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One-loop contributions to decays $e_b \to e_a\gamma$ and $(g-2)_{e_a}$ anomalies, and Ward identity

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Abstract

In this paper, we will present analytic formulas to express one-loop contributions to lepton flavor violating decays $e_b \to e_a\gamma$, which are also relevant to the anomalous dipole magnetic moments of charged leptons e_a. These formulas were computed in the unitary gauge, using the well-known Passarino-Veltman notations. We also show that our results are consistent with those calculated previously in the 't Hooft-Veltman gauge, or in the limit of zero lepton masses. At the one-loop level, we show that the appearance of fermion-scalar-vector type diagrams in the unitary gauge will violate the Ward Identity relating to an external photon. As a result, the validation of the Ward Identity guarantees that the photon always couples with two identical particles in an arbitrary triple coupling vertex containing a photon.

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1. Introduction

The lepton sector is one of the most interesting objects for experiments to search for new physics (NP) beyond the prediction of the standard model (SM). For example, the evidence of neutrino oscillation confirms that the SM must be extended. Recently, the experimental data of anomalous magnetic moments (AMM) of charged leptons $(g-2)_{e_a}/2 \equiv a_{e_a}$ has been updated, where the deviation between SM prediction and the lasted experiment data for muon is [1]

$$\Delta a_\mu^{\rm NP} \equiv a_\mu^{\rm exp} - a_\mu^{\rm SM} = (251 \pm 59) \times 10^{-11}, \quad (1)$$

corresponding to the 4.2σ deviation from standard model (SM) prediction [2] combined from various contributions [3–23]. For the electron anomaly, the deviation between SM and experiment is 1.6σ discrepancy [24].

On the other hand, $\Delta a_{e,\mu}$ are strongly constrained by the experimental data obtained from searching for the charged lepton flavor violating (cLFV) decays $e_b \to e_a\gamma$ are [25,26]:

$$\text{Br}(\tau \to \mu\gamma) < 4.4 \times 10^{-8},\ \text{Br}(\tau \to e\gamma) < 3.3 \times 10^{-8},\ \text{Br}(\mu \to e\gamma) < 4.2 \times 10^{-13}. \quad (2)$$

This important property was discussed previously, for example see discussions for a general estimation in Ref. [27], and many particular models beyond the standard model (BSM) [28–33]. General formulas expressing simultaneously both one-loop contributions to AMM and cLFV amplitudes were introduced in the limits of new heavy scalar and/or gauge boson exchanges $m_B^2 \gg m_a^2$ with m_a being the mass of a charged lepton $e_a = e, \mu, \tau$ [27]. Other calculations in the unitary gauge were discussed [34,35] for the one-loop contributions to a_{e_a} with $m_a \neq 0$, without the relations with the cLFV amplitudes. The analytic one-loop formulas for cLFV amplitudes calculated in the 't Hooft Feynman (HF) gauge were also shown in Ref. [36], using the notations of the Passarino-Veltman (PV) functions [37,38] with $m_a \neq m_b$. The approximate formulas with $m_a = m_b = 0$ were introduced and consistent with those given in Ref. [27], as shown particularly in Ref. [39] for 3-3-1 models. The general analytic formulas of these PV functions were introduced for numerical investigations. They are consistent with the results generated by LoopTools [40], which can be transformed into other PV notations implemented in the Fortran numerical package *Collier* [41], used to investigate cLFV decays in a two Higgs doublet model (2HDM) [42]. Many particular expressions to compute the AMM and/or cLFV decay amplitudes predicted by different particular BSM were constructed [28]. The relations among them can be checked by using suitable transformations, starting from the set of particular PV notations in this work. On the other hand, in a discussion on analytic formulas for one-loop contributions to AMM, a class of fermion-scalar-vector (FSV) diagrams consisting of a photon coupling with two different physical particles, namely one scalar and one gauge boson, were considered even in the unitary gauge [34]. It leads us to whether the Ward identity (WI) for the external photon is still valid with the presence of this diagram type. We emphasize that the general results for one-loop contributions to decays $e_b \to e_a\gamma$ and AMM of leptons introduced in many previous works do not include these FSV diagrams. Moreover, they imply the existence of the triple photon coupling with two distinguishable physical particles that has never been mentioned previously. In particular, many works introducing general one-loop contributions for AMM of charged leptons [27,28,35], or decays relating with photon such as cLFV decays $e_b \to e_a\gamma$ [27,28,36], loop-induced Higgs decays $h \to \gamma\gamma$ [43,44], $h \to Z\gamma$, $f\bar{f}\gamma$ [44–47], quark decays $q \to q'\gamma$, ... Excluding the FSV vertex type will reduce a huge number of related one- and two-loop diagrams as well as confirm the validation of general one-loop calculation introduced previously.

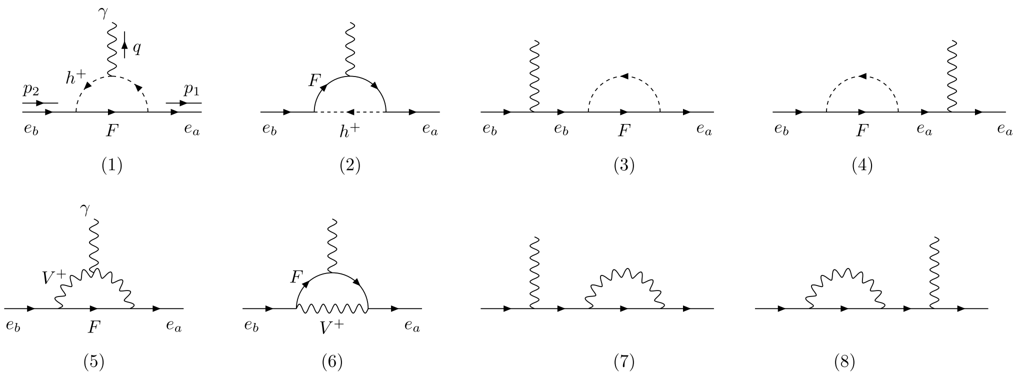

Fig. 1. Feynman diagrams for one-loop contribution to a_{e_a} and cLFV amplitudes $e_b \to e_a\gamma$ in the unitary gauge.

In this work, we will show precisely the important steps to derive the one-loop contributions to both AMM and cLFV decays. The calculation is performed by hand, which is consistent with another cross-checking using FORM package [48]. The final formulas are expressed exactly in terms of the PV functions defined by LoopTools. The results are then easy to change into all the other available forms using suitable transformations. The conventions of the PV-functions are very convenient to derive the exact formulas before solving particular pure mathematical problems. We also determine contributions arising from a new form of photon coupling with vector bosons such as leptoquarks and confirm the consistency between our results and those introduced in Ref. [44,49,50].

Our paper is organized as follows. Section 1 explains our aim of this work. Section 2 introduces notations and important formulas to establish the relations between AMM and cLFV amplitudes. Section 3 shows discussions to confirm the consistency of our results and previous works, and the validation of the WI for the relevant analytic formulas. Section 4 summarizes main features of our work. Finally, we provide many appendices showing precisely many intermediate steps and notations to derive the final results mentioned in this work, including the analytic formulas of the PV functions consistent with LoopTools given in appendix A.

2. General amplitudes and notations

It is well-known that analytic formulas of one-loop contributions to the cLFV amplitudes $e_b(p_2) \to e_a(p_1)\gamma(q)$ and AMM of SM charged leptons e_a can be presented in the same expressions, see for example Ref. [27] corresponding to the presence of new heavy particles in BSM. Possible one-loop Feynman diagrams contributing to a_{e_a} and cLFV decay amplitudes $e_b \to e_a\gamma$ in BSM are shown in Fig. 1, where F is a fermion coupling with the SM charged lepton $e_a = e, \mu, \tau$; and the boson $B = h, V$ is a scalar or gauge boson, respectively. For a detailed calculation, precise conventions for external momenta and propagators are presented in appendix C. We note here that Ref. [34] argues another type of FSV one-loop diagrams giving new contributions to the AMM. They will be discussed in detail in this work.

Firstly, we adopt the Lagrangian generating one-loop diagrams in Fig. 1, namely [27]

$$\mathcal{L}_h = \overline{F}(g^L_{a,Fh}P_L + g^R_{a,Fh}P_R)e_a h + \text{h.c.}, \tag{3}$$

$$\mathcal{L}_V = \overline{F}\gamma^\mu(g^L_{a,FV}P_L + g^R_{a,FV}P_R)e_a V_\mu + \text{h.c.}, \tag{4}$$

Table 1
Feynman rules for cubic couplings of photon A^μ, where $p_{0,\pm}$ are incoming momenta into the relevant vertex.

Vertex	Coupling	Vertex	Couplings	Vertex	Couplings
$A^\mu(p_0)V^\nu(p_+)V^{*\lambda}(p_-)$	$-ieQ_V\Gamma_{\mu\nu\lambda}(p_0,p_+,p_-)$	$A^\mu h(p_+)h^*(p_-)$	$ieQ_h(p_+-p_-)_\mu$	$A^\mu\overline{F}F$	$ieQ_F\gamma_\mu$

where the fermion F and the boson $B=V_\mu, h$ have electric charges Q_F and Q_B, and masses m_F and m_B, respectively. These Lagrangians (3) and (4) are consistent with those in Ref. [36]. Moreover, the photon couplings with all physical particles should be mentioned clearly, as given in Ref. [36], i.e., we will adopt the Feynman rules that the photon always couples with two identical physical particles, as given in Table 1, where $\Gamma_{\mu\nu\lambda}(p_0,p_+,p_-)=g_{\mu\nu}(p_0-p_+)_\lambda+g_{\nu\lambda}(p_+-p_-)_\mu+g_{\lambda\mu}(p_--p_0)_\nu$ is the standard form. The more general form of $\Gamma_{\mu\nu\lambda}(p_0,p_+,p_-)$ introduced in Refs. [44,49,50] will be discussed in detail later.

All couplings listed in Lagrangians (3), (4), and Table 1 result in the following form factors relevant with one-loop contributions:

$$c_{RB}^{ab}=\frac{e}{16\pi^2}g_{a,FB}^{L*}g_{b,FB}^{R}m_F\times\frac{f_B(x_B)+Q_Fg_B(x_B)}{m_B^2}+\frac{e}{16\pi^2}\left(m_bg_{a,FB}^{L*}g_{b,FB}^{L}+m_ag_{a,FB}^{R*}g_{b,FB}^{R}\right)\times\frac{\tilde{f}_B(x_B)+Q_F\tilde{g}_B(x_B)}{m_B^2},\tag{5}$$

where $x_B\equiv m_F^2/m_B^2$. The four scalar functions $f_B(x)$, $g_B(x)$, $\tilde{f}_B(x)$, and $\tilde{g}_B(x)$ are listed in Eq. (A.24) of appendix A, as the approximate formulas in the limit $m_a, m_b\ll m_B$. The formula in Eq. (5) does not contain contributions from the FSV diagrams mentioned in Ref. [34], because of the absence of photon coupling AVh. The corresponding formulas of AMM and cLFV decay rates are:

$$a_{e_a}\equiv-\frac{2m_a}{e}\left(c_R^{aa}+c_R^{aa*}\right)=-\frac{4m_a}{e}\mathrm{Re}[c_R^{aa}],\tag{6}$$

$$\mathrm{Br}(e_b\to e_a\gamma)=\frac{m_b^3}{4\pi\Gamma_b}\left(\left|c_R^{ab}\right|^2+\left|c_R^{ba}\right|^2\right),\tag{7}$$

where m_a, m_b, and Γ_b are the masses and total decay width of the leptons e_a, e_b, and

$$c_R^{ab}\equiv\sum_{B,F}c_{RB}^{ab}.\tag{8}$$

The amplitude for a vertex $\bar{e}_ae_aA_\mu$ in Ref. [51] is consistent with the following form presenting both AMM and cLFV amplitudes [52,53]

$$i\mathcal{M}=-ie\overline{u_a}(p_1)\left[\gamma^\mu F_1-\frac{\sigma^{\mu\nu}q_\nu}{2m_a}\left(iF_2+\gamma^5F_3\right)\right]u_b(p_2)\varepsilon_\mu^*,\tag{9}$$

where $\sigma^{\mu\nu}\equiv\frac{i}{2}\left[\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right]$; $F_{1,2,3}$ are scalar form factors; ε_μ^* and q_ν is the polarized vector of the external photon. The derivation of Eq. (9) respecting the WI from the most general form was explained clearly in Ref. [53]. The form factors $F_{2,3}$ get contributions only from loop corrections. They relate with the well-known experimental quantities called the anomalous magnetic moment a_{e_a} and electric dipole moment d_{e_a} for $b=a$, respectively. Specifically, we have $F_1=1$ for the on-shell photon, and

$$a_{e_a}=F_2;\quad d_{e_a}=-\frac{e}{2m_a}F_3.\tag{10}$$

Regarding the LFV decay $e_b \to e_a\gamma$ the amplitude can also be written in the same form [36, 54], suggesting that F_2 can be calculated based on the one-loop corrections to LFV decays. In particular, the second term of the amplitude (9) can be expanded as follows [39]

$$\mathcal{M} = (2p_1.\varepsilon^*)\overline{u_a}\left(C_{(ab)L}P_L + C_{(ab)R}P_R\right)u_b + \overline{u_a}\left[D_{(ab)L}\not{\varepsilon}^*P_L + D_{(ab)R}\not{\varepsilon}^*P_R\right]u_b, \quad (11)$$

where $m_a = m_b$ and we can prove that $C_{(ab)L}P_L + C_{(ab)R}P_R = \frac{e}{2m_a}(F_2 - i\gamma^5 F_3)$. The WI for the external photon gives

$$D_{(ab)L} = -(m_b C_{(ab)R} + m_a C_{(ab)L}),\; D_{(ab)R} = -(m_b C_{(ab)L} + m_a C_{(ab)R}). \quad (12)$$

We note that although WI does not require the condition of on-shell photon $q^2 = 0$ in general, it was also used to derive the two relations given in Eq. (12), which simplify our calculation in the unitary gauge.[1] The general case of $q^2 = 0$ is beyond our scope, see Ref. [42] for a detailed discussion of this case in the 2HDM framework. The hermiticity that $C_{(aa)R} = C^*_{(aa)L}$ [53] gives

$$\begin{aligned} a_{e_a} &= \frac{m_a(C_{(aa)L} + C_{(aa)R})}{e} = \frac{2m_a\mathrm{Re}[C_{(aa)L,R}]}{e}, \\ d_{e_a} &= i(C_{(aa)R} - C_{(aa)L}) = \mathrm{Im}[C_{(aa)L}] = -\mathrm{Im}[C_{(aa)R}]. \end{aligned} \quad (13)$$

Hence, the following relations between two different notations must be satisfied:

$$c_R^{ab} = -\frac{1}{2}C_{(ab)R} \text{ and } c_R^{ba} = -\frac{1}{2}C_{(ab)L}. \quad (14)$$

From the above discussion, we see that one-loop contributions to the a_{e_a} and d_{e_a} can be written in terms of well-known PV functions, see detailed discussions in Ref. [39] or general formula introduced for calculations of the cLFV decay rates [36], with the identification that $\sigma_{L,R} \equiv -C_{(ab)L,R}$. In the limit of $0 \simeq m_a, m_b \ll m_B$, the numerical values of a_{e_a} can be evaluated using the numerical packages such as LoopTools [40] or Collier [41]. Although the exact analytic formulas of one-loop three-point functions presented in Ref. [39] can not be applied to calculate a_{e_a}, the limit of $m_b \to m_a$ can be used to solve this problem. The analytic formulas of a_{e_a} were introduced completely in Ref. [34].

Because of the relations in Eq. (12), only $C_{(ab)L,R}$ is needed to determine a_{e_a} and $\mathrm{Br}(e_b \to e_a\gamma)$. Because all two-point diagrams give contributions to just $D_{(ab)L,R}$, $C_{(ab)L,R}$ are calculated by considering only three-point diagrams. In this work, the analytic formulas of $D_{(ab)L,R}$ will be determined directly from all diagrams in Fig. 1 to check the validation of the WI in the presence of the FSV.

The analytic formulas for one-loop contributions to the cLFV decay amplitudes presented in this work are more general than the results introduced in Ref. [39] for general 3-3-1 models. Many important steps in our calculations were shown in appendix C. Using this unitary gauge, the assumption for a particular form of the Goldstone boson couplings given in Ref. [36] is unnecessary. In contrast, we use the same photon couplings to other physical particles in an arbitrary BSM, as given in Table 1. Namely, a tree-level photon coupling always contains two identical physical particles. This implies that the contributions from the FSV diagrams are not included.

Using the notations of PV-functions defined in appendix A, the Fhh contributions from diagram (1) in Fig. 1 are:

[1] We thank the referee for reminding us this point.

$$
\begin{aligned}
C^{Fhh}_{(ab)L} =&\frac{-eQ_h}{16\pi^2}\left[m_a g^{L*}_{a,Fh} g^{L}_{b,Fh} X^f_1 + m_b g^{R*}_{a,Fh} g^{R}_{b,Fh} X^f_2 - m_F g^{R*}_{a,Fh} g^{L}_{b,Fh} X^f_0\right],\\
C^{Fhh}_{(ab)R} =&\frac{-eQ_h}{16\pi^2}\left[m_a g^{R*}_{a,Fh} g^{bR}_{Fh} X^f_1 + m_b g^{L*}_{a,Fh} g^{L}_{b,Fh} X^f_2 - m_F g^{L*}_{a,Fh} g^{R}_{b,Fh} X^f_0\right],
\end{aligned}
\tag{15}
$$

where $X^f_0, X^f_1, \ldots$ are linear combinations of the PV-functions $C_{0,00,i,ij}(m_F^2, m_h^2, m_h^2)$ defined precisely in appendix A.

The diagram (2) in Fig. 1 gives hFF contributions as follows:

$$
\begin{aligned}
C^{hFF}_{(ab)L} =&\frac{-eQ_F}{16\pi^2}\left[m_a g^{L*}_{a,Fh} g^{bL}_{b,Fh} X^h_1 + m_b g^{R*}_{a,Fh} g^{R}_{b,Fh} X^h_2 + m_F g^{R*}_{a,Fh} g^{L}_{b,Fh} X^h_3\right],\\
C^{hFF}_{(ab)R} =&\frac{-eQ_F}{16\pi^2}\left[m_a g^{R*}_{a,Fh} g^{R}_{b,Fh} X^h_1 + m_b g^{L*}_{a,Fh} g^{L}_{b,Fh} X^h_2 + m_F g^{L^*}_{a,Fh} g^{R}_{b,Fh} X^h_3\right],
\end{aligned}
\tag{16}
$$

where $X^h_{1,2,3}$ are linear combinations of $C_{0,i,ij}(m_h^2, m_F^2, m_F^2)$. The above results are completely consistent with those introduced in Ref. [36], except an overall sign and the signs before the PV-functions $\bar{c}_{1,2}$, arising from the different definitions of the external momenta p_i in the denominators of the one-loop integrals. We also give the analytic formulas of $D^{Fhh}_{(ab)L,R}$ and $D^{hFF}_{(ab)L,R}$, used to confirm the WI given in Eq. (12) for the only-scalar contributions. The PV-functions derived from diagram (2) defined as X^h_i are different from X^f_i defined for three diagrams (1), (3), and (4). In contrast, the equal functions are denoted as follows:

$$
B^{(i)}_0 \equiv B^{(i)f}_0 = B^{(i)h}_0 = B_0(p_i^2, m_h^2, m_F^2),\ X_0 \equiv X^f_0 = X^h_0,\ i = 1, 2.
$$

The form factors $D_{(ab)L,R}$ originated from scalar contributions are:

$$
\begin{aligned}
D^{Fhh}_{(ab)L} =&\frac{-eQ_H}{16\pi^2}\left\{g^{L*}_{a,Fh} g^{L}_{b,Fh} \times 2C^f_{00}\right\}\\
&+\frac{-eQ_e}{16\pi^2(m_a^2-m_b^2)}\left\{\left(m_b g^{L*}_{a,Fh} g^{R}_{b,Fh} + m_a g^{R*}_{a,Fh} g^{L}_{b,Fh}\right) m_F\left(B^{(1)}_0 - B^{(2)}_0\right)\right.\\
&\left.- g^{L*}_{a,Fh} g^{L}_{b,Fh}\left(m_a^2 B^{(1)f}_1 - m_b^2 B^{(2)f}_1\right) - m_a m_b g^{R*}_{a,Fh} g^{R}_{b,Fh}\left(B^{(1)f}_1 - B^{(2)f}_1\right)\right\},\\
D^{Fhh}_{(ab)R} =&D^{FHH}_{(ab)L}\left[g^{L}_{a,Fh} \leftrightarrow g^{R}_{a,Fh}, g^{L}_{b,Fh} \leftrightarrow g^{R}_{b,Fh}\right],\\
D^{hFF}_{(ab)L} =&-\frac{eQ_F}{16\pi^2}\left\{g^{L*}_{a,Fh} g^{L}_{b,Fh}\left[m_F^2 C^h_0 + (2-d)C^h_{00} - m_a^2 X^h_1 - m_b^2 X^h_2\right]\right.\\
&\left.+ g^{R*}_{a,Fh} g^{R}_{b,Fh} m_a m_b X_0 + \left[g^{R*}_{a,Fh} g^{L}_{b,Fh} m_a + g^{L*}_{a,Fh} g^{R}_{b,Fh} m_b\right] m_F C^h_0\right\},\\
D^{hFF}_{(ab)R} =&D^{hFF}_{(ab)L}\left[g^{L}_{a,Fh} \leftrightarrow g^{R}_{a,Fh}, g^{L}_{b,Fh} \leftrightarrow g^{R}_{b,Fh}\right],
\end{aligned}
\tag{17}
$$

where $X^h_{1,2,3}$ are linear combinations of $C^h_{0,i,ij} \equiv C_{0,i,ij}(m_h^2, m_F^2, m_F^2)$, $C^f_{00} \equiv C_{00}(m_F^2, m_h^2, m_h^2)$, and $B^{(i)f}_1 \equiv B^{(i)}_1(m_F^2, m_h^2)$ given in Eq. (A.3).

