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Joint remote preparation of a single-photon hyper-state with two pairs of hyper-Bell states

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Abstract. Hyper-entanglement is gaining more and more attentions thanks to its outstanding applications in processing quantum information. It is considered as a quantum resource in many important tasks. In this work, we deal with a tripartite task for three remote parties Alice, Bob and Charlie. Namely, Alice and Bob are two senders who independently share the full information of an arbitrary single-photon state which is encoded simultaneously in two different degrees of freedom, the spatial-mode degree of freedom and the polarization degree of freedom (hereafter called photon's hyper-state). We show that Alice and Bob can jointly prepare for the receiver Charlie such a hyper-state of the photon when they share beforehand two pairs of hyper-Bell states.

1. Introduction

Quantum entanglement [1] is the most significant and oddest trait in quantum mechanics and quantum information. Photons have been shown to be the best carriers of quantum information over long distances. In addition, the entangled state of them have been used to experimentally demonstrate quantum dense coding, quantum metrology, quantum cryptography, etc. They can store quantum information in some different degrees of freedom (DOF) includings the spatial-mode, the polarization, the frequency and so on. Different DOFs can be used at the same time to encode quantum states and to construct quantum entanglement, which are referred to as hyper-states and hyper-entanglement, respectively. The hyper-entangled state (see, e.g., [2]) has been exploited by many scientists around the world. It is used as the indispensable quantum sources in quantum communication for single photon in only one DOF, such as quantum superdense coding using linear-optical elements devices [3], the complete Bell-state discrimination in the polarization DOF [4, 5, 6, 7, 8], the deterministic entanglement purification [9, 10, 11], and the efficient quantum repeater [12].

A quantum state can be conveyed from one to another location by not physically sending the state itself through space thanks to the principles of quantum mechanics. This is known as a wonderful task that cannot be performed by classical tools. In 1993, a protocol called quantum teleportation (QT) [13] which permits to faithfully and securely transfer an unknown qubit state from from one location to another by using an Einstein-Podolsky-Rosen (EPR) state pair

[14], were proposed by Bennett et al. for the first time. As the extended versions of quantum teleporation method, there are two established versions called remote state preparation (RSP) protocol [15] and joint remote state preparation (JRSP) protocol [16]. In RSP scheme, there is only one sender and she/he knows the full set of parameters characterizing the state to be prepared. As for JRSP protocol, it involves several senders and each of them only knows a subset of the parameters of the state to be prepared. In JRSP task, the senders must cooperate with each other in such a way that no one can obtain the information of another one, leaving the to-be-prepared state confidential to anybody. Thus, the security of JRSP schemes will have level higher than that in RSP ones. Furthermore, by using adaptive measurements which are done in sequence and the actions of later stage depend on the measurement results of the previous stages [17], JRSP scheme can be obtained with unit probability [18, 19, 20, 21].

With hyper-entanglement, transmitting quantum state remotely has attracted a lot of interest since a large amount of information contained in photons can be exchanged. In [22], the authors proposed the parallel QT of an arbitrary single-qubit state by using the hyper-entangled state between two different DOFs which are spatial mode and polarization DOFs. In [23], with only linear optic elements, two perfect remote preparation protocols for a single-qubit pure and mixed state were presented. In [24], by using hyper-entangled states, quantum teleportation of more than one DOF of a single photon has been experimentally constructed. In [25], the authors presented a protocol to prepare a single photon in two-qubit states via hyper-entangled states basing on only linear-optical elements devices. In this work, we propose a new scheme in which two senders can jointly prepare a single-photon in two-qubit states for a receiver by using two hyper-Bell state pairs as quantum channels.

The structure of this paper is presented as follows. In Sect. 2, we take two hyper Bell state pairs as the quantum channel to prepare an arbitrary single photon in two DOFs. Conclusion is given in Sect. 3.

