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Controlled remote implementation of operators via hyperentanglement

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Abstract
Distributed quantum computation is a good solution for salable quantum computation within a quantum network each node of which just contains reasonably a few number of qubits. Controlled implementation of operators on states of a remote node is thus necessary. In this paper we propose protocols for three kinds of tasks of controlled implementation of operators on remote photon states via one hyperentangled Greenberger–Horne–Zeilinger state assisted with cross-Kerr nonlinearities: one with general operators and photon states in spatial degree of freedom (DOF), another one also with general operators but the photon state being in polarization DOF and the third one with a limited subset of operators acting on photon state in both spatial and polarization degrees of freedom. All the protocols are deterministic and performed in two steps under quantum control in each step.

Keywords: controlled remote implementation of operators, hyperentanglement, cross-Kerr nonlinearities, X-quadrature measurement

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum computers exploit weird traits of quantum mechanics like state superposition, entanglement and quantum interference to achieve quantum supremacy [1]. To be scalable a quantum computer must contain a large qubit number as well as be fault-tolerant. Besides the obstacles

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due to decoherence, storing safely and manipulating reliably a large number of qubits are challenging since the qubits may influence each other in an undesirable manner causing erroneous calculations. A promising solution is to deploy a 'distributed quantum computer' in such a way that not all the number of relevant qubits are kept in one computer but there is a network of quantum computers located in different places, each keeps only a small number of qubits and the computation is performed over all the qubits among the network [2–4] (see also [5–9] for multi-stage roadmap for quantum internet). Since quantum computation is a physical process dealing with implementation of unitary operators on quantum states followed by quantum measurements, when distributed quantum computation is concerned, two problems arise: how by means of local operations and classical communication (LOCC) (i) to securely and faithfully transfer quantum states from one place to another and (ii) to optimally implement quantum operators on a remote state or on spatially separated states? Apart from their fundamental interest, these problems constitute an essential leap to realize the quantum networking [10, 11] and quantum programing [12, 13]. Problem (i) has been judiciously tackled by quantum state teleportation (QST) invented in 1993 by Bennett *et al* [14]. As for problem (ii), Huelga *et al* [15] put forward a nonlocal task referred to as remote implementation of operator (RIO). In RIO Bob possesses an operator U, which is aimed to apply via LOCC on an arbitrary state $|\psi\rangle$ of a qubit held by his remote partner Alice. Naturally, such task can be fulfilled by 'bidirectional quantum state teleportation' (BQST) in which Alice teleports $|\psi\rangle$ to Bob who, upon receiving the state, applies U on it and then teleports $|\psi'\rangle = U\,|\psi\rangle$ back to Alice. The total cost of the BQST-based RIO is two ebits (i.e., two Einstein–Podolsky–Rosen states [16]) plus four cbits (i.e., classical bits). This total amount of resource is maximum [15]. Any more efficient method should expense less than that maximum cost. Next, Huelga *et al* [17] proved that the minimum total resource needed for RIO with an arbitrary operator U is two ebits plus just three cbits. The reduction in the number of cbits gives rise to the possibility of devising new methods evading the BQST techniques. Actually, a nontrivial protocol was devised in [17] that exhausts the said minimum total resource, yet succeeds just in half of the cases if U is arbitrarily unknown. Remarkably, the authors of [17] also found out that if U belongs to one of the two restricted sets of operators then it can be remotely implemented with unit probability consuming just one ebit and two cbits that is significantly economical compared to the general BQST method. An experiment for teleporting a rotation angle was demonstrated in [18]. The idea of RIO in [15] was extended in [19] to a multiparty setting that allows a group of n parties to execute, albeit probabilistically, a certain class of rotations on a remote qubit state if the $(n+1)$ involved parties share an $(n+1)$ -partite Greenberger–Horne–Zeilinger (GHZ) state [20]. Multiparty setting also applies to a simplified quantum secret-sharing scheme [21]. Reference [22] dealt with implementing a two-qubit nonlocal operator U_{AB} on two far apart qubits A and B, one in state $|\Psi\rangle_A$ at Alice's place and one in state $|\Phi\rangle_B$ at Bob's place. Again in this case BQST techniques prove helpful. Concretely, Alice (Bob) teleports her (his) qubit to Bob (Alice), who first locally performs U_{AB} on $|\Psi\rangle_A|\Phi\rangle_B$ to obtain $|\Theta\rangle_{AB} = U_{AB}|\Psi\rangle_A|\Phi\rangle_B$ and then teleports qubit A (B) of $|\Theta\rangle_{AB}$ back to Alice (Bob). Hence, the maximum overall resource needed for such nonlocal implementation of U_{AB} is the same as that needed for remote implementation of a single-qubit operator U. The authors of [22] demonstrated that in general not so much resources are actually required. For example, one ebit plus two cbits are necessary and sufficient to implement a controlled-NOT gate (CNOT) on two distant qubits. An experiment for teleporting a CNOT with an average fidelity of 0.84 was reported in [23]. From another angle, Reznik *et al* [24] was interested in implementing a class of operations whose full set of characteristics is split into subsets to be delivered to remote partners. To implement such 'splitted' operators the authors of [24] developed an unusual approach by introducing an object, called 'stator', which expresses quantum correlations between states of one partner and operators of the other partner. Given a suitably prepared stator, a desired operation on Alice's system is remotely brought about by Bob's local actions. In [25] nonmaximally entangled states are used to implement desired nonlocal operators nearly deterministically despite the amount of shared entanglement is vanishingly small. Furthermore, RIO was modified in [26] to the situation when a unitary operator is remotely implemented on two replicas of a quantum state. If the two replicas are located in one place then this task can be accomplished consuming fewer resources than those required for two independent implementations, each on a replica of the state. A more difficult situation when two identical states are located in different places was also discussed in [26]. Another possible way of extending RIO is going to multi-qubits [27]. Actually, for remote implementation on N qubits of a partially unknown operator belonging to the specific restricted sets the needed quantum resource costs just N ebits which is half of that for the BQST-based scheme. The so-called controlled RIO (CRIO) with partially unknown multi-qubit operators was studied in [28] using multipartite GHZ states (see also [29]). The BQST method and the method in [27] can be hybridized to form a hybrid protocol for RIO [30]. From the above-cited references the BQST technique seems universal. Yet, there are kinds of RIO that are not benefited from that technique. An example is the kind associated with 'splitted' operators which should be tackled by a state-operator approach via stators [24]. Remote implementation of hidden operators [31] is another example.