It is noted that the Fhh contributions are the sum of three diagrams (1), (3), and (4), while the hFF contributions are from only diagram (2). We emphasize that the electric charge conservation $Q_F = Q_h + Q_e$ is one of the necessary requirements to guarantee the WI given in Eq. (12), see a detailed proof in appendix C. We can see this crudely from the necessary condition that $\mathrm{div}[D^{hFF}_{(ab)L}] + \mathrm{div}[D^{Fhh}_{(ab)L}] \sim g^{L*}_a g^L_b (Q_e + Q_h - Q_F) = 0$ and $\mathrm{div}[D^{hFF}_{(ab)R}] + \mathrm{div}[D^{Fhh}_{(ab)R}] \sim g^{R*}_a g^R_b (Q_e + Q_h - Q_F) = 0$. This conclusion supports completely the only case of electric conservation among the remaining ones mentioned in Ref. [36].

Regarding Lagrangian (4), which results in four diagrams in the second line of Fig. 1, diagram (5) gives the following FVV contributions:

$$C^{FVV}_{(ab)L} = -\frac{eQ_V}{16\pi^2}\left\{ g^{R*}_{a,FV} g^{L}_{b,FV} m_F \left[3X^f_3 + \frac{1}{2m_V^2}\right] - g^{L*}_{a,FV} g^{R}_{b,FV} m_F \times \frac{m_a m_b}{m_V^2} X^f_{012} \right.$$
$$+ g^{L*}_{a,FV} g^{L}_{b,FV} m_a \left[2(X^f_1 - X^f_3) + \frac{m_F^2 X^f_{01} + m_b^2 X^f_2}{m_V^2}\right]$$
$$\left. + g^{R*}_{a,FV} g^{R}_{b,FV} m_b \left[2(X^f_2 - X^f_3) + \frac{m_F^2 X^f_{02} + m_a^2 X^f_1}{m_V^2}\right]\right\}, \tag{18}$$

where X^f_i is the linear combinations of $C_{0,ij}(m_F^2, m_V^2, m_V^2)$, given in Eq. (A.6), and

$$C^{FVV}_{(ab)R} = -\frac{eQ_V}{16\pi^2}\left\{ g^{L*}_{a,FV} g^{R}_{b,FV} m_F \left[3X^f_3 + \frac{1}{2m_V^2}\right] - g^{R*}_{a,FV} g^{L}_{b,FV} m_F \times \frac{m_a m_b}{m_V^2} X^f_{012} \right.$$
$$+ g^{R*}_{a,FV} g^{R}_{b,FV} m_a \left[2(X^f_1 - X^f_3) + \frac{m_F^2 X^f_{01} + m_b^2 X^f_2}{m_V^2}\right]$$
$$\left. + g^{L*}_{a,FV} g^{L}_{b,FV} m_b \left[2(X^f_2 - X^f_3) + \frac{m_F^2 X^f_{02} + m_a^2 X^f_1}{m_V^2}\right]\right\}. \tag{19}$$

Diagram (6) gives VFF contributions:

$$C^{VFF}_{(ab)L} = -\frac{eQ_F}{16\pi^2}\left\{ m_a g^{L*}_{a,FV} g^{L}_{b,FV} \left[2X^v_{01} + \frac{m_F^2\left(X^v_1 - X^v_3\right) + m_b^2 X^v_2}{m_V^2}\right]\right.$$
$$+ m_b g^{R*}_{a,FV} g^{R}_{b,FV} \left[2X^v_{02} + \frac{m_F^2\left(X^v_2 - X^v_3\right) + m_a^2 X^v_1}{m_V^2}\right]$$
$$- g^{R*}_{a,FV} g^{L}_{b,FV} m_F \left[4X_0 + \frac{m_a^2 X^v_1 + m_b^2 X^v_2 - m_F^2 X^v_3}{m_V^2}\right]$$
$$\left. - g^{L*}_{a,FV} g^{R}_{b,FV} \frac{m_a m_b}{m_V^2} \times m_F (X^v_{12} - X^v_3)\right\}, \tag{20}$$

where all X^v_i are expressed in terms of PV functions $C^{VFF}_{0,i,ij} = C_{0,i,ij}(m_V^2, m_F^2, m_F^2)$, and

$$C^{VFF}_{(ab)R} = -\frac{eQ_F}{16\pi^2}\left\{ m_a g^{R*}_{a,FV} g^{R}_{b,FV} \left[2X^v_{01} + \frac{m_F^2\left(X^v_1 - X^v_3\right) + m_b^2 X^v_2}{m_V^2}\right]\right.$$
$$+ m_b g^{L*}_{a,FV} g^{L}_{b,FV} \left[2X^v_{02} + \frac{m_F^2\left(X^v_2 - X^v_3\right) + m_a^2 X^v_1}{m_V^2}\right]$$
$$- g^{L*}_{a,FV} g^{R}_{b,FV} m_F \left[4X^v_0 + \frac{m_a^2 X^v_1 + m_b^2 X^v_2 - m_F^2 X^v_3}{m_V^2}\right]$$
$$\left. - g^{R*}_{a,FV} g^{L}_{b,FV} \frac{m_b m_a}{m_V^2} \times m_F (X^v_{12} - X^v_3)\right\}. \tag{21}$$

Finally, using the simple notations $g_a^{L,R} \equiv g_{a,FV}^{L,R}$, the formulas of $D_{(ab)L}$ and $D_{(ab)R}$ are

$$
\begin{aligned}
D^{(78)}_{(ab)L} =& D^{(7)}_{(ab)L} + D^{(8)}_{(ab)L} \\
=& \frac{eQ_e}{16\pi^2(m_a^2 - m_b^2)} \left\{ \left(g_a^{L*} g_b^R m_b + g_a^{R*} g_b^L m_a \right) 3m_F \left[B_0^{(1)} - B_0^{(2)} \right] \right. \\
& - m_b \left(m_a g_a^{R*} g_b^R + m_b g_a^{L*} g_b^L \right) \\
& \times \left[\left(2 + \frac{m_F^2 + m_b^2}{m_V^2} \right) B_1^{(2)v} + \frac{A_0(m_V^2) + 2m_F^2 B_0^{(1)}}{m_V^2} + 1 \right] \\
& + m_a \left(m_b g_a^{R*} g_b^R + m_a g_a^{L*} g_b^L \right) \\
& \left. \times \left[\left(2 + \frac{m_F^2 + m_a^2}{m_V^2} \right) B_1^{(1)v} + \frac{A_0(m_V^2) + 2m_F^2 B_0^{(2)}}{m_V^2} + 1 \right] \right\}, \qquad (22) \\
D^{(78)}_{(ab)R} =& D^{(78)}_{(ab)L} \left[g_a^L \leftrightarrow g_a^R,\ g_b^L \leftrightarrow g_b^R \right]. \\
D^{FVV}_{(ab)L} =& -\frac{eQ_V}{16\pi^2} \left\{ g_a^{L*} g_b^L \left[2(d-2)C^f_{00} + 2(m_a^2 + m_b^2) X_3^f \right. \right. \\
& \left. - \frac{1}{m_V^2} \left(m_F^2 (B_0^{(1)} + B_0^{(2)} - 2C^f_{00}) + A_0(m_V^2) + m_a^2 B_1^{(1)f} + m_b^2 B_1^{(2)f} \right) \right] \\
& + g_a^{R*} g_b^R m_a m_b \left[4X_3^f + \frac{2C^f_{00}}{m_V^2} \right] + g_a^{R*} g_b^L \times m_a m_F \left[3C_0^f - \frac{1}{2m_V^2} + \frac{m_b^2 X^f_{012}}{m_V^2} \right] \\
& \left. + g_a^{L*} g_b^R \times m_b m_F \left[3C_0^f - \frac{1}{2m_V^2} + \frac{m_a^2 X^f_{012}}{m_V^2} \right] \right\}, \\
D^{FVV}_{(ab)R} =& C^{FVV}_{(ab)L} \left[g_a^L \leftrightarrow g_a^R, g_b^L \leftrightarrow g_a^R \right], \qquad (23)
\end{aligned}
$$

where all X_i^f are expressed in terms of PV functions $C^f_{0,ij} \equiv C_{0,ij}(m_F^2, m_V^2, m_V^2)$ and $B_1^{(i)f}$ is given in Eq. (A.3).

The remaining formulas of $D_{(ab)L,R}$ from diagram (6) of Fig. 1 are

$$
\begin{aligned}
D^{VFF}_{(ab)L} =& \frac{eQ_F}{16\pi^2} \left\{ g_a^{L*} g_b^L \left[-2m_F^2 C_0 + (d-2)^2 C^v_{00} + 2m_a^2 X^v_{01} + 2m_b^2 X^v_{02} \right. \right. \\
& - \frac{1}{m_V^2} \left[(2-d) m_F^2 C^v_{00} + A_0(m_V^2) + m_F^2 \left(B_0^{(1)} + B_0^{(2)} \right) \right. \\
& - m_a^2 \left(B_0^{(1)v} + B_1^{(1)} \right) - m_b^2 \left(B_0^{(2)v} + B_1^{(2)v} \right) + m_a^2 m_b^2 X_0 \\
& \left. \left. - m_F^2 \left((m_a^2 + m_b^2 - m_F^2) C_0 + m_a^2 X_1^v + m_b^2 X_2^v \right) \right] \right] \\
& + g_a^{R*} g_b^R m_a m_b \left[2X_0 - \frac{1}{m_V^2} \left((2-d) C^v_{00} + m_F^2 X_3^v - m_a^2 X_1^v - m_b^2 X_2^v \right) \right] \\
& + \frac{g_a^{R*} g_b^L m_a m_F}{m_V^2} \left[-B_1^{(1)v} + (2-d) C_{00} - m_a^2 X_1^v + m_b^2 (X_3^v - X_2^v) \right]
\end{aligned}
$$

$$+\frac{g_a^{L*}g_b^R m_b m_F}{m_V^2}\left[-B_1^{(2)v}+(2-d)C_{00}^v-m_b^2X_2^v+m_a^2(X_3^v-X_1^v)\right]\Bigg\},$$

$$D_{(ab)R}^{VFF}=D_{(ab)L}^{VFF}\left[g_a^L\leftrightarrow g_a^R,\ g_b^L\leftrightarrow g_b^R\right], \tag{24}$$

where all X_i^v are expressed in terms of PV functions $C_{0,ij}^v\equiv C_{0,ij}(m_V^2,m_F^2,m_F^2)$ and $B_1^{(i)v}$ is given in Eq. (A.3).

We note that all results presented here are crosschecked by FORM package [48], using intermediate steps given in appendix C. There is a property that $C_{(ab)R}^X=C_{(ab)L}^X\left[g_a^L\leftrightarrow g_a^R, g_b^L\leftrightarrow g_b^R\right]$ for all $X=Fhh, hFF, FVV, VFF$. The above results of one-loop contribution to $C_{(ab)L,R}$ are totally consistent with those introduced in Ref. [36], after some transformations of notations presented in appendix B. In the limit of $m_h^2, m_V^2\gg m_a^2, m_b^2$, i.e., $m_a^2/m_B^2, m_b^2/m_h^2\simeq 0$ with $B=h,V$, we get consistent results with those given in Ref. [27,55,56]. To derive the above results for gauge boson exchanges, we start with many important features different from those mentioned in Ref. [36], namely: i) we do not use the typical form of couplings relating to Goldstone bosons going along with the presence of new gauge bosons, ii) we have to use the massless property of the on-shell photon $q^2=0$, iii) to confirm the WI for all diagrams given in Fig. 1, we need the charge conservation law corresponding to the Lagrangian (1): $Q_F=Q_V+Q_e$. Therefore, our calculation is another independent approach to confirm the result given in Ref. [36]. The details of the calculation to confirm the WI for all one-loop contributions are given in appendix C. We remind that our results are derived from the photon couplings listed in the Table 1, and do not contain the contributions from the FSV diagrams. In the following, we pay attention to the possibility of adding the FSV diagrams or the new forms of the photon couplings.

3. Discussion on WI and previous results

3.1. WI to constrain the form of photon couplings

Now we focus on the feature that the WI of the on-shell photon will constrain strongly the forms of the cubic photon couplings with two physical particles in a renormalized Lagrangian. Now we consider the existence of the photon coupling types at tree level:

$$\begin{aligned}\mathcal{L}^{\gamma XX}=&eQ_FA^\mu\left[\overline{F_1}\gamma^\mu F_2+\text{h.c.}\right]+ieQ_hA^\mu\left[\left(h_1^*\partial_\mu h_2-h_2\partial_\mu h_1^*\right)+\text{h.c.}\right]\\&-\left[eQ_VA^\mu V_1^\nu V_2^{\lambda*}\Gamma_{\mu\nu\lambda}(p_0,p_+p_-)+\text{h.c.}\right]+\left[g_{\gamma hV}g_{\mu\nu}h^{-Q}A^\mu V^{Q\nu}+\text{h.c.}\right],\end{aligned} \tag{25}$$

where all couplings are more general than those well-known as the standard forms given in Table 1. In addition, the last term corresponds to the photon coupling to a scalar $h\equiv S$ and a gauge boson V mentioned in Eq. (D.1). The above Lagrangian results in the following decays from the heavy particle to the lighter one: i) $F_2\to F_1\gamma$, ii) $h_2\to h_1\gamma$, iii) $V_2\to V_1\gamma$, and iv) $V\to h\gamma$. The WI for these decay amplitudes at tree level is $\mathcal{M}^\mu(X_1\to X_2\gamma)p_{0\mu}=0$ with $p_{0\mu}$ being the external photon momentum. It can be derived that:

- Using the same convention of external momenta given in Fig. 1, we have $\mathcal{M}^\mu(F_2\to F_1\gamma)q_\mu\sim(m_{F_2}-m_{F_1})\overline{u}_{F_2}(p_2)u_{F_1}(p_1)=0$, where $p_0\equiv -q$. Therefore, $m_{F_2}=m_{F_1}$. This case is automatically satisfied for the tree-level AMM amplitude.

- $\mathcal{M}^{\mu}(h_2 \to h_1\gamma) p_{0\mu} \sim (p_2 - p_1).(p_2 + p_1) = (m^2_{h_2} - m^2_{h_1}) = 0$, where all on-shell momenta are incoming the vertex $A^{\mu} h_1^* h_2$, implying that $p_0 = -(p_1 + p_2)$ and $p^2_{1,2} = m^2_{h_{1,2}}$. The consequence is $m_{h_1} = m_{h_2}$.
- $\mathcal{M}^{\mu}(V \to h\gamma) p_{0\mu} \sim \varepsilon_v . p_0 = 0$, where ε_v and p_0 are the polarization of gauge boson V and the external momentum of the photon A_{μ}. Hence, the presence of a AhV vertex does not automatically satisfy the WI. One-loop contributions for all diagrams arising from this vertex must be checked for the validation of WI. In Ref. [34], the presence of these vertices was mentioned in a Higgs triplet model (HTM). A detailed calculation in appendix E shows an opposite conclusion that this vertex vanishes at tree level.[2]
- $\mathcal{M}_{\mu}(V_1 \to V_2\gamma) p_0^{\mu} \sim \varepsilon_1^{\nu} \varepsilon_2^{\lambda *} p_0^{\mu} \Gamma_{\mu\nu\lambda}(p_0, p_1, p_2) = 0$, where $\varepsilon_{1,2}$, and $p_{1,2,0}$ are the polarization of the gauge boson $V_{1,2}$ and the external momentum of the gauge bosons $V_{1,2}$ and photon A_{μ}, respectively. We will use the following properties of the external gauge bosons $V_i (i = 1, 2)$ and photon: $\varepsilon_i . p_i = 0,\ p_0^2 = 0,\ p_i^2 = m^2_{V_i}$, and the momentum conservation $p_0 + p_1 + p_2 = 0$ following notations in Table 1. After some intermediate calculating steps, we have:

$$\begin{aligned}\mathcal{M}_{\mu}(V_1 \to V_2\gamma) p_0^{\mu} \sim & (p_0.\varepsilon_1) \left[(p_0 - p_1).\varepsilon_2^*\right] + (\varepsilon_1.\varepsilon_2^*) \left[(p_1 - p_2).p_0\right] \\ & + (p_0.\varepsilon_2^*) \left[(p_2 - p_0).\varepsilon_1\right] \\ = & (\varepsilon_1.\varepsilon_2^*) \left[m^2_{V_2} - m^2_{V_1}\right] = 0. \end{aligned} \tag{26}$$

Hence, $m_{V_1} = m_{V_2}$ is necessary. From this, we consider the more general photon coupling with a gauge boson [49] describing the couplings of a leptoquark field [50]

$$\begin{aligned}\Gamma'_{\mu\nu\lambda}(p_0, p_1, p_2) = & g_{\mu\nu}(k_v p_0 - p_1)_{\lambda} + g_{\nu\lambda}(p_1 - p_2)_{\mu} + g_{\lambda\mu}(p_2 - k_v p_0)_{\nu} \\ = & \Gamma_{\mu\nu\lambda}(p_0, p_1, p_2) + \delta k_v \left(g_{\mu\nu} p_{0\lambda} - g_{\lambda\mu} p_{0\nu}\right), \end{aligned} \tag{27}$$

with $\delta k_v = k_v - 1$ showing the deviation from the standard vertex listed in Table 1. This may change the one-loop contributions of the diagram (5) in Fig. 1, hence change the formulas of $C^{FVV}_{(ab)L,R}$ given in Eqs. (18) and (19), respectively. One can prove immediately that the vertex deviation

$$\delta\Gamma_{\mu\nu\lambda}(p_0, p_1, p_2) \equiv \Gamma'_{\mu\nu\lambda}(p_0, p_1, p_2) - \Gamma_{\mu\nu\lambda}(p_0, p_1, p_2) = \delta k_v \left(g_{\mu\nu} p_{0\lambda} - g_{\lambda\mu} p_{0\nu}\right) \tag{28}$$

guarantees the WI. The new one-loop contributions arising from $\delta\Gamma$ are also satisfied the WI, see analytic formulas given in Eq. (C.36).

