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Bidirectional remote hyperstate preparation under common quantum control using hyperentanglement

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In this paper, we propose a new, to the best of our knowledge, protocol that enables two distant parties to prepare a photon hyperstate for each other encoded at the same time in both polarization and spatial-mode degrees of freedom. The bidirectional remote hyperstate preparation is demanded so that it is remotely controllable by a common supervisor. Such a task appears possible using a shared quantum channel made of five photons entangled simultaneously in the two corresponding degrees of freedom, the so-called hyperentanglement. We first design a near-deterministic scheme to produce a relevant five-photon hyperentanagled state to be served as the working nonlocal channel and then present our protocol for controlled bidirectional remote hyperstate preparation, which always is successful.

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1. INTRODUCTION

Quantum entanglement [1], the most celebrated trait of quantum mechanics, has widely been recognized as exhibiting nonlocal correlations in the weird quantum world that have no counterparts in the intuitive classical world. It plays an utmost vital role in quantum information processing and quantum computing. In a quantum paradigm, information is encoded in quantum states that obey the laws of quantum nature. Entanglement between photons, which has been demonstrated experimentally, is the key resource for tasks such as quantum secure direct communications [2–5], quantum secret sharing [6], and quantum key distribution [7–10]. Conventionally, the encoding is made using only one degree of freedom (DOF). However, a quantum system is essentially characterized at the same time by many different DOFs, such as the polarization DOF (P-DOF), spatial-mode DOF (S-DOF), orbital-angular-momentum DOF, frequency, and time-bin DOFs. Simultaneously exploiting multiple DOFs for information encoding promises high capacity and high security as well as allowing quantum interference, which gives rise to parallel operations to exponentially speed up intractable computations. Here, a single-system state characterized by more than one DOF is referred to as hyperstate while a multisystem entangled state with more than one DOF is called a hyperentangled one [11,12]. Recently, hyperentanglement has been employed for important tasks such as hyper teleportation [13–15], hyper dense coding [16–18], hyper remote state preparation [19–21], and hyper joint remote state preparation [22].

Quantum state teleportation (QST) [23] invented by Bennett *et al.* and remote state preparation (RSP) proposed by Lo [24] and Bennett *et al.* [25] are considered as two quantum communications methods for the transmission of quantum information. By these methods, a quantum state at one location can be found at another distant location in a secure, faithful way by means of local operations and classical communications without violating the principles of quantum mechanics and the relativity theory. Controlled quantum protocols [26–32] allow the addition of a supervisor whose role is to decide the completion of a task without the need to know the details of the state under processing. In particular, controlled bidirectional QST [33–43] and controlled bidirectional RSP [44–54], which are two-way quantum communications protocols, have been dealt with by several authors. In those protocols, by using multi-qubit entangled states, two parties under the control of a controller are able to simultaneously transmit to or prepare a quantum state for to each other. If the information to be processed is encoded in photon hyperstates, hyperentangled states between photons are the indispensable resources. Production of hyperentangled resources is generally complicated, yet possible, by means of modern quantum technologies. One of the most widely used techniques is the spontaneous parametric down-conversion (SPDC) process in nonlinear crystals [55–57]. Production of hyperentangled states can be realized by a combination of the techniques to generate conventional entanglement in a single DOF [58–63]. There are several quantum resource types that are useful in certain intriguing tasks of one-way quantum communications protocols such as a hyper Einstein–Podolsky–Rosen pair state [13–15,19–21,64–67], and a hyper Greenberger–Horne–Zeilinger state [22]. In [67], the authors offered bidirectional quantum teleportation protocols that allow a sender and a receiver to exchange two single-photon states, one encoded in P-DOF and the other encoded in S-DOF, via the quantum channel in terms of a two-photon Bell-type state entangled simultaneously in the two DOFs.

Our concern in this paper is to devise a protocol so that Alice can prepare for Bob a photon hyperstate, while Bob can also prepare for Alice another photon hyperstate, and the bidirectional remote hyperstate preparation is put under the common control of a third-party Charlie. To that purpose, a suitable multiphoton state should be successfully produced and properly distributed among the three authorized parties. In Section 2, we design a near-deterministic scheme to produce a five-photon hyperentangled state. Next, in Section 3, we use that hyperentangled state as the working quantum channel to perform the controlled bidirectional remote hyperstate preparation mentioned above. Finally, we conclude with a relevant discussion in Section 4.

2. WORKING QUANTUM CHANNEL

Our task involves three parties who are far apart from each other: the two hyperstate preparers Alice and Bob and the supervisor Charlie. Suppose that Alice has a photon whose state $|\psi\rangle$ is encoded in both P-DOF and S-DOF; i.e.,

$$|\psi\rangle = \alpha_{00}|Ha_0\rangle + \alpha_{01}|Ha_1\rangle + \alpha_{10}|Va_0\rangle + \alpha_{11}|Va_1\rangle, \quad (1)$$

with real parameters α_{ij} satisfying the normalization condition $\sum_{i,j=0}^{1} \alpha_{ij}^2 = 1$, while Bob's photon state $|\phi\rangle$ has the form

$$|\phi\rangle = \beta_{00}|Hb_0\rangle + \beta_{01}|Hb_1\rangle + \beta_{10}|Vb_0\rangle + \beta_{11}|Vb_1\rangle, \quad (2)$$

with real parameters β_{ij} satisfying the normalization condition $\sum_{i,j=0}^{1} \beta_{ij}^2 = 1$. The notation $|Hc_j\rangle(|Vc_j\rangle)$, with $c = a, b$ and $j = 0, 1$, implies a hyperstate of a horizontally (vertically) polarized photon that travels along spatial path c_j. Although $|\psi\rangle$ and $|\phi\rangle$ are single-photon states, the information amount contained in each of them is worth two qubits. Therefore, here we formally refer to them as hyperstates to distinguish them from conventional states, which are encoded in a single DOF with an information amount of just one qubit. Alice, who knows α_{ij} but has no ideas about β_{ij}, wishes to prepare her hyperstate $|\psi\rangle$ for Bob, and Bob, who knows β_{ij} but has no ideas about α_{ij}, also wishes to prepare his hyperstate $|\phi\rangle$ for Alice. In this two-way controlled information processing protocol, Alice and Bob play an equal role; i.e., the mutual hyperstate preparations could take place at the same time under the quantum control of Charlie, who does not need to know the values of the parameters α_{ij} and β_{ij} of $|\psi\rangle$ and $|\phi\rangle$, but is able to command whether the task should be accomplished or not.