Many publications use abstract qubits represented by a superposition of logical states $|0\rangle$ and $|1\rangle$ with the CNOT tacitly assumed available. Practically, in any real protocol $|0\rangle$ and $|1\rangle$ are physical states. In the optical paradigm good physical qubits are photons which are robust against decoherence and propagate fastest [32]. Single-photon states are easily manipulated but the weak point is the lack of direct photon–photon interaction making photon-based quantum computation (PBQC) inefficient by means of linear-optics toolbox. Although in principle scalable PBQC is possible using linear-optics devices and photodetectors [33], a huge unacceptable overhead is requested. To realize PBQC nonlinear-optics elements such as cross-Kerr nonlinearities can be resorted to. Despite strong nonlinearities are still challenging, marvelous advances in the field are optimistic and numerous works have used them in diverse contexts [34–41].

The entangled states employed in the protocols cited above are all conventional in the sense that only one degree of freedom (DOF) is exploited. Actually, a physical object can at the same time be characterized in multiple DOFs, so entanglement can be too. Entanglement existing simultaneously in more than one DOF is called hyperentanglement (see, e.g. [42, 43]). Hyperentangled states feature several advantages over the conventional one as they carry more qubits, have higher data rates, boost the channel capacity and enhance security level in quantum communication. During the last years hyperentanglement has attracted much attention as it provides a high-capacity resource for quantum tasks. Applications include hyperentangled-Bell-state analysis [44–46], hyperentanglement concentration/purification [47–49], hyper CNOT gate [50], hyper quantum key distribution [51], hyper teleportation [52–54], hyper dense coding [55–57], hyper remote state preparation [58–60], hyper joint remote state preparation [61], hyper quantum secure direct communication [62–64] (notably, in [64] a *single-step* protocol for quantum secure direct communication has been designed which proves very useful for quantum networking), hyper quantum dialogue [65] and so on.