2. Our protocol

We propose that the problem is given as follows: Alice and Bob wish to transmit to Charlie a single-photon hyper-state, which can written in the form

$$|\psi\rangle = \lambda_{00}e^{i\phi_{00}}\,|Ha_0\rangle + \lambda_{01}e^{i\phi_{01}}\,|Ha_1\rangle + \lambda_{10}e^{i\phi_{10}}\,|Va_0\rangle + \lambda_{11}e^{i\phi_{11}}\,|Va_1\rangle\,. \tag{1}$$

The parameters λ_{ij} and ϕ_{ij} satisfy the normalization condition $\sum_{i,j=0}^{1}\lambda_{ij}^2 = 1$. In the state $|\psi\rangle$, $|H\rangle$ and $|V\rangle$ indicate two polarization states of the photon $|\psi\rangle$, while $|a_0\rangle$ and $|a_1\rangle$ denote two spatial-mode states of $|\psi\rangle$. In order to do that the parameters of the state, which contains all information of the photon $|\psi\rangle$ is splitted into two sub-sets $S_A = \{\lambda_{ij}\}$ and $S_B = \{\phi_{ij}\}$, then let Alice know only one subset (S_A) and Bob know the other one (S_B). In order to achieve this goal, the three participants must beforehand share two hyper-EPR states as quantum channel, which is written as

$$|\Phi_1\rangle_{ACBD} = |\Phi_1\rangle_{AC} \otimes |\Phi_1\rangle_{BD}\,, \tag{2}$$

with

$$|\Phi_1\rangle_{AC} = \frac{1}{2}[(|HH\rangle + |VV\rangle)_{AC}^{D_1} \otimes (|a_0c_0\rangle + |a_1c_1\rangle)_{AC}^{D_2}] \tag{3}$$

and

$$|\Phi_1\rangle_{BD} = \frac{1}{2}[(|HH\rangle + |VV\rangle)_{BD}^{D_1} \otimes (|b_0d_0\rangle + |b_1d_1\rangle)_{BD}^{D_2}]. \tag{4}$$

In the quantum channel, Alice holds photon A, Bob holds B and two photons C and D belong to Charlie. D_1 is the polarization DOF and D_2 is the spatial-mode DOF. Our protocol is composed of four sequential stages.

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Stage 1: The actions of Charlie and Bob. Charlie performs the following actions. He first performs two hyper-$CNOT_{CD}$ on two photon C and D in two DOFs D_1 and D_2, respectively. Note that, by using the nonlinear elements, the generation of the hyper-CNOT gate on the polarization and spatial-mode DOFs of a two-photon system were described in [26]. In that work, a CNOT gate has been nearly perfectly implemented on the spatial-mode DOF as well as polarization DOF of two photons. The action of CNOT can be presented as $CNOT_{CD}\left|H\right\rangle_C\left|H\right\rangle_D = \left|H\right\rangle_C\left|H\right\rangle_D$, $CNOT_{CD}\left|H\right\rangle_C\left|V\right\rangle_D = \left|H\right\rangle_C\left|V\right\rangle_D$, $CNOT_{CD}\left|V\right\rangle_C\left|H\right\rangle_D = \left|V\right\rangle_C\left|V\right\rangle_D$, $CNOT_{CD}\left|V\right\rangle_C\left|V\right\rangle_D = \left|V\right\rangle_C\left|H\right\rangle_D$,$CNOT_{CD}\left|c_0\right\rangle_C\left|d_0\right\rangle_D = \left|c_0\right\rangle_C\left|d_0\right\rangle_D$, $CNOT_{CD}\left|c_0\right\rangle_C\left|d_1\right\rangle_D = \left|c_0\right\rangle_C\left|d_1\right\rangle_D$, $CNOT_{CD}\left|c_1\right\rangle_C\left|d_0\right\rangle_D = \left|c_1\right\rangle_C\left|d_1\right\rangle_D$ and $CNOT_{CD}\left|c_1\right\rangle_C\left|d_1\right\rangle_D = \left|c_1\right\rangle_C\left|d_0\right\rangle_D$. After performing hyper-CNOT on two photons C and D, he measures photon D by using photon detectors (PD) and publicly announces the obtained outcomes kl. Assume that the polarization measurement of photon D gives outcome k with $k = 0$ if photon D is in the H state and $k = 1$ if photon D is in the V state. Similarly, l is outcome of spatial-mode measurement with $l = 0$ if photon D is in path d_0 and $l = 1$ if photon D is in path d_1. There are 4 possible cases of kl for two states of photon D. If $kl = 00, 01, 10$ and 11, photons A, B and C can be found in the state

$$|H_{00}\rangle_{ABC} = \frac{1}{2}[(|HHH\rangle + |VVV\rangle)_{ABC} \otimes (|a_0b_0c_0\rangle + |a_1b_1c_1\rangle)_{ABC}], \quad (5)$$

