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Controlled-joint remote implementation of operators and its possible generalization

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Abstract. The existing notion of the shared entangled state-assisted remote preparation of unitary operator (equivalently the existing notion of quantum remote control) using local operation and classical communication is generalized to a scenario where under the control of a supervisor two users can jointly implement arbitrary unitaries (one unknown unitary operation by each or equivalently a single unitary decomposed into two unitaries of the same dimension and given to two users) on an unknown quantum state available with a geographically separated user. It is explicitly shown that the task can be performed using a four-qubit hyperentangled state, which is entangled simultaneously in both spatial and polarization degrees of freedom of photons. The proposed protocol which can be viewed as primitive for distributed photonic quantum computing is further generalized to the case that drops the restrictions on the number of controllers and the number of parties performing unitaries and allows both the numbers to be arbitrary. It is also shown that all the existing variants of quantum remote control schemes can be obtained as special cases of the present scheme.

1 Introduction

Entanglement is known to be an important resource for quantum computing and communication. The importance of the entangled state underlies in the fact that it (along with a slightly stronger version of it called nonlocal states) can be used to perform various tasks that cannot be done in the classical world. For example, restricting us to the context of the present work, we may mention that entanglement can be used as a resource to realize the so-called quantum teleportation—an idea introduced by Bennett et al. in 1993 [1]. In a conventional teleportation scheme, an unknown single-qubit quantum state is transferred from one place to another faraway place without physically sending the qubit itself with the help of a shared bipartite entangled state, local operations and 2 bits of classical communication. Later, Pati et al. [2] showed that a known quantum states can be teleported to a receiver using an entangled state and a classical bit only. Such a scheme for teleportation of a known quantum state is known as quantum remote state preparation (RSP). Subsequently, the introduction of the concept of RSP in 2000 led to two different kinds of research interests. On the one hand, several variants of RSP (e.g., controlled remote state preparation, joint remote state preparation, controlled-joint remote state preparation, bidirectional remote state preparation) have been proposed (see [3] and references therein); on the other hand, a dedicated effort has been made to address the question: In analogy to RSP can we remotely prepare a quantum operation? The question was answered in the affirmative by Huelga et al. in 2001 [4], and it was shown that quantum operation can be prepared remotely using shared entanglement along with local operation and classical communication (LOCC). Such a remote realization of quantum operations using shared entanglement is referred to as the quantum remote control (some authors have referred to it as the remote implementation of an operator (RIO), too.) Before we proceed further it would be apt to note that any scheme for bidirectional quantum state teleportation [5,6] can be trivially used for implementing a scheme for RIO. This can be visualized easily if we consider that Bob wishes to implement an arbitrary operator $U_{\rm B}$ remotely on a quantum state $|\psi\rangle$ available with Alice. Now, Alice may teleport the state $|\psi\rangle$ to Bob and he may apply his operation on the state received to yield $|\psi'\rangle = U_{\rm B}|\psi\rangle$ and teleport the state $|\psi'\rangle$ to Alice. This trivial scheme would require

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at least two copies of Bell states and 4 bits of classical communication. This sets a kind of upper limit on the resource requirement as the intentional use of a higher amount of resource will make no sense. Now, a scheme of efficient RIO would require a lesser amount of resources, and in Ref. [7], it is shown that RIO can be implemented using two copies of Bell states and 4 bits of classical communication, and the same is a minimal requirement. In [7], it was also shown that if U_{B} mentioned above belongs to certain classes of unitary operators, then the task (i.e., RIO) can be implemented using 1 Bell state and 2 bits of classical communication only. In 2006, one of the present authors came up with an idea of remote implementation of a hidden operator [8]: The necessary operator is hidden in a lump operator given to the implementer who can locally manipulate the lump operator as a whole only. Several variants of such remote implementation of hidden operator have also been proposed such as controlled remote implementation of partially unknown quantum operation [9], cyclic controlled remote implementation of partially unknown quantum operations [10] and double-direction cyclic controlled remote implementation of partially known quantum operations [11]. It is interesting to note that a specific version of RIO is experimentally realized. Specifically, remote implementation of a rotation angle was experimentally demonstrated in [12].

Since the introduction of the concept of quantum remote control, several variants of it have been proposed that are analogous to the variants of RSP. For example, controlled remote implementation of operator (CRIO)[1] [13] and joint remote implementation of operator (JRIO) [14] have been proposed. The proposed schemes utilize different types of quantum resources. For example, in [13–15] hyperentanglement is used for RIO, CRIO and JRIO and in [16] graph state is used for CRIO. Thus, the schemes for RIO, CRIO and JRIO have already been studied with reasonable rigor, but no scheme for controlled-joint remote implementation of operator (CJRIO) has yet been proposed. This has motivated us to look into the possibility of designing such a protocol. Also, there is an additional motivation, and the scheme for CJRIO can be easily reduced to the schemes for JRIO, RIO and CRIO. Further, such a scheme can be of use in distributed quantum computing requiring nonlocal operation (see [17] and references therein) as well as in the quantum networks in general and quantum internet in particular. For example, in Ref. [18] a device architecture for distributed quantum computing is proposed which is very apt for the present situation where only noisy intermediate quantum computers are available. As the available quantum computers are small in size, RIO will be essential in all such situations where the number of qubits required to perform a specific computational task exceeds the number of qubits that can be stored and compiled in a single quantum computer [18]. Now, as the task can be distributed over a large number of small quantum computers, JRIO is a requirement. Further, a master–slave architecture is often used in traditional distributed computing, where a master node (user) acts as the central control unit that receives tasks from clients and distributes the task among slave nodes. In our situation, slave nodes are located at different quantum computers and the master node is referred to as the controller, leading to an analogous situation in the quantum world leading to the requirement of CJRIO. Now, in the classical world, remote operations mentioned here are usually referred to as teleoperations and there exist schemes for teleoperations that involve multiple masers (i.e., controllers in our case) [19]. A quantum analog of such a scheme would be a generalized version of CJRIO allowing multiple controllers (master nodes). Interestingly, this need and the fact that no scheme for CJRIO (independent of the number of controllers) exists, motivated us to design a scheme for CJRIO with a single controller first and then to generalize that to a multiple controller situation.

In this paper, we have first proposed a scheme for CJRIO using an entangled state and LOCC. It is explicitly shown that CJRIO can be realized using a four-qubit hyperentangled state, which is entangled in spatial and polarization degree of freedom of photons. The preparation of such a hyperentangled state can be found in references [20–23]. The proposed protocol is also generalized to the case that allows an arbitrary number of controllers and an arbitrary number of parties to perform unitaries. This is not only the most generalized version of quantum remote control, it can also be reduced to all the existing variants of quantum remote control schemes. Specifically, schemes for RIO, CRIO, JRIO, etc., can be obtained as special cases of the CJRIO schemes proposed here.

The rest of the paper is organized as follows. In Sect. 2, we briefly describe the task that we wish to perform here. Thereafter, in Sect. 3, we propose a scheme for CJRIO using a four-qubit hyperentangled state, which is entangled at the same time in double degrees of freedom—the spatial and the polarization ones. Subsequently, in Sect. 4 we have generalized our protocol to the case where any number of parties can jointly prepare the quantum unitary in the supervision of an arbitrary number of controllers. The process for reducing our proposed scheme for CJRIO into the existing variants of RIO is described in Sect. 5 with specific attention to the efficiency of the CJRIO scheme and the schemes that can be obtained as special cases of CJRIO. Finally, in Sect. 6 the relevance of the proposed protocols is discussed and the paper is concluded.

[1] Note that if we use particle order permutation technique as described and utilized in [5] and the scheme of [7], a scheme of CRIO of an arbitrary operator would require 2 Bell states and 4 bits of classical communication, whereas a trivial scheme for CRIO obtained by modifying an efficient scheme for controlled bidirectional quantum teleportation [5] would require one more classical bit.