We find out that the task mentioned above could be achieved if the three participants are connected beforehand by a quantum channel whose state reads

$$|\Gamma\rangle_{12345} = |\Gamma^{(S)}\rangle_{12345}|\Gamma^{(P)}\rangle_{12345}, \quad (3)$$

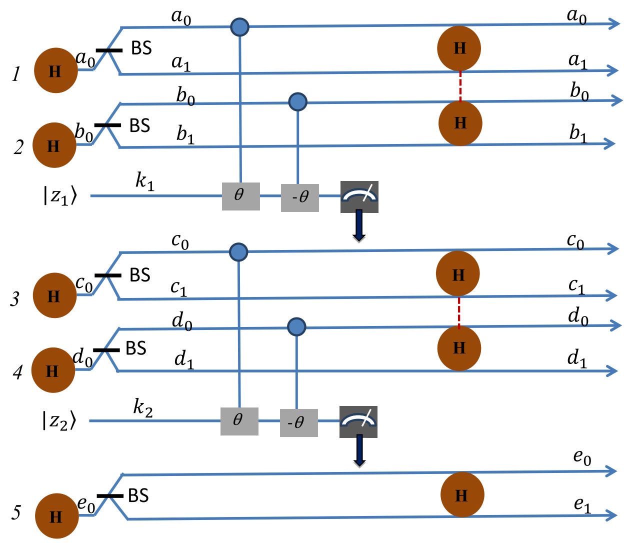

Fig. 1. Step 1 of the scheme for production of the state $|\Gamma^{(S)}\rangle_{12345}$ [Eq. (4)]. This step entangles photon 1 with photon 2 and photon 3 with photon 4 in S-DOF. A dot with H implies a photon in horizontal polarization state $|H\rangle$, while $|z\rangle_1$ and $|z\rangle_2$ are coherent states of real positive amplitudes z_1 and z_2, respectively. θ and $-\theta$ are the dimensionless amplitudes of cross-Kerr interactions. BS is a balanced beam splitter. Photons entangled in S-DOF are connected by red dashed lines.

with

$$|\Gamma^{(S)}\rangle_{12345} = \frac{1}{2}[|a_0b_0\rangle(|c_0d_0e_0\rangle + |c_1d_1e_1\rangle) + |a_1b_1\rangle(|c_0d_0e_1\rangle + |c_1d_1e_0\rangle)]_{12345} \quad (4)$$

and

$$|\Gamma^{(P)}\rangle_{12345} = \frac{1}{2}[|HH\rangle(|HHH\rangle + |VVV\rangle) + |VV\rangle(|HHV\rangle + |VVH\rangle)]_{12345}. \quad (5)$$

Obviously, $|\Gamma\rangle_{12345}$ is a five-photon hyperentangled state. Indeed, each of the five photons is at the same time both horizontally and vertically polarized and simultaneously propagates along two paths: photon 1 along paths a_0 and a_1, photon 2 along paths b_0 and b_1, photon 3 along paths c_0 and c_1, photon 4 along paths d_0 and d_1, and photon 5 along paths e_0 and e_1.

The production of $|\Gamma\rangle_{12345}$ starts from the initial state

$$|\Phi_0\rangle_{12345} = |\Phi_0^{(S)}\rangle_{12345}|\Phi_0^{(P)}\rangle_{12345}, \quad (6)$$

where

$$|\Phi_0^{(S)}\rangle_{12345} = |a_0b_0c_0d_0e_0\rangle_{12345} \quad (7)$$

and

$$|\Phi_0^{(P)}\rangle_{12345} = |HHHHH\rangle_{12345}, \quad (8)$$

which can be treated separately because manipulating the S-DOF cause no effects on the P-DOF and vice versa. Let us first deal with $|\Phi_0^{(S)}\rangle_{12345}$ following two main steps. The operations in step 1 are sketched in Fig. 1.

In step 1, we first send photons 1, 2, 3, 4, and 5 through five balanced beam splitters (BSs). As the BS transforms $|x_0\rangle$ to $(|x_0\rangle + |x_1\rangle)/\sqrt{2}$, with $x = a, b, c, d$ or e, $|\Phi_0^{(S)}\rangle_{12345}$ is transformed to

$$|\Phi_1^{(S)}\rangle_{12345} = \frac{1}{4\sqrt{2}}[(|a_0\rangle + |a_1\rangle)_1(|b_0\rangle + |b_1\rangle)_2(|c_0\rangle + |c_1\rangle)_3 \times (|d_0\rangle + |d_1\rangle)_4(|e_0\rangle + |e_1\rangle)_5]. \quad (9)$$

Second, an ancillary coherent state $|z_1\rangle_{k_1}(|z_2\rangle_{k_2})$ of real positive amplitude $z_1(z_2)$ and propagation path $k_1(k_2)$ is allowed to sequentially interact with mode a_0 of photon 1 and mode b_0 of photon 2 (mode c_0 of photon 3 and mode d_0 of photon 4) via cross-Kerr nonlinearities [68–74] with dimensionless amplitudes θ and $-\theta$, respectively. Cross-Kerr interactions $U_{nk}(\pm\theta)$ with amplitudes $\pm\theta$ between a photon n along path x_j and a coherent state $|z\rangle_k$ leaves the photon state unchanged but adds a phase $\pm\theta$ to the coherent state; *i.e.*, $U_{nk}(\pm\theta)|x_j\rangle_n|z\rangle_k = |x_j\rangle_n|ze^{\pm i\theta}\rangle_k$. Hence, under the cross-Kerr interaction, $|\Phi_1^{(S)}\rangle_{12345}|z_1\rangle_{k_1}|z_2\rangle_{k_2}$ changes to

$$\begin{aligned}|\Phi_2^{(S)}\rangle_{12k_134k_25} = \frac{1}{4\sqrt{2}}&[(|a_0b_0\rangle + |a_1b_1\rangle)_{12}|z_1\rangle_{k_1} \\ &+ |a_0b_1\rangle_{12}|z_1e^{i\theta}\rangle_{k_1} + |a_1b_0\rangle_{12}|z_1e^{-i\theta}\rangle_{k_1}] \\ &\times [(|c_0d_0\rangle + |c_1d_1\rangle)_{34}|z_2\rangle_{k_2} \\ &+ |c_0d_1\rangle_{34}|z_2e^{i\theta}\rangle_{k_2} + |c_1d_0\rangle_{34}|z_2e^{-i\theta}\rangle_{k_2}] \\ &\times (|e_0\rangle + |e_1\rangle)_5. \end{aligned} \quad (10)$$

Third, the resulting coherent states are measured by homodyne detections [75], which are also called X quadrature measurements. If $|z_1\rangle_{k_1}$ and $|z_2\rangle_{k_2}$ are found (i.e., no shifts in phase space are observed), then state of the photons collapses to

$$|Q_0^{(S)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|a_0b_0\rangle + |a_1b_1\rangle)_{12}(|c_0d_0\rangle + |c_1d_1\rangle)_{34} \times (|e_0\rangle + |e_1\rangle)_5. \quad (11)$$

Another possibility is that the coherent states, which emerge as $|z_1\rangle_{k_1}, |z_2e^{\pm i\theta}\rangle_{k_2}$ ($|ze^{+i\theta}\rangle$, and $|ze^{-i\theta}\rangle$, are indistinguishable by homodyne detections. In this case, the photon states appear as

$$|Q_1^{(S)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|a_0b_0\rangle + |a_1b_1\rangle)_{12}(|c_0d_1\rangle + |c_1d_0\rangle)_{34} \times (|e_0\rangle + |e_1\rangle)_5. \quad (12)$$

The coherent states may also emerge as $|z_1e^{\pm i\theta}\rangle_{k_1}$ and $|z_2\rangle_{k_2}$, projecting the photons onto

$$|Q_2^{(S)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|a_0b_1\rangle + |a_1b_0\rangle)_{12}(|c_0d_0\rangle + |c_1d_1\rangle)_{34} \times (|e_0\rangle + |e_1\rangle)_5. \quad (13)$$