Of more practice is developing RIO with real photons and hyperentanglement than with abstract qubits and conventional entanglement [15, 17]. Indeed, such deployment has been done for remote implementation of single-photon operations via hyperentangled state with cross-Kerr nonlinearity [66]. Here, we raise the level of security for the RIO protocol in [66] by adding a supervisor who serves as a controller having the right to decide on completion of the RIO task. Such controlled protocols concerning QST was first proposed in [67] for discrete qubits and then in [68] for coherent-state superpositions. In section 2 we consider CRIO when the operator has the most general form and acts on photon state in spatial DOF (S-DOF). The CRIO with the general operator but acting on photon state in polarization DOF (P-DOF) is the content of section 3. Section 4 investigates CRIO for a restricted subset of operators (CRISO) acting on photon state encoded simultaneously in both S-DOF and P-DOF. Section 5 is the conclusion.

2. CRIO on photon state in S-DOF

Let alice and Bob be two partners who are under control of Charlie. The three people are in remote places and can communicate via classical means only. Alice has a photon a with certain polarization which propagates simultaneously along two distinct spatial paths x_0 and x_1. Without loss of generality we assume that the photon polarization is vertical (V), thus Alice's photon state is of the form

$$|\psi\rangle_a = \left|\psi^{(S)}\right\rangle_a |V\rangle_a, \tag{1}$$

$$\left|\psi^{(S)}\right\rangle_a = (\alpha\,|x_0\rangle + \beta\,|x_1\rangle)_a, \tag{2}$$

with unknown complex coefficients α, β satisfying the normalization $|\alpha|^2 + |\beta|^2 = 1$. The superindex ($S$) implies that the photon is encoded in S-DOF, $|x_j\rangle_a$ with $j = 0, 1$ denotes the state of photon a propagating along path x_j and $|V\rangle_a$ signals that photon a is vertically polarized. Bob is equipped with an apparatus that executes a general unitary operator

$$U^{(S)} = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \tag{3}$$

on any single-photon state in S-DOF:

$$U^{(S)}\left|\psi^{(S)}\right\rangle = \left|\psi'^{(S)}\right\rangle = \alpha'\,|x_0\rangle + \beta'\,|x_1\rangle\,, \tag{4}$$

$$\alpha' = \alpha u - \beta v^*, \quad \beta' = \alpha v + \beta u^*. \tag{5}$$

Charlie is the controller who is decisive for completion of concerned tasks.

We shall consider three tasks. The first task is to design a protocol for Alice and Bob to cooperate under Charlie's control so that at the end, upon approval of Charlie, Alice will hold a photon in the state

$$U^{(S)}\,|\psi\rangle = (U^{(S)}\left|\psi^{(S)}\right\rangle)\,|V\rangle = \left|\psi'^{(S)}\right\rangle|V\rangle \tag{6}$$

only by means of LOCC. This task can be looked upon as a CRIO in which Bob remotely implements his operator $U^{(S)}$ on Alice's state $\left|\psi^{(S)}\right\rangle$ under control of Charlie. Such CRIO is possible if Alice, Bob and Charlie share proper entanglement resource in advance. As learnt from [15, 67], the usual quantum channel could be two GHZ states, which are made of six photons (three photons per GHZ state). It is desirable to cut down the consumption of photon number in the shared quantum channel, especially when many CRIO protocols are demanded. If, instead of usual entanglement, hyperentanglement is utilized the entanglement amount contained in two GHZ states can be provided just from three (not six) photons that are entangled simultaneously in two different DOFs. Obviously, use of hyperentanglement considerably saves the number of photons that must be distributed among the participants, thus, reducing the overhead. A remarkable record for creating hyperentanglement has been reported [69]: eighteen qubits have successfully been packed into an entangled state of only six photons (i.e., each photon carries information of three qubits) by taking advantage of the photon's three kinds of DOFs at the same time.