2 The task of interest

The idea of CJRIO is to jointly and controllably operate an unknown quantum operation on an unknown quantum state at different nodes. The task can be visualized as one involving four spatially separated parties named Alice, Bob^1, Bob^2 and Charlie. Here, we consider that an arbitrary unitary U which can be decomposed as $U = U_{\rm B}^1.U_{\rm B}^2$ is implemented by Bob^1 and Bob^2 jointly on an arbitrary quantum state $|\psi\rangle_X$ available with Alice under supervision of a controller Charlie. For a generalized view, we consider that the operators which Bob^1 and Bob^2 wish to operate are $U_{\rm B}^1$ and $U_{\rm B}^2$, respectively, the form of which can be given as follows:

$$U_{\rm B}^1 = \begin{pmatrix} u_B^1 & v_B^1 \\ -v_B^{1*} & u_B^{1*} \end{pmatrix}, \qquad (1)$$

$$U_{\rm B}^2 = \begin{pmatrix} u_B^2 & v_B^2 \\ -v_B^{2*} & u_B^{2*} \end{pmatrix}. \qquad (2)$$

Note that the above operators are the most general in the sense that they describe all possible rotations of the qubit (the state in possession of Alice on which Bob's operator is to be implemented remotely). This is so as the arbitrary rotation of a qubit can be represented by an unimodular matrix of the form $U = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}$ such that $|u|^2 + |v|^2 = 1, u, v \in \mathbb{C}$. A set of all such unitaries forms the SU(2) group, and in the present work unitary operations of the same form are used. As mentioned above, Alice who is spatially separated from Bob^1 and Bob^2 has an unknown quantum state $|\psi\rangle_X$ of the following form:

$$|\psi\rangle_X = (\alpha|x_0\rangle + \beta|x_1\rangle)_X|{\rm V}\rangle_X \qquad (3)$$

where α and β are unknown coefficients which satisfies the normalization condition $|\alpha|^2 + |\beta|^2 = 1$, and $|V\rangle_X$ describing the polarization state of photon. Physically, it can be viewed as if Alice has photon indexed by X which is vertically polarized and is in spatial superposition state of $|x_0\rangle$ and $|x_1\rangle$.

The action of unitary operator $U_{\rm B}^1$ on $|\psi\rangle_X$ can be described as $|\psi_{B^1}\rangle = U_{\rm B}^1|\psi\rangle_X = \alpha_{B^1}|x_0\rangle + \beta_{B^1}|x_1\rangle$ with $\alpha_{B^1} = \alpha u_B^1 + \beta v_B^1$ and $\beta_{B^1} = -\alpha v_B^{*1} + \beta u_B^{*1}$. Further, the action of unitary operator $U_{\rm B}^2$ on $|\psi\rangle_X$ can be described as $|\psi_{B^2}\rangle = U_{\rm B}^2|\psi\rangle_X = \alpha_{B^2}|x_0\rangle + \beta_{B^2}|x_1\rangle$ with $\alpha_{B^2} = \alpha u_B^2 + \beta v_B^2$ and $\beta_{B^2} = -\alpha v_B^{*2} + \beta u_B^{*2}$.

The task of concern is that Bob^1 and Bob^2 should remotely apply their operators on Alice's state, which can be mathematically represented as

$$\begin{aligned}|\psi_{B^1B^2}\rangle &= U_{\rm B}^1 U_{\rm B}^2|\psi\rangle_X \\ &= (\alpha_{B^1B^2}|x_0\rangle + \beta_{B^1B^2}|x_1\rangle)|{\rm V}\rangle_X\end{aligned} \qquad (4)$$

where $\alpha_{B^1B^2} = \alpha_{B^2}u_B^1 + \beta_{B^2}v_B^1$ and $\beta_{B^1B^2} = -\alpha_{B^2}v_B^{*1} + \beta_{B^2}u_B^{*1}$.

The quantum channel used here to perform the task of concern is a four-qubit hyperentangled state given as

$$|Q\rangle_{AB^1B^2C} = |Q^S\rangle_{AB^1B^2C}|Q^P\rangle_{AB^1B^2C} \qquad (5)$$

where

$$\begin{aligned}|Q^S\rangle_{AB^1B^2C} &= |a_0\rangle_A|b_0^1\rangle_{B^1}|b_0^2\rangle_{B^2}|c_0\rangle_C \\ &\quad + |a_1\rangle_A|b_1^1\rangle_{B^1}|b_1^2\rangle_{B^2}|c_1\rangle_C \end{aligned} \qquad (6)$$

$$\begin{aligned}|Q^P\rangle_{AB^1B^2C} &= |{\rm H}\rangle_A|{\rm H}\rangle_{B^1}|{\rm H}\rangle_{B^2}|{\rm H}\rangle_C \\ &\quad + |{\rm V}\rangle_A|{\rm V}\rangle_{B^1}|{\rm V}\rangle_{B^2}|{\rm V}\rangle_C \end{aligned} \qquad (7)$$

with a_j, b_j^1, b_j^2, c_j ($j = 0, 1$) the spatial paths, while H and V the horizontal and vertical polarization. The superscript S denotes the spatial degree of freedom (S-DOF), and P denotes the polarization degree of freedom (P-DOF). It is to be noted that in Eqs. (6) and (7) the factor of normalization $1/\sqrt{2}$ is omitted. The labeling A, B^1, B^2 and C in the subscript denotes photon state with Alice, Bob^1, Bob^2, and Charlie, respectively. Bob^1 and Bob^2 try to implement an arbitrary unitary operation on an unknown state at Alice's node with all the participants physically far apart from each other.

3 Protocols for controlled-joint remote implementation of operators

The combined state of Alice's state and quantum channel can be written as

$$|\psi\rangle_X|Q^{SP}\rangle_{AB^1B^2C} = |\phi^S\rangle_{XAB^1B^2C}|{\rm V}\rangle_X|Q^P\rangle_{AB^1B^2C} \qquad (8)$$

where

$$\begin{aligned}|\phi^S\rangle &= (\alpha|x_0\rangle + \beta|x_1\rangle)_X \otimes (|a_0\rangle_A|b_0^1\rangle_{B^1}|b_0^2\rangle_{B^2}|c_0\rangle_C \\ &\quad + |a_1\rangle_A|b_1^1\rangle_{B^1}|b_1^2\rangle_{B^2}|c_1\rangle_C)\end{aligned} \qquad (9)$$

We will now only consider S-DOF of the combined state and will come back to P-DOF in Sect. 3.2.

3.1 Utilizing S-DOF

Step 1 The first step is to entangle photon X with the remaining photons of the quantum channel. To do so, Alice prepares an auxiliary coherent state (CS) $|z\rangle$ and lets it interact with one of the path of photon X (here $|x_0\rangle$) and photon A (here $|a_0\rangle$) via cross-Kerr nonlinear interaction[2] [24,25] with interaction parameters θ and

[2] The cross-Kerr nonlinear interaction between an auxiliary coherent state $|z\rangle$ ($|z\rangle = \exp(-|z|^2/2)\sum_{n=0}^{\infty}(z^n/\sqrt{n!}|n\rangle)$ where $|n\rangle$ is a Fock state containing n photons) and a photon path lets say $|b\rangle$ with interaction parameters θ and $-\theta$ is mathematically represented as $K_b(\pm\theta)|z\rangle|b\rangle = |ze^{\pm i\theta}\rangle|b\rangle$. The X-quadrature homodyne detection technique is used to

$-\theta$ ($K_{x_0}(\theta)$ and $K_{a_0}(-\theta)$), respectively. This transformation can be shown with the following equation.

$$\begin{aligned}|\xi\rangle &= (\alpha|x_0\rangle_X|a_0\rangle_A|b_0^1\rangle_{B^1}|b_0^2\rangle_{B^2}|c_0\rangle_C \\ &+ \beta|x_1\rangle_X|a_1\rangle_A|b_1^1\rangle_{B^1}|b_1^2\rangle_{B^2}|c_1\rangle_C)|z\rangle \\ &+ (\alpha|x_0\rangle_X|a_1\rangle_A|b_1^1\rangle_{B^1}|b_1^2\rangle_{B^2}|c_1\rangle_C \\ &+ \beta|x_1\rangle_X|a_0\rangle_A|b_0^1\rangle_{B^1}|b_0^2\rangle_{B^2}|c_0\rangle_C)|ze^{\pm i\theta}\rangle\end{aligned} \tag{10}$$

Now, the measurement of the coherent state gives two possible outcomes $k = 0$ (1) corresponding to $|z\rangle$ ($|ze^{\pm i\theta}\rangle$). Alice's measurement outcome changes the state in Eq. (9) to the following:

$$\begin{aligned}|\xi_k\rangle &= \alpha|x_0\rangle_X|a_k\rangle_A|b_k^1\rangle_{B^1}|b_k^2\rangle_{B^2}|c_k\rangle_C \\ &+\beta|x_1\rangle_X|a_{k\oplus1}\rangle_A|b_{k\oplus1}^1\rangle_{B^1}|b_{k\oplus1}^2\rangle_{B^2}|c_{k\oplus1}\rangle_C\end{aligned} \tag{11}$$

with $\oplus$ denoting an addition mod 2. Now, one can see from Eq. (11) that the photon X is entangled with the remaining photons in the quantum channel.