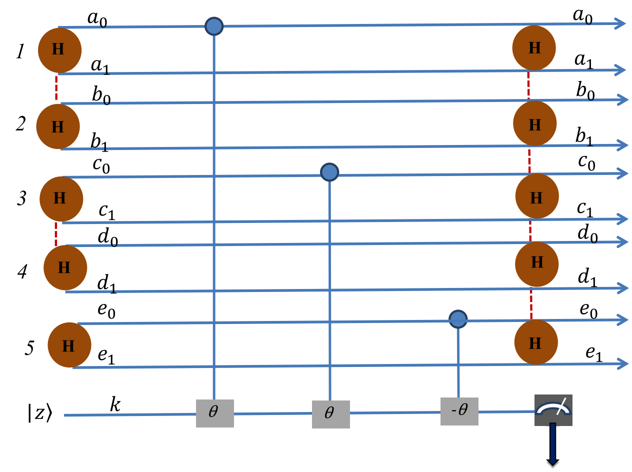

Fig. 2. Step 2 of the scheme for production of the S-DOF part of the working quantum channel. $|z\rangle$ is a coherent state of real positive amplitude z. In this step, the desired state $|\Gamma^{(S)}\rangle_{12345}$ [Eq. (4)], is obtained if the homodyne detection measurement outcome is $|ze^{\pm i\theta}\rangle$; otherwise, appropriate operations can be applied to return the initial state $|\Phi_0^{(S)}\rangle_{12345}$ [Eq. (7)], allowing the process to be repeated.

The last possibility is finding $|z_1e^{\pm i\theta}\rangle_{k_1}$ and $|z_2e^{\pm i\theta}\rangle_{k_2}$, in which case one has the following state of the photons:

$$|Q_3^{(S)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|a_0b_1\rangle + |a_1b_0\rangle)_{12}(|c_0d_1\rangle + |c_1d_0\rangle)_{34} \times (|e_0\rangle + |e_1\rangle)_5. \quad (14)$$

Note that $|Q_1^{(S)}\rangle_{12345}$, $|Q_2^{(S)}\rangle_{12345}$, and $|Q_3^{(S)}\rangle_{12345}$ can be converted to $|Q_0^{(S)}\rangle_{12345}$ by path-flipping photon 4, photon 2, or both. Therefore, without a loss of generality, we shall execute step 2, from $|Q_0^{(S)}\rangle_{12345}$. The operations in step 2 are sketched in Fig. 2.

In step 2, another ancillary coherent state $|z\rangle_k$ interacts consecutively with states $|a_0\rangle_1$, $|c_0\rangle_3$, and $|e_0\rangle_5$ via cross-Kerr operations $U_{1k}(\theta)$, $U_{3k}(\theta)$, and $U_{5k}(-\theta)$, respectively. This entangles the photons and the coherent state as

$$\begin{aligned}|\Omega^{(S)}\rangle_{12345k} = \frac{1}{2\sqrt{2}}&\{[(|a_0b_0c_0d_0e_0\rangle + |a_0b_0c_1d_1e_1\rangle \\ &+ |a_1b_1c_0d_0e_1\rangle)_{12345}|ze^{i\theta}\rangle_k \\ &+ |a_1b_1c_1d_1e_0\rangle_{12345}|ze^{-i\theta}\rangle_k] \\ &+ |a_0b_0c_0d_0e_1\rangle_{12345}|ze^{2i\theta}\rangle_k \\ &+ (|a_0b_0c_1d_1e_0\rangle + |a_1b_1c_0d_0e_0\rangle \\ &+ |a_1b_1c_1d_1e_1\rangle)_{12345}|z\rangle_k\}. \end{aligned} \quad (15)$$

Homodyne detection measurement is then performed on the coherent state of $|\Omega^{(S)}\rangle_{12345k}$. There are three possible types of the measurement outcome: (i) If the outcome is $|ze^{\pm i\theta}\rangle_k$, then the five photons turn out to be in the desired state $|\Gamma^{(S)}\rangle_{12345}$. (ii) Alternatively, if the outcome is $|ze^{2i\theta}\rangle_k$, then the photons get

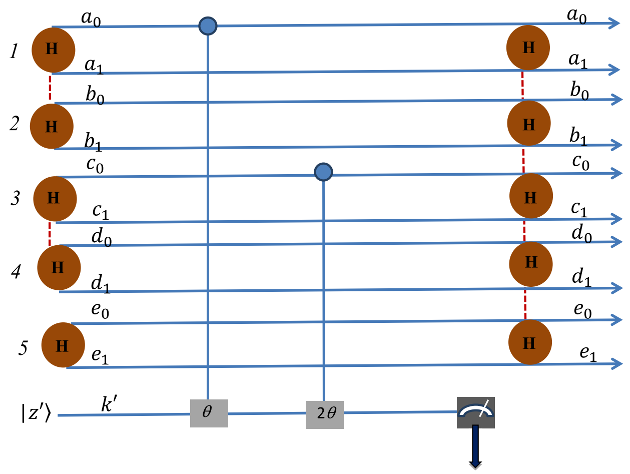

Fig. 3. In step 2 of the scheme for the production of the S-DOF part of the working quantum channel if the outcome is $|z\rangle_k$ after the homodyne detection measurement, then the photons' state becomes $(|a_0b_0c_1d_1e_0\rangle + |a_1b_1c_0d_0e_0\rangle + |a_1b_1c_1d_1e_1\rangle)_{12345}/\sqrt{3}$. In this case, the appropriate operations can be applied to return the initial state $|\Phi_0^{(S)}\rangle_{12345}$ [Eq. (7)], allowing the process to be repeated. $|z'\rangle$ is a coherent state of real positive amplitude z'.

separated in the product state $|a_0b_0c_0d_0e_1\rangle_{12345}$. This does not at all mean a failure because $|a_0b_0c_0d_0e_1\rangle_{12345}$ can be brought to $|a_0b_0c_0d_0e_0\rangle_{12345}$ by path-flipping photon 5 and the whole process is to be re-initiated from step 1. (iii) Finally, if the outcome is $|z\rangle_k$, then the photon states become $(|a_0b_0c_1d_1e_0\rangle + |a_1b_1c_0d_0e_0\rangle + |a_1b_1c_1d_1e_1\rangle)_{12345}/\sqrt{3}$, which is not a failure either. In this situation, we can figure out each of the three components $|a_0b_0c_1d_1e_0\rangle_{12345}$, $|a_1b_1c_0d_0e_0\rangle_{12345}$, and $|a_1b_1c_1d_1e_1\rangle_{12345}$ by switching on a cross-Kerr interaction with amplitude θ between $|a_0\rangle_1$ and a coherent state $|z'\rangle_{k'}$, followed by another cross-Kerr interaction with amplitude 2θ between $|c_0\rangle_3$ and the coherent state $|z'\rangle_{k'}$, as shown in Fig. 3. It is straightforward to verify that if the homodyne detection measurement yields the outcome $|z'\rangle_{k'}$, $|z'e^{i\theta}\rangle_{k'}$, or $|z'e^{2i\theta}\rangle_{k'}$, then the photons are projected on state $|a_1b_1c_1d_1e_1\rangle_{12345}$, $|a_0b_0c_1d_1e_0\rangle_{12345}$, or $|a_1b_1c_0d_0e_0\rangle_{12345}$. Clearly, each of these three states can be transformed to $|a_0b_0c_0d_0e_0\rangle_{12345}$ by appropriate path-flip operations on the relevant photons, allowing the whole process be done again from step 1.