Now we start from the point that all results of one loop contributions given from Eq. (15) to Eq. (24) based on the standard forms of photon couplings given in Table 1, where a photon always couples with two identical physical fields. On the other hand, a recent work [34] assumed the existence of a new photon coupling kind ASV, which may appear in some BSM, in which the photon couples with one gauge boson V and one scalar S. The appearance of a boson V or S will generate by itself the one-loop contributions that always guarantee the WI by the respective set of four diagrams given in Fig. 1. Hence, the two FSV diagrams must give contributions satisfying the WI themselves, namely

[2] To the best of our knowledge, we have not seen any UV models beyond the SM that have violated $U(1)_{em}$ couplings of the form S-V-γ, which is the necessary source for generating FSV-type diagrams.

$$D^{FSV}_{(ab)_L}+m_aC^{FSV}_{(ab)_L}+m_bC^{FSV}_{(ab)_R}=D^{FSV}_{(ab)_R}+m_aC^{FSV}_{(ab)_R}+m_bC^{FSV}_{(ab)_L}=0. \quad (29)$$

As a result, the divergent parts of $h\equiv S$ given in appendix D for both L and R parts give:

$$\begin{aligned}0=&g_{\gamma hV}\left[2g^{L*}_{ah}g^{L}_{bV}m_F-g^{L*}_{ah}g^{R}_{bV}m_b-g^{R*}_{ah}g^{L}_{bV}m_a\right]\\=&g_{\gamma hV}\left[2g^{R*}_{ah}g^{R}_{bV}m_F-g^{R*}_{ah}g^{L}_{bV}m_b-g^{L*}_{ah}g^{R}_{bV}m_a\right].\end{aligned} \quad (30)$$

Considering the case of $g_{\gamma hV}\neq 0$. Then, all quantities g^L_{ah}, g^R_{ah}, g^L_{bV}, and g^R_{bV} are zeros if at least one of them is zero. More strictly, we require that the two Eqs. (29) must be held for both divergent and finite parts arising from $D_{(ab)L,R}$ and $C_{(ab)L,R}$ given in appendix D. Consequently, $g_{\gamma hV}=0$, i.e., the FSV diagram type does not satisfy the WI.

Regarding the vertex deviation of the AVV couplings defined in Eq. (28), the new one-loop contributions relating to $C^{FVV}_{(ab)L,R}$ and $D^{FVV}_{(ab)L,R}$ are shown in Eq. (C.36) of appendix C. Our results are consistent with previous works [49,50]. Although they satisfy the WI, they contain divergences. For example, the divergent part of δC^{FVV}_L is

$$\mathrm{div}\left[-\delta C^{FVV}_L\right]=\frac{\delta k_v eQ_V}{32\pi^2m_V^2}\left[g^{L*}_a g^L_b m_a+g^{R*}_a g^R_b m_b-2g^{R*}_a g^L_b m_F\right]. \quad (31)$$

Hence, $\delta k_v=0$ is equivalent to the renormalizable condition of the theory, see a more detailed explanation in Ref. [49]. This confirms that the AVV coupling listed in Table 1 is still valid for a general UV-complete model. Consequently, $\delta C^{FVV}_L=0$, implying that the results of $C^{FVV}_{(ab)L,R}$ given in Eqs. (18) and (19) are unchanged for many renormalizable theories.

3.2. Discussions on previous results

It is easy to derive that $C_{(ab)L,R}=\sigma_{L,R}$ corresponding to the notations given in Ref. [36], see a detailed explanation in appendix B. This confirms a perfect consistency of the two results obtained from different original assumptions that we have indicated above. In addition, these results are also consistent with those given in Ref. [27] in the limit of heavy boson masses in the loops, which are very useful for studying the correlations of AMM and cLFV decays.

In some BSM, SM light quark may play the role of the light fermions $u,d\equiv F$ in the Yukawa couplings [29], hence the condition $m_F^2\gg m_a^2, m_b^2$ is not held. But numerical illustrations [39] to investigate cLFV decays $e_b\to e_a\gamma$ with very light neutrinos show that the case of $m_F^2\ll m_a^2$ are also valid for approximation formulas with $m_a^2=m_b^2=0$, provided $m_a^2, m_b^2\ll m_h^2, m_V^2$. An analytic approximation to explain this result was given in, for example, Ref. [58].

For analytic formulas of cLFV and a_{e_a} introduced in Ref. [28], they can be changed into the form of PV-functions consistent with our results. An exceptional case mentioned is the coupling of a doubly charged boson with two identical leptons. For example, the Lagrangian containing couplings of a doubly charged Higgs boson is [28]:

$$\mathcal{L}_{\rm int}=g^{ij}_{s3}\phi^{++}\overline{\ell^C_i}\ell_j+g^{ij}_{p3}\phi^{++}\overline{\ell^C_i}\gamma^5\ell_j+\mathrm{h.c.}, \quad (32)$$

where we can identify that $g^R_{a,Fh}=g^{ij}_{s3}+g^{ij}_{p3}$ and $g^R_{a,Fh}=g^{ij}_{s3}-g^{ij}_{p3}$. But the Feynman rules for the vertex $\overline{\ell^C_i}\ell_j\phi^{++}$ containing two identical leptons give an extra factor 2, implying that $C_{(ab)L,R}$ given in Eqs. (15) and (16) must be added a factor 4. Instead of many particular formulas to calculate one-loop contributions relating to different charged particles, the one-loop results for $(g-2)_{e_a}$ and $e_b\to e_a\gamma$ decays can be generalized for a_{e_a} with an arbitrary electric charge Q_F

of a new fermion and the boson with $Q_B = Q_F - Q_e$ with $B = h, V$. Namely, the a_{e_a} formulas are

$$a_{e_a}(h) = \frac{Q_h m_a}{16\pi^2} \int_0^1 dx \times \frac{x(x-1)\left[2\mathrm{Re}[g^{RL}]m_F + (g^{LL}+g^{RR})m_a x\right]}{(1-x)m_F^2 + x\left[m_h^2 + m_a^2(x-1)\right]}$$
$$+ \frac{Q_F m_a}{16\pi^2} \int_0^1 dx \times \frac{x^2\left[-2g^{RL}[g^{RL}]m_F + (g^{LL}+g^{RR})m_a(x-1)\right]}{(1-x)m_h^2 + x\left[m_F^2 + m_a^2(x-1)\right]}, \quad (33)$$

$$a_{e_a}(V) = -\frac{Q_V m_a}{16\pi^2 m_V^2} \int_0^1 dx$$
$$\times \left[\frac{\mathrm{Re}[g^{RL}]m_F\left[m_F^2(x-1) + m_V^2 x(6x-1) + m_a^2 x(3-5x+2x^2)\right]}{(1-x)m_F^2 + x\left[m_V^2 + m_a^2(x-1)\right]}\right.$$
$$\left. - \frac{m_a(g^{LL}+g^{RR})\left[m_F^2(2-3x+x^2) + m_V^2 2x(x+1) + m_a^2 x(x-1)\right]}{(1-x)m_F^2 + x\left[m_V^2 + m_a^2(x-1)\right]}\right]$$
$$+ \frac{Q_F m_a}{16\pi^2 m_V^2} \int_0^1 dx \left[\frac{2g^{RL}[g^{RL}]m_F x\left[m_F^2 x - 4m_V^2(1-x) + m_a^2 x(2x-1)\right]}{(1-x)m_V^2 + x\left[m_F^2 + m_a^2(x-1)\right]}\right.$$
$$\left. + \frac{(g^{LL}+g^{RR})m_a x\left[m_F^2 x(1+x) + 2m_V^2(2-3x+x^2) + m_a^2 x(x-1)\right]}{(1-x)m_V^2 + x\left[m_F^2 + m_a^2(x-1)\right]}\right], \quad (34)$$

where $g^{RL} = g^{R*}_{a,FB} g^{L}_{a,FB}$, $g^{LL} = g^{L*}_{a,FB} g^{L}_{a,FB}$, and $g^{RR} = g^{R*}_{a,FB} g^{R}_{a,FB}$ with $B = h, V$. The coupling identifications are $g^{R}_{a,Fh} = g^{aa}_{sk} + g^{aa}_{pk}$ and $g^{R}_{a,Fh} = g^{aa}_{sk} - g^{aa}_{pk}$ for $k = 1, 2, 3$ relating to neutral, singly, and doubly charged Higgs bosons. Similarly to the gauge bosons, $g^{R}_{a,FV} = g^{aa}_{vk} + g^{aa}_{ak}$ and $g^{R}_{a,FV} = g^{aa}_{vk} - g^{aa}_{ak}$ for $Q_V = 1, 0, -1, 2$ corresponding $k = 1, 2, 3, 4$. The two formulas (33) and (34) are derived by inserting the PV functions given in appendix A in the limit $p_1^2 = p_2^2 = m_a^2$ into $C_{(ab)L,R}$. We have checked that our results are consistent with all HFF, FHH, and VFF contributions relating to the diagrams (1), (2), and (6) in Fig. 1, respectively. For the one-loop FVV contributions arising from the diagram (5), there is a difference between our result and that in Ref. [28], namely

$$\delta(a_{e_a})(FVV) = \frac{Q_V m_a m_F}{16\pi^2 m_V^2}(|g^{aa}_{vk}|^2 - |g^{aa}_{ak}|^2) \int_0^1 dx(2x+1) = \frac{Q_V m_a m_F}{8\pi^2 m_V^2}(|g^{aa}_{vk}|^2 - |g^{aa}_{ak}|^2).$$

It shows that the two results are consistent if $g^{aa}_{vk} = \pm g^{aa}_{ak}$, i.e., $g^{L}_{a,FB} g^{R}_{a,FB} = 0$, which appears in many BSM such as the SM, 3-3-1 models,... We also see that the FVV contribution to a_{e_a} of the doubly gauge boson given in Ref. [28] has an opposite sign with our result.

We note that our results are also valid as the exact solutions for studying the AMM and $e_b \to e_a\gamma$ decay in BSM consisting of very light bosons $m_B \ll m_a^2, m_b^2$ such as an axion-like particle (ALP) [59,60], or a new scalar singlet [61].

4. Conclusion

Using the unitary gauge, we confirm the exact results of analytic formulas in terms of PV functions for one-loop contributions to the cLFV decay rates $e_b \to e_a\gamma$ given in Ref. [36], which are also applicable to compute the AMM of charged leptons. These results are consistent with those given in Ref. [27] in the limit of heavy bosons $m_B \gg m_a, m_b$. The general expressions in terms of PV-functions are very convenient to change into available forms. Our calculations here have many new features as follows. Our calculation is independent of the Goldstone boson couplings of the new gauge bosons. The Ward Identity of the external photon allows only the couplings of a photon with two identical physical particles, as given in Table 1. At tree-level, the ASV couplings do not satisfy the WI if $\varepsilon_v.p_0 \neq 0$, where ε_v and p_0 are the polarization of gauge boson V and the external momentum of the photon, respectively. The one-loop FSV contributions arising from this vertex type to cLFV amplitudes and AMM do violate the WI. Therefore, the results given in Ref. [27,36] are valid in all renormalizable BSM respecting the WI. They are still applied for other similar decays of quarks $q \to q'\gamma$. The photon-scalar-vector ASV vertex does not appear in BSM satisfying the WI. Our conclusion is very useful for constructing loop calculations relating to photon couplings, where only the vertex types listed in Table 1 are valid.

CRediT authorship contribution statement

L.T. Hue: Formal analysis, Investigation, Writing – original draft. **H.N. Long:** Writing – review & editing. **V.H. Binh:** Formal analysis, Investigation. **H.L.T. Mai:** Formal analysis, Investigation. **T. Phong Nguyen:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data that has been used is confidential.

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Appendix A. PV functions for one loop contributions defined by LoopTools

A.1. General notations

The PV-functions used here were listed in Ref. [39], namely

$$A_0(m^2) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \frac{d^dk}{k^2 - m^2 + i\delta},$$

$$B_{\{0,\mu\}}(p_i^2, M_1^2, M_2^2) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \frac{d^dk \times \{1, k_\mu, k_\mu k_\nu\}}{D_0 D_i},\ i = 1, 2,$$
$$C_{\{0,\mu,\mu\nu\}} = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \frac{d^dk\{1, k_\mu, k_\mu k_\nu\}}{D_0 D_1 D_2},$$
$$B_\mu(p_i^2, M_1^2, M_2^2) = \left(-p_{i\mu}\right) B_1^{(i)},$$
$$C_\mu = \left(-p_{1\mu}\right) C_1 + \left(-p_{2\mu}\right) C_2,$$
$$C_{\mu\nu} = g_{\mu\nu} C_{00} + p_{1\mu}p_{1\nu}C_{11} + p_{2\mu}p_{2\nu}C_{22} + (p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu})C_{12}, \tag{A.1}$$

where $D_0 \equiv k^2 - M_1^2 + i\delta$, $D_i \equiv (k-p_i)^2 - M_2^2 + i\delta$, $C_{0,\mu,\mu\nu} = C_{0,\mu,\mu\nu}(p_1^2, 0, p_2^2; M_1^2, M_2^2, M_2^2)$, μ is an arbitrary mass parameter introduced via dimensional regularization [57]. In this work, we discuss only the case of external photon $q^2 = (p_2 - p_1)^2 = 0$. The scalar functions $A_0, B_0, B_1^{(i)}, C_0, C_{00}, C_i$, and C_{ij} $(i, j = 1, 2)$ are well-known PV functions, which are consistent with those defined by LoopTools [40]. The well-known relations are:

$$B_0^{(i)} \equiv B_0^{(i)}(p_i^2; M_1^2, M_2^2) = B_0^{(i)}(p_i^2; M_2^2, M_1^2),$$
$$B_1^{(i)} \equiv B_1^{(i)}(p_i^2; M_1^2, M_2^2) = -\frac{1}{2p_i^2}\left[A_0(M_2^2) - A_0(M_1^2) + f_i B_0^{(i)}\right], \tag{A.2}$$

where $f_i = p_i^2 + M_2^2 - M_1^2$. Depending on the particle exchanges in Feynman diagrams, the $B_1^{(i)}$-function given in Eq. (A.2) is denoted more precisely as follows:

$$B_1^{(i)f} \equiv B_1^{(i)}(p_i^2; m_F^2, m_B^2),\ B_1^{(i)v} \equiv B_1^{(i)}(p_i^2; m_V^2, m_F^2),\ B_1^{(i)h} \equiv B_1^{(i)}(p_i^2; m_h^2, m_F^2). \tag{A.3}$$

The scalar functions A_0, B_0, and C_0 can be calculated using the techniques of [38]. Other PV functions needed in this work are

$$B_{0,\mu,\mu\nu}(M_2) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \frac{d^dk\ \{1,\ k_\mu,\ k_\mu k_\nu\}}{D_1 D_2}. \tag{A.4}$$

For simplicity, we define the following notations appearing in many important formulas:

$$X_0 \equiv C_0 + C_1 + C_2,$$
$$X_1 \equiv C_{11} + C_{12} + C_1,$$
$$X_2 \equiv C_{12} + C_{22} + C_2,$$
$$X_3 \equiv C_1 + C_2 = X_0 - C_0,$$
$$X_{012} \equiv X_0 + X_1 + X_2,\ X_{ij} = X_i + X_j. \tag{A.5}$$

Depending on the form of the PV-functions, we have

$$X_i^f = X_i(m_F^2, m_B^2, m_B^2),\ X_i^h \sim X_i(m_B^2, m_F^2, m_F^2),\ X_i^v \sim X_i(m_V^2, m_F^2, m_F^2) \tag{A.6}$$

corresponding to the diagram types of FBB, HFF, and VFF with $B = h, V$.

A.2. $p_1^2 \neq p_2^2 \neq 0$ and $p_1^2, p_2^2 \neq 0$

From the definitions of PV-functions given in Eq. (A.1), it can be proved that:

$$B_0^{(0)} \equiv B_0^{(0)}(M_2) \equiv B_0(0; M_2, M_2) = C_{UV} - \ln(M_2^2) + \mathcal{O}(\epsilon),$$

$$A_0(M) = M^2\left[B_0^{(0)}(M)+1\right], \tag{A.7}$$

$$B^{\mu}(M_2) = \frac{1}{2}B_0^{(0)}(p_1^{\mu}+p_2^{\mu}), \tag{A.8}$$

$$B^{\mu\nu}(M_2) = \frac{g^{\mu\nu}}{2}M_2^2\left[B_0^{(0)}+1\right]+\frac{1}{6}B_0^{(0)}\left(2p_1^{\mu}p_1^{\nu}+p_1^{\mu}p_2^{\nu}+p_2^{\mu}p_1^{\nu}+2p_2^{\mu}p_2^{\nu}\right),$$

$$C_{00} = \frac{1}{4}\left[2M_2^2C_0+\frac{\left(M_2^2-M_1^2+m_a^2\right)B_0^{(1)}-\left(M_2^2-M_1^2+m_b^2\right)B_0^{(2)}}{m_a^2-m_b^2}+1\right], \tag{A.9}$$

where C_{UV} is defined as the divergent part of the PV functions when $D \to 4$, $C_{UV} = 1/\epsilon - \gamma_E + \log(4\pi\mu^2)$ with γ_E being Euler's constant and $D = 4-2\epsilon$. It is well-known that the PV-functions having non-zero divergent parts are:

$$\mathrm{div}\left[B_0^{(0)}\right] = \mathrm{div}\left[B_0^{(1)}\right] = \mathrm{div}\left[B_0^{(2)}\right] = -2\mathrm{div}\left[B_1^{(1)}\right] = -2\mathrm{div}\left[B_1^{(2)}\right] = 4\mathrm{div}\left[C_{00}\right] = C_{UV},$$
$$\mathrm{div}\left[A_0(M)\right] = M^2C_{UV}. \tag{A.10}$$

As mentioned in Ref. [39], we can derive all formulas of C_i, and C_{ij} as functions of A_0, $B_0^{(i)}$, and C_0 consistent with Ref. [39], using the following relations:

$$\begin{aligned}
&2m_a^2C_1+(m_a^2+m_b^2)C_2 = -f_aC_0-B_0^{(0)}+B_0^{(2)},\\
&(m_a^2+m_b^2)C_1+2m_b^2C_2 = -f_bC_0-B_0^{(0)}+B_0^{(1)},\\
&2C_{00}+2m_a^2C_{11}+(m_a^2+m_b^2)C_{12} = \frac{1}{2}B_0^{(0)}-f_aC_1,\\
&2m_a^2C_{12}+(m_a^2+m_b^2)C_{22} = \frac{1}{2}B_0^{(0)}+B_1^{(2)}-f_aC_2\\
&2C_{00}+(m_a^2+m_b^2)C_{12}+2m_b^2C_{22} = \frac{1}{2}B_0^{(0)}-f_bC_2,\\
&(m_a^2+m_b^2)C_{11}+2m_b^2C_{12} = \frac{1}{2}B_0^{(0)}+B_1^{(1)}-f_bC_1,\\
&4C_{00}-\frac{1}{2}+m_a^2C_{11}+(m_a^2+m_b^2)C_{12}+m_b^2C_{22} = B_0^{(0)}+M_1^2C_0,
\end{aligned} \tag{A.11}$$

where $f_{a,b} = M_2^2-M_1^2+m_{a,b}^2$, and $C_{12} = C_{21}$ is used. In this work, we need just combinations of these PV-functions for our immediate steps. In particular, we can prove that:

$$\begin{aligned}
X_0 &= -\frac{B_0^{(1)}-B_0^{(2)}}{m_a^2-m_b^2},\\
X_{12} &= -\frac{B_1^{(1)}-B_1^{(2)}}{m_a^2-m_b^2}\\
&= \frac{A_0(M_1^2)-A_0(M_2^2)}{2m_a^2m_b^2}+\frac{(M_1^2-M_2^2)}{2(m_a^2-m_b^2)}\left(\frac{B_0^{(1)}}{m_a^2}-\frac{B_0^{(2)}}{m_b^2}\right)-\frac{1}{2}X_0,\\
m_a^2B_1^{(1)}-m_b^2B_1^{(2)} &= -\frac{1}{2}\left[\left(m_a^2+M_2^2-M_1^2\right)B_0^{(1)}-\left(m_b^2+M_2^2-M_1^2\right)B_0^{(2)}\right],\\
\mathbf{b}_1 \equiv \frac{m_a^2B_1^{(1)}-m_b^2B_1^{(2)}}{(m_a^2-m_b^2)} &= -(2C_{00}+m_a^2X_1+m_b^2X_2),
\end{aligned}$$

$$
\begin{aligned}
(2-d)C_{00}+M_2^2C_0 &= -2C_{00}+\frac{1}{2}+M_2^2C_0\\
&= -\frac{\left(m_a^2+M_1^2-M_2^2\right)B_0^{(1)}-\left(m_b^2+M_1^2-M_2^2\right)B_0^{(2)}}{2(m_a^2-m_b^2)}\\
&= \mathbf{b}_1+(M_2^2-M_1^2)X_0,
\end{aligned} \tag{A.12}
$$

where $A_0(M_2^2)=M_2^2(B_0^{(0)}+1)$ and $A_0(M_1^2)=M_1^2(B_0^{(0)}+1+\ln(M_2^2/M_1^2))$.