This step can be visualized in pictorial form as shown in Fig. 1 as Step 1. Photons are labeled as X, A, B^1, B^2 and C. Photons X and A are with Alice, and photons B^1, B^2 and C are with Bob^1, Bob^2 and Charlie, respectively.

Step 2 Then, Alice tries to disentangle her photons X and A from the remaining photons in S-DOF. To do so, Alice mixes the two spatial paths $|a_k\rangle$ and $|x_1\rangle$ ($|a_k\rangle$ and $|a_{k\oplus1}\rangle$) of her photon X (A) on a balanced beam splitter (BBS) (for more details, see Sect. 2 of [26] or that of [13].) The BBS transformation rule is (up to a normalization factor) $|\sigma_j\rangle \rightarrow |\sigma_j\rangle + (-1)^j|\sigma_{j\oplus1}\rangle$. After mixing the photons on the BBS, the state in Eq. (11) reduces to:

$$\begin{aligned}|\xi_k^{'}\rangle &= (|x_0\rangle|a_k\rangle + (-1)^k|x_1\rangle|a_{k+1}\rangle) \otimes (\alpha|b_k^1\rangle||b_k^2\rangle|c_k\rangle \\ &+ (-1)^k\beta|b_{k+1}^1\rangle|b_{k+1}^2\rangle|c_{k+1}\rangle) \\ &+ (|x_0\rangle|a_{k+1}\rangle + (-1)^k|x_1\rangle|a_k\rangle) \\ &\otimes (\alpha|b_k^1\rangle||b_k^2\rangle|c_k\rangle \\ &- (-1)^k\beta|b_{k+1}^1\rangle|b_{k+1}^2\rangle|c_{k+1}\rangle)\end{aligned} \tag{12}$$

It can be seen from Eq. (12) that photons X and A are still not separated from the remaining photons in the channel. To get them separated, Alice again uses an auxiliary CS $|z\rangle$ and turns on the cross-Kerr nonlinear interaction with photon X on path $|x_0\rangle$ and photon A on path $|a_k\rangle$ with interaction parameters θ and 2θ, respectively. Alice then measures the X-quadrature of the CS whose measurement outcomes are $mn = 00, 01, 10$ and 11 corresponding to $|z\rangle$, $|ze^{i\theta}\rangle$, $|ze^{i2\theta}\rangle$ and $|ze^{i3\theta}\rangle$, respectively. After the measurement, the state becomes:

$$\begin{aligned}|\xi_{kmn}\rangle &= |x_{n\oplus1}\rangle|a_{k\oplus m\oplus1}\rangle(\alpha|b_k^1\rangle|b_k^2\rangle|c_k\rangle \\ &+(-1)^{k\oplus m\oplus n}\beta|b_{k\oplus1}^1\rangle|b_{k\oplus1}^2\rangle|c_{k\oplus1}\rangle)\end{aligned} \tag{13}$$

The photons X and A are now separated in S-DOF from the remaining photons in the channel, which can be seen from Eq. (13). To avoid complexity, we may forget photon X and Eq. (13) can be written as:

$$\begin{aligned}|\Xi_{kmn}\rangle &= |a_{k\oplus m\oplus1}\rangle(\alpha|b_k^1\rangle|b_k^2\rangle|c_k\rangle \\ &+(-1)^{k\oplus m\oplus n}\beta|b_{k\oplus1}^1\rangle|b_{k\oplus1}^2\rangle|c_{k\oplus1}\rangle)\end{aligned} \tag{14}$$

This step can be visualized in pictorial form as shown in Fig. 1 as Step 2.

Step 3 Now, let us understand the role of controller Charlie. If Charlie wants to stop the joint operation, then she does nothing; otherwise, she mixes her photon path states $|c_k\rangle$ and $|c_{k\oplus1}\rangle$ on a BBS. It is to be noted that until Charlie mixes her photon paths, the joint parties are unable to apply the correct unitary operation. After mixing photon paths, Eq. (14) transforms into the following:

$$\begin{aligned}|\Xi_{kmn}^{'}\rangle &= |a_{k\oplus m\oplus1}\rangle[(\alpha|b_k^1\rangle|b_k^2\rangle \\ &+ (-1)^{m\oplus n}\beta|b_{k\oplus1}^1\rangle|b_{k\oplus1}^2\rangle)|c_k\rangle \\ &+ (-1)^k(\alpha|b_k^1\rangle|b_k^2\rangle \\ &- (-1)^{m\oplus n}\beta|b_{k\oplus1}^1\rangle|b_{k\oplus1}^2\rangle)|c_{k\oplus1}\rangle]\end{aligned} \tag{15}$$

She will then take an arbitrary CS $|z\rangle$ and let it interact with one of her photon path states $|c_k\rangle$ via cross-Kerr nonlinear interaction with interaction parameter θ and then measure it. Let the measurement outcome be $s = 0$ (1) corresponding to $|z\rangle$ ($|ze^{i\theta}\rangle$). After the measurement, the state in Eq. (15) transforms into the following:

$$\begin{aligned}|\Xi_{kmns}\rangle &= |a_{k\oplus m\oplus1}\rangle(\alpha|b_k^1\rangle|b_k^2\rangle \\ &-(-1)^{m\oplus n\oplus s}\beta|b_{k\oplus1}^1\rangle|b_{k\oplus1}^2\rangle)|c_{k\oplus s\oplus1}\rangle\end{aligned} \tag{16}$$

Charlie's photon path is now separated from the remaining photon paths. By doing so, Charlie is allowing the joint parties Bob^1 and Bob^2 to perform the joint operations.

This step can be visualized in pictorial form as shown in Fig. 2 as Step 3

measure whether the coherent state is in $|z\rangle$ or $|ze^{\pm i\theta}\rangle$. It is to be noted that $|ze^{i\theta}\rangle$ and $|ze^{-i\theta}\rangle$ are indistinguishable with this kind of measurement.

Step 4 At this stage, Bob^1 and Bob^2 jointly decide who will implement the operation first. Let Bob^2 will implement the operation first. Bob^2 is able to implement his operation $U_{\rm B}^2$ if Bob^1 mixes the spatial states $|b_k^1\rangle$ and $|b_{k\oplus 1}^1\rangle$ of his photon B^1 and lets one of the path (say $|b_k^1\rangle$) to interact with an arbitrary CS $|z\rangle$ via cross-Kerr nonlinear interaction with interaction parameter θ and measures it. Consider the measurement outcome be $l=0$ (1) corresponding to $|z\rangle$ ($|ze^{i\theta}\rangle$). The new state after the measurement of Bob^1 is given as follows:

$$|\Xi_{kmnsl}\rangle = |a_{k\oplus m\oplus 1}\rangle|b_{k\oplus l\oplus 1}^1\rangle(\alpha|b_k^2\rangle +(-1)^{k\oplus m\oplus n\oplus s\oplus l}\beta|b_{k\oplus 1}^2\rangle)|c_{k\oplus s\oplus 1}\rangle \tag{17}$$

It can be clearly seen from Eq. (17) that the coefficient α and β which was initially with photon X has shifted toward photon B^2. Bob^2 will now recover the state $\alpha|b_0^2\rangle+\beta|b_1^2\rangle$ by applying the appropriate unitary operation $Z_S^{k\oplus m\oplus n\oplus s\oplus l}X_S^k$, where $X_S=|b_0^2\rangle\langle b_1^2|+|b_1^2\rangle\langle b_0^2|$ and $Z_S=|b_0^2\rangle\langle b_0^2|-|b_1^2\rangle\langle b_1^2|$. Once Bob^2 recovers the state, then he will implement the operation $U_{\rm B}^2$ on it, which will transform the state into the following:

$$|\Lambda_{kmsl}\rangle = |a_{k\oplus m\oplus 1}\rangle|b_{k\oplus l\oplus 1}^1\rangle(\alpha_{B^2}|b_0^2\rangle +\beta_{B^2}|b_1^2\rangle)|c_{k\oplus s\oplus 1}\rangle \tag{18}$$

This step can be visualized in pictorial form as shown in Fig. 2 as Step 4.