As described above, given five photons in the initial state $|\Phi_0^{(S)}\rangle_{12345} = |a_0b_0c_0d_0e_0\rangle_{12345}$, [Eq. (7)], the process of production of the desired S-DOF entangled state $|\Gamma^{(S)}\rangle_{12345}$ defined by Eq. (4) can be repeated again and again until it succeeds.

We now turn to the production process for the P-DOF entangled state $|\Gamma^{(P)}\rangle_{12345}$ defined by Eq. (5), starting from the product state $|\Phi_0^{(P)}\rangle_{12345} = |HHHHH\rangle_{12345}$ [Eq. (8)]. This process also consists of two main steps.

In step 1 shown in Fig. 4, 10 quarter-wave plates (QWPs) are first placed on the 10 paths of the five photons. Since QWP transforms $|H\rangle$ to $(|H\rangle + |V\rangle)/\sqrt{2}$, the initial state $|\Phi_0^{(P)}\rangle_{12345}$ is transformed to

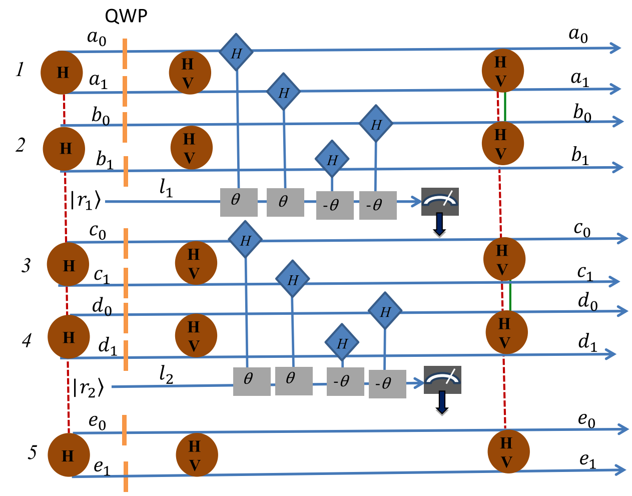

Fig. 4. Step 1 of the scheme for production of the state $|\Gamma^{(P)}\rangle_{12345}$ [Eq. (5)]. This step entangles photon 1 with photon 2 and photon 3 with photon 4 in P-DOF. A dot with H,V implies a photon in superposition of horizontal $|H\rangle$ and vertical $|V\rangle$ polarization states, while $|r\rangle_1$ and $|r\rangle_2$ are the coherent states of real positive amplitudes r_1 and r_2, respectively. QWP is a quarter-wave plate. Photons entangled in P-DOF are connected by green solid lines. Here, cross-Kerr interactions take place between a coherent state and the $|H\rangle$ component of a photon.

$$|\Phi_1^{(P)}\rangle_{12345} = \frac{1}{4\sqrt{2}}[(|H\rangle + |V\rangle)_1(|H\rangle + |V\rangle)_2 \times (|H\rangle + |V\rangle)_3(|H\rangle + |V\rangle)_4(|H\rangle + |V\rangle)_5]. \tag{16}$$

Second, let the $|H\rangle$ component of photon 1 interact with a coherent state $|r_1\rangle_{l_1}$ via cross-Kerr nonlinearity with amplitude θ. Then the modified coherent state is further interacts with the $|H\rangle$ component of photon 2 via another cross-Kerr nonlinearity with amplitude $-\theta$. Likewise, the $|H\rangle$ components of photon 3 and photon 4 allowed to interact with a coherent state, which is initially in state $|r_2\rangle_{l_2}$, via cross-Kerr nonlinearities with amplitudes θ and $-\theta$, respectively. Note that the $|H\rangle$ component of a photon can be split from the $|V\rangle$ component by a polarization beam splitter (PBS), which transmits the $|H\rangle$ component, but reflects the $|V\rangle$ component. After the cross-Kerr interactions mentioned above, $|\Phi_1^{(P)}\rangle_{12345}|r_1\rangle_{l_1}|r_2\rangle_{l_2}$ becomes

$$|\Phi_2^{(P)}\rangle_{1234l_1l_25} = \frac{1}{2\sqrt{2}}(|\lambda_1\rangle_{1234l_1l_2} + |\lambda_2\rangle_{1234l_1l_2} + |\lambda_3\rangle_{1234l_1l_2} + |\lambda_4\rangle_{1234l_1l_2})(|H\rangle + |V\rangle)_5, \tag{17}$$

where

$$|\lambda_1\rangle_{1234l_1l_2} = \frac{1}{2}(|HHHH\rangle + |HHVV\rangle + |VVHH\rangle + |VVVV\rangle)_{1234}|r_1\rangle_{l_1}|r_2\rangle_{l_2}, \tag{18}$$

$$|\lambda_2\rangle_{1234l_1l_2} = \frac{1}{2}[(|HHHV\rangle + |VVHV\rangle)_{1234}|r_1\rangle_{l_1}|r_2e^{i\theta}\rangle_{l_2} + (|HHVH\rangle + VVVH\rangle)_{1234}|r_1\rangle_{l_1}|r_2e^{-i\theta}\rangle_{l_2}], \tag{19}$$

$$|\lambda_3\rangle_{1234l_1l_2} = \frac{1}{2}[(|HVHH\rangle + |HVVV\rangle)_{1234}|r_1e^{i\theta}\rangle_{l_1}|r_2\rangle_{l_2} + (|VHHH\rangle + VHVV\rangle)_{1234}|r_1e^{-i\theta}\rangle_{l_1}|r_2\rangle_{l_2}] \quad (20)$$

and

$$|\lambda_4\rangle_{1234l_1l_2} = \frac{1}{2}[|HVHV\rangle_{1234}|r_1e^{i\theta}\rangle_{l_1}|r_2e^{i\theta}\rangle_{l_2} + |HVVH\rangle_{1234}|r_1e^{i\theta}\rangle_{l_1}|r_2e^{-i\theta}\rangle_{l_2} + |VHHV\rangle_{1234}|r_1e^{-i\theta}\rangle_{l_1}|r_2e^{i\theta}\rangle_{l_2} + |VHVH\rangle_{1234}|r_1e^{-i\theta}\rangle_{l_1}|r_2e^{-i\theta}\rangle_{l_2}]. \quad (21)$$