$$|H_{01}\rangle_{ABC} = \frac{1}{2}[(|HHH\rangle + |VVV\rangle)_{ABC} \otimes (|a_0b_1c_0\rangle + |a_1b_0c_1\rangle)_{ABC}], \quad (6)$$

$$|H_{10}\rangle_{ABC} = \frac{1}{2}[(|HVH\rangle + |VHV\rangle)_{ABC} \otimes (|a_0b_0c_0\rangle + |a_1b_1c_1\rangle)_{ABC}] \quad (7)$$

and

$$|H_{11}\rangle_{ABC} = \frac{1}{2}[(|HVH\rangle + |VHV\rangle)_{ABC} \otimes (|a_0b_1c_0\rangle + |a_1b_0c_1\rangle)_{ABC}]. \quad (8)$$

Depending on k and l, Bob then applies the operator $X_P^k \otimes X_S^l$ on photon B, the state of three photons A, B and C will become $|\Phi_2\rangle_{ABC}$ which sames as in Eq. (5). Here $X_P = |V\rangle\langle H| + |H\rangle\langle V|$ and $X_S = |b_0\rangle\langle b_1| + |b_1\rangle\langle b_0|$.

Stage 2: Action of Alice, Bob and Charlie. Firstly, Alice lets the spatial-mode state a_0 and a_1 of photon A pass through two unbalanced beamsplitters UBS_1 and UBS_2 which have the reflection coefficients $r_1 = \sqrt{\lambda_{01}^2 + \lambda_{11}^2}$ and $r_2 = \sqrt{\lambda_{00}^2 + \lambda_{10}^2}$, respectively. After this action, the state of photons' Alice, Bob and Charlie will become $|\Phi_3\rangle_{ABC}$.

$$\begin{aligned}|\Phi_3\rangle_{ABC} &= \frac{1}{2}(|HHH\rangle + |VVV\rangle)_{ABC} \\ &\otimes [(\sqrt{\lambda_{00}^2 + \lambda_{10}^2}|a_{00}\rangle_A + \sqrt{\lambda_{01}^2 + \lambda_{11}^2}|a_{01}\rangle_A)|b_0c_0\rangle_{BC} \\ &+ (\sqrt{\lambda_{01}^2 + \lambda_{11}^2}|a_{11}\rangle_A + \sqrt{\lambda_{00}^2 + \lambda_{10}^2}|a_{10}\rangle_A)|b_1c_1\rangle_{BC}]. \quad (9)\end{aligned}$$

Secondly, the polarization state of photon A in modes $a_{000}, a_{001}, a_{010}, a_{011}, a_{110}, a_{111}, a_{100}, a_{101}$ are allowed to pass via wave plates R_i (i=1,2,3,4) with angle θ_i

$$\theta_1 = arccos\frac{\lambda_{00}}{\sqrt{\lambda_{00}^2 + \lambda_{10}^2}}, \quad (10)$$

$$\theta_2 = \pi - arcsin\frac{\lambda_{10}}{\sqrt{\lambda_{00}^2 + \lambda_{10}^2}}, \tag{11}$$

$$\theta_3 = arccos\frac{\lambda_{01}}{\sqrt{\lambda_{01}^2 + \lambda_{11}^2}} \tag{12}$$

and

$$\theta_4 = 1 - arcsin\frac{\lambda_{11}}{\sqrt{\lambda_{01}^2 + \lambda_{11}^2}}. \tag{13}$$

After this action, the state of three photons A, B and C will be $|\Phi_4\rangle_{ABC}$ as follows

$$\begin{aligned} |\Phi_4\rangle_{ABC} &= \frac{1}{2}[(\lambda_{00}|Ha_{000}\rangle + \lambda_{10}|Va_{000}\rangle + \lambda_{01}|Ha_{010}\rangle + \lambda_{11}|Va_{010})|Hb_0\rangle|Hc_0\rangle \\ &+ (\lambda_{01}|Ha_{110}\rangle + \lambda_{11}|Va_{110}\rangle + \lambda_{00}|Ha_{100}\rangle + \lambda_{10}|Va_{100}\rangle)|Hb_1\rangle|Hc_1\rangle \\ &+ (\lambda_{10}|Ha_{001}\rangle + \lambda_{00}|Va_{001}\rangle + \lambda_{11}|Ha_{011}\rangle + \lambda_{01}|Va_{011}\rangle)|Vb_0\rangle|Vc_0\rangle \\ &+ (\lambda_{11}|Ha_{111}\rangle + \lambda_{01}|Va_{111}\rangle + \lambda_{10}|Ha_{101\rangle} + \lambda_{00}|Va_{101}\rangle)|Vb_1\rangle|Vc_1\rangle]. \end{aligned} \tag{14}$$