Step 5 The path of photon B^1 got separated in the previous step which we need to bring back into the spatial superposition by passing through a BBS. When $|b_{k\oplus l\oplus 1}^1\rangle$ passes through a BBS, it triggers a new path $|b_{k\oplus l}^1\rangle$. Behind the BBS, Bob^1 picks one path (say $|b_{k\oplus l\oplus 1}^1\rangle$) and lets it interact with an auxiliary CS $|z\rangle$ via cross-Kerr interaction with interaction parameter θ and forwards the CS to Bob^2, which interacts with path $|b_0^2\rangle$ via cross-Kerr interaction with interaction parameter $-\theta$. After the interaction, Bob^2 measures the CS, whose measurement outcome is $r=0$ (1) corresponding to $|z\rangle$ ($|ze^{\pm i\theta}\rangle$). The measurement of Bob^2 here transforms the state in Eq. (18) into a new state $|\Lambda_{kmslr}\rangle$ which can be written as follows:

$$|\Lambda_{kmslr}\rangle = |a_{k\oplus m\oplus 1}\rangle(\alpha_{B^2}|b_{k\oplus l\oplus r\oplus 1}^1\rangle|b_0^2\rangle +(-1)^{k\oplus l\oplus 1}\beta_{B^2}|b_{k\oplus l\oplus r}^1\rangle|b_1^2\rangle)|c_{k\oplus s\oplus 1}\rangle \tag{19}$$

This step can be visualized in pictorial form as shown in Fig. 3 as Step 5.

Step 6 In Step 4, Bob^1 cooperates with Bob^2 to implement his operation $U_{\rm B}^2$. Now, its time for Bob^2 to cooperate with Bob^1 by mixing his photon path states $|b_0^2\rangle$ and $|b_1^2\rangle$ on a BBS and turning on the cross-Kerr interaction between $|b_1^2\rangle$ and an auxiliary CS $|z\rangle$ with interaction parameter θ. After the interaction, Bob^2 measures the CS whose measurement outcomes are $g=0$ (1) corresponding to $|z\rangle$ ($|ze^{i\theta}\rangle$). The measurement of Bob^2 transforms the state $|\Lambda_{kmslr}\rangle$ to $|\Lambda_{kmslrg}\rangle$ which is given as follows:

$$|\Lambda_{kmslrg}\rangle = |a_{k\oplus m\oplus 1}\rangle(\alpha_{B^2}|b_{k\oplus l\oplus r\oplus 1}^1\rangle +(-1)^{k\oplus l\oplus g\oplus 1}\beta_{B^2}|b_{k\oplus l\oplus r}^1\rangle)|b_g^2\rangle|c_{k\oplus s\oplus 1}\rangle \tag{20}$$

The coefficient α_{B^2} and β_{B^2}, which was initially with Bob^2, has now shifted to Bob^1. To recover the original path of photon B^1, Bob^1 will apply an appropriate unitary $Z_S^{k\oplus l\oplus g\oplus 1}X_S^{k\oplus l\oplus r\oplus 1}$ on his photon. After the recovery, the state will become

$$|\Lambda'_{kmslrg}\rangle = |a_{k\oplus m\oplus 1}\rangle(\alpha_{B^2}|b_0^1\rangle +\beta_{B^2}|b_1^1\rangle)|b_g^2\rangle|c_{k\oplus s\oplus 1}\rangle \tag{21}$$

Bob^1 can now implement his unitary operation $U_{\rm B}^1$ on his photon B^1 that transforms the state to $|\Lambda''_{kmslrg}\rangle$ which is given as

$$|\Lambda''_{kmslrg}\rangle = |a_{k\oplus m\oplus 1}\rangle(\alpha_{B^1B^2}|b_0^1\rangle +\beta_{B^1B^2}|b_1^1\rangle)|b_g^2\rangle|c_{k\oplus s\oplus 1}\rangle \tag{22}$$

It is noted that $\alpha_{B^1B^2}|b_0^1\rangle+\beta_{B^1B^2}|b_1^1\rangle = U_{\rm B}^1U_{\rm B}^2(\alpha|b_0^1+\beta|b_1^1)$. But the task is not completed yet, the coefficients $\alpha_{B^1B^2}$ and $\beta_{B^1B^2}$ need to be transferred to Alice's node. To do so, the communicating parties will now use their P-DOF which is described in Sect. 3.2.
This step can be visualized in pictorial form as shown in Fig. 3 as Step 6.

3.2 Utilizing P-DOF

Taking into account the P-DOF, the combined state after Step 6 can now be written as follows:

$$|\phi^P\rangle = |\Lambda''_{kmslrg}\rangle|Q^P\rangle \tag{23}$$

The expanded form of Eq. (23) can be written as

$$\begin{aligned}|\phi^P\rangle = |a_{k\oplus m\oplus 1}\rangle[&\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H},b_0^1\rangle_{B^1}|\mathrm{H},b_g^2\rangle_{B^2}|\mathrm{H}\rangle_C\\ &+\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V},b_0^1\rangle_{B^1}|\mathrm{V},b_g^2\rangle_{B^2}|\mathrm{V}\rangle_C\\ &+\beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H},b_1^1\rangle_{B^1}|\mathrm{H},b_g^2\rangle_{B^2}|\mathrm{H}\rangle_C\\ &+\beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V},b_1^1\rangle_{B^1}|\mathrm{V},b_g^2\rangle_{B^2}|\mathrm{V}\rangle_C]|c_{k\oplus s\oplus 1}\rangle\end{aligned} \tag{24}$$