Third, two homodyne detection measurements, one for the coherent state propagating along path l_1 and the other for that propagating along path l_2, are made. If the measurement outcomes are $|r_1\rangle_{l_1}|r_2\rangle_{l_2}$, $|r_1\rangle_{l_1}|r_2e^{\pm i\theta}\rangle_{l_2}$, $|r_1e^{\pm i\theta}\rangle_{l_1}|r_2\rangle_{l_2}$, or $|r_1e^{\pm i\theta}\rangle_{l_1}|r_2e^{\pm i\theta}\rangle_{l_2}$, the state of the five photons is projected onto

$$|Q_0^{(P)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|HH\rangle + |VV\rangle)_{12}(|HH\rangle + |VV\rangle)_{34} \times (|H\rangle + |V\rangle)_5, \quad (22)$$

$$|Q_1^{(P)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|HH\rangle + |VV\rangle)_{12}(|HV\rangle + |VH\rangle)_{34} \times (|H\rangle + |V\rangle)_5, \quad (23)$$

$$|Q_2^{(P)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|HV\rangle + |VH\rangle)_{12}(HH\rangle + |VV\rangle)_{34} \times (|H\rangle + |V\rangle)_5, \quad (24)$$

or

$$|Q_3^{(P)}\rangle_{12345} = \frac{1}{2\sqrt{2}}(|HV\rangle + |VH\rangle)_{12}(HV\rangle + |VH\rangle)_{34} \times (|H\rangle + |V\rangle)_5, \quad (25)$$

respectively. Using half-wave plates (HWPs) to flip the polarization of photon 4, photon 2, or both, we can obtain $|Q_0^{(P)}\rangle_{12345}$ from $|Q_1^{(P)}\rangle_{12345}$, $|Q_2^{(P)}\rangle_{12345}$, or $|Q_3^{(P)}\rangle_{12345}$. Thus, we shall execute step 2 of the process of production of the P-DOF state $|\Gamma^{(P)}\rangle_{12345}$ in Eq. (5) from $|Q_0^{(S)}\rangle_{12345}$, as displayed in Fig. 5.

In step 2, a single coherent state $|r\rangle_l$ first interacts sequentially with the $|H\rangle$ components of photon 1, photon 3, and photon 5 of state $|Q_0^{(S)}\rangle_{12345}$ via cross-Kerr nonlinearities with amplitudes θ, $-\theta$, and θ, respectively. The product state $|Q_0^{(P)}\rangle_{12345}|r\rangle_l$ thus becomes entangled as

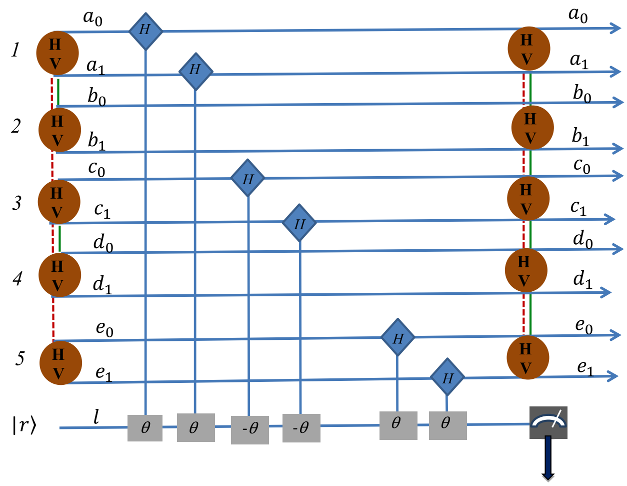

Fig. 5. Step 2 of the scheme for the production of the P-DOF part of the working quantum channel. $|r\rangle$ is a coherent state of real positive amplitude r. In this step, the desired state $|\Gamma^{(P)}\rangle_{12345}$ [Eq. (5)], is obtained if the homodyne detection measurement outcome is $|re^{\pm i\theta}\rangle$; otherwise, appropriate operations can be applied to return the initial state $|\Phi_0^{(P)}\rangle_{12345}$ [Eq. (8)], allowing the process to be repeated.

$$|\Omega^{(P)}\rangle_{12345l} = \frac{1}{2\sqrt{2}}[2|\Gamma^{(P)}\rangle_{12345}|re^{\pm i\theta}\rangle_l + |HHVVH\rangle_{12345}|re^{2i\theta}\rangle_l + (|HHHHV\rangle + |VVHHH\rangle + |VVVVV\rangle)_{12345}|r\rangle_l]. \quad (26)$$

Second, the coherent state of $|\Omega^{(P)}\rangle_{12345l}$ is measured by the homodyne detection technique yielding one of the three outcomes, which is $|re^{\pm i\theta}\rangle_l$, $|re^{2i\theta}\rangle_l$, or $|r\rangle_l$. If it is $|re^{\pm i\theta}\rangle_l$, the process is successful; i.e., $|\Gamma^{(P)}\rangle_{12345}$ is obtained. However, if the outcome is $|re^{2i\theta}\rangle_l$, the photons' state appears as $|HHVVH\rangle_{12345}$, which can be transformed to $|\Phi_0^{(P)}\rangle_{12345} = |HHHHH\rangle_{12345}$ by flipping the polarization of both photon 3 and photon 4. Finally, if the outcome is $|r\rangle_l$, the photons' state collapses to $(|HHHHV\rangle + |VVHHH\rangle + |VVVVV\rangle)_{12345}/\sqrt{3}$. In this case, the H components of photons 4 and photon 5 interact in sequence with a coherent state $|r'\rangle_{l'}$ via two cross-Kerr interactions with the same amplitude θ, as shown in Fig. 6. The resulting (unnormalized) state of the photons and the coherent state turns out to be $|HHHHV\rangle_{12345}|r'e^{i\theta}\rangle_{l'} + |VVHHH\rangle_{12345}|r'e^{2i\theta}\rangle_{l'} + |VVVVV\rangle_{12345}|r'\rangle_{l'}$. When the coherent state is measured, the photons are projected onto either $|HHHHV\rangle_{12345}$, $|VVHHH\rangle_{12345}$, or $|VVVVV\rangle_{12345}$, depending on the outcome $|r'e^{i\theta}\rangle_{l'}$, $|r'e^{2i\theta}\rangle_{l'}$, or $|r'\rangle_{l'}$, respectively. Each of the three above states of the photons can be converted to $|\Phi_0^{(P)}\rangle_{12345}$ [Eq. (8)], by flipping the polarization of photon 5, photons 1 and 2, or all of the five photons. This means that when the two last possible outcomes have occurred, we can restart the whole process and this procedure can be repeated until it succeeds.

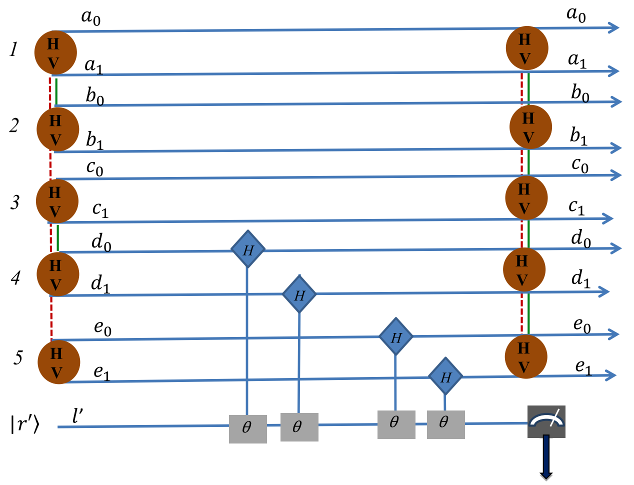

Fig. 6. In the step 2 of the scheme for production of the P-DOF part of the working quantum channel, if the outcome is $|r\rangle_l$ after the homodyne detection measurement, then the photons' state becomes $(|HHHHV\rangle + |VVHHH\rangle + |VVVVV\rangle)_{12345}/\sqrt{3}$. In this case, the appropriate operations can be applied to return to the initial state $|\Phi_0^{(P)}\rangle_{12345}$ [Eq. (8)], allowing the process to be repeated. $|r'\rangle$ is a coherent state of real positive amplitude r'.