It was proved previously, for example [39], that

$$
\begin{aligned}
&B_0(p^2;M_1^2,M_2^2)=B_0(p^2;M_2^2,M_1^2)=C_{UV}-\ln(M_2^2)+2-\sum_{\sigma=\pm}(1-\frac{1}{x_\sigma})\ln(1-x_\sigma),\\
&C_0(m_a^2,0,m_b^2;M_1^2,M_2^2,M_2^2)=-\frac{1}{m_a^2-m_b^2}\sum_{\sigma=\pm}\left[\mathrm{Li}_2(y_{a\sigma})-\mathrm{Li}_2(y_{b\sigma})\right],
\end{aligned} \tag{A.13}
$$

where $p=p_a,p_b$; and

$$
\begin{aligned}
x_\pm &= \frac{1}{2M_2^2}\left[(M_2^2-M_1^2+p^2)\pm\sqrt{(M_2^2-M_1^2+p^2)^2-4M_2^2p^2}\right],\\
y_{a\pm} &= \frac{1}{2M_2^2}\left[(M_2^2-M_1^2+m_a^2)\pm\Lambda\right],\\
y_{b\pm} &= x_{a\pm}\,[b\to a]
\end{aligned} \tag{A.14}
$$

with $\Lambda=(M_1^4+M_2^4+m_a^4-2M_1^2M_2^2-2M_1^2m_a^2-2M_2^2m_a^2)^{1/2}$. The above formula of C_0 is also consistent with that introduced in loop-induced decay amplitude of $h\to Z\gamma$ [62].

A.3. $m_a^2=p_a^2=p_b^2\neq 0$

Formulas for AMM in Ref. [34] require that analytic formulas of PV functions with $m_b=m_a$. It seems that the results of PV-functions listed in Ref. [39] are not valid. But the limit $m_b=m_a$ can be derived mathematically. For example, the result of C_0 given in Eq. (A.13) leads to a consequence that

$$
\begin{aligned}
C_0(m_a^2,0,m_a^2;M_1^2,M_2^2,M_2^2) &= \lim_{m_b\to m_a}C_0(m_a^2,0,m_b^2;M_1^2,M_2^2,M_2^2)\\
&= -\frac{\partial}{\partial(m_a^2)}\sum_{\sigma=\pm}\mathrm{Li}_2(y_{a\sigma})=\sum_{\sigma=\pm}\frac{y'_{a\sigma}\ln(1-y_{a\sigma})}{y_{a\sigma}}\\
&= \sum_{\sigma=\pm}\frac{\ln(1-y_{a\sigma})}{2M_2^2y_{a\sigma}}\times\left[1-\sigma\times\frac{M_1^2+M_2^2-m_a^2}{\Lambda}\right],
\end{aligned} \tag{A.15}
$$

where $f'\equiv\partial f/(\partial m_a^2)$ denotes a well-known derivative notation. In addition, $B_0^{(1)}=B_0^{(2)}$ and $B_1^{(1)}=B_1^{(2)}$ is automatically satisfied. Many formulas containing $(m_a^2-m_b^2)$ in the denominators corresponding a derivative in the limit $m_a\to m_b$:

$$
\begin{aligned}
X_0 &= -B_0^{(1)\prime}=\sum_{\sigma=\pm}\frac{y'_{a\sigma}\left[y_{a\sigma}+\ln(1-y_{a\sigma})\right]}{y_{a\sigma}^2},\\
X_{12} &= -B_1^{(1)\prime},\ldots.
\end{aligned} \tag{A.16}
$$

In this way, we can confirm all results introduced in Ref. [34]. There is another way to calculate form factors, using the Feynman trick:

$$\frac{1}{D_0 D_1 D_2} = \Gamma(3) \int_0^1 \frac{dx\, dy\, dz\, \delta(1-x-y-z)}{D^3}, \tag{A.17}$$

where

$$\begin{aligned} &D = [k-(yp_1+zp_2)]^2 - M^2 + i\delta, \\ &M^2 = y(y+z-1)p_1^2 + z(y+z-1)p_2^2 + xM_1^2 + (1-x)M_2^2. \end{aligned} \tag{A.18}$$

With $M_0^2 = (p_2^2 - p_1^2)xy - x(1-x)p_2^2 + xM_1^2 + (1-x)M_2^2$, the PV functions are:

$$\begin{aligned} C_{\{0,1,2,11,22,12\}} &= -\int_0^1 dx \int_0^{1-x} \frac{dy\left\{1, -y, -(1-x-y), y^2, (1-x-y)y, (1-x-y)^2\right\}}{M_0^2}, \\ X_{0,1,2,3} &= -\int_0^1 dx \int_0^{1-x} \frac{dy \times \{x, -xy, -x(1-x-y), -(1-x)\}}{M_0^2}. \end{aligned} \tag{A.19}$$

The expressions of X_i in Eq. (A.19) are very convenient for the case of $(g-2)$ anomaly, where $p_1^2 = p_2^2 = m_a^2$ results in $M_0^2 = -x(1-x)m_a^2 + xM_1^2 + (1-x)M_2^2$, which is independent with y. Consequently, the

$$\begin{aligned} X_{0,1,2,3} &= -\int_0^1 dx \frac{\{x(1-x), -x(1-x)^2/2, -x(1-x)^2/2, -(1-x)^2\}}{M_0^2} \\ &= -\int_0^1 dx \frac{\{x(1-x), -(1-x)x^2/2, -(1-)x^2/2, -x^2\}}{M_0^2}. \end{aligned} \tag{A.20}$$

Formulas of Eq. (A.20) are enough to check the consistency between our results with those of $(g-2)$ anomalies and cLFV amplitudes mentioned in ref. [28]. Using the second line of Eq. (A.20), we can write the general formulas of a_μ as shown in Eqs. (33) and (34).

Indeed, all integrals in Eqs. (33) and (34) can be solved analytically. Starting from the general formulas of M_0^2 as functions of x: $M_0^2(x) = m_a^2(x-x_+)(x-x_-)$ corresponding to the two solutions $x_\pm$. All numerators in Eqs. (33) and (34) are always written in the following forms:

$$ax^2 + bx^2 + c = a_1 M_0^2 + b_1 \frac{dM_0^2}{dx} + c_1. \tag{A.21}$$

The consequence is

$$\int_0^1 dx \times \frac{ax^2 + bx^2 + c}{M_0^2} = a_1 + b_1 \ln \frac{M_1^2}{M_2^2} + \frac{c_1}{\sqrt{\Lambda}} \ln\left[\frac{(1-x_-)x_+}{(1-x_+)x_-}\right]. \tag{A.22}$$

The result in this way must be consistent with those discussed in Ref. [34], hence we do not show precisely here.

A.4. $p_a^2 = p_b^2 = 0$

Results for the case of $p_a^2 = p_b^2 = 0$ were provided in Ref. [36], namely

$$C_0 = a = \frac{M_1^2 - M_2^2 + M_1^2 \ln\left[\frac{M_2^2}{M_1^2}\right]}{(M_1^2 - M_2^2)^2},$$
$$C_1 = C_2 = c = -\frac{3M_1^4 - 4M_1^2M_2^2 + M_2^4 + 2M_1^4 \ln\left[\frac{M_2^2}{M_1^2}\right]}{4(M_1^2 - M_2^2)^3},$$
$$C_{11} = C_{22} = 2C_{12} = d = \frac{11M_1^6 - 18M_1^4M_2^2 + 9M_1^2M_2^4 - 2M_2^6 + 6M_1^6 \ln\left[\frac{M_2^2}{M_1^2}\right]}{18(M_1^2 - M_2^2)^4}. \quad (A.23)$$

This approximate formulas of PV functions give results consistent with those given in Ref. [27], namely

$$f_h(x) = 2\tilde{g}_h(x) = \frac{x^2 - 1 - 2x\log x}{4(x-1)^3},$$
$$g_h(x) = \frac{x - 1 - \log x}{2(x-1)^2},$$
$$\tilde{f}_h(x) = \frac{2x^3 + 3x^2 - 6x + 1 - 6x^2\log x}{24(x-1)^4}, \quad (A.24)$$
$$f_V(x) = \frac{x^3 - 12x^2 + 15x - 4 + 6x^2\log x}{4(x-1)^3},$$
$$g_V(x) = \frac{x^2 - 5x + 4 + 3x\log x}{2(x-1)^2},$$
$$\tilde{f}_V(x) = \frac{-4x^4 + 49x^3 - 78x^2 + 43x - 10 - 18x^3\log x}{24(x-1)^4},$$
$$\tilde{g}_V(x) = \frac{-3(x^3 - 6x^2 + 7x - 2 + 2x^2\log x)}{8(x-1)^3},$$

where $x \equiv m_F^2/m_B^2$. The diagrams FBB and BFF corresponds to different identifications that $\{M_1, M_2\} = \{m_F, m_B\}$ or and $\{M_1, M_2\} = \{m_B, m_F\}$.

Appendix B. Notations in Ref. [36]

Here we give a brief review of the approach of Ref. [36]. Apart from the general couplings of physical Higgs and gauge bosons given in Eqs. (3) and (4), the photon couplings were assumed to be the standard forms given in Table 1. Furthermore, the couplings of the Goldstone boson G_V corresponding to V are assumed to be the following forms:

$$\mathcal{L}_{G_V} = \left\{ G_V \frac{i}{m_V} \overline{F} \left[\left(g^R_{a,FV} m_a - g^L_{a,FV} m_F \right) P_L + \left(g^L_{a,FV} m_a - g^R_{a,FV} m_F \right) P_R \right] e_a + \text{h.c.} \right\}$$
$$+ eQ_V m_V A_\mu V^{*\mu} G_V - eQ_V m_V A_\mu V^\mu G_V^*. \quad (B.1)$$

The above assumptions of the G_V couplings are necessary for the calculation of one-loop gauge contributions that were done in the 't Hoof Feynman gauge. These final results introduced in Ref. [36] were the sum of all diagrams consisting of gauge and Goldstone boson exchanges. Corresponding to the two one-loop diagram classes FVV and VFF, we have the following equivalence between two classes of notations

$$\{a, c_1, c_2, d_1, d_2, f, g\} \equiv \{C_0, C_2, C_1, C_{22}, C_{11}, C_{12}, C_{00}\}^B,$$
$$\{\bar{a}, -\bar{c}_1, -\bar{c}_2, \bar{d}_1, \bar{d}_2, \bar{f}, \bar{g}\} \equiv \{C_0, C_2, C_1, C_{11}, C_{22}, C_{12}, C_{00}\}^f,$$

where $B = h, V$ are gauge bosons in the loop. In addition, the different notations in the definitions of the one-loop integrals given in Eq. (A.1), we have $\{m_1, m_2\} \equiv \{m_b, m_a\}$ while $\{p_1, p_2\} \equiv \{-p_2, -p_1\}$ and $\{p_1, p_2\} \equiv \{p_2, p_1\}$ for the diagrams VFF and FVV respectively. The couplings in the Yukawa Lagrangian of physical bosons are $L_1 \equiv g_b^L$, $R_1 \equiv g_b^R$, $L_2 \equiv g_a^L$, and $R_2 \equiv g_R^a$, which result in the following equivalences: $\lambda \equiv g_a^{L*} g_b^L = g^{LL}$, $\rho \equiv g_a^{R*} g_b^R = g^{RR}$, $\zeta \equiv g_a^{L*} g_b^R = g^{LR}$, and $\upsilon \equiv g_a^{R*} g_b^L = g^{RL}$. As a result, we can identify that:

$$k_1 = m_b X_2^B,\ k_2 = m_a X_1^B,\ k_3 = m_F(c_1 + c_2) = m_F X_3^B,$$
$$\bar{k}_1 = m_b X_2^f,\ \bar{k}_2 = m_b X_1^f,\ k_3 = -m_F X_3^f. \tag{B.2}$$

For a gauge boson B_μ, the one-loop form factors relate to the following notations:

$$y_1 = m_b \left[2X_{02}^f + \frac{m_F^2 (X_2^f - X_3^f) + m_a^2 X_1^f}{m_B^2} \right],$$
$$y_2 = m_a \left[2X_{01}^f + \frac{m_F^2 (X_1^f - X_3^f) + m_b^2 X_2}{m_B^2} \right],$$
$$y_3 = m_F \left[-4X_0^f + \frac{m_F^2 X_3^f + m_a^2 X_1^f + m_b^2 X_2^f}{m_B^2} \right],\ y_4 = -\frac{m_a m_b m_F (X_{12}^f - X_3^f)}{m_B^2}, \tag{B.3}$$

and

$$\bar{y}_1 = m_b \left[2(X_2^f - X_3^f) + \frac{m_F^2 X_{02}^f + m_a^2 X_1^f}{m_B^2} \right],$$
$$\bar{y}_2 = m_a \left[2(X_1^f - X_3^f) + \frac{m_F^2 X_{01}^f + m_b^2 X_2^f}{m_B^2} \right],$$
$$\bar{y}_3 = m_F \left[4X_3^f + \frac{-m_F^2 X_0 - m_a^2 X_1 - m_b^2 X_2}{m_B^2} \right],\ \bar{y}_4 = -\frac{m_a m_b m_F}{m_B^2} X_{012}. \tag{B.4}$$

Appendix C. Important steps to derive $C_{(ab)L,R}$ and $D_{(ab)L,R}$ by hand

The notations for calculating the amplitude corresponding to all diagrams of both Higgs and gauge boson exchanges in Fig. 1 are shown in Fig. 2. All diagrams in the same class will have the same conventions of external momenta and propagators. There are three classes of diagrams: i) The first class consists of four diagrams (1), (2), (5), and (6) in Fig. 1, and the two diagrams (1) and (2) in Fig. 2; ii) the second class consists of three diagrams: (3) and (7) in Fig. 1, and (3) in Fig. 2; iii) the last class consists of the remaining diagrams in the two Figs. 1 and 2.

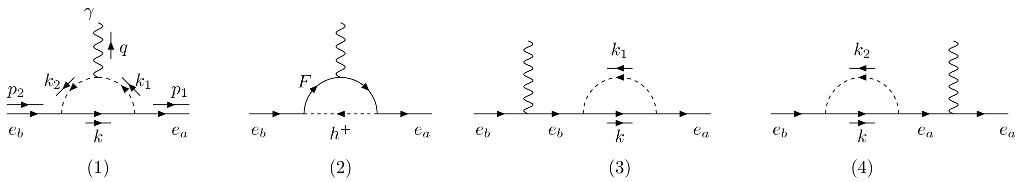

Fig. 2. Momneta notations to derive the one-loop contributions.

Although all the internal momenta have opposite signs with those denoted following LoopTools, the PV-functions are defined with the same values. The relations relevant to momenta are:

$$k_i = k - p_i,\ p_1^2 = m_a^2,\ p_2 = q + p_1,\ p_2^2 = m_b^2,\ q^2 = 0,$$
$$q.\varepsilon^* = 0,\ p_1.\varepsilon^* = p_2.\varepsilon^*. \tag{C.1}$$

Only four diagrams (1), (2), (5), and (6) in Fig. 1 give non-zero contributions to $C_{(ab)L,R}$, hence we firstly derive $C_{(ab)L,R}$ as the factors of $(2p_1.\varepsilon^*)$ in the amplitudes arising from these diagrams. For convenience in detailed calculations, we use simple notations for all the coupling factors $g^{aL,R}_{FB} \to g_a^{L,R}$. For integrals containing divergences, we use the regular dimensional regularization defined by the following replacement:

$$\int \frac{d^4k}{(2\pi)^4} \to \frac{i}{16\pi^2} \times \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk \equiv \int Dk.$$

The final results now are written in terms of the PV functions. In many intermediate steps, we use many results for products of gamma matrices in the dimension d [51], namely

$$\begin{aligned}
&\gamma^\mu\gamma_\mu = d,\\
&\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu \to \gamma^\mu \not{p}\gamma^\mu = (2-d)\not{p},\\
&\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} + (d-4)\gamma^\nu\gamma^\rho \to \gamma^\mu \not{p}_1\not{p}_2\gamma^\mu = 4p_1.p_2 + (d-4)\not{p}_1\not{p}_2,\\
&\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu - (d-4)\gamma^\nu\gamma^\rho\gamma^\sigma \to \gamma^\mu \not{p}_1\not{p}_2\not{p}_3\gamma_\mu\\
&\qquad = -2\not{p}_3\not{p}_2\not{p}_1 - (d-4)\not{p}_1\not{p}_2\not{p}_3, \ldots
\end{aligned}$$

C.1. Scalar contributions

We list here 8 formulas of amplitudes corresponding to 8 particular diagrams shown in Fig. 1. Namely, for three diagrams (1), (3), and (4) we have

$$i\mathcal{M}_1 = -eQ_H \int Dk \times \overline{u_a}[g_a^{R*}P_L + g_a^{L*}P_R]\frac{(m_F + \not{k})}{D_0D_1D_2}[g_b^L P_L + g_b^R P_R]u_b \times (2k_1.\varepsilon^*), \tag{C.2}$$

$$i\mathcal{M}_3 = \frac{-eQ_e}{m_a^2 - m_b^2}\int Dk \times \overline{u_a}[g_a^{R*}P_L + g_a^{L*}P_R]\frac{(m_F + \not{k})}{D_0D_1}[g_b^L P_L + g_b^R P_R](m_b + \not{p}_1)\not{\varepsilon}^* u_b, \tag{C.3}$$

$$i\mathcal{M}_4 = \frac{eQ_e}{m_a^2 - m_b^2}\int Dk \times \overline{u_a}\not{\varepsilon}^*(m_a + \not{p}_2)[g_a^{R*}P_L + g_a^{L*}P_R]\frac{(m_F + \not{k})}{D_0D_2}[g_b^L P_L + g_b^R P_R]u_b, \tag{C.4}$$

where $D_0=k^2-m_F^2$ and $D_i=k_i^2-m_h^2$. The amplitude for the diagram (2) is:

$$i\mathcal{M}_2=-eQ_F\int Dk\times\overline{u_a}[g_a^{R*}P_L+g_a^{L*}P_R]\frac{(m_F-\not{k}_1)\not{\varepsilon}^*(m_F-\not{k}_2)}{D_0D_1D_2}[g_b^LP_L+g_b^RP_R]u_b, \tag{C.5}$$

where $D_0=k^2-m_h^2$ and $D_i=k_i^2-m_F^2$.