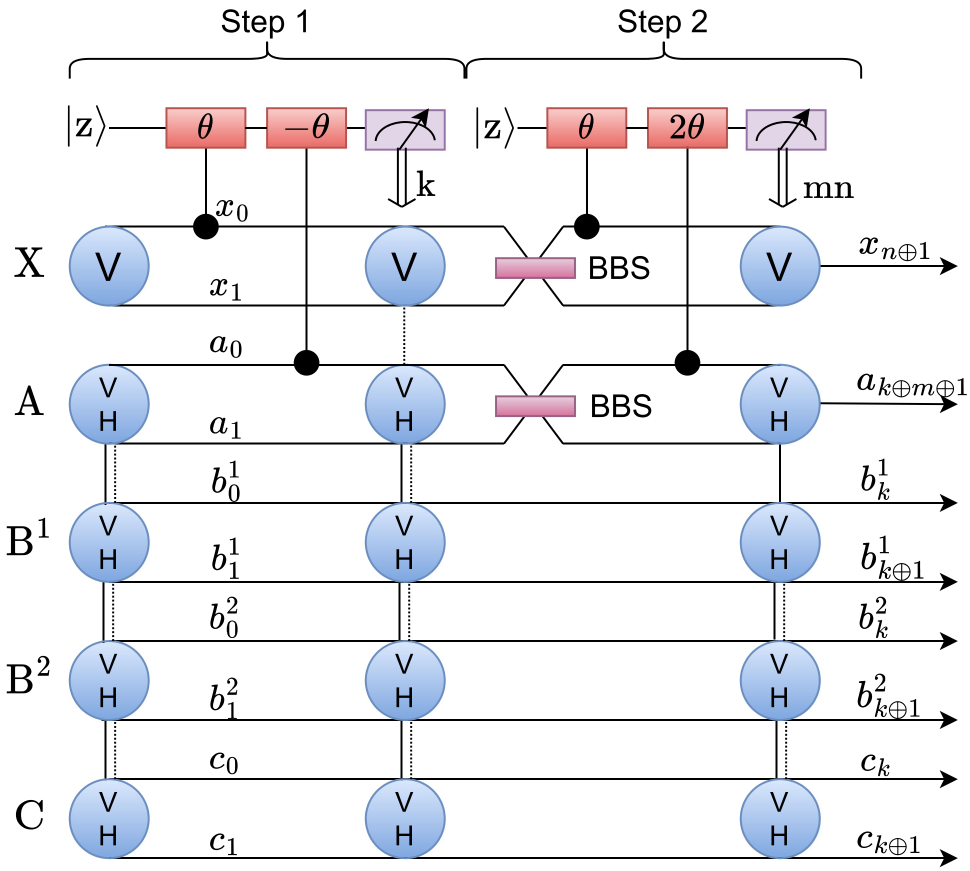

Fig. 1 A schematic which represents first two steps of the CJRIO protocol. A circle with V, H represents photon simultaneously in vertical and horizontal polarization, and circle with V only represents photon in vertical polarization. The two (one) lines attached with circles represent photons having two spatial paths simultaneously (photons having one path only). The cross-Kerr nonlinear interaction between a photon path and the coherent state is attached by a line with a bold dot on the photon path. The dimensionless parameter θ determines the change of phase of the CS brought about by the cross-Kerr interaction. The double arrow from the coherent state represents measurement outcomes. Vertical solid (dashed) line represents entanglement in P-DOF (S-DOF). BBS here is a balanced beam splitter. Here, the photon X first gets entangled with remaining photons by allowing nonlinear interaction $K_{x_0}(\theta)|z\rangle|x_0\rangle$ and $K_{a_0}(-\theta)|z\rangle|a_0\rangle$. The measurement of the CS gives outcome k. Once photon X gets entangled, then Alice tries to pass the coefficient of $|\psi\rangle_X$ to joint parties by mixing her photon paths on BBSs and allowing the interaction $K_{x_0}(\theta)|z\rangle|x_0\rangle$ and $K_{a_k}(2\theta)|z\rangle|a_0\rangle$ and then measures the CSs, whose measurement outcomes are m and n

Here, $|H, b_0^1\rangle$ denotes state of horizontally polarized photon propagating along path b_0^1 and similarly for $|H, b_g^2\rangle$, $|V, b_0^1\rangle$, $|V, b_g^2\rangle$, $|H, b_1^1\rangle$, $|H, b_g^2\rangle$, $|V, b_1^1\rangle$ and $|V, b_g^2\rangle$.

The initial goal was to implement an operator jointly at Alice's node. To achieve the goal, the coefficients $\alpha_{B^1B^2}$ and $\beta_{B^1B^2}$ have to be shifted toward Alice. For that, the joint parties (Bob^1, Bob^2) and the controller (Charlie) will have to measure their photons in appropriate bases.

Step 7 First, the joint parties Bob^1 and Bob^2 measure their photons in an appropriate bases. Before the measurement, Bob^1 puts a half-wave plate (HWF) on path b_1^1 to exchange the photon polarization $|H, b_1^1\rangle \rightleftarrows |V, b_1^1\rangle$ and then mixes the two paths $|b_0^1\rangle$ and $|b_1^1\rangle$ of his photon on a BBS, which transforms the state to the following:

$$\begin{aligned}|\Omega\rangle = |a_{k\oplus m\oplus 1}\rangle[&\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}, b_0^1\rangle_{B^1}|\mathrm{H}, b_g^2\rangle_{B^2}|\mathrm{H}\rangle_C\\ &+\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}, b_1^1\rangle_{B^1}|\mathrm{H}, b_g^2\rangle_{B^2}|\mathrm{H}\rangle_C\\ &+\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}, b_0^1\rangle_{B^1}|\mathrm{V}, b_g^2\rangle_{B^2}|\mathrm{V}\rangle_C\\ &+\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}, b_1^1\rangle_{B^1}|\mathrm{V}, b_g^2\rangle_{B^2}|\mathrm{V}\rangle_C\\ &+\beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{V}, b_0^1\rangle_{B^1}|\mathrm{H}, b_g^2\rangle_{B^2}|\mathrm{H}\rangle_C\\ &-\beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{V}, b_1^1\rangle_{B^1}|\mathrm{H}, b_g^2\rangle_{B^2}|\mathrm{H}\rangle_C\\ &+\beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{H}, b_0^1\rangle_{B^1}|\mathrm{V}, b_g^2\rangle_{B^2}|\mathrm{V}\rangle_C\\ &-\beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{H}, b_1^1\rangle_{B^1}|\mathrm{V}, b_g^2\rangle_{B^2}|\mathrm{V}\rangle_C]|c_{k\oplus s\oplus 1}\rangle\end{aligned} \tag{25}$$

and Bob^2 now puts a quarter wave plate (QWP) on path $|b_g^2\rangle$ to, up to the normalization factor, transform $|H, b_g^2\rangle$ to $(|H, b_g^2\rangle + |V, b_g^2\rangle)$ and $|V, b_g^2\rangle$ to $|H, b_g^2\rangle - |V, b_g^2\rangle$. The transformed state

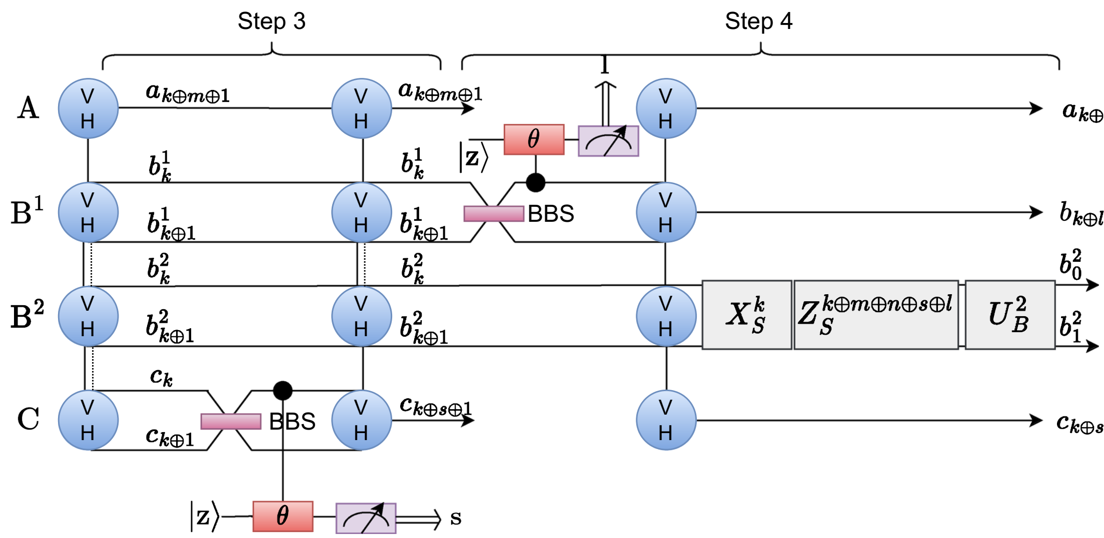

Fig. 2 A schematic which represents Step 3 and Step 4 of the CJRIO protocol. Here, controller Charlie first mixes her photon paths on a BBS and allows the interaction $K_{c_k}(\theta)|z\rangle|c_k\rangle$ and measures the CS, whose measurement outcome is s, which disentangles photon C from remaining photons in S-DOF. After that Bob^1 mixes his photon paths on a BBS and allows the interaction $K_{b^1_k}(\theta)|z\rangle|b^1_k\rangle$ and measures the CS, whose measurement outcome is l, which allows Bob^2 to apply appropriate unitaries to get $\alpha_{B^2}|b^2_0\rangle + \beta_{B^2}|b^2_1\rangle$