3. CONTROLLED BIDIRECTIONAL REMOTE HYPERSTATE PREPARATION

Once the five-photon hyperentangled state in Eq. (3) has been produced, it can be employed as a working quantum channel to perform the controlled bidirectional remote hyperstate preparation protocol mentioned previously. The hyperentangled state $|\Gamma\rangle_{12345}$ should be shared in such a way that Alice holds photons 1 and 3, Bob holds photons 2 and 4, while Charlie holds photon 5. To achieve unit success probability Alice, Bob,

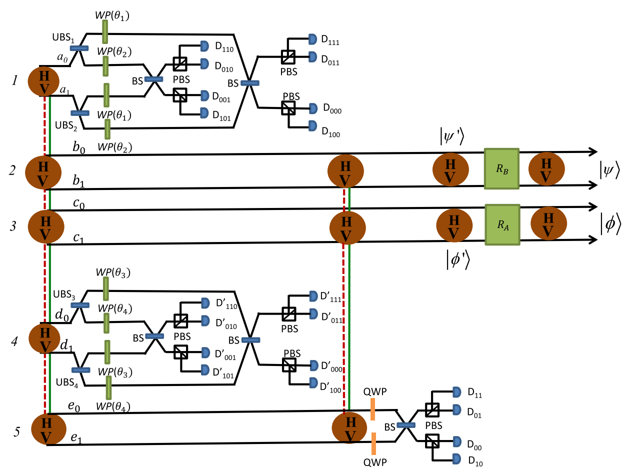

Fig. 7. The scheme for the controlled bidirectional remote hyperstate preparation. UBS_i $(i = 1, 2, 3, 4)$ is an unbalanced beam splitter with the reflection (transmission) coefficient $r_i(t_i)$. $WP(\theta_j)$ $(j = 1, 2, 3, 4)$ is a θ_j wave plate that rotates the polarization state by an angle θ_j. PBS is a polarization beam splitter that transmits the horizontal polarization state and reflects the vertical polarization state. D_{mkl}, $D'_{m'k'l'}$, and D_{pq} with $m, k, l, m', k', l', p, q \in \{0, 1\}$ are 20 photodetectors. $|\psi'\rangle$, R_B, $|\phi'\rangle$, and R_A are defined in Eqs. (51), (52), (53), and (54), respectively.

and Charlie should agree in proper co-operations, as detailed below and shown in Fig. 7.

First, Alice and Bob independently perform their actions. Alice places two unbalanced beam splitters UBS_1 and UBS_2, one on path a_0 and the other on path a_1 of her photon 1. The UBS_1 on path a_0 has the reflection (transmission) coefficient $r_1 = \sqrt{\alpha_{01}^2 + \alpha_{11}^2} (t_1 = \sqrt{\alpha_{00}^2 + \alpha_{10}^2})$, while that of UBS_2 on path a_1 is $r_2 = t_1 (t_2 = r_1)$. Bob also uses two unbalanced beam splitters UBS_3 and UBS_4, with UBS_3 placed on path d_0 and UBS_4 on path d_1 of his photon 4. Moreover, the reflection (transmission) coefficient of Bob's unbalanced beam splitter UBS_3 is chosen to be $r_3 = \sqrt{\beta_{01}^2 + \beta_{11}^2}$ $(t_3 = \sqrt{\beta_{00}^2 + \beta_{10}^2})$ and the choice for UBS_4 is $r_4 = t_3 (t_4 = r_3)$. These unbalanced beam splitters modify $|\Gamma\rangle_{12345}$ to

$$\begin{aligned}|\Gamma_1\rangle_{12345} = \frac{1}{2}[&(t_1|a_{00}\rangle + r_1|a_{01}\rangle)_1|b_0c_0\rangle_{23}(t_3|d_{00}\rangle \\ &+ r_3|d_{01}\rangle)_4|e_0\rangle_5 + (t_1|a_{00}\rangle + r_1|a_{01}\rangle)_1|b_0c_1\rangle_{23} \\ &\times (r_3|d_{11}\rangle + t_3|d_{10}\rangle)_4|e_1\rangle_5 + (r_1|a_{11}\rangle + t_1|a_{10}\rangle)_1 \\ &\times |b_1c_0\rangle_{23}(t_3|d_{00}\rangle + r_3|d_{01}\rangle)_4|e_1\rangle_5 + (r_1|a_{11}\rangle \\ &+ t_1|a_{10}\rangle)_1|b_1c_1\rangle_{23}(r_3|d_{11}\rangle + t_3|d_{10}\rangle)_4|e_0\rangle_5] \\ &\otimes|\Gamma^{(P)}\rangle_{12345}. \end{aligned} \tag{27}$$

Second, a pair of θ_1 wave plates denoted by $WP(\theta_1)$ are placed on path a_{00} and path a_{10} of photon 1, while another pair of θ_2 wave plates denoted by $WP(\theta_2)$ on path a_{01} and path a_{11} of the same photon 1. The angles θ_1 and θ_2 of the wave plates are chosen as

$$\theta_1 = \arccos\frac{\alpha_{00}}{t_1} \tag{28}$$

and

$$\theta_2 = \arccos\frac{\alpha_{01}}{r_1}. \tag{29}$$

The choices of r_1, r_2 (t_1, t_2), θ_1, and θ_2 can be made by Alice because she knows α_{ij}. The θ wave plate $WP(\theta)$ itself acts on the photon polarization states:

$$WP(\theta)|H\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle, \tag{30}$$

$$WP(\theta)|V\rangle = -\sin\theta|H\rangle + \cos\theta|V\rangle. \tag{31}$$

Similarly, two pairs of wave plates are also used by Bob. One pair of $WP(\theta_3)$ are placed on paths d_{00} and d_{10} of photon 4, and the other pair of $WP(\theta_4)$ are placed on paths d_{01} and d_{11} of the same photon 4. The angles of Bob's wave plates are chosen as

$$\theta_3 = \arccos\frac{\beta_{00}}{t_3} \tag{32}$$

and

$$\theta_4 = \arccos\frac{\beta_{01}}{r_3}. \tag{33}$$

The choices of r_3, r_4 (t_3, t_4), θ_3, and θ_4 can be made by Bob because he knows β_{ij}. Due to actions of the four wave plates $WP(\theta_1)$, $WP(\theta_2)$, $WP(\theta_3)$, and $WP(\theta_4)$, state $|\Gamma_1\rangle_{12345}$ changes to

$$|\Gamma_2\rangle_{12345} = \frac{1}{4}\sum_i^{16} |\Psi_i\rangle_{12345}, \tag{34}$$

where

$$\begin{aligned}|\Psi_1\rangle_{12345} = {} & (\alpha_{00}|Ha_{00}\rangle + \alpha_{10}|Va_{00}\rangle + \alpha_{01}|Ha_{01}\rangle \\ & + \alpha_{11}|Va_{01}\rangle)_1|Hb_0\rangle_2|Hc_0\rangle_3(\beta_{00}|Hd_{00}\rangle \\ & + \beta_{10}|Vd_{00}\rangle + \beta_{01}|Hd_{01}\rangle + \beta_{11}|Vd_{01}\rangle)_4|He_0\rangle_5,\end{aligned} \tag{35}$$