In the next calculation, we use the following simple notations:

$$\begin{aligned}&g^{LL}\equiv g_a^{L*}g_b^L,\ g^{RR}\equiv g_a^{R*}g_b^R,\ g^{RL}\equiv g_a^{R*}g_b^L,\ g^{LR}\equiv g_a^{L*}g_b^R,\\ &A_1=g_a^{L*}g_b^RP_R+g_a^{R*}g_b^LP_L,\ A_2=g_a^{L*}g_b^LP_L+g_a^{R*}g_b^RP_R,\end{aligned} \tag{C.6}$$

where $g_a^{L,R}\equiv g_{a,Fh}^{L,R}$ and $g_b^{L,R}\equiv g_{b,Fh}^{L,R}$ without any confusions with the gauge boson couplings $g_{a,FV}^{L,R}$. It is not hard to write all amplitudes in terms of PV-functions as follows:

$$\mathcal{M}_1=\frac{-eQ_H}{16\pi^2}\overline{u_a}\left\{-2p_1.\varepsilon^*[A_1]m_FX_0+\left[2C_{00}^f\not{\varepsilon}^*+\left(X_1^f\not{p}_1+X_2^f\not{p}_2\right)(2p_1.\varepsilon^*)\right][A_2]\right\}u_b, \tag{C.7}$$

$$\mathcal{M}_3=\frac{-eQ_e}{16\pi^2}\times\frac{\overline{u_a}[g_a^{R*}P_L+g_a^{L*}P_R](m_FB_0^{(1)}-B_1^{(1)f}\not{p}_1)[g_b^LP_L+g_b^RP_R](m_b+\not{p}_1)\not{\varepsilon}^*u_b}{(m_a^2-m_b^2)}, \tag{C.8}$$

$$\mathcal{M}_4=\frac{eQ_e}{16\pi^2}\times\frac{\overline{u_a}\not{\varepsilon}^*(m_a+\not{p}_2)[g_a^{R*}P_L+g_a^{L*}P_R](m_FB_0^{(2)}-B_1^{(2)f}\not{p}_2)[g_b^LP_L+g_b^RP_R]u_b}{(m_a^2-m_b^2)}, \tag{C.9}$$

and

$$\begin{aligned}\mathcal{M}_2=&-eQ_F\int Dk\times\overline{u_a}\left\{m_F^2\not{\varepsilon}^*+\not{k}_1\not{\varepsilon}^*\not{k}_2\right\}[A_2]u_b\\ &-eQ_F(-1)m_F\int Dk\times\overline{u_a}\left\{2k.\varepsilon^*-\not{p}_1\not{\varepsilon}^*-\not{\varepsilon}^*\not{p}_2\right\}[A_1]u_b\\ =&\frac{-eQ_F}{16\pi^2}\overline{u_a}\left\{\left[m_F^2C_0+(2-d)C_{00}\right]\not{\varepsilon}^*+(C_{11}+C_1)\not{p}_1\not{\varepsilon}^*\not{p}_1+(C_{22}+C_2)\not{p}_2\not{\varepsilon}^*\not{p}_2\right.\\ &\left.+(X_0+C_{12})\not{p}_1\not{\varepsilon}^*\not{p}_2+C_{12}\not{p}_2\not{\varepsilon}^*\not{p}_1\right\}\times[A_2]u_b\\ &-\frac{eQ_Fm_F}{16\pi^2}\overline{u_a}\left\{(2p_1.\varepsilon^*)(C_1+C_2)+\left(\not{p}_1\not{\varepsilon}^*+\not{\varepsilon}^*\not{p}_2\right)C_0\right\}[A_1]u_b.\end{aligned} \tag{C.10}$$

The validation of the WI given in Eq. (12) implies whether $f_L^{WI}=0$ is correct with:

$$\begin{aligned}f_L^{WI}\equiv&D_{(ab)L}+m_aC_{(ab)L}+m_bC_{(ab)R}\\ =&g^{LL}\left[\frac{Q_e\left(m_a^2B_1^{(1)}-m_b^2B_1^{(2)}\right)^f}{m_a^2-m_b^2}-\left(\frac{1}{2}-2C_{00}^h+m_F^2C_0^h\right)Q_f\right.\\ &\left.-Q_h\left(m_a^2X_1+m_b^2X_2+2C_{00}\right)^f\right]\end{aligned}$$

$$+ g^{RR} m_a m_b \left(\frac{Q_e \left(B_1^{(1)} - B_1^{(2)}\right)^f}{m_a^2 - m_b^2} - Q_f X_{012}^h - Q_h X_{12}^f \right). \tag{C.11}$$

We have used many formulas listed in Eqs. (A.2) and (A.12) to show that

$$\begin{aligned} 0 &= X_{12}^f + X_{12}^h + X_0 \to X_{012}^h = -X_{12}^f, \\ \mathbf{b}_1^f &= -\left(m_a^2 X_1 + m_b^2 X_2 + 2C_{00}\right)^f = \frac{1}{2} - 2C_{00}^h + m_F^2 C_0^h. \end{aligned} \tag{C.12}$$

Finally, the electric charge conservation $Q_F = Q_e + Q_h$ must be satisfied so that Eq. (C.11) resulting in $f_L^{WI} = 0$. On the other word, the WI is valid for only one-loop Higgs contributions arising from the set of four diagrams (1)-(4) in Fig. 1.

C.2. Vector contributions

To calculate the one-loop contributions from gauge boson exchanges corresponding to Lagrangian (4), we denote $g_a^{L,R} \equiv g_{a,FV}^{L,R}$ and $g_b^{L,R} \equiv g_{b,FV}^{L,R}$ then use the notations given in Eq. (C.6). The amplitudes relevant with gauge boson exchanges are:

$$\begin{aligned} i\mathcal{M}_5 =& \int Dk \times \overline{u_a} i\gamma_\alpha [g_a^{L^*} P_L + g_a^{R^*} P_R] \frac{i(m_F + \not{k})}{D_0} i\gamma_\beta [g_b^L P_L + g_b^R P_R] u_b \\ & \times \frac{-i}{D_1} \left(g^{\alpha\alpha'} - \frac{k_1^\alpha k_1^{\alpha'}}{m_V^2} \right) \left[-ieQ_V \Gamma_{\mu\alpha'\beta'}(-q, k_1, -k_2) \varepsilon^{*\mu} \right] \frac{-i}{D_2} \left(g^{\beta\beta'} - \frac{k_2^\beta k_2^{\beta'}}{m_V^2} \right) \\ =& eQ_V \int \frac{d^4k}{(2\pi)^4} \overline{u_a} \gamma_\alpha [g_a^{L^*} P_L + g_a^{R^*} P_R] \frac{(m_F + \not{k})}{D_0 D_1 D_2} \gamma_\beta [g_b^L P_L + g_b^R P_R] u_b \\ & \times \left[\Gamma_{\mu\alpha'\beta'}(-q, k_1, -k_2) \varepsilon^{*\mu} \right] \left(g^{\alpha\alpha'} - \frac{k_1^\alpha k_1^{\alpha'}}{m_V^2} \right) \left(g^{\beta\beta'} - \frac{k_2^\beta k_2^{\beta'}}{m_V^2} \right), \end{aligned} \tag{C.13}$$

$$\begin{aligned} i\mathcal{M}_7 =& \frac{eQ_e}{m_a^2 - m_b^2} \int Dk \times \frac{1}{D_0 D_1} \times \left(g^{\alpha\beta} - \frac{k_1^\alpha k_1^\beta}{m_V^2} \right) \\ & \times \overline{u_a} \gamma_\alpha [g_a^{L^*} P_L + g_a^{R^*} P_R](m_F + \not{k}) \gamma_\beta [g_b^L P_L + g_b^R P_R] \left(m_b + \not{p}_1\right) \not{\varepsilon}^* u_b, \end{aligned} \tag{C.14}$$

$$\begin{aligned} i\mathcal{M}_8 =& -\frac{eQ_e}{m_a^2 - m_b^2} \int Dk \times \frac{1}{D_0 D_2} \times \left(g^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{m_V^2} \right) \\ & \times \overline{u_a} \not{\varepsilon}^* \left(m_a + \not{p}_2\right) \gamma_\alpha [g_a^{L^*} P_L + g_a^{R^*} P_R](m_F + \not{k}) \gamma_\beta [g_b^L P_L + g_b^R P_R] u_b, \end{aligned} \tag{C.15}$$

where $D_0 = k^2 - m_F^2$, $D_i = k_i^2 - m_V^2$, and

$$\Gamma_{\mu\alpha'\beta'}(-q, k_1, -k_2) = g_{\alpha'\beta'}(k_1 + k_2)_\mu + g_{\beta'\mu}(-k_2 + q)_{\alpha'} + g_{\mu\alpha'}(-q - k_1)_{\beta'}. \tag{C.16}$$

The amplitude for the diagram (6) is:

$$\begin{aligned} i\mathcal{M}_6 =& eQ_F \int Dk \times \frac{1}{D_0 D_1 D_2} \times \left(g^{\alpha\beta} - \frac{k^\alpha k^\beta}{m_V^2} \right) \\ & \times \overline{u_a} \gamma_\alpha [g_a^{L^*} P_L + g_a^{R^*} P_R](m_F - \not{k}_1) \not{\varepsilon}^* (m_F - \not{k}_2) \gamma_\beta [g_b^L P_L + g_b^R P_R] u_b, \end{aligned} \tag{C.17}$$

where $D_0 = k^2 - m_V^2$ and $D_i = k_i^2 - m_F^2$.

Considering diagram (7), we have:

$$\begin{aligned} i\mathcal{M}_7 =& \frac{eQ_e}{m_a^2 - m_b^2} \int Dk \times \frac{\overline{u_a}\left[\gamma_\alpha \gamma_\beta m_F [A_1] + \gamma_\alpha \not{k} \gamma_\beta [A_2]\right]\left(m_b + \not{p}_1\right)\not{\varepsilon}^* u_b}{D_0 D_1} \\ & \times \left(g^{\alpha\beta} - \frac{k_1^\alpha k_1^\beta}{m_V^2}\right) \\ =& \frac{eQ_e}{m_a^2 - m_b^2} \int Dk \times \frac{1}{D_0 D_1} \\ & \times \overline{u_a}\left[m_F\left(d - \frac{k_1^2}{m_V^2}\right)[A_1] + \left((2-d)\not{k} - \frac{\not{k}_1 \not{k} \not{k}_1}{m_V^2}\right)[A_2]\right]\left(m_b + \not{p}_1\right)\not{\varepsilon}^* u_b \\ =& \frac{ieQ_e}{16\pi^2(m_a^2 - m_b^2)} \overline{u_a}\left\{ m_F [A_1]\left[(d-1)B_0^{(1)} - \frac{A_0(m_F^2)}{m_V^2}\right]\right. \\ & \left. + m_a\left[\left(-(2-d) + \frac{m_F^2 + m_a^2}{m_V^2}\right)B_1^{(1)} + \frac{A_0(m_V^2) + 2m_F^2 B_0^{(1)}}{m_V^2}\right][A_2]\right\} \\ & \times \left(m_b + \not{p}_1\right)\not{\varepsilon}^* u_b, \end{aligned} \tag{C.18}$$

where we have used the following results

$$\not{k}_1 \not{k} \not{k}_1 = \left(D_0 + m_F^2\right)\not{k} - 2\left(D_0 + m_F^2\right)\not{p}_1 + \not{p}_1 \not{k} \not{p}_1,$$

$$\int \frac{d^4k}{(2\pi)^4} \times \frac{k_\mu}{D_1} = A_0(m_V^2) p_{1\mu}.$$

Then the one-loop contribution form factors from diagram (7) are:

$$\begin{aligned} D_{(ab)L,7} =& \frac{eQ_e}{16\pi^2(m_a^2 - m_b^2)} \left\{\left(g^{RL} m_a + g^{LR} m_b\right) m_F \left[(d-1)B_0^{(1)} - \frac{A_0(m_F^2)}{m_V^2}\right]\right. \\ & + m_a\left(m_a g^{LL} + m_b g^{RR}\right) \\ & \left. \times \left[\left(-(2-d) + \frac{m_F^2 + m_a^2}{m_V^2}\right)B_1^{(1)} + \frac{A_0(m_V^2) + 2m_F^2 B_0^{(1)}}{m_V^2}\right]\right\}, \\ D_{(ab)R,7} =& D_{(ab)L,7}\left[g_a^L \leftrightarrow g_a^R, g_b^L \leftrightarrow g_b^R\right]. \end{aligned} \tag{C.19}$$

The same calculation for diagram (8) gives the following one-loop contribution form factor:

$$\begin{aligned} D_{(ab)L,8} =& -\frac{eQ_e}{16\pi^2(m_a^2 - m_b^2)} \left\{\left(g^{RL} m_a + g^{LR} m_b\right) m_F \left[(d-1)B_0^{(2)} - \frac{A_0(m_F^2)}{m_V^2}\right]\right. \\ & + m_b\left(m_a g^{RR} + m_b g^{LL}\right) \\ & \left. \times \left[\left(-(2-d) + \frac{m_F^2 + m_b^2}{m_V^2}\right)B_1^{(2)} + \frac{A_0(m_V^2) + 2m_F^2 B_0^{(2)}}{m_V^2}\right]\right\}, \\ D_{(ab)R,8} =& D_{(ab)L,8}\left[g_a^L \leftrightarrow g_a^R, g_b^L \leftrightarrow g_b^R\right]. \end{aligned} \tag{C.20}$$

Using $d = 4 - 2\epsilon$ and the divergent parts of PV-functions given in Eq. (A.10), we get the formulas of $D_{(ab)L,78}$ given in Eq. (22).

Diagram (5)

From the equalities $q^2 = 0$, $q.\varepsilon^* = 0$, and $k_1 = q + k_2$, it is easy to prove that

$$\begin{aligned}&\left[\Gamma_{\mu\alpha'\beta'}(-q, k_1, -k_2)\,\varepsilon^{*\mu}\right] k_1^{\alpha} k_1^{\alpha'} k_2^{\beta} k_2^{\beta'}\\&= k_1^{\alpha} k_2^{\beta}\left\{(k_1.k_2)\left[(k_1 + k_2).\varepsilon^*\right] + (k_2.\varepsilon^*)\left[k_1.(-k_2 + q)\right] + (k_1.\varepsilon^*)\left[k_2.(-q - k_1)\right]\right\}\\&\sim (k_1.k_2)\left[2k_1.\varepsilon^*\right] + (k_1.\varepsilon^*)\left[q^2 - k_2^2\right] + (k_1.\varepsilon^*)\left[q^2 - k_1^2\right] = 0.\end{aligned} \tag{C.21}$$

As a result, the amplitude (C.13) is written as follows:

$$\begin{aligned}i\mathcal{M}_5 = eQ_V \int Dk \frac{\overline{u_a}\gamma_\alpha\,[A]\,\gamma_\beta u_b}{D_0 D_1 D_2}\left[\Gamma_{\mu\alpha'\beta'}(-q, k_1, -k_2)\,\varepsilon^{*\mu}\right]\\ \times\left(g^{\alpha\alpha'} g^{\beta\beta'} - \frac{g^{\beta\beta'} k_1^{\alpha} k_1^{\alpha'} + g^{\alpha\alpha'} k_2^{\beta} k_2^{\beta'}}{m_V^2}\right),\end{aligned} \tag{C.22}$$

where

$$A = m_F\left(g_a^{L*} g_b^{R} P_L + g_a^{R*} g_b^{L} P_R\right) + [A_2]\not{k}. \tag{C.23}$$

The first term in the integrand is

$$\begin{aligned}(1) =&\overline{u_a}\left\{4(k_1 + k_2).\varepsilon^* + (-\not{k} + 2\not{p}_2 - \not{p}_1)\not{\varepsilon}^* + \not{\varepsilon}^*(-\not{k} + 2\not{p}_1 - \not{p}_2)\right\} \times m_F\,[A_1]\,u_b\\&+ \overline{u_a}\left\{(2 - d)(2k_1.\varepsilon^*)\not{k} + (-\not{k} + 2\not{p}_2 - \not{p}_1)\not{k}\not{\varepsilon}^* + \not{\varepsilon}^*\not{k}(-\not{k} + 2\not{p}_1 - \not{p}_2)\right\} \times [A_2]\,u_b\\=&\overline{u_a}\left\{6k.\varepsilon^* - 3(\not{p}_1\not{\varepsilon}^* + \not{\varepsilon}^*\not{p}_2)\right\} m_F\,[A_1]\,u_b\\&+ \overline{u_a}\left\{(2 - d)(2k_1.\varepsilon^*)\not{k} + (2\not{p}_2 - \not{p}_1)\not{k}\not{\varepsilon}^* + \not{\varepsilon}^*\not{k}(2\not{p}_1 - \not{p}_2) - 2k^2\not{\varepsilon}^*\right\}[A_2]\,u_b.\end{aligned} \tag{C.24}$$

After integrating out, the formula is

$$\begin{aligned}(1) =&\overline{u_a}\left\{(2p_1.\varepsilon^*) \times (-3m_F)X_3 - 3m_F C_0(\not{p}_1\not{\varepsilon}^* + \not{\varepsilon}^*\not{p}_2)\right\}[A_1]\,u_b\\&+ \overline{u_a}\left\{(2 - d)2\varepsilon^{\alpha*}(C_{\alpha\beta} - C_\beta p_{1\alpha})\gamma^\beta + C_\alpha\left[(2\not{p}_2 - \not{p}_1)\gamma^\alpha\not{\varepsilon}^* + \not{\varepsilon}^*\gamma^\alpha(2\not{p}_1 - \not{p}_2)\right]\right.\\&\left.-2\left(B_0^{(0)} + m_F^2 C_0\right)\not{\varepsilon}^*\right\} \times [A_2]\,u_b\\=&\overline{u_a}(-3m_F) \times \left\{(2p_1.\varepsilon^*)X_3 + C_0(\not{p}_1\not{\varepsilon}^* + \not{\varepsilon}^*\not{p}_2)\right\}[A_1]\,u_b\\&+ \overline{u_a}\left\{\not{\varepsilon}^*\left[2(2 - d)C_{00} - 2(B_0^{(0)} + m_F^2 C_0) - (3m_a^2 + 2m_b^2)C_1 - (2m_a^2 + 3m_b^2)C_2\right]\right.\\&\left.+\not{p}_1\not{\varepsilon}^*\not{p}_2(-3X_3)\right\}[A_2]\,u_b\\&+ \overline{u_a}(2p_1.\varepsilon^*)\left\{[-2(C_{11} + C_{12}) + C_2]\,\not{p}_1 + [-2(C_{12} + C_{22}) + C_1]\,\not{p}_2\right\}[A_2]\,u_b.\end{aligned} \tag{C.25}$$

The second term in the integrand is

$$\left(-\frac{1}{m_V^2}\right)^{-1} \times (2)$$

$$
\begin{aligned}
=&\Gamma_{\mu\alpha'\beta'}(-q,k_1,-k_2)\,\varepsilon^{*\mu}\left(g^{\beta\beta'}k_1^{\alpha}k_1^{\alpha'}+g^{\alpha\alpha'}k_2^{\beta}k_2^{\beta'}\right)\times\overline{u_a}\gamma_\alpha[A]\gamma_\beta u_b\\
=&\overline{u_a}\not{k}_1[A]\left[(k_1.\varepsilon^*)\not{k}_2-\not{\varepsilon}^*k_2^2\right]u_b+\overline{u_a}\left[(k_1.\varepsilon^*)\not{k}_1-\not{\varepsilon}^*k_1^2\right][A]\not{k}_2u_b\\
=&\overline{u_a}m_F[A_1]\left[2(k_1.\varepsilon^*)\not{k}_1\not{k}_2-k_2^2\not{k}_1\not{\varepsilon}^*-k_1^2\not{\varepsilon}^*\not{k}_2\right]u_b\\
&+\overline{u_a}\left[2(k_1.\varepsilon^*)\not{k}_1\not{k}\not{k}_2-k_2^2\not{k}_1\not{k}\not{\varepsilon}^*-k_1^2\not{\varepsilon}^*\not{k}\not{k}_2\right][A_2]u_b\\
=&\overline{u_a}m_F[A_1]\left[\not{k}_1\not{\varepsilon}^*\not{q}\not{k}_2\right]u_b+\overline{u_a}\left[2(k_1.\varepsilon^*)\not{k}_1\not{k}\not{k}_2-k_2^2\not{k}_1\not{k}\not{\varepsilon}^*-k_1^2\not{\varepsilon}^*\not{k}\not{k}_2\right][A_2]u_b.
\end{aligned}\tag{C.26}
$$