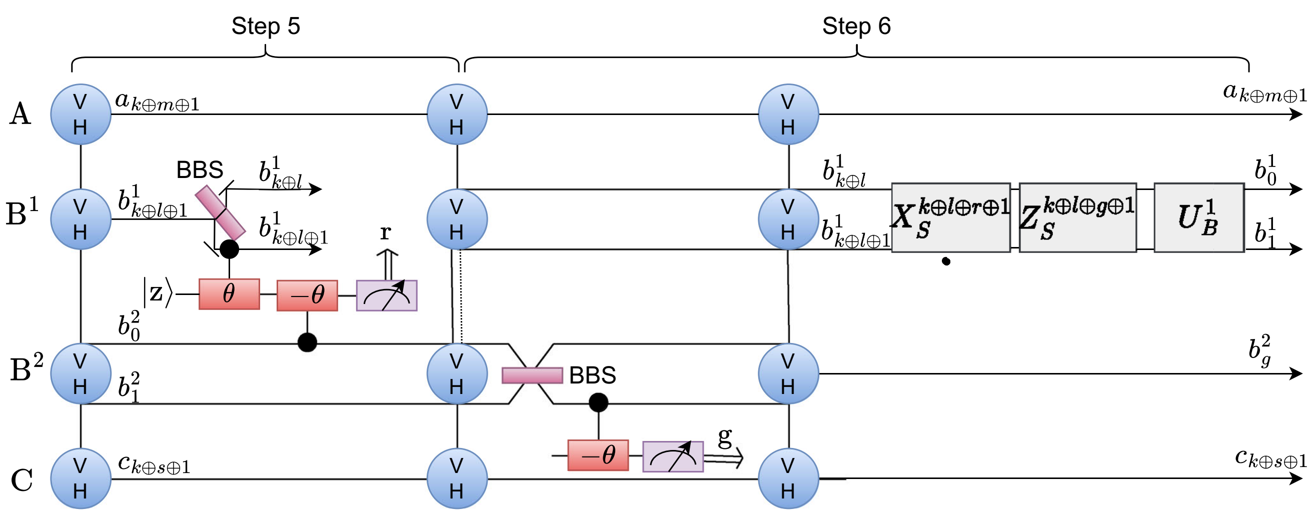

Fig. 3 A schematic which represents Step 5 and Step 6 of the CJRIO protocol. Here, Bob^1 first triggers a new path using a BBS and allows the nonlinear interaction $K_{b^1_{k\oplus l\oplus 1}}(\theta)|z\rangle|b^1_{k\oplus l\oplus 1}\rangle$ and forwards it to Bob^2 which allows the interaction $K_{b^2_0}(-\theta)|z\rangle|b^2_0\rangle$ and measures the CS, whose measurement outcome is r. Bob^2 then mixes two spatial paths of his photon and turns on proper cross-Kerr interaction between a CS and one path of his photon followed by measuring the CS with outcome g as shown in the figure that allows Bob^1 to implement an appropriate unitaries to get $\alpha_{B^1B^2}|b^1_0\rangle + \beta_{B^1B^2}|b^1_1\rangle$

is given as follows:

$$
\begin{aligned}
|\Omega^{'}\rangle = {} & |a_{k\oplus m\oplus 1}\rangle[|\mathrm{H}, b^1_0\rangle_{B^1}|\mathrm{H}, b^2_g\rangle_{B^2} \\
& (\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C + \beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C) \\
& +|\mathrm{H}, b^1_0\rangle_{B^1}|\mathrm{V}, b^2_g\rangle_{B^2} \\
& (\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C - \beta_{B^1B^2}\mathrm{V}\rangle_A|\mathrm{V}\rangle_C) \\
& +|\mathrm{H}, b^1_1\rangle_{B^1}|\mathrm{H}, b^2_g\rangle_{B^2} \\
& (\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C - \beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C) \\
& +|\mathrm{H}, b^1_1\rangle_{B^1}|\mathrm{V}, b^2_g\rangle_{B^2} \\
& (\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C + \beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C) \\
& +|\mathrm{V}, b^1_0\rangle_{B^1}|\mathrm{H}, b^2_g\rangle_{B^2} \\
& (\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C + \beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C) \\
& -|\mathrm{V}, b^1_0\rangle_{B^1}|\mathrm{V}, b^2_g\rangle_{B^2} \\
& (\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C - \beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C) \\
& +|\mathrm{V}, b^1_1\rangle_{B^1}|\mathrm{H}, b^2_g\rangle_{B^2} \\
& (\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C - \beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C)
\end{aligned}
$$

$$-|\mathrm{V}, b_1^1\rangle_{B^1}|\mathrm{V}, b_g^2\rangle_{B^2} (\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C + \beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C)]|c_{k\oplus s\oplus 1}\rangle \tag{26}$$

Now, Bob^1 and Bob^2 will measure their photons in an appropriate basis. Bob^1 (Bob^2) measures his photon B^1 (B^2) in the basis $\{|\mathrm{H}, b_0^1\rangle, |\mathrm{H}, b_1^1\rangle, |\mathrm{V}, b_0^1\rangle, |\mathrm{V}, b_1^1\rangle\}$ ($\{|\mathrm{H}, b_g^2\rangle, |\mathrm{V}, b_g^2\rangle\}$), whose corresponding measurement results are $pq = 00,\ 01,\ 10,\ 11$ ($w = 0,\ 1$). The collapsed state after the measurement is given as follows:

$$|\Omega_{kmpqw}\rangle = |a_{k\oplus m\oplus 1}\rangle \begin{cases} (\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C + \beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C)|c_{k\oplus s\oplus 1}\rangle \text{ for } pqw = 000, 011 \\ (\alpha_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C - \beta_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C)|c_{k\oplus s\oplus 1}\rangle \text{ for } pqw = 010, 001 \\ (\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C + \beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C)|c_{k\oplus s\oplus 1}\rangle \text{ for } pqw = 100, 111 \\ (\alpha_{B^1B^2}|\mathrm{V}\rangle_A|\mathrm{V}\rangle_C - \beta_{B^1B^2}|\mathrm{H}\rangle_A|\mathrm{H}\rangle_C)|c_{k\oplus s\oplus 1}\rangle \text{ for } pqw = 110, 101 \end{cases} \tag{27}$$

This step can be visualized in pictorial form as shown in Fig. 4 as Step 7.

Step 8 The controller Charlie once again uses her power. If she wants to stop the protocol, she does nothing otherwise she places a QWP on her photon path $c_{k\oplus s\oplus 1}$ to rotate the polarization state of her photon state from $|H\rangle_C \rightarrow |H\rangle_C + |V\rangle_C$ and $|V\rangle_C \rightarrow |H\rangle_C - |V\rangle_C$ (normalization is omitted). After that, Charlie lets her photon pass through a polarizing beam splitter (PBS), which transmits photon of horizontal polarization but reflects that of vertical one and measures it in the basis $\{|\mathrm{H}, c_{k\oplus s\oplus 1}\rangle, |\mathrm{V}, c_{k\oplus s\oplus 1}\rangle$, whose corresponding measurement outcome is $v = 0,\ 1$. The collapsed state after Charlie's measurement is given as follows:

$$|\Omega_{kmpqwv}\rangle = |a_{k\oplus m\oplus 1}\rangle \begin{cases} (\alpha_{B^1B^2}|\mathrm{H}\rangle + \beta_{B^1B^2}|\mathrm{V}\rangle) \text{ for } pqwv = 0000, 0101, 0011, 0110 \\ (\alpha_{B^1B^2}|\mathrm{H}\rangle - \beta_{B^1B^2}|\mathrm{V}\rangle) \text{ for } pqwv = 0001, 0100, 0010, 0111 \\ (\alpha_{B^1B^2}|\mathrm{V}\rangle + \beta_{B^1B^2}|\mathrm{H}\rangle) \text{ for } pqwv = 1000, 1101, 1011, 1110 \\ (\alpha_{B^1B^2}|\mathrm{V}\rangle - \beta_{B^1B^2}|\mathrm{H}\rangle) \text{ for } pqwv = 1001, 1100, 1010, 1111 \end{cases} \tag{28}$$

It can be seen that the coefficients $\alpha_{B^1B^2}$ and $\beta_{B^1B^2}$ are finally shifted to Alice. Now, Alice applies $Z_P^{q\oplus w\oplus v}X_P^p$ on her photon, where $X_P = |H\rangle\langle V| + |V\rangle\langle H|$ and $Z_P = |H\rangle\langle H| - |V\rangle\langle V|$, to obtain a new state given as follows:

$$|\Omega_{km}\rangle = (\alpha_{B^1B^2}|H\rangle + \beta_{B^1B^2}|V\rangle)|a_{k\oplus m\oplus 1}\rangle \tag{29}$$

This step can be visualized in pictorial form as shown in Fig. 4 as Step 8.

Now, the next step is to transform Alice's photon state from P-DOF to the photon's state in S-DOF.