$$\begin{aligned}|\Psi_2\rangle_{12345} = {} & (\alpha_{00}|Ha_{00}\rangle + \alpha_{10}|Va_{00}\rangle + \alpha_{01}|Ha_{01}\rangle \\ & + \alpha_{11}|Va_{01}\rangle)_1|Hb_0\rangle_2|Vc_0\rangle_3(\beta_{00}|Vd_{00}\rangle \\ & - \beta_{10}|Hd_{00}\rangle + \beta_{01}|Vd_{01}\rangle - \beta_{11}|Hd_{01}\rangle)_4|Ve_0\rangle_5,\end{aligned} \tag{36}$$

$$\begin{aligned}|\Psi_3\rangle_{12345} = {} & (\alpha_{00}|Va_{00}\rangle - \alpha_{10}|Ha_{00}\rangle + \alpha_{01}|Va_{01}\rangle \\ & - \alpha_{11}|Ha_{01}\rangle)_1|Vb_0\rangle_2|Hc_0\rangle_3(\beta_{00}|Hd_{00}\rangle \\ & + \beta_{10}|Vd_{00}\rangle + \beta_{01}|Hd_{01}\rangle + \beta_{11}|Vd_{01}\rangle)_4|Ve_0\rangle_5,\end{aligned} \tag{37}$$

$$\begin{aligned}|\Psi_4\rangle_{12345} = {} & (\alpha_{00}|Va_{00}\rangle - \alpha_{10}|Ha_{00}\rangle + \alpha_{01}|Va_{01}\rangle \\ & - \alpha_{11}|Ha_{01}\rangle)_1|Vb_0\rangle_2|Vc_0\rangle_3(\beta_{00}|Vd_{00}\rangle \\ & - \beta_{10}|Hd_{00}\rangle + \beta_{01}|Vd_{01}\rangle - \beta_{11}|Hd_{01}\rangle)_4|He_0\rangle_5,\end{aligned} \tag{38}$$

$$\begin{aligned}|\Psi_5\rangle_{12345} = {} & (\alpha_{00}|Ha_{00}\rangle + \alpha_{10}|Va_{00}\rangle + \alpha_{01}|Ha_{01}\rangle \\ & + \alpha_{11}|Va_{01}\rangle)_1|Hb_0\rangle_2|Hc_1\rangle_3(\beta_{01}|Hd_{00}\rangle \\ & + \beta_{11}|Vd_{11}\rangle + \beta_{00}|Hd_{10}\rangle + \beta_{10}|Vd_{10}\rangle)_4|He_1\rangle_5,\end{aligned} \tag{39}$$

$$\begin{aligned}|\Psi_6\rangle_{12345} = {} & (\alpha_{00}|Ha_{00}\rangle + \alpha_{10}|Va_{00}\rangle + \alpha_{01}|Ha_{01}\rangle \\ & + \alpha_{11}|Va_{01}\rangle)_1|Hb_0\rangle_2|Vc_1\rangle_3(\beta_{01}|Vd_{11}\rangle \\ & - \beta_{11}|Hd_{11}\rangle + \beta_{00}|Vd_{10}\rangle - \beta_{10}|Hd_{10}\rangle)_4|Ve_1\rangle_5,\end{aligned} \tag{40}$$

$$\begin{aligned}|\Psi_7\rangle_{12345} = {} & (\alpha_{00}|Va_{00}\rangle - \alpha_{10}|Ha_{00}\rangle + \alpha_{01}|Va_{01}\rangle \\ & - \alpha_{11}|Ha_{01}\rangle)_1|Vb_0\rangle_2|Hc_1\rangle_3(\beta_{01}|Hd_{00}\rangle \\ & + \beta_{11}|Vd_{11}\rangle + \beta_{00}|Hd_{10}\rangle + \beta_{10}|Vd_{10}\rangle)_4|Ve_1\rangle_5,\end{aligned} \tag{41}$$

$$\begin{aligned}|\Psi_8\rangle_{12345} = {} & (\alpha_{00}|Va_{00}\rangle - \alpha_{10}|Ha_{00}\rangle + \alpha_{01}|Va_{01}\rangle \\ & - \alpha_{11}|Ha_{01}\rangle)_1|Vb_0\rangle_2|Vc_1\rangle_3(\beta_{01}|Vd_{11}\rangle \\ & - \beta_{11}|Hd_{11}\rangle + \beta_{00}|Vd_{10}\rangle - \beta_{10}|Hd_{10}\rangle)_4|He_1\rangle_5,\end{aligned} \tag{42}$$

$$\begin{aligned}|\Psi_9\rangle_{12345} = {} & (\alpha_{01}|Ha_{11}\rangle + \alpha_{11}|Va_{11}\rangle + \alpha_{00}|Ha_{10}\rangle \\ & + \alpha_{10}|Va_{10}\rangle)_1|Hb_1\rangle_2|Hc_0\rangle_3(\beta_{00}|Hd_{00}\rangle \\ & + \beta_{10}|Vd_{00}\rangle + \beta_{01}|Hd_{01}\rangle + \beta_{11}|Vd_{01}\rangle)_4|He_1\rangle_5,\end{aligned} \tag{43}$$

$$\begin{aligned}|\Psi_{10}\rangle_{12345} = {} & (\alpha_{01}|Ha_{11}\rangle + \alpha_{11}|Va_{11}\rangle + \alpha_{00}|Ha_{10}\rangle \\ & + \alpha_{10}|Va_{10}\rangle)_1|Hb_1\rangle_2|Vc_0\rangle_3(\beta_{00}|Vd_{00}\rangle \\ & - \beta_{10}|Hd_{00}\rangle + \beta_{01}|Vd_{01}\rangle - \beta_{11}|Hd_{01}\rangle)_4|Ve_1\rangle_5,\end{aligned} \tag{44}$$

$$\begin{aligned}|\Psi_{11}\rangle_{12345} = {} & (\alpha_{01}|Va_{11}\rangle - \alpha_{11}|Ha_{11}\rangle + \alpha_{00}|Va_{10}\rangle \\ & - \alpha_{10}|Ha_{10}\rangle)_1|Vb_1\rangle_2|Hc_0\rangle_3(\beta_{00}|Hd_{00}\rangle \\ & + \beta_{10}|Vd_{00}\rangle + \beta_{01}|Hd_{01}\rangle + \beta_{11}|Vd_{01}\rangle)_4|Ve_1\rangle_5,\end{aligned} \tag{45}$$

$$\begin{aligned}|\Psi_{12}\rangle_{12345} = {} & (\alpha_{01}|Va_{11}\rangle - \alpha_{11}|Ha_{11}\rangle + \alpha_{00}|Va_{10}\rangle \\ & - \alpha_{10}|Ha_{10}\rangle)_1|Vb_1\rangle_2|Vc_0\rangle_3(\beta_{00}|Vd_{00}\rangle \\ & - \beta_{10}|Hd_{00}\rangle + \beta_{01}|Vd_{01}\rangle - \beta_{11}|Hd_{01}\rangle)_4|He_1\rangle_5,\end{aligned} \tag{46}$$