Thirdly, Alice uses some balance beam splitters (BS) which are used as the Hadamard operators on spatial-mode DOF to perform a measurement on photon A. After that, she publicly announces the obtained outcomes $\{mnrt : m, n, p, t = 0, 1\}$ if photon A is found on mode a_{npt}. Note that $m = 0$ when photon A is H state and $m = 1$ when photon A is V state. The actions of Alice on photon A can be plotted in Fig. 1. There are 16 possible cases of the state of photon B and C after Alice's measurements. If outcome of measurement of photon A is $mnpt = 0000$, $mnpt = 0001$, $mnpt = 0110$, $mnpt = 0111$, $mnpt = 0010$, $mnpt = 0011$, $mnpt = 0100$, $mnpt = 0101$, $mnpt = 1010$, $mnpt = 1011$, $mnpt = 1100$, $mnpt = 1101$, $mnpt = 1000$, $mnpt = 1001$, $mnpt = 1110$ and $mnpt = 1111$ the state of photon B and C will be

$$\begin{aligned} |L_{0000}\rangle_{BC} &= \lambda_{00}\,|Hb_0\rangle_B\,|Hc_0\rangle_C + \lambda_{01}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &+ \lambda_{10}\,|Vb_0\rangle_B\,|Vc_0\rangle_C + \lambda_{11}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,, \end{aligned} \tag{15}$$

$$\begin{aligned} |L_{0001}\rangle_{BC} &= \lambda_{00}\,|Hb_0\rangle_B\,|Hc_0\rangle_C - \lambda_{01}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &+ \lambda_{10}\,|Vb_0\rangle_B\,|Vc_0\rangle_C - \lambda_{11}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,, \end{aligned} \tag{16}$$

$$\begin{aligned} |L_{0110}\rangle_{BC} &= \lambda_{00}\,|Hb_0\rangle_B\,|Hc_0\rangle_C + \lambda_{01}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &- \lambda_{10}\,|Vb_0\rangle_B\,|Vc_0\rangle_C - \lambda_{11}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,, \end{aligned} \tag{17}$$

$$\begin{aligned} |L_{0111}\rangle_{BC} &= \lambda_{00}\,|Hb_0\rangle_B\,|Hc_0\rangle_C - \lambda_{01}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &- \lambda_{10}\,|Vb_0\rangle_B\,|Vc_0\rangle_C + \lambda_{11}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,, \end{aligned} \tag{18}$$

$$\begin{aligned} |L_{0010}\rangle_{BC} &= \lambda_{01}\,|Hb_0\rangle_B\,|Hc_0\rangle_C + \lambda_{00}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &+ \lambda_{11}\,|Vb_0\rangle_B\,|Vc_0\rangle_C + \lambda_{10}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,, \end{aligned} \tag{19}$$

$$\begin{aligned} |L_{0011}\rangle_{BC} &= \lambda_{01}\,|Hb_0\rangle_B\,|Hc_0\rangle_C - \lambda_{00}\,|Hb_2\rangle_B\,|Hc_2\rangle_C \\ &+ \lambda_{11}\,|Vb_0\rangle_B\,|Vc_0\rangle_C - \lambda_{10}\,|Vb_2\rangle_B\,|Vc_2\rangle_C\,, \end{aligned} \tag{20}$$