Step 9 In this step, Alice first applies PBS on the state $|\Omega_{km}\rangle$, which triggers a new path and turns the state into $(\alpha_{B^1B^2}|H, a_{k\oplus m\oplus 1}\rangle + \beta_{B^1B^2}|V, a_{k\oplus m}\rangle)$. Further, a HWP is placed in one of the path (say $a_{k\oplus m\oplus 1}$) which generates a new state $(\alpha_{B^1B^2}|a_{k\oplus m\oplus 1}\rangle + \beta_{B^1B^2}|a_{k\oplus m}\rangle)|V\rangle$. Alice finally applies an operator $X_S^{k\oplus m\oplus 1}$, which recovers the required state $(\alpha_{B^1B^2}|a_0\rangle + \beta_{B^1B^2}|a_1\rangle) = U_\mathrm{B}^1U_\mathrm{B}^2|\psi\rangle_A$. The task of CJRIO has been successfully achieved now.

This step can be visualized in pictorial form as shown in Fig. 4 as Step 9.

4 A possible generalization for CJRIO

The proposed scheme for CJRIO can be generalized to M-joint parties (say Bob^1, Bob^2,..., Bob^M) and N-controllers (say Charlie^1, Charlie^2,...,Charlie^N). The operator of the respective (say ith) joint party is U_B^i which is given as follows:

$$U_B^i = \begin{pmatrix} u_B^i & v_B^i \\ -v_B^{*i} & u_B^{*i} \end{pmatrix} \tag{30}$$

The task here is to jointly prepare an arbitrary operation on Alice's photon by all M parties which is mathematically represented as

$$\begin{aligned} |\psi_{B^1B^2...B^M}\rangle &= U_B^1U_B^2...U_B^M|\psi\rangle_X \\ &= (\alpha_{B^1B^2...B^M}|x_0\rangle + \beta_{B^1B^2...B^M}|x_1\rangle) \end{aligned} \tag{31}$$

The quantum channel used to complete the task of CJRIO for M-joint parties and N-controllers can be given as:

$$|Q\rangle_{AB^1B^2...B^MC^1C^2...C^N} = |Q^S\rangle_{AB^1B^2...B^MC^1C^2...C^N} |Q^P\rangle_{AB^1B^2...B^MC^1C^2...C^N} \tag{32}$$

where (up to the normalization factor)

$$|Q^S\rangle_{AB^1B^2...B^MC^1C^2...C^N}$$

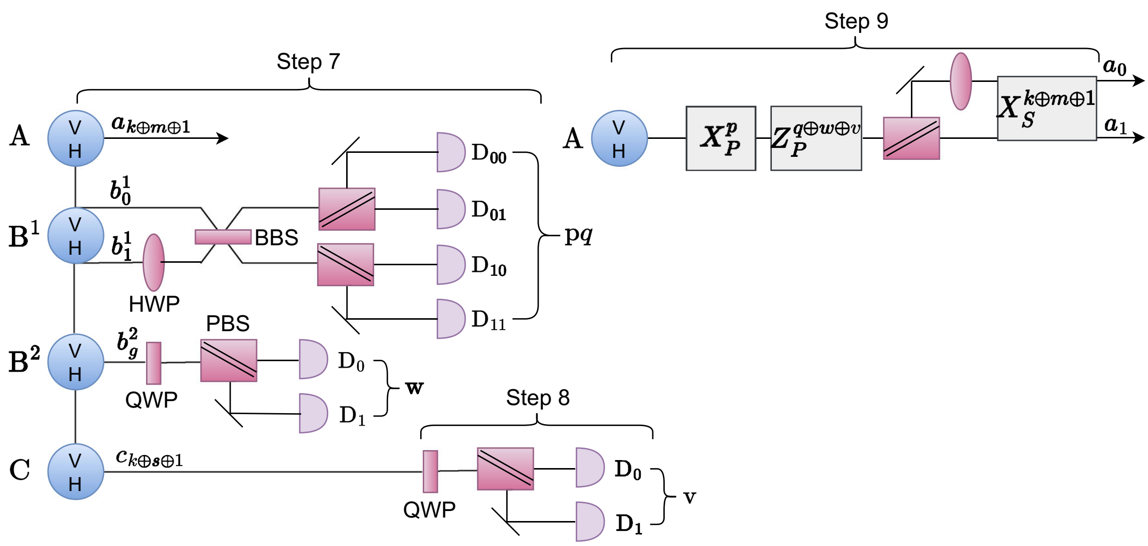

Fig. 4 A schematic which represents Step 7 to Step 9 of the CJRIO protocol. Here, the joint parties Bob^1 and Bob^2 and the controller Charlie measure their photons in an appropriate basis using HWP, QWP, BBS and PBS. After the measurement photons B^1, B^2 and C collapsed and we are only left with photon A. Alice then applies appropriate unitary followed by PBS and HWP in one path to obtain the desired state $\alpha_{B^1B^2}|a_0\rangle + \beta_{B^1B^2}|a_1\rangle$

$$= |a_0\rangle_A|b_0^1\rangle_{B^1}|b_0^2\rangle_{B^2}...|b_0^M\rangle_{B^M}|c_0^1\rangle_{C^1}|c_0^2\rangle_{C^2}...|c_0^N\rangle_{C^N} + |a_1\rangle_A|b_1^1\rangle_{B^1}|b_1^2\rangle_{B^2}...|b_1^M\rangle_{B^M}|c_1^1\rangle_{C^1}|c_1^2\rangle_{C^2}...|c_1^N\rangle_{C^N} \tag{33}$$

$$|Q^P\rangle_{AB^1B^2...B^MC^1C^2...C^N} = |H\rangle_A|H\rangle_{B^1}|H\rangle_{B^2}...|H\rangle_{B^M}|H\rangle_{C^1}|H\rangle_{C^2}...|H\rangle_{C^N} + |V\rangle_A|V\rangle_{B^1}|V\rangle_{B^2}...|V\rangle_{B^M}|V\rangle_{C^1}|V\rangle_{C^2}...|V\rangle_{C^N} \tag{34}$$

The steps involved to achieve the task of CJRIO for M-joint parties and N-controllers are described as follows.

Step 1 This step is same as Step 1 of Sect. 3.1 where Alice's unknown state $|\psi\rangle_X$ gets entangled with the quantum channel $|Q^S\rangle_{AB^1B^2...B^MC^1C^2...C^N}$ in S-DOF. The entangled state can be shown as follows:

$$|\Phi_k\rangle = \alpha|x_0\rangle|a_k\rangle \bigotimes_{i=1}^{M}|b_k^i\rangle \bigotimes_{j=1}^{N}|c_k^j\rangle + \beta|x_1\rangle|a_{k\oplus 1}\rangle \bigotimes_{i=1}^{M}|b_{k\oplus 1}^i\rangle \bigotimes_{j=1}^{N}|c_{k\oplus 1}^j\rangle \tag{35}$$

Step 2 In this step, Alice tries to disentangle her photons from the remaining photons in the quantum channel in a similar manner as Step 2 of Sect. 3.1. The collapsed state is given as follows:

$$|\Phi_{kmn}\rangle = |a_{k\oplus m\oplus 1}\rangle(\alpha \bigotimes_{i=1}^{M}|b_k^i\rangle \bigotimes_{j=1}^{N}|c_k^j\rangle + \beta \bigotimes_{i=1}^{M}|b_{k\oplus 1}^i\rangle \bigotimes_{j=1}^{N}|c_{k\oplus 1}^j\rangle) \tag{36}$$

Step 3 Each controller mixes her photon paths on a BBS and then lets one path of photon interact with a CS via cross-Kerr interaction with interaction parameter θ and measures it, whose measurement outcomes are $s_j = 0$ (1) corresponding to $|z\rangle$ ($|ze^{i\theta}\rangle$). After the measurement of each controller, the state turns into the following:

$$|\Phi_{kmns}\rangle = |a_{k\oplus m\oplus 1}\rangle \left(\alpha \bigotimes_{i=1}^{M}|b_k^i\rangle - (-1)^{m\oplus n\oplus s_1\oplus s_2\oplus ...\oplus s_N} \beta \bigotimes_{i=1}^{M}|b_{k\oplus 1}^i\rangle\right) \bigotimes_{j=1}^{N}|c_{k\oplus s_j\oplus 1}\rangle \tag{37}$$

Step 4 In the previous step, controllers allow joint parties to complete the task. Here comes the role of joint parties to implement their respective operation U_B^i, where ($i = 1, 2, \ldots, M$). The joint parties decide among themselves who will implement the operation first. Let them decide that Bob^M will implement his operation first, then each of the remaining joint parties follows the Step 4 of Sect. 3.1, which gives the measurement outcomes $l_i = 0$ (1). Bob^M then applies an appropriate unitary to get $\alpha|b_0^M\rangle + \beta|b_1^M\rangle$.