The first term in Eq. (C.26) gives

$$
\begin{aligned}
\not{k}_1\not{\varepsilon}^*\not{q}\not{k}_2=&\left(\not{k}-\not{p}_1\right)\not{\varepsilon}^*\not{q}\left(\not{k}-\not{p}_2\right)=\not{k}\not{\varepsilon}^*\not{q}\not{k}-\not{p}_1\not{\varepsilon}^*\not{q}\not{k}-\not{k}\not{\varepsilon}^*\not{q}\not{p}_2+\not{p}_1\not{\varepsilon}^*\not{q}\not{p}_2\\
=&C_{\alpha\beta}\gamma^\alpha\not{\varepsilon}^*\not{q}\gamma^\beta-C_\alpha\not{p}_1\not{\varepsilon}^*\not{q}\gamma^\alpha-C_\alpha\gamma^\alpha\not{\varepsilon}^*\not{q}\not{p}_2+C_0\not{p}_1\not{\varepsilon}^*\not{q}\not{p}_2\\
&+\left(C_1\not{p}_1+C_2\not{p}_2\right)\not{\varepsilon}^*\not{q}\not{p}_2+\not{p}_1\not{\varepsilon}^*\not{q}\left(C_1\not{p}_1+C_2\not{p}_2\right)+C_0\not{p}_1\not{\varepsilon}^*\not{q}\not{p}_2\\
=&C_{00}\left[\varepsilon^*.q-(4-d)\not{\varepsilon}^*\not{q}\right]+\left(C_{12}\not{p}_2+C_{11}\not{p}_1+C_1\not{p}_1\right)\not{\varepsilon}^*\not{q}\not{p}_1\\
&+\left(C_{12}\not{p}_1+C_{22}\not{p}_2+C_1\not{p}_1+C_2\not{p}_2+C_2\not{p}_1+C_0\not{p}_1\right)\not{\varepsilon}^*\not{q}\not{p}_2.
\end{aligned}\tag{C.27}
$$

Because the divergent part $C_{00}=\Delta_\epsilon/4=1/(4\epsilon)$, which $d=4-2\epsilon$, hence $C_{00}(4-d)=1/2$. The result is:

$$
\begin{aligned}
\not{k}_1\not{\varepsilon}^*\not{q}\not{k}_2=&-\frac{1}{2}\not{\varepsilon}\not{q}+\left[C_{12}\left(\not{p}_1+\not{q}\right)+(C_{11}+C_1)\not{p}_1\right]\not{\varepsilon}^*\not{q}\left(\not{p}_2-\not{q}\right)\\
&+\left[(C_{12}+X_0)\not{p}_1+(C_{22}+C_2)\left(\not{p}_1+\not{q}\right)\right]\not{\varepsilon}^*\not{q}\not{p}_2\\
=&-\frac{1}{2}\not{\varepsilon}^*\not{q}+X_{012}\not{p}_1\not{\varepsilon}^*\not{q}\not{p}_2,
\end{aligned}\tag{C.28}
$$

where we have used $\varepsilon^*.q=q^2=0$ and $\not{q}\not{\varepsilon}^*\not{q}=2\varepsilon^*.q\not{q}-q^2\not{\varepsilon}^*=0$. The final result is

$$
\begin{aligned}
\overline{u_a}m_F[A_1]\left[\not{k}_1\not{\varepsilon}^*\not{q}\not{k}_2\right]u_b=&\overline{u_a}m_F\left\{\not{p}_1\not{\varepsilon}^*\left[-\frac{1}{2}+m_b^2X_{012}\right]+\not{\varepsilon}^*\not{p}_2\left[-\frac{1}{2}+m_a^2X_{012}\right]\right.\\
&\left.+(2p_1.\varepsilon^*)\left[\frac{1}{2}-X_{012}\not{p}_1\not{p}_2\right]\right\}[A_1]u_b.
\end{aligned}\tag{C.29}
$$

Consider the last two terms in the last line of the formula (C.26)

$$
\begin{aligned}
&-k_2^2\not{k}_1\not{k}\not{\varepsilon}^*-k_1^2\not{\varepsilon}^*\not{k}\not{k}_2\\
=&-k_2^2\left(k^2-\not{p}_1\not{k}\right)\not{\varepsilon}^*-\not{\varepsilon}^*\left(k^2-\not{k}\not{p}_2\right)k_1^2\\
=&-k^2\left(k_1^2+k_2^2\right)\not{\varepsilon}^*+\left(D_2+m_V^2\right)\not{p}_1\not{k}\not{\varepsilon}^*+\left(D_1+m_V^2\right)\not{\varepsilon}^*\not{k}\not{p}_2\\
\to&-\not{\varepsilon}^*\frac{(D_0+m_F^2)(D_1+D_2+2m_V^2)}{D_0D_1D_2}+\frac{\not{p}_1\not{k}\not{\varepsilon}^*}{D_0D_1}+\frac{\not{\varepsilon}^*\not{k}\not{p}_2}{D_0D_2}+m_V^2\left(\frac{\not{p}_1\not{k}\not{\varepsilon}^*}{D_0D_1D_2}+\frac{\not{\varepsilon}^*\not{k}\not{p}_2}{D_0D_1D_2}\right)\\
=&-\not{\varepsilon}^*\left[2m_V^2(B_0^{(0)}+1)+2m_V^2B_0^{(0)}+m_F^2(B_0^{(1)}+B_0^{(2)}+2m_V^2C_0)\right]\\
&-\not{p}_1\left[B_1^{(1)}\not{p}_1+m_V^2\left(C_1\not{p}_1+C_2\not{p}_2\right)\right]\not{\varepsilon}^*-\not{\varepsilon}^*\left[B_1^{(2)}\not{p}_2+m_V^2\left(C_1\not{p}_1+C_2\not{p}_2\right)\right]\not{p}_2\\
=&-\not{\varepsilon}^*\left[m_V^2(4B_0^{(0)}+2+2m_F^2C_0+m_a^2C_1+m_b^2C_2)+m_a^2B_1^{(1)}+m_b^2B_1^{(2)}+m_F^2(B_0^{(1)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ B_0^{(2)})\Big] - m_V^2 \left(C_2 \not p_1 \not p_2 \not\varepsilon^* + C_1 \not\varepsilon^* \not p_1 \not p_2\right) \\
&= - \not\varepsilon^* \Big[m_V^2 (4B_0^{(0)} + 2 + 2m_F^2 C_0 + m_a^2 C_1 + m_b^2 C_2) + m_a^2 B_1^{(1)} + m_b^2 B_1^{(2)} + m_F^2 (B_0^{(1)} \\
&+ B_0^{(2)})\Big] + m_V^2 X_3 \not p_1 \not\varepsilon^* \not p_2 + (2p_1.\varepsilon^*)(-m_V^2) \left[C_2 \not p_1 + C_1 \not p_2\right]. \qquad \text{(C.30)}
\end{aligned}
$$

Lastly, consider the first term in the last line of the formula (C.26):

$$
\begin{aligned}
&2(k_1.\varepsilon^*)\not k_1 \not k \not k_2 = \left(k.\varepsilon^* - 2p_1.\varepsilon^*\right) \times \left(\not k - \not p_1\right) \not k \left(\not k - \not p_2\right) \\
&= \left(-2p_1.\varepsilon^* + 2k.\varepsilon^*\right) \times \left(k^2 \not k - k^2 \not p_1 - k^2 \not p_2 + \not p_1 \not k \not p_2\right) \\
&\to \left(-2p_1.\varepsilon^* + 2k.\varepsilon^*\right) \times \left[\left(\frac{1}{D_1 D_2} + \frac{m_F^2}{D_0 D_1 D_2}\right)\left(\not k - \not p_1 - \not p_2\right) + \frac{\not p_1 \not k \not p_2}{D_0 D_1 D_2}\right] \\
&= \left(-2p_1.\varepsilon^*\right) \\
&\times \left\{\left[-\frac{1}{2} B_0^{(0)} - m_F^2 C_0\right](\not p_1 + \not p_2) - m_F^2 (C_1 \not p_1 + C_2 \not p_2) - \not p_1 (C_1 \not p_1 + C_2 \not p_2) \not p_2\right\} \\
&+ \left(2\varepsilon_\mu^*\right)\left[\left(\frac{1}{D_1 D_2} + \frac{m_F^2}{D_0 D_1 D_2}\right) \not k k^\mu - \left(\frac{1}{D_1 D_2} + \frac{m_F^2}{D_0 D_1 D_2}\right)(\not p_1 + \not p_2) k^\mu \right. \\
&\left. + \frac{\not p_1 \not k \not p_2 k^\mu}{D_0 D_1 D_2}\right] \\
&= \left(2p_1.\varepsilon^*\right)\left\{\left[\frac{B_0^{(0)}}{2} + m_F^2 C_0\right]\left(\not p_1 + \not p_2\right) + \left(m_F^2 C_1 + m_b^2 C_2\right) \not p_1 + \left(m_F^2 C_2 + m_a^2 C_1\right) \not p_2\right\} \\
&+ \left(2\varepsilon_\mu^*\right) \times \left\{\left(B^{\mu\nu} + m_F^2 C^{\mu\nu}\right)\gamma_\nu - \left(B^\mu + m_F^2 C^\mu\right)\left(\not p_1 + \not p_2\right) + C^{\mu\nu} \not p_1 \gamma_\nu \not p_2\right\}, \qquad \text{(C.31)}
\end{aligned}
$$

where $B^\mu = B^\mu(0, m_V^2, m_V^2)$ and $B^{\mu\nu} = B^{\mu\nu}(0, m_V^2, m_V^2)$. The last line in Eq. (C.31) is expressed in terms of the PV functions as follows

$$
\begin{aligned}
&\left(2\varepsilon_\mu^*\right)\left\{\gamma_\nu \left[\frac{g^{\mu\nu}}{2}\left(B_0^{(0)} + 1\right) + \frac{1}{6} B_0^{(0)} \left(2p_1^\mu p_1^\nu + p_1^\mu p_2^\nu + p_2^\mu p_1^\nu + 2p_2^\mu p_2^\nu\right)\right]\right. \\
&\qquad + m_F^2 \gamma_\nu \left[C_{00} g^{\mu\nu} + C_{11} p_1^\mu p_1^\nu + C_{12} p_1^\mu p_2^\nu + C_{12} p_2^\mu p_1^\nu + C_{22} p_2^\mu p_2^\nu\right] \\
&\qquad - \left[\frac{1}{2} B_0^{(0)} (p_1 + p_2)^\mu - m_F^2 \left(C_1 p_1^\mu + C_2 p_2^\mu\right)\right]\left(\not p_1 + \not p_2\right) \\
&\qquad \left. + \left[C_{00} g^{\mu\nu} + C_{11} p_1^\mu p_1^\nu + C_{12} p_1^\mu p_2^\nu + C_{12} p_2^\mu p_1^\nu + C_{22} p_2^\mu p_2^\nu\right] \not p_1 \gamma_\nu \not p_2\right\} \\
&= m_V^2 \not\varepsilon^* \left(B_0^{(0)} + 1\right) + (p_1.\varepsilon^*) B_0^{(0)} \left(\not p_1 + \not p_2\right) \\
&+ m_F^2 \left[2C_{00} \not\varepsilon^* + \left(2p_1.\varepsilon^*\right)\left(C_{11} \not p_1 + C_{12} \not p_2 + C_{12} \not p_1 + C_{22} \not p_2\right)\right] \\
&- \left[B_0^{(0)} \left(2p_1.\varepsilon^*\right) - m_F^2 \left(2p_1.\varepsilon^*\right)\left(C_1 + C_2\right)\right]\left(\not p_1 + \not p_2\right) \\
&+ \not p_1 \left[2\not\varepsilon^* C_{00} + \left(2p_1.\varepsilon^*\right)\left(C_{11} \not p_1 + C_{12} \not p_1 + C_{12} \not p_2 + C_{22} \not p_2\right)\right] \not p_2. \qquad \text{(C.32)}
\end{aligned}
$$

Hence the final result of Eq. (C.31) is

$$
\begin{aligned}
2(k_1.\varepsilon^*)\not{k}_1\not{k}\not{k}_2 =&\not{\varepsilon}^*\left[m_V^2(B_0^{(0)}+1)+2m_F^2C_{00}\right]+\not{p}_1\not{\varepsilon}^*\not{p}_2\,(2C_{00})\\
&+(2p_1.\varepsilon^*)\left[m_F^2X_{01}+m_b^2X_2\right]\not{p}_1+(2p_1.\varepsilon^*)\left[m_F^2X_{01}+m_a^2X_1\right]\not{p}_2.
\end{aligned}
\tag{C.33}
$$

The sum of three terms given in Eqs. (C.25), (C.30), and (C.33) gives $C_{(ab)L,R}$ corresponding to the diagrams (5) given in Eqs. (18) and (19). The formulas of $D_{(ab)L,5}$ and $D_{(ab)R,5}$ are given in Eq. (23).

Regarding to the case of photon couplings in Eq. (27), the equality given in Eq. (C.21) is still valid because the new part $\Delta\Gamma_{\mu\alpha'\beta'}=\Gamma_{\mu\alpha'\beta'}-\Gamma'_{\mu\alpha'\beta'}=\delta k_v\left(g_{\mu\alpha'}q_{\beta'}-g_{\beta'\mu}q_{\alpha'}\right)$ satisfies $\left(g_{\mu\alpha'}q_{\beta'}-g_{\beta'\mu}q_{\alpha'}\right)\varepsilon^{*\mu}k_1^{\alpha'}k_2^{\beta'}=q^2(\varepsilon^*.k_2)-(q.k_2)(\varepsilon^*.q)=0$. The other relevant part of $\mathcal{M}_5$ is:

$$
\begin{aligned}
&-\gamma_\alpha\,[A]\,\gamma_\beta\times\Delta\Gamma_{\mu\alpha'\beta'}\varepsilon^{\mu*}\left(g^{\alpha\alpha'}g^{\beta\beta'}-\frac{g^{\beta\beta'}k_1^\alpha k_1^{\alpha'}+g^{\alpha\alpha'}k_2^\beta k_2^{\beta'}}{m_V^2}\right)\\
=&\left(\not{q}\not{\varepsilon}^*-\not{\varepsilon}^*\not{q}\right)m_FA_1+\left(\not{q}\not{k}\not{\varepsilon}^*-\not{\varepsilon}^*\not{k}\not{q}\right)A_2\\
&-\frac{1}{m_V^2}\left\{\left[(k_1.q)(\not{k}_1\not{\varepsilon}^*-\not{\varepsilon}^*\not{k}_2)+(k_1.\varepsilon^*)(\not{q}\not{k}_2-\not{k}_1\not{q})\right]m_FA_1\right.\\
&\qquad\left.+\left[(k_1.q)(-\not{p}_1\not{k}\not{\varepsilon}^*+\not{\varepsilon}^*\not{k}\not{p}_2)+(k_1.\varepsilon^*)(\not{p}_1\not{k}\not{q}-\not{q}\not{k}\not{p}_2)\right]A_2\right\}.
\end{aligned}
\tag{C.34}
$$

The final result of new contributions to $i\delta\mathcal{M}_5$ is:

$$
\begin{aligned}
i\delta\mathcal{M}_5=&\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u_a}\gamma_\alpha\,[A]\,\gamma_\beta u_b}{D_0D_1D_2}\times\Delta\Gamma_{\mu\alpha'\beta'}\left(g^{\alpha\alpha'}g^{\beta\beta'}-\frac{g^{\beta\beta'}k_1^\alpha k_1^{\alpha'}+g^{\alpha\alpha'}k_2^\beta k_2^{\beta'}}{m_V^2}\right)\\
=&-\frac{ieQ_V\delta_v}{16\pi^2}\left\{\overline{u_a}\left[\left(4p_1.\varepsilon^*-2\not{p}_1\not{\varepsilon}^*-2\not{\varepsilon}^*\not{p}_2\right)C_0m_FA_1\right]u_b\right.\\
&+\overline{u_a}\left[(2p_1.\varepsilon^*)(\not{p}_1+\not{p}_2)-(m_a^2+m_b^2)\not{\varepsilon}^*-2\not{p}_1\not{\varepsilon}^*\not{p}_2\right]X_3A_2u_b\\
&-\frac{1}{m_V^2}\overline{u_a}\left[C_{00}\left(\not{q}\not{\varepsilon}^*-\not{\varepsilon}^*\not{q}\right)+(C_{11}+C_{12})\left[-2(p_1.q)\not{\varepsilon}^*\not{p}_1+2(p_1.\varepsilon^*)\not{q}\not{p}_1\right]\right.\\
&+(C_{22}+C_{12})\left[-2(p_2.q)\not{\varepsilon}^*\not{p}_2+2(p_2.\varepsilon^*)\not{q}\not{p}_2\right]\\
&+(p_1.q)\left[-X_0\left(-\not{p}_1\not{\varepsilon}^*+\not{\varepsilon}^*\not{p}_2\right)+(2p_1.\varepsilon^*)(C_2-C_1)+2\left(C_1\not{p}_1\not{\varepsilon}^*-C_2\not{\varepsilon}^*\not{p}_2\right)\right]\\
&\left.-(p_1.\varepsilon^*)\left(2\not{p}_1\not{p}_2-m_a^2-m_b^2\right)(2X_3+C_0)\right]m_FA_1u_b\\
&-\frac{1}{m_V^2}\overline{u_a}\left[C_{00}\left(2(m_a^2+m_b^2)\not{\varepsilon}^*+4\not{p}_1\not{\varepsilon}^*\not{p}_2-2(\not{p}_1+\not{p}_2)\times(2p_1.\varepsilon^*)\right)\right.\\
&\qquad+\frac{m_b^2-m_a^2}{2}\times X_1\left(-m_a^2\not{\varepsilon}^*-\not{p}_1\not{\varepsilon}^*\not{p}_2+(2p_1.\varepsilon^*)\not{p}_1\right)\\
&\qquad\left.\left.+\frac{m_b^2-m_a^2}{2}\times X_2\left(m_b^2\not{\varepsilon}^*+\not{p}_1\not{\varepsilon}^*\not{p}_2-(2p_1.\varepsilon^*)\not{p}_2\right)\right]A_2u_b\right\}.
\end{aligned}
\tag{C.35}
$$

Ignoring the factor $\frac{eQ_V\delta k_v}{16\pi^2}$, the form factors are:

$$
-\delta C^{FVV}_{(ab)L}=g^{LL}m_a\left[X_3+\frac{4C_{00}-(m_b^2-m_a^2)X_1}{2m_V^2}\right]
$$

$$
\begin{aligned}
&+g^{RR} m_b\left[X_3+\frac{4C_{00}+(m_b^2-m_a^2)X_2}{2m_V^2}\right]\\
&+g^{RL} m_F\left[2C_0-\frac{8C_{00}+m_a^2(2X_1+X_0)+m_b^2(2X_2+X_0)}{2m_V^2}\right]\\
&+g^{LR}\frac{m_F m_a m_b X_{012}}{m_V^2},\\
-\delta C^{FVV}_{(ab)R}=&\delta C_{L,5}\left[g_a^L\leftrightarrow g_a^R,\, g_b^L\leftrightarrow g_b^R\right],\\
-\delta D^{FVV}_{(ab)L}=&g^{LL}\left[-(m_a^2+m_b^2)X_3-\frac{4(m_a^2+m_b^2)C_{00}+(m_b^2-m_a^2)(-m_a^2X_1+m_b^2X_2)}{2m_V^2}\right]\\
&+g^{RR} m_a m_b\left[-2X_3-\frac{8C_{00}+(m_b^2-m_a^2)(-X_1+X_2)}{2m_V^2}\right]\\
&+g^{RL} m_a m_F\left[-2C_0-\frac{-8C_{00}+(m_b^2-m_a^2)(2X_1+X_0)}{2m_V^2}\right]\\
&+g^{LR} m_b m_F\left[-2C_0+\frac{8C_{00}+(m_b^2-m_a^2)(2X_2+X_0)}{2m_V^2}\right],\\
\delta D^{FVV}_{(ab)R}=&\delta D^{FVV}_{(ab)L}\left[g_a^L\leftrightarrow g_a^R,\, g_b^L\leftrightarrow g_b^R\right].
\end{aligned}
\tag{C.36}
$$

All results given in Eq. (C.36) were cross checked using FORM package [48]. All formulas in Eq. (C.36) satisfy automatically the WI, namely $\delta D^{FVV}_{(ab)L}+m_a\delta C^{FVV}_{(ab)L}+m_b\delta C^{FVV}_{(ab)R}=0$.

