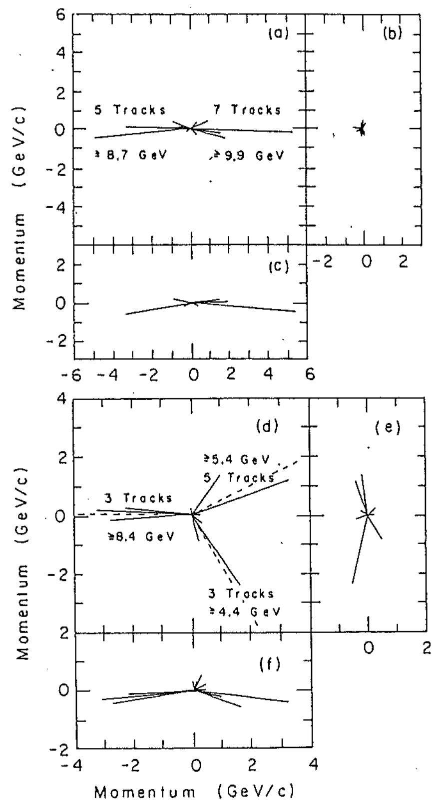

Fig. 6. Momentum space representation of a two-jet event (a)–(c) and a three-jet event (d)–(f) in each of three projections. (a), (d) $\hat{n}_2$–$\hat{n}_3$ plane; (b), (e) $\hat{n}_1$–$\hat{n}_2$ plane; (c), (f) $\hat{n}_1$–$\hat{n}_3$ plane.

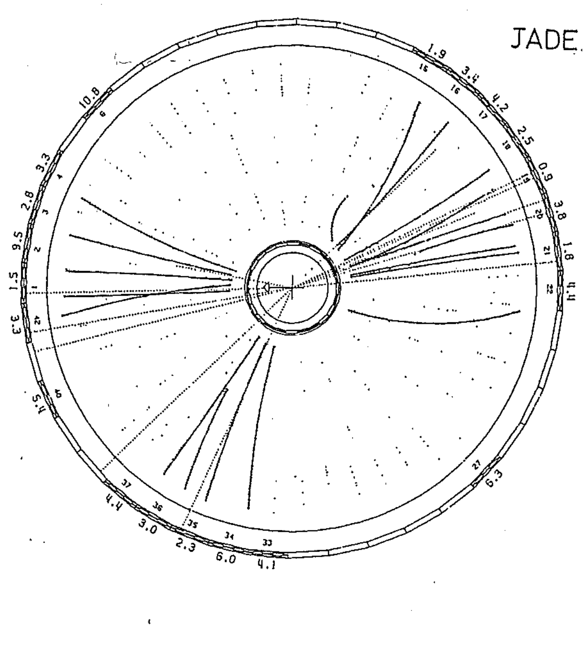

Figure 10.4: A three-jet event measured by the JADE Collaboration, viewed along the beam axis. [P. Söding and G. Wolf, *Ann. Rev. Nucl. Part. Sci.*, **31**, 231 (1981).]