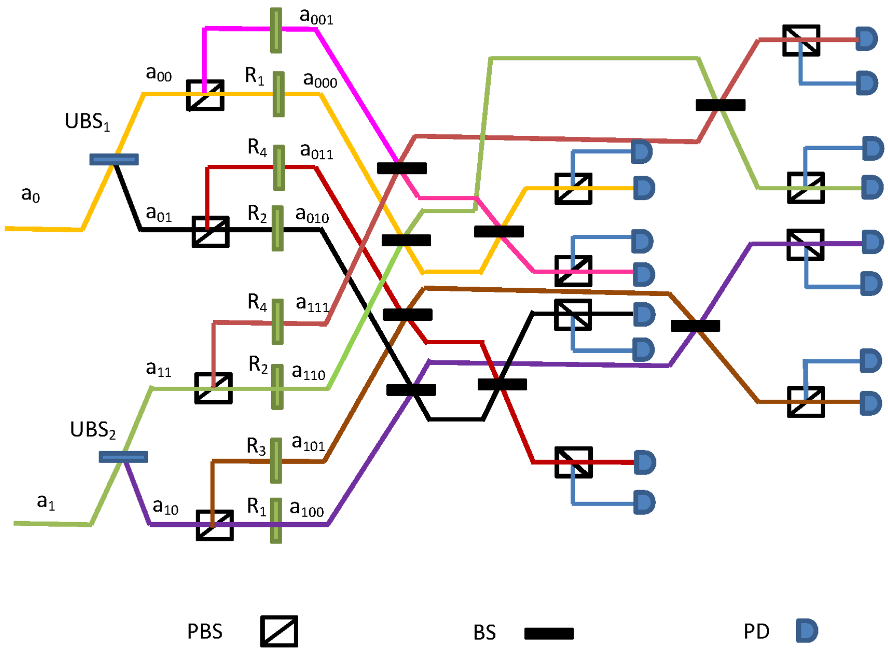

Figure 1. The actions of Alice on photon A. UBS_1 and UBS_2 are the unbalanced beam splitters with reflection r_1 and r_2, respectively. R_i $(i = 1, 2, 3, 4)$ performs perform an angle rotation θ_i on the polarizational states of photon. The polarization beam splitter (PBS) allows to reflect the vertical polarization state and to transmit horizontal polarization state. PD is single-photon detector, while BS is a balance beam splitter which works in spatial-mode state as Hadamard gate.

$$\begin{aligned} |L_{0100}\rangle_{BC} &= \lambda_{01} |Hb_0\rangle_B |Hc_0\rangle_C + \lambda_{00} |Hb_1\rangle_B |Hc_1\rangle_C \\ &- \lambda_{11} |Vb_0\rangle_B |Vc_0\rangle_C - \lambda_{10} |Vb_1\rangle_B |Vc_1\rangle_C \,, \end{aligned} \tag{21}$$

$$\begin{aligned} |L_{0101}\rangle_{BC} &= \lambda_{01} |Hb_0\rangle_B |Hc_0\rangle_C - \lambda_{00} |Hb_1\rangle_B |Hc_1\rangle_C \\ &- \lambda_{11} |Vb_0\rangle_B |Vc_0\rangle_C + \lambda_{10} |Vb_1\rangle_B |Vc_1\rangle_C \,, \end{aligned} \tag{22}$$

$$\begin{aligned} |L_{1000}\rangle_{BC} &= \lambda_{10} |Hb_0\rangle_B |Hc_0\rangle_C + \lambda_{11} |Hb_1\rangle_B |Hc_1\rangle_C \\ &+ \lambda_{00} |Vb_0\rangle_B |Vc_0\rangle_C + \lambda_{01} |Vb_1\rangle_B |Vc_1\rangle_C \,, \end{aligned} \tag{23}$$

$$\begin{aligned} |L_{1001}\rangle_{BC} &= \lambda_{10} |Hb_0\rangle_B |Hc_0\rangle_C - \lambda_{11} |Hb_1\rangle_B |Hc_1\rangle_C \\ &+ \lambda_{00} |Vb_0\rangle_B |Vc_0\rangle_C - \lambda_{01} |Vb_1\rangle_B |Vc_1\rangle_C \,, \end{aligned} \tag{24}$$

$$\begin{aligned} |L_{1111}\rangle_{BC} &= \lambda_{10} |Hb_0\rangle_B |Hc_0\rangle_C - \lambda_{11} |Hb_1\rangle_B |Hc_1\rangle_C \\ &- \lambda_{00} |Vb_0\rangle_B |Vc_0\rangle_C + \lambda_{01} |Vb_1\rangle_B |Vc_1\rangle_C \,, \end{aligned} \tag{25}$$

$$\begin{aligned}|L_{1110}\rangle_{BC} &= \lambda_{10}\,|Hb_0\rangle_B\,|Hc_0\rangle_C + \lambda_{11}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &- \lambda_{00}\,|Vb_0\rangle_B\,|Vc_0\rangle_C - \lambda_{01}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,,\end{aligned} \tag{26}$$

$$\begin{aligned}|L_{1010}\rangle_{BC} &= \lambda_{11}\,|Hb_0\rangle_B\,|Hc_0\rangle_C + \lambda_{10}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &+ \lambda_{01}\,|Vb_0\rangle_B\,|Vc_0\rangle_C + \lambda_{00}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,,\end{aligned} \tag{27}$$