Diagram (6)

After using the property of chiral operators $P_{L,R}$, the amplitude (C.17) is written as

$$
\begin{aligned}
i\mathcal{M}_6=&eQ_F\int\frac{d^4k}{(2\pi)^4}\frac{1}{D_0D_1D_2}\times\overline{u_a}\Big[\left(m_F^2\gamma_\alpha\not{\varepsilon}^*\gamma_\beta+\gamma_\alpha\not{k}_1\not{\varepsilon}^*\not{k}_2\gamma_\beta\right)[A_2]\\
&-m_F[A_1]\left(g_a^{L^*}g_b^R P_R+g_a^{R^*}g_b^L P_L\right)\left(\gamma_\alpha\not{k}_1\not{\varepsilon}^*\gamma_\beta+\gamma_\alpha\not{\varepsilon}^*\not{k}_2\gamma_\beta\right)\Big]u_b.
\end{aligned}
\tag{C.37}
$$

The numerator is divided into the two parts $N_1\sim g^{\alpha\beta}$ and $N_2\sim -k^\alpha k^\beta/m_V^2$. After extracting $g^{\alpha\beta}$, the first part is

$$
\begin{aligned}
N_1=&\overline{u_a}\Big\{\left[(2-d)m_F^2\not{\varepsilon}^*-2\not{k}_2\not{\varepsilon}^*\not{k}_1+(4-d)\not{k}_1\not{\varepsilon}^*\not{k}_2\right][A_2]\\
&-m_F[A_1]\left[4\varepsilon^*.(k_1+k_2)-(4-d)\left(\not{k}_1\not{\varepsilon}^*+\not{\varepsilon}^*\not{k}_2\right)\right]\Big\}u_b.
\end{aligned}
\tag{C.38}
$$

Ignoring the overall factor $eQ_F/(16\pi^2)$, the formula in terms of tensor notations is

$$
\begin{aligned}
N_1=&\overline{u_a}\not{\varepsilon}^*[A_2]u_b\left[-2m_F^2C_0+(d-4)(d-2)C_{00}\right]+\left(2p_1.\varepsilon^*\right)\overline{u_a}[A_1](4m_FX_0)u_b\\
&+\overline{u_a}\left[(2-d)C_{\alpha\beta}\gamma^\alpha\not{\varepsilon}^*\gamma^\beta+2C_\alpha\left(\not{p}_2\not{\varepsilon}^*\gamma^\alpha+\gamma^\alpha\not{\varepsilon}^*\not{p}_1\right)-2C_0\not{p}_2\not{\varepsilon}^*\not{p}_1\right]\times[A_2]u_b.
\end{aligned}
\tag{C.39}
$$

After expanding the tensors in terms of scalar PV-functions, the final result is

$$
\begin{aligned}
N_1 =& \overline{u_a} \not{\varepsilon}^* [A_2] u_b \Big[-2m_F^2 C_0 + (d-2)^2 C_{00} + 2m_a^2 X_{01} + m_b^2 X_{02} \Big] \\
&+ \overline{u_a} \not{p}_1 \not{\varepsilon}^* \not{p}_2 [A_2] u_b \times (2X_0) \\
&+ \left(2\varepsilon^*.p_1\right) \overline{u_a} (-2) \left[X_{01} \not{p}_1 + X_{02} \not{p}_2 \right] [A_2] u_b + \left(2\varepsilon^*.p_1\right) \overline{u_a} \left\{ 4m_F X_0 [A_1] \right\} u_b .
\end{aligned} \tag{C.40}
$$

Considering the second term proportional to $k^\alpha k^\beta$, we have

$$
-m_V^2 N_2 = \overline{u_a} \left(m_F^2 \not{k} \not{\varepsilon}^* \not{k} + \not{k} \not{k}_1 \not{\varepsilon}^* \not{k}_2 \not{k} \right) [A_2] u_b - m_F \overline{u_a} \left(\not{k} \not{k}_1 \not{\varepsilon}^* \not{k} + \not{k} \not{\varepsilon}^* \not{k}_2 \not{k} \right) [A_1] u_b . \tag{C.41}
$$

The two relations $\not{k}\not{k}_1 = D_1 + m_F^2 - m_a^2 + \not{p}_1 \not{k}$ and $\not{k}_2 \not{k} = D_2 + m_F^2 - m_b^2 + \not{k} \not{p}_2$ give

$$
\begin{aligned}
N_2 \sim & \overline{u_a} \left(m_F^2 \not{k} \not{\varepsilon}^* \not{k} \right) [A_2] u_b \\
&+ \overline{u_a} \left(D_1 + m_F^2 - m_1^2 + \not{p}_1 \not{k} \right) \not{\varepsilon}^* \left(D_2 + m_F^2 - m_2^2 + \not{k} \not{p}_2 \right) [A_2] u_b \\
&- m_F \overline{u_a} \Big[\left(D_1 + m_F^2 - m_a^2 + \not{p}_1 \not{k} \right) \not{\varepsilon}^* \not{k} + \not{k} \not{\varepsilon}^* \left(D_2 + m_F^2 - m_b^2 + \not{k} \not{p}_2 \right) \Big] [A_1] u_b \\
\equiv & \overline{u_a} \left[(L_1 + L_2) [A_2] - m_F [A_1] L_3 \right] u_b ,
\end{aligned} \tag{C.42}
$$

where

$$
\begin{aligned}
L_1 =& m_F^2 \Big\{ \not{\varepsilon}^* \Big[(2-d) C_{00} - m_a^2 (C_{11} + C_{12}) - m_b^2 (C_{12} + C_{22}) \Big] \\
& \quad + (2p_1.\varepsilon^*) \left[(C_{11} + C_{12}) \not{p}_1 + (C_{12} + C_{22}) \not{p}_2 \right] \Big\} , \\
L_2 =& \frac{1}{D_0 D_1 D_2} \left(D_1 + m_F^2 - m_a^2 + \not{p}_1 \not{k} \right) \not{\varepsilon}^* \left(D_2 + m_F^2 - m_b^2 + \not{k} \not{p}_2 \right) \\
=& \not{\varepsilon}^* \left[\frac{1}{D_0} + \frac{m_F^2 - m_b^2 + \not{k}\not{p}_2}{D_0 D_2} + \frac{m_F^2 - m_a^2}{D_0 D_1} + \frac{\left(m_F^2 - m_a^2\right)\left(m_F^2 - m_b^2\right)}{D_0 D_1 D_2} \right] + \frac{\not{p}_1 \not{k} \not{\varepsilon}^*}{D_0 D_1} \\
&+ \frac{\not{p}_1 \not{k} \not{\varepsilon}^* \not{k} \not{p}_2}{D_0 D_1 D_2} + \frac{\not{p}_1 \not{k} \not{\varepsilon}^* \left(m_F^2 - m_b^2\right)}{D_0 D_1 D_2} + \frac{\left(m_F^2 - m_a^2\right) \not{\varepsilon}^* \not{k} \not{p}_2}{D_0 D_1 D_2} \\
=& \not{\varepsilon}^* \Big[A_0(m_V^2) + \left(m_F^2 - m_b^2 \right) B_0^{(2)} - m_b^2 B_1^{(2)} + \left(m_F^2 - m_a^2 \right) B_0^{(1)} - m_a^2 B_1^{(1)} \\
&+ \left(m_F^2 - m_a^2 \right) \left(m_F^2 - m_b^2 \right) C_0 \Big] \\
&+ C_{\alpha\beta} (\not{p}_1 \gamma^\alpha \not{\varepsilon}^* \gamma^\beta \not{p}_2) + C_\alpha (\not{p}_1 \gamma^\alpha \not{\varepsilon}^*) \left(m_F^2 - m_b^2 \right) + C_\alpha (\not{\varepsilon}^* \gamma^\alpha \not{p}_2) \left(m_F^2 - m_a^2 \right) , \\
L_3 =& \frac{\not{\varepsilon}^* \not{k}}{D_0 D_2} + \frac{\not{k} \not{\varepsilon}^*}{D_0 D_1} + \frac{m_F^2 (2k.\varepsilon^*) - m_a^2 \not{\varepsilon}^* \not{k} - m_b^2 \not{k} \not{\varepsilon}^*}{D_0 D_1 D_2} + \frac{\not{p}_1 \not{k} \not{\varepsilon}^* \not{k} + \not{k} \not{\varepsilon}^* \not{k} \not{p}_2}{D_0 D_1 D_2} \\
=& - B_1^{(2)} \not{\varepsilon}^* \not{p}_2 - B_1^{(1)} \not{p}_1 \not{\varepsilon}^* - (2p_1.\varepsilon^*)(C_1 + C_2) m_F^2 \\
&- C_\alpha \left(m_a^2 \not{\varepsilon}^* \gamma^\alpha + m_b^2 \gamma^\alpha \not{\varepsilon}^* \right) + C_{\alpha\beta} \left(\not{p}_1 \gamma^\alpha \not{\varepsilon}^* \gamma^\beta + \gamma^\alpha \not{\varepsilon}^* \gamma^\beta \not{p}_2 \right) .
\end{aligned}
$$

It can be proved that:

$$
\begin{aligned}
C_{\alpha\beta} (\not{p}_1 \gamma^\alpha \not{\varepsilon}^* \gamma^\beta \not{p}_2) =& \not{p}_1 \not{\varepsilon}^* \not{p}_2 \Big[(2-d) C_{00} - m_a^2 (C_{11} + C_{12}) - m_b^2 (C_{22} + C_{12}) \Big] \\
&+ (2p_1.\varepsilon^*) \Big[m_b^2 (C_{22} + C_{12}) \not{p}_1 + m_a^2 (C_{11} + C_{12}) \not{p}_2 \Big] ,
\end{aligned}
$$

$$
\begin{aligned}
&C_\alpha(\not{p}_1\gamma^\alpha\not{\varepsilon}^*) = -m_a^2 C_1\not{\varepsilon}^* - C_2\left[(2p_1.\varepsilon^*)\not{p}_1 - \not{p}_1\not{\varepsilon}^*\not{p}_2\right],\\
&C_\alpha(\not{\varepsilon}^*\gamma^\alpha\not{p}_2) = -m_b^2 C_2\not{\varepsilon}^* - C_1\left[(2p_1.\varepsilon^*)\not{p}_2 - \not{p}_1\not{\varepsilon}^*\not{p}_2\right],\\
&C_\alpha\left(m_a^2\not{\varepsilon}^*\gamma^\alpha + m_b^2\gamma^\alpha\not{\varepsilon}^*\right)\\
=&(2p_1.\varepsilon^*)\left[-m_a^2C_1 - m_b^2C_2\right] + \not{p}_1\not{\varepsilon}^*(m_a^2-m_b^2)C_1 + \not{\varepsilon}^*\not{p}_2(m_b^2-m_a^2)C_2,\\
&C_{\alpha\beta}\left(\not{p}_1\gamma^\alpha\not{\varepsilon}^*\gamma^\beta + \gamma^\alpha\not{\varepsilon}^*\gamma^\beta\not{p}_2\right)\\
=&(2p_1.\varepsilon^*)\left[m_a^2(C_{11}+C_{12}) + m_b^2(C_{22}+C_{12}) + \not{p}_1\not{p}_2(C_{11}+2C_{12}+C_{22})\right]\\
&+(\not{p}_1\not{\varepsilon}^* + \not{\varepsilon}^*\not{p}_2)\left[(2-d)C_{00} - m_a^2(C_{11}+C_{12}) - m_b^2(C_{22}+C_{12})\right]. \qquad \text{(C.43)}
\end{aligned}
$$

Final results are:

$$
\begin{aligned}
L_1 =& m_F^2\left\{\not{\varepsilon}^*\left[(2-d)C_{00} - m_a^2(C_{11}+C_{12}) - m_b^2(C_{12}+C_{22})\right]\right.\\
&\left.+(2p_1.\varepsilon^*)\left[(C_{11}+C_{12})\not{p}_1 + (C_{12}+C_{22})\not{p}_2\right]\right\},\\
L_2 =& \not{\varepsilon}^*\left\{m_V^2(B_0^{(0)}+1) + m_F^2(B_0^{(1)}+B_0^{(2)}) - m_a^2(B_0^{(1)}+B_1^{(1)}) - m_b^2(B_0^{(2)}+B_1^{(2)})\right.\\
&\left.+m_F^4C_0 - m_F^2\left[(m_a^2+m_b^2)C_0 + m_a^2C_1 + m_b^2C_2\right] + m_a^2m_b^2X_0\right\}\\
&+\not{p}_1\not{\varepsilon}^*\not{p}_2\left[(2-d)C_{00} + m_F^2X_3 - m_a^2X_1 - m_b^2X_2\right]\\
&+(2p_1.\varepsilon^*)\left[\left(m_b^2X_2 - m_F^2C_2\right)\not{p}_1 + \left(m_a^2X_1 - m_F^2C_1\right)\not{p}_2\right],\\
L_3 =& \not{p}_1\not{\varepsilon}^*\left[-B_1^{(1)} + (2-d)C_{00} - m_a^2X_1 + m_b^2(X_3-X_2)\right]\\
&+\not{\varepsilon}^*\not{p}_2\left[-B_1^{(2)} + (2-d)C_{00} - m_b^2X_2 + m_a^2(X_3-X_1)\right]\\
&+(2p_1.\varepsilon^*)\left[m_a^2X_1 + m_b^2X_2 - m_F^2X_3 + \not{p}_1\not{p}_2(X_1+X_2-X_3)\right]. \qquad \text{(C.44)}
\end{aligned}
$$

The above calculation is enough to derive relevant contributions to $C_{L,R}^{VFF}$ given in Eqs. (20) and (21), and $D_{L,R}^{VFF}$ given in (24).

Ward identity for the only gauge boson exchanges

Before coming to discuss the WI, we use the relations given in Eq. (A.12) to write all the one-loop factors (22), (23), and (24) from gauge boson exchanges in the following simple forms, ignoring the overall factor $e/(16\pi^2)$:

$$
\begin{aligned}
D_{(ab)L,78}^{FV} =& Q_e\left(g^{RL}m_a + g^{LR}m_b\right)(-3m_FX_0)\\
&+Q_eg^{RR}m_am_b\left[-2X_{12}^f + \frac{\mathbf{b}_1^f - m_F^2(2X_0+X_{12}^f)}{m_V^2}\right]\\
&+Q_eg^{LL}\left\{\left(2+\frac{m_F^2+m_a^2+m_b^2}{m_V^2}\right)\mathbf{b}_1^f + 1 + \frac{A_0(m_V^2)+m_a^2m_b^2X_{12}^f}{m_V^2}\right.\\
&\left.+\frac{2m_F^2(m_a^2B_0^{(1)} - m_b^2B_0^{(2)})}{(m_a^2-m_b^2)m_V^2}\right\}. \qquad \text{(C.45)}
\end{aligned}
$$

The WI for the FVV and FVV diagrams are $f^{WI}_{FVV} \equiv D^{FVV}_{(ab)L} + m_a C^{FVV}_{(ab)L} + m_b C^{FVV}_{(ab)R}$ and $f^{WI}_{VFF} \equiv D^{VFF}_{(ab)L} + m_a C^{VFF}_{(ab)L} + m_b C^{VFF}_{(ab)R}$, respectively. The relations given in Eq. (A.12) give:

$$
\begin{aligned}
&2C^f_{00} + m_a^2 X^f_1 + m_b^2 X^f_2 = -\mathbf{b}^f_1,\\
&X^v_{012} = -X^f_{12},\\
&m_F^2(X^v_{12} - X^v_3) + m_a^2 X^v_1 + m_b^2 X^v_2 + \mathbf{b}^v_1 + 1/2 = m_F^2(X^v_{12} - X^v_3) - m_F^2 C^v_0\\
&= m_F^2(X^v_{012} - 2X_0) = -m_F^2(X^f_{12} + 2X_0),\\
&1/2 + m_a^2 X^v_1 + m_b^2 X^v_2 - m_F^2 X^v_3 = (m_F^2 - m_V^2)X_0 - m_F^2 C^v_0 - m_F^2 X^v_3 = -m_V^2 X_0.
\end{aligned} \tag{C.46}
$$

Combining the above formulas and results of $C_{i,ij}$ functions listed in Ref. [39], the WI of all diagrams with boson exchanges is derived as follows

$$
\begin{aligned}
f^{WI}_V =& D_{(ab)L,78} + f^{WI}_{FVV} + f^{WI}_{VFF}\\
&\sim (Q_e + Q_V - Q_F)\\
&\times \Bigg\{ g^{LL}\Bigg[\frac{3 - B_0^{(0)}(m_V^2)}{2} + \frac{A_0(m_F^2)}{2m_V^2} - \frac{(m_a^2 + m_F^2 - 2m_V^2)m_V^2 + (m_a^2 - m_F^2)^2}{2(m_a^2 - m_b^2)m_V^2} \times B_0^{(1)}\\
&\qquad + \frac{(m_b^2 + m_F^2 - 2m_V^2)m_V^2 + (m_b^2 - m_F^2)^2}{2(m_a^2 - m_b^2)m_V^2} \times B_0^{(2)} \Bigg]\\
&+ g^{RR}\Bigg[\frac{(m_V^2 B_0^{(0)} - A_0(m_F^2)) \times (m_F^2 + 2m_V^2) - (m_F^2 - 4m_V^2)m_V^2}{2m_V^2}\\
&\qquad - \frac{m_b^2\left[(m_a^2 + m_F^2 - 2m_V^2)m_V^2 + (m_a^2 - m_F^2)^2\right]}{2(m_a^2 - m_b^2)m_V^2} \times B_0^{(1)}\\
&\qquad + \frac{m_a^2\left[(m_b^2 + m_F^2 - 2m_V^2)m_V^2 + (m_b^2 - m_F^2)^2\right]}{2(m_a^2 - m_b^2)m_V^2} \times B_0^{(2)} \Bigg]\\
&+ \left(g^{RL} m_a + g^{LR} m_b\right)(3m_F X_0) \Bigg\}.
\end{aligned} \tag{C.47}
$$

The final result is $f^{WI}_V \sim Q_F - (Q_e + Q_V) = 0$. In conclusion, the contributions from the four diagrams with only gauge boson exchanges satisfy the WI when the electric charge conservation is valid.