Bob^M is now ready to implement his operator U_B^M to get $\alpha_{B^M}|b_0^M\rangle + \beta_{B^M}|b_1^M\rangle$.

Step 5 In this step, the coefficient α_{B^M} and β_{B^M}, which are with Bob^M, are gradually shifted toward Bob^{M-1}, Bob^{M-2},...and so on. The two alternate joint parties work together at once here. First, suppose Bob^M and Bob^{M-1} work together. Bob^{M-1} places a BBS on his photon path which triggers a new path and then allows one of the paths to interact with an auxiliary CS $|z\rangle$ via cross-Kerr nonlinear interaction of parameter θ and forwards it to Bob^M, which he allows to interact with his one of photon path and measures it, whose measurement outcome is $r_M = 0$ (1). Bob^M now mixes two paths of his photon on a BBS and allows one path to interact with an auxiliary CS via cross-Kerr nonlinear interaction with interaction parameter θ and measures it, whose measurement outcome is $g_M = 0$ (1). Depending on the measurement outcomes, Bob^{M-1} will apply an appropriate unitary to get $\alpha_{B^M}|b_0^{M-1}\rangle + \beta_{B^M}|b_1^{M-1}\rangle$, on which he will operate U_B^{M-1} to get $\alpha_{B^{M-1}B^M}|b_0^{M-1}\rangle + \beta_{B^{M-1}B^M}|b_1^{M-1}\rangle$. This process is repeated now for Bob^{M-1} and Bob^{M-2}, and so on till Bob^1 and Bob^2. The final state will now become

$$|\Phi_{kmns_jl_ir_ig_i}\rangle = |a_{k\oplus m\oplus 1}\rangle(\alpha_{B^1B^2\ldots B^M}|b_0^1\rangle + \beta_{B^1B^2\ldots B^M}|b_1^1\rangle)\bigotimes_{i=2}^{M} |b_{g_i}^i\rangle \bigotimes_{j=1}^{N}|c_{k\oplus s_j\oplus 1}\rangle \tag{38}$$

Step 6 The joint parties take part in this step. They measure their photons in an appropriate bases and hand over the task to the sender and controllers. As we know from the previous step only Bob^1's photon is in spatial superposition and the rest joint parties' photons are spatially separated. So, Bob^1 places a HWF on path b_0^1 and mixes the superimposed path on a BBS, rest of the joint parties put a QWP on their photons path. All the joint parties now measure their photons in an appropriate basis. The role of joint parties ended here.

Step 7 All the controllers now put a QWP in their photons path and pass it through PBS and measure it in an appropriate bases. The controllers roles are ended here. Now, Alice will apply an appropriate Pauli operations to get $(\alpha_{B^1B^2\ldots B^M}|\mathrm{H}\rangle + \beta_{B^1B^2\ldots B^M}|\mathrm{V}\rangle)|a_{k\oplus m\oplus 1}\rangle$, which is same as Eq. (29).

Step 8 This step is same as Step 9 of Sect. 3.2.

5 Existing variants of RIO as a special case of CJRIO

The proposed scheme for CJRIO can be seen as a generalized scheme, and all existing variants of RIO scheme can be obtained as a special case of the proposed scheme, e.g., if one removes Charlie and the corresponding steps from our proposed scheme, then our scheme reduces to the existing scheme for JRIO reported in reference [14]. Removing Charlie will first reduce the quantum channel in Eq. (5) to three-qubit hyperentangled state and then remove the corresponding steps that involve Charlie, which are Step 3 and Step 8 of Sect. 3.1. Similarly, if one removes either of the joint parties Bob^1 or Bob^2 from our proposed scheme, then our scheme reduces to the existing scheme for CRIO reported in reference [13]. Let's remove Bob^2, this changes the quantum channel in Eq. (5) and the corresponding Steps which are Step 4, Step 5, Step 6 and Step 7. Step 4 is removed which retains the spatial superposition of photon B^1 from the previous step, on which Bob^1 applies an appropriate unitary and his operation to get $\alpha_{B^1}|b_0^1\rangle + \beta_{B^1}|b_1^1\rangle$. Now, there is no need for Step 5 and Step 6 and the role of Bob^2 is removed from Step 7, which ended up with a scheme for CRIO.

As for the task efficiency, to the best of our knowledge there has not yet been available a fully satisfactory formula for computing the efficiency of a quantum scheme. However, any definition of efficiency will be a function of the resources used and the amount of tasks done. As the task is the remote implementation of an operator, we may quantify the efficiency as follows:

$$\eta = \frac{c}{b+e} \tag{39}$$

where c is the number of 2×2 unitary operations done (it can be easily generalized for the case of multi-qubit operations, but the multi-qubit operations are not of concern to this manuscript), and b represents the amount of classical communication in the unit of bits required to achieve the task (this automatically captures the effect of the number of cross-Kerr interaction as the announcement of classical bit(s) happens only after each cross-Kerr interaction), whereas e represents the number of e-bits required to achieve the task. This is analogous to the efficiency quantifier introduced by Cabello [27] and frequently used in the context of secure quantum communication (e.g., see [28] and [29]). Now in the most general case CJRIO involving M Bobs (implementing M operators jointly) under the supervision of N controllers (Charlies), we will have $1+M+N$ parties including Alice and a bit of calculation would reveal that in this case, $c = M$, $b = 4M + 2N + 1$ and $e = M + N + 1$. This will imply that

$$\eta_{\mathrm{CJRIO}} = \frac{M}{5M+3N+2}. \tag{40}$$

Now, the impact of the increase in the number of the preparers and the controllers can be better understood by considering the special cases of the above general formula. For example, if we consider a special case, where there is no controller (i.e., $N = 0$), then we will obtain $\eta_{\text{JRIO}} = \frac{M}{5M+2}$. Clearly, for large values of M (i.e., for $M >> \frac{2}{5} = 0.4$), η_{JRIO} would approach the value $\frac{1}{5}$ (implying 20% efficiency) which is independent of M. Similarly, if we consider the case where there is only one Bob (the operator is not implemented jointly), but there are N controllers, we will obtain the efficiency of the corresponding CRIO scheme as $\eta_{\text{CRIO}} = \frac{1}{7N+3}$, which will imply that for moderate values of N, η_{CRIO} will be inversely proportional to N with highest efficiency of $\frac{1}{10}$ for the simplest case when $N = 1$. Further, we can see that η_{CRIO} will vanish asymptotically (for large values of N). The same will be the case (i.e., the vanishing of efficiency) for fixed values of $M << N$. However, such cases involving very large number of users are not practical. In all practical situations, the efficiency of the proposed protocol will be computed through $\eta_{\text{CJRIO}} = \frac{M}{5M+3N+2}$. Finally, it is worth noting that, for whatever efficiency, our schemes prove to be effective in the sense that they always succeed with unit probability.

6 Discussion and conclusion

In this work, we provide 2 interesting schemes for CJRIO which can be viewed as the basic building blocks for distributed photonic quantum computing and deserves particular use in the noisy intermediate-scale quantum era when scalable quantum computers have not yet been available. Specifically, here we first propose a scheme for CJRIO that allows two users to jointly prepare an arbitrary unitary operation on an unknown state at a remote node in the presence of a controller. The proposed scheme is completed using a four-qubit hyperentangled state, which is entangled in both S-DOF and P-DOF of photons. Finally, the idea is generalized to propose a scheme that allows an arbitrary number of joint parties as well as controllers to perform the CJRIO task. As all these schemes are designed considering their realization using photonic quantum states and as it is described in the introduction that the distributed computing requires CJRIO in the implementations involving master–slave architecture, the present work seems to be very useful in distributed photonic quantum computing. Further, before we conclude this paper, it may be apt to note that the proposed schemes are the first set of schemes for CJRIO and seem to be experimentally realizable with the existing technology. Keeping the above in mind, we conclude this paper with the hope that the work will be experimentally realized and found applications in distributed photonic quantum computing in the near future.

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Author contributions

AP and NBA conceptualized the problem and edited the final manuscript. AP supervised the work and verified the calculations. SK performed most of the computation and initial analysis. He has also prepared the first draft of the manuscript.

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