$$\begin{aligned}|L_{1011}\rangle_{BC} &= \lambda_{11}\,|Hb_0\rangle_B\,|Hc_0\rangle_C - \lambda_{10}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &+ \lambda_{01}\,|Vb_0\rangle_B\,|Vc_0\rangle_C - \lambda_{00}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\,,\end{aligned} \tag{28}$$

$$\begin{aligned}|L_{1100}\rangle_{BC} &= \lambda_{11}\,|Hb_0\rangle_B\,|Hc_0\rangle_C + \lambda_{10}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &- \lambda_{01}\,|Vb_0\rangle_B\,|Vc_0\rangle_C - \lambda_{00}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\end{aligned} \tag{29}$$

and

$$\begin{aligned}|L_{1101}\rangle_{BC} &= \lambda_{11}\,|Hb_0\rangle_B\,|Hc_0\rangle_C - \lambda_{10}\,|Hb_1\rangle_B\,|Hc_1\rangle_C \\ &- \lambda_{01}\,|Vb_0\rangle_B\,|Vc_0\rangle_C + \lambda_{00}\,|Vb_1\rangle_B\,|Vc_1\rangle_C\end{aligned} \tag{30}$$

Fourthly, Bob and Charlie use unitary operation $R_B = [(X_P^m Z_P^n) \otimes (X_S^{(p+n)} Z_S^t)]$ and $R_C = [X_P^m \otimes X_S^{(p+n)}]$ to apply on photon B and C, respectively. As this result, the state of photon B and C in from Eq. (16) to Eq. (30) can be converted to Eq. (15).

Stage 3: Action of Bob. After stage 2, the state of photon B and C will become $|\Phi_5\rangle_{BC} = |L_{0000}\rangle_{BC}$ in Eq. (15). Bob first uses a PBS to rotate the spatial-mode states b_0 and b_1 of photon B, then he rotates the polarization state of photon B by using four wave plates R_{00}, R_{01}, R_{10} and R_{11} as in Figure. 2 with

$$R_{00} = \begin{pmatrix} e^{i\phi_{00}} & 0 \\ 0 & e^{-i\phi_{00}} \end{pmatrix}, \tag{31}$$

$$R_{01} = \begin{pmatrix} e^{i\phi_{01}} & 0 \\ 0 & e^{-i\phi_{01}} \end{pmatrix}, \tag{32}$$

$$R_{10} = \begin{pmatrix} e^{-i\phi_{10}} & 0 \\ 0 & e^{i\phi_{10}} \end{pmatrix}, \tag{33}$$

and

$$R_{11} = \begin{pmatrix} e^{-i\phi_{11}} & 0 \\ 0 & e^{i\phi_{11}} \end{pmatrix}. \tag{34}$$

Photons B and C will be found in the state

$$\begin{aligned}|\Phi_6\rangle_{BC} &= \lambda_{00}e^{i\phi_{00}}\,|Hb_0\rangle\,|Hc_0\rangle + \lambda_{01}e^{i\phi_{01}}\,|Hb_1\rangle\,|Hc_1\rangle \\ &+ \lambda_{10}e^{i\phi_{10}}\,|Vb_0\rangle\,|Vc_0\rangle + \lambda_{11}e^{i\phi_{11}}\,|Vb_1\rangle\,|Vc_1\rangle\,.\end{aligned} \tag{35}$$

After that, photon B will be passed through the BSs and wave plates H_P. Bob makes a measurement on photon B by detection photon. Here, BS is balance beam splitter which works on spatial-mode state as a Hadamard gate, while H_P is the wave plate which works in polarization state with the role of Hadamard operator. He reported the results rs to Charlie. Namely, he publicly announces $rs = 00, 01, 10$ or 11 if photon A is found in $|Hb_0\rangle$, $|Hb_1\rangle$, $|Vb_0\rangle$ and $|Vb_1\rangle$, respectively. The actions of Bob on photon B is shown in Fig. 2. If photon B was