For the first task of CRIO mentioned above, we employ one (not two) three-photon hyperentangled GHZ state with respect to two kinds of DOFs, which can be produced by several schemes (see, e.g. [70–72]). Since photon a on which operator $U^{(S)}$ will act is encoded in S-DOF, one of the two DOFs of the hyperentangled GHZ state should be the S-DOF. For the second kind of DOF we can choose the P-DOF. Hence, we shall work with the following hyperentangled GHZ state

$$\left|Q^{(SP)}\right\rangle_{ABC} = \left|Q^{(S)}\right\rangle_{ABC}\left|Q^{(P)}\right\rangle_{ABC}, \tag{7}$$

$$\left|Q^{(S)}\right\rangle_{ABC} = |a_0\rangle_A|b_0\rangle_B|c_0\rangle_C + |a_1\rangle_A|b_1\rangle_B|c_1\rangle_C, \tag{8}$$

$$\left|Q^{(P)}\right\rangle_{ABC} = |H\rangle_A|H\rangle_B|H\rangle_C + |V\rangle_A|V\rangle_B|V\rangle_C, \tag{9}$$

where the superindices (P) and (SP) indicate P-DOF alone and S-DOF and P-DOF together, while $|H\rangle_{A(B,C)}$ ($|V\rangle_{A(B,C)}$) represents state of photon A (B, C) which is horizontally (vertically) polarized. The same hyperentangled GHZ state (7) was already used, e.g., for quantum private comparison protocol in [73]. Note that in states (8) and (9) we, for simplicity, omit the normalization coefficients $1/\sqrt{2}$ and in all the formulae that follow we shall also ignore any common factors. This lightens the mathematical formulation and affects only probability of a certain measurement event, but not the total success probability which is 100% in our protocol. The state (7) is made of three photons but its information capacity is as much as of six qubits because each photon is worth of two qubits thanks to being simultaneously encoded in two distinct DOFs. In other words, the entanglement amount of three photons in state (7) is hyped as that of six photons in two usual entangled GHZ states. That is why state (7) is called hyperentangled GHZ state.

To perform the CRIO task, photons A, B and C of state (7) should necessarily be distributed among the three people. Let photons A, B and C be distributed to Alice, Bob and Charlie, respectively. After the photons' distribution, which is assumed successful (possibly with the aid of a purification/distillation procedure), two photons a and A are held by Alice, while photon B by Bob and photon C by Charlie. Initially, the total state is

$$|\psi\rangle_a\left|Q^{(SP)}\right\rangle_{ABC} = \left|\Phi^{(S)}\right\rangle_{aABC}|V\rangle_a\left|Q^{(P)}\right\rangle_{ABC}, \tag{10}$$

$$\begin{aligned}\left|\Phi^{(S)}\right\rangle_{aABC} = {} & \alpha|x_0\rangle_a|a_0\rangle_A|b_0\rangle_B|c_0\rangle_C + \alpha|x_0\rangle_a|a_1\rangle_A|b_1\rangle_B|c_1\rangle_C \\ & + \beta|x_1\rangle_a|a_0\rangle_A|b_0\rangle_B|c_0\rangle_C + \beta|x_1\rangle_a|a_1\rangle_A|b_1\rangle_B|c_1\rangle_C.\end{aligned} \tag{11}$$

The CRIO task consists of two steps. In the first step only the S-DOF part of the quantum channel (i.e., $\left|\Phi^{(S)}\right\rangle_{aABC}$) is exploited so, to evade cumbersomeness, we shall not write the P-DOF part (i.e., $|V\rangle_a\left|Q^{(P)}\right\rangle_{ABC}$) until when the second step begins. Each participant should fulfill his/her local operations correctly. Alice is the first to act and her operations are displayed in figure 1. First of all she uses an auxiliary coherent state (CS) $|z\rangle_d$ of real positive amplitude z and propagation direction along path d to entangle photon a with photons A, B, C by letting the CS interact with states $|x_0\rangle_a$ and $|a_0\rangle_A$ via cross-Kerr nonlinearity with (dimensionless) strengths θ