$$\begin{aligned}|\Psi_{13}\rangle_{12345} = {} & (\alpha_{01}|Ha_{11}\rangle + \alpha_{11}|Va_{11}\rangle + \alpha_{00}|Ha_{10}\rangle \\ & + \alpha_{10}|Va_{10}\rangle)_1|Hb_1\rangle_2|Hc_1\rangle_3(\beta_{01}|Hd_{00}\rangle \\ & + \beta_{11}|Vd_{11}\rangle + \beta_{00}|Hd_{10}\rangle + \beta_{10}|Vd_{10}\rangle)_4|He_0\rangle_5,\end{aligned} \tag{47}$$

$$\begin{aligned}|\Psi_{14}\rangle_{12345} = {} & (\alpha_{01}|Ha_{11}\rangle + \alpha_{11}|Va_{11}\rangle + \alpha_{00}|Ha_{10}\rangle \\ & + \alpha_{10}|Va_{10}\rangle)_1|Hb_1\rangle_2|Vc_1\rangle_3(\beta_{01}|Vd_{11}\rangle \\ & - \beta_{11}|Hd_{11}\rangle + \beta_{00}|Vd_{10}\rangle - \beta_{10}|Hd_{10}\rangle)_4|Ve_0\rangle_5,\end{aligned} \tag{48}$$

$$\begin{aligned}|\Psi_{15}\rangle_{12345} = {} & (\alpha_{01}|Va_{11}\rangle - \alpha_{11}|Ha_{11}\rangle + \alpha_{00}|Va_{10}\rangle \\ & - \alpha_{10}|Ha_{10}\rangle)_1|Vb_1\rangle_2|Hc_1\rangle_3(\beta_{01}|Hd_{00}\rangle \\ & + \beta_{11}|Vd_{11}\rangle + \beta_{00}|Hd_{10}\rangle + \beta_{10}|Vd_{10}\rangle)_4|Ve_0\rangle_5,\end{aligned} \tag{49}$$

$$\begin{aligned}|\Psi_{16}\rangle_{12345} = {} & (\alpha_{01}|Va_{11}\rangle - \alpha_{11}|Ha_{11}\rangle + \alpha_{00}|Va_{10}\rangle \\ & - \alpha_{10}|Ha_{10}\rangle)_1|Vb_1\rangle_2|Vc_1\rangle_3(\beta_{01}|Vd_{11}\rangle \\ & - \beta_{11}|Hd_{11}\rangle + \beta_{00}|Vd_{10}\rangle - \beta_{10}|Hd_{10}\rangle)_4|He_0\rangle_5.\end{aligned} \tag{50}$$

Third, Alice mixes the spatial modes a_{01} and a_{10} on a balanced beam splitter (BS) and the other spatial modes a_{00} and a_{11} on another BS, while Bob mixes the spatial modes d_{01} and d_{10} on a BS and the other spatial modes d_{00} and d_{11} on another BS. Fourth, behind the BSs Alice (Bob) arranges four polarization beam splitters and eight photodetectors $D_{mkl}(D'_{m'k'l'})$, with $m, k, l(m', k', l') \in \{0, 1\}$, as shown in Fig. 7, to detect the outgoing photon 1 (photon 4). Note that the photodetector label is made in such a way that m (m') signals the photon polarization:

$m = 0 (m' = 0)$ implies H polarization and $m = 1 (m' = 1)$ V-polarization, while k, $l (k', l')$ identify the path. One of the eight photodetectors $D_{mkl} (D'_{m'k'l'})$ should click and the values of m, k, $l (m', k', l')$ corresponding to the photodetector that clicked are to be announced publicly by Alice (Bob). It should be emphasized at this point that with the announced values of m, k, $l (m', k', l')$ Bob (Alice) is still unable to obtain the hyperstate $|\psi\rangle (|\phi\rangle)$ that Alice (Bob) wishes to prepare for him (her). As mentioned before, completion of the protocol depends on Charlie's decision. If Charlie does not want to complete the protocol, he does nothing. Otherwise, he will do something. Thus, Charlie superimposes the spatial modes e_0 and e_1 of his photon 5 on a BS, then passes the modes after the BS through two QWPs. Afterward, Charlie detects photon 5 using two PBSs and four photodetectors D_{pq}, for $p, q \in \{0, 1\}$, with $p = 0 (p = 1)$ implying H polarization (V-polarization) and q specifying the path. One of the four photodetectors D_{pq} with certain values of p, q should click and Charlie reveals those p, q via a public reliable classical media.

After Alice, Bob, and Charlie detected, respectively, the photons 1, 4, and 5 (i.e., photodetectors D_{mkl}, $D'_{m'k'l'}$, and D_{pq} with fixed m, k, l, m', k', l', p, and q clicked) the remaining photons 2 and 3 get separated from each other. The state of Bob's photon 2 is of the form

$$|\psi'\rangle_2 = R_B^+ |\psi\rangle_2, \tag{51}$$

with

$$R_B = (X_P^{(m)} Z_P^{(m \oplus p \oplus 1)}) \otimes (X_S^{(k \oplus l)} Z_S^{(k \oplus q)}), \tag{52}$$

while the state of Alice's photon 3 reads

$$|\phi'\rangle_3 = R_A^+ |\phi\rangle_3, \tag{53}$$

with

$$R_A = (X_P^{(m')} Z_P^{(m' \oplus p \oplus 1)}) \otimes (X_S^{(k' \oplus l')} Z_S^{(k' \oplus q)}). \tag{54}$$

As clearly shown in Eqs. (51) and (53), in the final step Bob just needs to apply R_B on his photon 2 and Alice applies R_A on her photon 3 to complete the mutual controlled remote preparation of the hyperstates $|\psi\rangle$ and $|\phi\rangle$ of the forms in Eqs. (1) and (2), respectively.

4. CONCLUSION

We have designed an optical scheme to produce a five-photon, 10-qubit hyperentangled state that can be used as the shared quantum channel for Alice and Bob to prepare for each other a single-photon, two-qubit hyperstate under the common control of Charlie. The two distinct degrees of freedom that are simultaneously exploited here are polarization and spatial-mode ones. The scheme of production of the hyperentangled state requires the use of cross-Kerr nonlinearities and would work if the outcomes of homodyne detection measurements on coherent states are resolvable. In practice, cross-Kerr nonlinearities are very weak. Fortunately, however, the measurement resolution depends not only on the strength of cross-Kerr nonlinearities but also on the intensity of the coherent states. The use of strong-enough coherent states can compensate for the weakness of the cross-Kerr nonlinearities so that the homodyne detection measurements outcomes can be resolved. Starting from a simple initial separable state, the hyperentangled state production process may not immediately be successful. In unlucky cases, however, it returns the initial state and thus can be repeated any number of times until it succeeds. In this sense, it is near-deterministic.

Given the five-photon hyperentangled state as the working quantum, the protocol for the controlled bidirectional remote hyperstate preparation itself is performed only by means of linear optic devices such as unbalanced/balanced beam splitters, polarization beam splitters, and wave plates together with photodetectors. In our protocol, the diagram is extremely simple and the number of linear optical devices used is significantly reduced compared to even the one-way protocols in [19–21]. Since only one photon at most would hit a photodetector, photon-number-resolving photodetectors are not needed here. Although our protocol carries out photon detection measurements that are probabilistic, the task succeeds with unit probability because, for each possible set of the measurement outcomes, corresponding recovery operators exist that Alice and Bob will use to obtain the desired hyperstates. We believe the protocol considered in this paper is beneficial for different high-capacity tasks in quantum information processing and quantum computing.

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