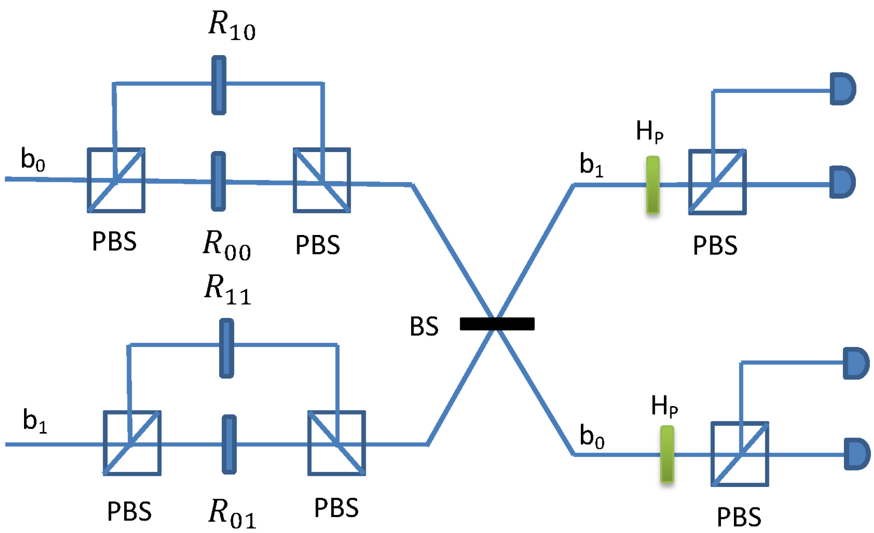

Figure 2. The actions of Bob on photon B. The wave plate $R_{00}, R_{01}, R_{10}, R_{11}$ rotates polarizational states of photon. H_P is the wave plate which works in polarization state as Hadamard operator. The polarization beam splitter allows to reflect the vertical polarization state and to transmit horizontal polarization state.

found in the state $|Hb_0\rangle$, the state of photon C will become

$$|T_{00}\rangle_C = \lambda_{00}e^{i\phi_{00}}|Hc_0\rangle + \lambda_{01}e^{i\phi_{01}}|Hc_1\rangle + \lambda_{10}e^{i\phi_{10}}|Vc_0\rangle + \lambda_{11}e^{i\phi_{11}}|Vc_1\rangle . \quad (36)$$

If photon B was found in the state $|Hb_1\rangle$, the state of photon C will become

$$|T_{01}\rangle_C = \lambda_{00}e^{i\phi_{00}}|Hc_0\rangle - \lambda_{01}e^{i\phi_{01}}|Hc_1\rangle + \lambda_{10}e^{i\phi_{10}}|Vc_0\rangle - \lambda_{11}e^{i\phi_{11}}|Vc_1\rangle . \quad (37)$$

If photon B was found in the state $|Vb_0\rangle$, the state of photon C will become

$$|T_{10}\rangle_C = \lambda_{00}e^{i\phi_{00}}|Hc_0\rangle + \lambda_{01}e^{i\phi_{01}}|Hc_1\rangle - \lambda_{10}e^{i\phi_{10}}|Vc_0\rangle - \lambda_{11}e^{i\phi_{11}}|Vc_1\rangle . \quad (38)$$

If photon B was found in the state $|Vb_1\rangle$, the state of photon C will become

$$|T_{11}\rangle_C = \lambda_{00}e^{i\phi_{00}}|Hc_0\rangle - \lambda_{01}e^{i\phi_{01}}|Hc_1\rangle - \lambda_{10}e^{i\phi_{10}}|Vc_0\rangle + \lambda_{11}e^{i\phi_{11}}|Vc_1\rangle . \quad (39)$$

Stage 4: Action of Charlie. After receiving all the results of the measurement from Bob, the receiver can transform the states of her photon to the target one by applying the recovery operators R'_C on photon C with $R'_C = [Z_P^r \otimes Z_S^s]$. Because Charlie always obtain the state to be prepared by using these recovery operators, so the fidelity and the total success probability of our scheme are both 100% when we assume that the hyper-CNOT operation in stage 1 can be generated perfectly.

3. Conclusion

To conclude, we have proposed a deterministic protocol for joint remote preparation of a single photon encoded at the same time in two spatial-mode and polarization DOFs via two hyper EPR pairs combined with classical communication of eight bits. In our protocol, in order to perform the task for JRSP via hyper-entangled states, the senders simply needs to perform single-qubit measurement through linear optical elements in different DOFs. However, the receiver has an active role in the task because she is required to perform a hyper controlled-NOT gate on her photons in the first stage. The adaptive measurement strategy for which the action in the later stage depends on the measurement outcome of the previous stage makes the probability become maximum.

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