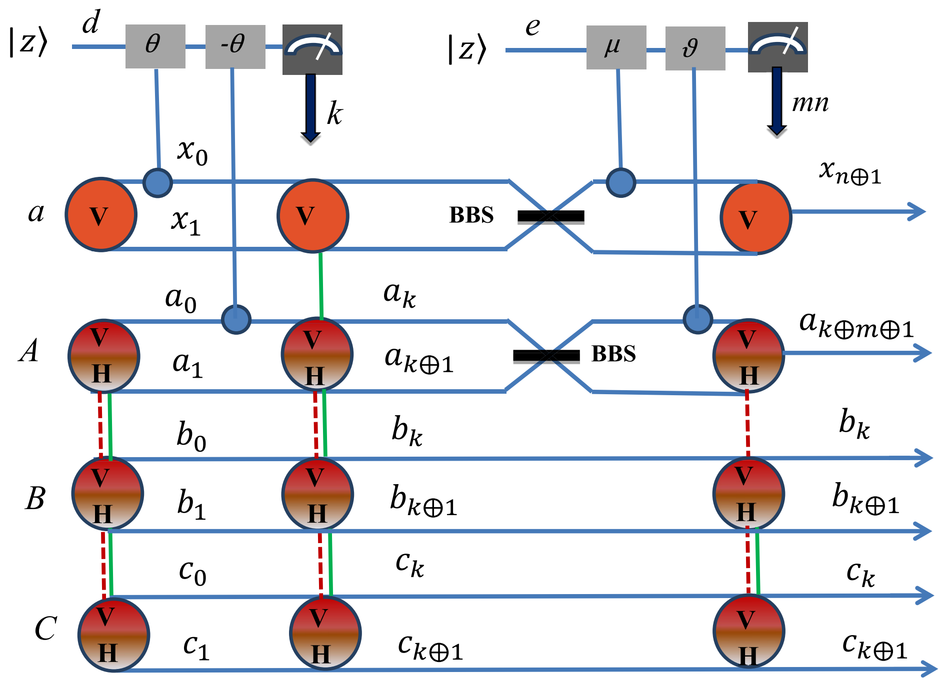

Figure 1. Alice's operations in the first step of CRIO on photon state in S-DOF. A circle with two attached lines represents a photon propagating at the same time along two paths, while that with one attached line a photon propagating along only one path. A circle with V, H implies a photon existing simultaneously in both vertical $|V\rangle$ and horizontal $|H\rangle$ polarization states, while that with V indicates a vertically polarized photon. $|z\rangle$ is CS of real positive amplitude z, BBS balanced beam splitter and $\pm\theta$, μ, ν dimensionless strengths of cross-Kerr interactions. Bold arrows are classical communication with letters (here k or mn) being measurement outcomes. Photons entangled in S-DOF (P-DOF) are connected by solid (dashed) lines.

and $-\theta$, respectively. The cross-Kerr nonlinear interaction with strengths $\pm\theta$ between a Fock state $|n\rangle_x$ and a CS $|z\rangle_y$ is represented by an operator $K_{xy}(\pm\theta)$ which adds a phase $\pm n\theta$ to the CS, i.e., $K_{xy}(\pm\theta)|n\rangle_x|z\rangle_y = |n\rangle_x\left|z\,\mathrm{e}^{\pm\mathrm{i}n\theta}\right\rangle_y$. Upon action of $K_{x_0d}(\theta)$ and $K_{a_0d}(-\theta)$ the state $\left|\Phi^{(S)}\right\rangle_{aABC}|z\rangle_d$ changes to

$$\begin{aligned}\left|\Phi^{(S)}\right\rangle_{aABCd} = &\left(\alpha|x_0\rangle_a|a_0\rangle_A|b_0\rangle_B|c_0\rangle_C\right.\\&\left.+\beta|x_1\rangle_a|a_1\rangle_A|b_1\rangle_B|c_1\rangle_C\right)|z\rangle_d\\&+\alpha|x_0\rangle_a|a_1\rangle_A|b_1\rangle_B|c_1\rangle_C\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_d\\&+\beta|x_1\rangle_a|a_0\rangle_A|b_0\rangle_B|c_0\rangle_C\left|z\,\mathrm{e}^{-\mathrm{i}\theta}\right\rangle_d.\end{aligned} \tag{12}$$