Appendix D. Ward identity for the diagrams of FSV-type in the unitary gauge

This type of diagrams were mentioned firstly in Ref. [34] for the general case of their contributions to BSM. The $\gamma - S - V$ vertices come the kinetic terms of the scalars:

$$
L^D(S) = \left(\partial_\mu S - iP_\mu S\right)^\dagger \left(\partial^\mu S - iP^\mu S\right) = \left[g_{\gamma SV} g_{\mu\nu} S^{-Q} A^\mu V^{Q\nu} + \text{h.c.}\right] + \ldots, \tag{D.1}
$$

where P_μ containing the photon A_μ and V_μ is the covariant part of the covariant derivative of the Higgs multiplets. The Feynman diagrams in the general gauge R_ξ are shown in Fig. 3. Here only two diagrams (1) and (2) give non-zeros contributions in the unitary gauge, which correspond to

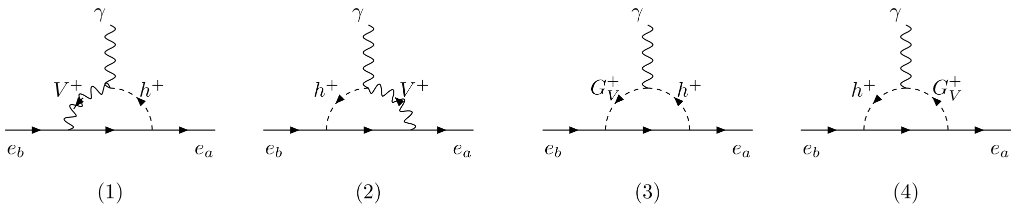

Fig. 3. One-loop three-point FSV diagrams in the gauge R_ξ.

the two diagrams (b) and (a) in Fig. 5 introduced in Ref. [34]. In this gauge, the contributions of these two diagrams are:

$$
\begin{aligned}
i\mathcal{M}_9 =& g_{\gamma SV}\int \frac{d^4k}{(2\pi)^4} \\
&\times \frac{\overline{u_a}[g_a^{L^*}P_R + g_a^{R^*}P_L](m_F + \not{k})\gamma_\alpha[g_b^L P_L + g_b^R P_R]u_b}{D_0D_1D_2}\left(\varepsilon^{*\alpha} - \frac{(\varepsilon^*.k_2)k_2^\alpha}{m_V^2}\right) \\
=& g_{\gamma SV}\int \frac{d^4k}{(2\pi)^4}\times\frac{1}{D_0D_1D_2} \\
&\times\overline{u_a}\left\{\not{\varepsilon}^*\left[m_F[A_2] + \not{k}[A_1]\right] - \frac{\not{k}_2(k_2.\varepsilon^*)}{m_V^2}m_F[A_2]\right. \\
&\left. - [A_1]\frac{\left(D_0 + m_F^2 - \not{k}\not{p}_2\right)(k_2.\varepsilon^*)}{m_V^2}\right\}u_b \\
=& \frac{ig_{\gamma SV}}{16\pi^2}\overline{u_a}\left\{\not{\varepsilon}^*\left[C_0m_F[A_2] - (C_1\not{p}_1 + C_2\not{p}_2)[A_1]\right]\right. \\
&-\frac{m_F}{m_V^2}\left[(\gamma^\mu\varepsilon^{*\nu})C_{\mu\nu} - (C_\mu\gamma^\mu)(p_2.\varepsilon^*) + \not{p}_2X_0(p_1.\varepsilon^*)\right][A_2] \\
&\left.+\frac{1}{m_V^2}[A_1]\left[C_{00}\not{\varepsilon}^*\not{p}_2 + (p_1.\varepsilon^*)\left(m_F^2X_0 + X_1\not{p}_1\not{p}_2 + m_b^2X_2\right)\right]\right\}u_b, \qquad \text{(D.2)}
\end{aligned}
$$

where we have used $k_2.\varepsilon^*/(D_1D_2) \to 0$. The formulas of $D_{L,R}$ and $C_{L,R}$ are:

$$
\begin{aligned}
eD^{Fhv}_{(ab)L,9}\times\left(\frac{g_{\gamma SV}}{16\pi^2}\right)^{-1} =& g^{LL}m_F\left(C_0 - \frac{C_{00}}{m_V^2}\right) - g^{RL}m_aC_1 + g^{LR}m_b\left(C_2 + \frac{C_{00}}{m_V^2}\right), \\
D^{Fhv}_{(ab)R,9} =& D^{Fhv}_{(ab)L,9}\left[g_a^L \leftrightarrow g_a^R, g_b^L \leftrightarrow g_b^R\right], \\
eC^{Fvh}_{(ab)L,9}\times\left(\frac{g_{\gamma SV}}{16\pi^2}\right)^{-1} =& -g^{RL}C_2 - \frac{m_F}{2m_V^2}\left[g^{LL}m_aX_1 + g^{RR}m_bX_{02}\right] \\
&+\frac{1}{2m_V^2}\left[g^{RL}(m_F^2X_0 + m_b^2X_2) + g^{LR}m_am_bX_1\right], \\
C^{Fhv}_{(ab)R,9} =& C_{(ab)L}\left[g_a^L \leftrightarrow g_a^R, g_b^L \leftrightarrow g_b^R\right], \qquad \text{(D.3)}
\end{aligned}
$$

where $X_i^{Fhv} \equiv X_i(m_a^2, 0, m_b^2; m_F^2, m_h^2, m_V^2)$. Similarly, the results for diagram (10) are:

$$
\begin{aligned}
eD^{Fvh}_{(ab)L,10}\times\left(\frac{g_{\gamma SV}}{16\pi^2}\right)^{-1} =& g^{LL}m_F\left(C_0-\frac{C_{00}}{m_V^2}\right)-g^{LR}m_bC_2+g^{RL}m_a\left(C_1+\frac{C_{00}}{m_V^2}\right),\\
D^{Fvh}_{(ab)R,10} =& D^{Fvh}_{(ab)L,10}\left[g^L_a\leftrightarrow g^R_a,\, g^L_b\leftrightarrow g^R_b\right],\\
eC^{Fvh}_{(ab)L,10}\times\left(\frac{g_{\gamma SV}}{16\pi^2}\right)^{-1} =& -g^{R*}_a g^L_b C_1-\frac{m_F}{2m_V^2}\left[g^{R*}_a g^R_b m_b X_2+g^{L*}_a g^L_b m_a X_{01}\right]\\
&+\frac{1}{2m_V^2}\left[g^{R*}_a g^L_b(m_F^2X_0+m_a^2X_1)+g^{L*}_a g^R_b m_a m_b X_2\right],\\
eC^{Fvh}_{(ab)R,10}\times\left(\frac{g_{\gamma SV}}{16\pi^2}\right)^{-1} =& C^{Fvh}_{(ab)L}\left[g^L_a\leftrightarrow g^R_a,\, g^L_b\leftrightarrow g^R_b\right],
\end{aligned}
\tag{D.4}
$$

where $X^{Fvh}_i\equiv X_i(m_a^2,0,m_b^2;m_F^2,m_V^2,m_h^2)$. The above formulas are consistent with calculation using FORM. The corresponding formulas of WI are

$$
\begin{aligned}
\frac{f^{Fhv}_{WI}}{k_{\gamma SV}} =& g^{LL}m_F\left[2(m_V^2C_0-C_{00})+m_am_bX_{012}\right]^{fhv}-g^{RR}m_F\left[m_a^2X_1+m_b^2X_{02}\right]^{fhv}\\
&+g^{RL}\left[-2m_V^2(m_aC_1+m_bC_2)+m_b\left(m_F^2X_0+m_a^2X_1+m_b^2X_2\right)\right]^{fhv}\\
&+g^{LR}\left[2m_V^2(m_bC_2-m_aC_1)+2m_bC_{00}+m_a\left(m_f^2X_0+m_b^2X_{12}\right)\right]^{fhv},\\
\frac{f^{Fvh}_{WI}}{k_{\gamma SV}} =& g^{LL}m_F\left[2(m_V^2C_0-C_{00})-m_am_bX_{012}\right]-g^{RR}m_F\left[m_a^2X_{01}+m_b^2X_2\right]^{fvh}\\
&+g^{RL}\left[2m_V^2(m_aC_1-m_bC_2)+2m_aC_{00}+m_b\left(m_f^2X_0+m_a^2X_{12}\right)\right]^{fvh}\\
&+g^{LR}\left[-2m_V^2(m_aC_1+m_bC_2)+m_a\left(m_f^2X_0+m_a^2X_1+m_b^2X_2\right)\right]^{fvh},
\end{aligned}
\tag{D.5}
$$

where $k_{\gamma SV}=g_{\gamma SV}/(32\pi^2m_V^2)$. The WI valid if only $f^{Fhv}_{WI}+f^{Fvh}_{WI}=0$. We can see crudely that all $C_{(ab)L,9}$, $C_{(ab)R,9}$, $C_{(ab)L,10}$, and $C_{(ab)R,10}$ are convergent. In contrast, all $D_{(ab)L,9}$, $D_{(ab)R,9}$, $D_{(ab)L,10}$, and $D_{(ab)R,10}$ contain divergent terms. Therefore, the necessary condition to guarantee the validation of the WI given in Eq. (12) is that all of these divergent terms must vanish. Strictly, the WI is valid if only $g_{\gamma SV}$=0 or $g^L_a=g^R_a=0$. Because at least one of g^L_a or g^R_a must be non-zero, the condition $g_{\gamma SV}$=0 is the only valid choice, i.e., the vertex-type γ-S-V does not appear in the all BSM guaranteeing the WI for the external photon. This conclusion is also true for the case $a=b$, corresponding to the one-loop contribution to the AMM of the leptons.

Finally, using the assumption of the Lagrangian for couplings of the Goldstone boson given in Eq. (B.1), we can determine the one-loop contributions of the FSV diagrams mentioned above, using the general gauge R_ξ. The propagator of the gauge boson V can be written in terms of two separated parts:

$$
\begin{aligned}
&\Delta^{(\xi)\mu\nu}_V(k^2)\equiv\Delta^{(u)\mu\nu}_V(k^2)+\Delta^{(T)\mu\nu}_{\xi,V}(k^2),\\
&\Delta^{(u)\mu\nu}_V(k^2)=\frac{-i}{k^2-m_V^2}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{m_V^2}\right),\\
&\Delta^{(T)\mu\nu}_{\xi,V}(k^2)=\frac{-i}{m_V^2}\times\frac{k^\mu k^\nu}{k^2-\xi\, m_V^2}=\frac{-k^\mu k^\nu}{m_V^2}\times i\Delta^{0\xi}_{G_V},
\end{aligned}
\tag{D.6}
$$

where $\Delta_V^{(u)\mu\nu}(k^2)$ is the propagator in the unitary gauge, and $\Delta^{0\xi}_{G_V}$ relates to the propagator of G_V as follows:

$$\Delta^{\xi}_{G_V} = i\Delta^{0\xi}_{G_V} = \frac{i}{k^2-\xi m_V^2} = \left[\begin{array}{ll} 0 & \xi \to \infty: \text{ Unitary}(u), \\ \frac{i}{k^2-m_V^2}, & \xi = 1: \text{'t Hooft} - \text{Feynman}(HF) \end{array}\right. . \tag{D.7}$$

For two diagrams (3) and (4) in Fig. 3, the Feynman rules for the couplings $\gamma - S - G_V$ are the same form as those given in Lagrangian (25), namely $S \equiv h_1$ and $G_V \equiv h_2$. The reason is that all mass eigenstates of the scalar with the same electric charges come from the same squared mass matrix. Therefore, $\mathcal{L}^{\gamma h G_V} = ieQ_H A^{\mu}\left[\left(h^*\partial_\mu G_V - G_V\partial_\mu h^*\right) + \text{h.c.}\right]$. Formulas corresponding to diagrams (1) and (3) of Fig. 3 in the general gauge R_ξ are

$$\begin{aligned} i\mathcal{M}_9^{(\xi)} =& i\mathcal{M}_9^{(u)} + g_{\gamma SV}\int \frac{d^4k}{(2\pi)^4} \\ &\times \frac{\overline{u_a}[g^{L*}_{a,Fh}P_R + g^{R*}_{a,Fh}P_L](m_F + \not{k})\gamma_\alpha[g^L_{b,FV}P_L + g^R_{b,FV}P_R]u_b(\varepsilon^*.k_2)k_2^\alpha}{D_0D_1D_2m_V^2}, \end{aligned}$$

where $D_0 = k^2 - m_F^2$, $D_1 = k_1^2 - m_h^2$, $D_2 = k_2^2 - \xi m_V^2$, and $\mathcal{M}_9^{(u)}$ is exactly the part given in Eq. (D.2), calculated in the unitary gauge. The results of the two diagrams (1) and (3) are:

$$\begin{aligned} i\Delta\mathcal{M}_9^{(\xi)} \equiv& i\mathcal{M}_9^{(\xi)} - i\mathcal{M}_9^{(u)} \\ =& \frac{ig_{\gamma SV}}{16\pi^2}\overline{u_a}\left\{\frac{m_F}{m_V^2}\left[(\gamma^\mu\varepsilon^{*\nu})C_{\mu\nu} - (C_\mu\gamma^\mu)(p_2.\varepsilon^*) + \not{p}_2X_0(p_1.\varepsilon^*)\right][A_2]\right. \\ &\left. + \frac{1}{m_V^2}[A_1]\left[C_{00}\not{\varepsilon}^*\not{p}_2 + (p_1.\varepsilon^*)\left(m_F^2X_0 + X_1\not{p}_1\not{p}_2 + m_b^2X_2\right)\right]\right\}u_b, \end{aligned} \tag{D.8}$$

$$\begin{aligned} i\mathcal{M}_3^{(\xi)} =& \frac{-ieQ_H}{16\pi^2}\overline{u_a}\left\{-2p_1.\varepsilon^*[A_1]m_FX_0\right. \\ &\left. + \left[2C_{00}^f\not{\varepsilon}^* + \left(X_1^f\not{p}_1 + X_2^f\not{p}_2\right)(2p_1.\varepsilon^*)\right][A_2]\right\}u_b, \end{aligned} \tag{D.9}$$

where $C_{00} = C_{00}(m_F^2, m_h^2, \xi m_V^2)$ and $X_{0,i} = X_{0,i}(m_F^2, m_h^2, \xi m_V^2)$.

As we showed clearly in Eq. (D.6), a propagator of an arbitrary internal gauge boson always consists of two parts: i) the first part is exactly the unitary propagator resulting in $\mathcal{M}_9^{(u)}$, and ii) the second is proportional to the propagator of the respective Goldstone boson, in which the parameter ξ defines a new mass value in the denominator, which results in $\Delta\mathcal{M}_9^{(\xi)} + \mathcal{M}_3^{(\xi)}$. Because ξ is arbitrary, the two one-loop contributions corresponding to the two mentioned parts are independent. As a result, the WI violation of the contributions relating to $\mathcal{M}_9^{(u)}$ is enough to guarantee that the contributions from the FSV-type diagrams always violate the WI.

Appendix E. Higgs gauge couplings in the Higgs triplet models

Here we summarize the HTM and derive precisely the Higgs gauge couplings. The Higgs sector consists of a Higgs triplet $\Delta \sim (3, 2)$ and a Higgs doublet $\Phi \sim (2, 1)$ in the electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ corresponding to the electric operator $Q = T^3 + Y/2$. Here we will use the notations from Ref. [63,64], the Higgs sector is

$$\Phi = \begin{pmatrix} \varphi^+ \\ \frac{1}{\sqrt{2}}(\varphi + v_\Phi + i\chi) \end{pmatrix}, \ \Delta = \begin{pmatrix} \frac{\Delta^+}{\sqrt{2}} & \Delta^{++} \\ \Delta^0 & -\frac{\Delta^+}{\sqrt{2}} \end{pmatrix} \text{ with } \Delta^0 = \frac{\delta + v_\Delta + i\eta}{\sqrt{2}}, \tag{E.1}$$

where v_Φ and v_Δ are the vacuum expectation values (VEV) of the neutral Higgs components. Because v_d has the lepton number 2, $v_d \ll v_\Delta$.

The Higgs gauge couplings appear in the following kinetic terms:

$$\mathcal{L}_{k,H} = \left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right) + \mathrm{Tr}\left[\left(D_\mu \Delta\right)^\dagger \left(D^\mu \Delta\right)\right], \tag{E.2}$$

where

$$D_\mu \Phi = \left(\partial_\mu + i\frac{g}{2}\tau^a W_\mu^a + i\frac{g'}{2}B_\mu\right)\Phi, \ D_\mu \Delta = \partial_\mu \Delta + i\frac{g}{2}\left[\tau^a W_\mu^a, \Delta\right] + i\frac{g'}{2}B_\mu \Delta. \tag{E.3}$$

The masses and mixing parameters of the gauge bosons are derived from the Eq. (E.2), with VEVs of Φ and Δ. A detailed calculation shows that the physical states $W^\pm$, neutral Z and photon A_μ are:

$$W_\mu^\pm = \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}}, \ W_\mu^3 = c_W Z_\mu + s_W A_\mu, \ B_\mu = -s_W Z_\mu + c_W A_\mu. \tag{E.4}$$

The respective masses are $m_W^2 = g^2(v_\Phi^2 + 2v_\Delta^2)/4$, $m_Z^2 = g^2(v_\Phi^2 + 2v_\Delta^2)/(4c_W^2)$ and the photon is massless. The relation $g'/g = t_W$ is well-known, the same as that in the SM. The covariant derivatives in Eq. (E.3) are written in the mass eigenstates as follows:

$$\begin{aligned} D_\mu \Delta &= \frac{ig}{2}\begin{pmatrix} v_\Delta W_\mu^+ + \sqrt{2} t_W B_\mu, & -2\Delta^+ W_\mu^+ \\ 2\Delta^+ W_\mu^- - \frac{\sqrt{2} v_\Delta Z_\mu}{c_W}, & -v_\Delta W_\mu^+ - \sqrt{2} t_W \Delta^+ B_\mu \end{pmatrix} + \ldots, \\ D_\mu \Phi &= \frac{ig}{2}\begin{pmatrix} \left(\frac{c_W^2 - s_W^2}{c_W} Z_\mu + 2 s_W A_\mu\right)\varphi^+ + v_\Phi W_\mu^+ \\ \sqrt{2} W_\mu^- \varphi^+ - \frac{v_\Phi}{\sqrt{2} c_W} Z_\mu \end{pmatrix} + \ldots, \end{aligned} \tag{E.5}$$

where we just focus on the couplings SVV relating to the vertex $H^\pm W^\mp \gamma$. Therefore, the relevant parts in the kinetic term are:

$$\begin{aligned} \mathcal{L}_{k,H} &= \frac{g^2}{4}\left(2 s_W A_\mu \varphi^- + v_\Phi W_\mu^-\right)\left(2 s_W A^\mu \varphi^+ + v_\Phi W^{+\mu}\right) \\ &+ \frac{g^2}{2}\left(v_\Delta W_\mu^- + \sqrt{2} t_W \Delta^- B_\mu\right)\left(v_\Delta W^{+\mu} + \sqrt{2} t_W \Delta^+ B^\mu\right) + \ldots \\ &= \frac{g^2 s_W}{2}\left[\left(v_\Phi \varphi^- + \sqrt{2} v_\Delta \Delta^-\right) W^{+\mu} + \text{h.c.}\right] A_\mu + \ldots \end{aligned} \tag{E.6}$$

The Higgs potential of all Higgs multiplets was investigated previously, for example, [63,64]. The results of masses and mixing parameters of all Higgs bosons are confirmed by our careful cross-check. We focus on the Higgs gauge couplings of the singly charged Higgs boson in this model, the mixing parameter $\beta_\pm$ relating to mass eigenstates and the original ones are:

$$\begin{pmatrix} \varphi^\pm \\ \Delta^\pm \end{pmatrix} = \begin{pmatrix} c_{\beta_\pm} & -s_{\beta_\pm} \\ s_{\beta_\pm} & c_{\beta_\pm} \end{pmatrix}\begin{pmatrix} G_W^\pm \\ H^\pm \end{pmatrix}, \ t_{\beta_\pm} = \frac{\sqrt{2} v_\Delta}{v_\Phi}. \tag{E.7}$$

Here $G_W^\pm$ is the Goldstone bosons of $W^\pm$, while $H^\pm$ is the only singly charged Higgs boson predicted by the HTM. Then the couplings $H^\pm W^\mp \gamma \sim \sqrt{2} c_{\beta_\pm} v_\Delta - s_{\beta_\pm} v_\Phi = 0$, and the couplings with $G_W^\pm$ are $(em_W)\left[W^+_\mu G^-_W + \text{h.c.}\right] A_\mu$, consistent with the SM. In contrast, Ref. [34] seems to take into account only the contribution of $\Delta^\pm$ to $H^\pm$, and ignored that of $\varphi^\pm$, although they have the same amplitude but opposite signs.

It is noted that the results derived from our calculation are consistent with those in recent works discussing all tree-level decays of Higgs and gauge bosons predicted by the HTM at LHC [63,64]. The decays $H^\pm \to W^\pm\gamma$ do not appear in the decay lists of these works.

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