After the nonlinear interactions Alice carries out a homodyne detection that determines X-quadrature of the CS. Since the resolution of the X-quadrature values regarding $|z\rangle_d$ and $\left|z\,\mathrm{e}^{\pm\mathrm{i}\theta}\right\rangle_d$ ($\left|z\,\mathrm{e}^{+\mathrm{i}\theta}\right\rangle_d$ and $\left|z\,\mathrm{e}^{-\mathrm{i}\theta}\right\rangle_d$ are unresolved) is about $z\theta^2$ for small θ, $|z\rangle_d$ and $\left|z\,\mathrm{e}^{\pm\mathrm{i}\theta}\right\rangle_d$ are resolvable if z is chosen big enough. Let the outcome of finding $|z\rangle_d$ ($\left|z\,\mathrm{e}^{\pm\mathrm{i}\theta}\right\rangle_d$) be labeled by a cbit $k = 0 (k = 1)$, then photon a gets entangled in S-DOF with photons A, B and C in the state

$$\left|\Gamma_k^{(S)}\right\rangle_{aABC} = \alpha|x_0\rangle_a|a_k\rangle_A|b_k\rangle_B|c_k\rangle_C + \beta|x_1\rangle_a|a_{k\oplus 1}\rangle_A|b_{k\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C, \tag{13}$$

with $\oplus$ an addition mod 2. The value of k is disclosed by Alice via a public reliable classical channel. As seen from (13), the coefficients α, β before belonged only to photon a are now spread out among the four photons. Alice continues by superimposing $|x_0\rangle_a$ and $|x_1\rangle_a$ on a balanced beam splitter (BBS) while $|a_k\rangle_A$ and $|a_{k\oplus 1}\rangle_A$ on another BBS, transforming $\left|\Gamma_k^{(S)}\right\rangle_{aABC}$ to

$$\begin{aligned}\left|\Delta_k^{(S)}\right\rangle_{aABC} = {} & \left(|x_0\rangle_a|a_k\rangle_A + (-1)^k|x_1\rangle_a|a_{k\oplus 1}\rangle_A\right) \\ & \otimes \left(\alpha|b_k\rangle_B|c_k\rangle_C + (-1)^k\beta|b_{k\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C\right) \\ & + \left(|x_0\rangle_a|a_{k\oplus 1}\rangle_A + (-1)^k|x_1\rangle_a|a_k\rangle_A\right) \\ & \otimes \left(\alpha|b_k\rangle_B|c_k\rangle_C - (-1)^k\beta|b_{k\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C\right).\end{aligned} \tag{14}$$

Equation (14) comes up thanks to the BBS transformation rule $|x_k\rangle_a \to |x_k\rangle_a + (-1)^k|x_{k\oplus 1}\rangle_a$ and $|a_k\rangle_A \to |a_k\rangle_A + (-1)^k|a_{k\oplus 1}\rangle_A$. Next, Alice uses another CS $|z\rangle_e$ and turns on cross-Kerr interactions $K_{x_0 e}(\mu)$ and $K_{a_k e}(\nu)$ to bring $\left|\Delta_k^{(S)}\right\rangle_{aABC}|z\rangle_e$ to

$$\begin{aligned}\left|\Theta_k^{(S)}\right\rangle_{aABCe} = {} & \left(|x_0\rangle_a|a_k\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}(\mu+\nu)}\right\rangle_e + (-1)^k|x_1\rangle_a|a_{k\oplus 1}\rangle_A|z\rangle_e\right) \\ & \otimes \left(\alpha|b_k\rangle_B|c_k\rangle_C + (-1)^k\beta|b_{k\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C\right) \\ & + (|x_0\rangle_a|a_{k\oplus 1}\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}\mu}\right\rangle_e + (-1)^k|x_1\rangle_a|a_k\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}\nu)}\right\rangle_e) \\ & \otimes \left(\alpha|b_k\rangle_B|c_k\rangle_C - (-1)^k\beta|b_{k\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C\right),\end{aligned} \tag{15}$$

and then measures X-quadrature of the CS. The values of μ and ν are chosen so that the four possible outcomes $mn = 00$, 01, 10 or 11 corresponding respectively to finding $|z\rangle_e$, $\left|z\,\mathrm{e}^{\mathrm{i}\mu}\right\rangle_e$, $\left|z\,\mathrm{e}^{\mathrm{i}\nu}\right\rangle_e$ or $\left|z\,\mathrm{e}^{\mathrm{i}(\mu+\nu)}\right\rangle_e$ are distinguishable. For any possible mn, $\left|\Theta_k^{(S)}\right\rangle_{aABCe}$ collapses into

$$\left|\Lambda_{kmn}^{(S)}\right\rangle_{aABC} = |x_{n\oplus 1}\rangle_a|a_{k\oplus m\oplus 1}\rangle_A\left(\alpha|b_k\rangle_B|c_k\rangle_C + (-1)^{m+n}\beta|b_{k\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C\right), \tag{16}$$

indicating that instead of simultaneously traveling along two paths at the beginning now photon a travels along only one path $x_{n\oplus 1}$ and so does photon A along path $a_{k\oplus m\oplus 1}$. With respect to S-DOF, photons a and A get disentangled from photons B and C, while photons B and C are still entangled. That means that photon a disjoins the game so we shall forget it from now on and work with

$$\left|\Lambda_{kmn}^{(S)}\right\rangle_{ABC} = |a_{k\oplus m\oplus 1}\rangle_A\left(\alpha|b_k\rangle_B|c_k\rangle_C + (-1)^{m+n}\beta|b_{k\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C\right) \tag{17}$$

instead of $\left|\Lambda_{kmn}^{(S)}\right\rangle_{aABC}$ in (16).

The next person to act is Charlie. If Charlie decides to stop the task then she does nothing, leaving Alice and Bob guideless towards achieving the target. Otherwise, she will do some operations as in figure 2. She first mixes states $|c_k\rangle_C$ and $|c_{k\oplus 1}\rangle_C$ on a BBS to transform

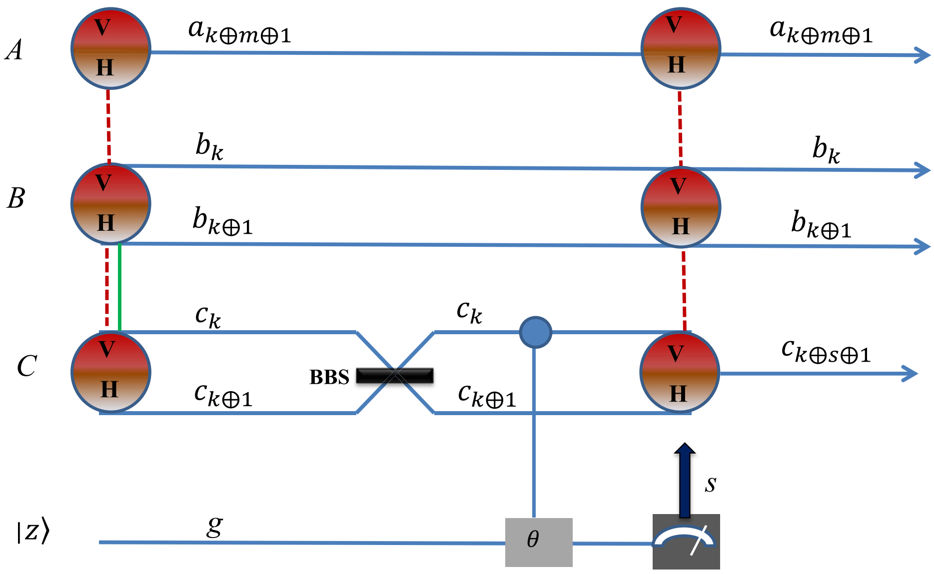

Figure 2. Charlie's operations in the first step of CRIO on photon state in S-DOF.

$\left|\Lambda^{(S)}_{kmn}\right\rangle_{ABC}$ to

$$\left|\Xi^{(S)}_{kmn}\right\rangle_{ABC} = |a_{k\oplus m\oplus 1}\rangle_A \left[(\alpha\,|b_k\rangle - (-1)^{k+m+n}\beta\,|b_{k\oplus 1}\rangle)_B|c_k\rangle_C \right. \\ \left. + (-1)^k(\alpha\,|b_k\rangle + (-1)^{k+m+n}\beta\,|b_{k\oplus 1}\rangle)_B|c_{k\oplus 1}\rangle_C\right]. \tag{18}$$

Subsequently, she takes a CS $|z\rangle_g$ and lets it interact with state $|c_k\rangle_C$ via cross-Kerr nonlinearity of strength θ. The resulting state is

$$\left|\Pi^{(S)}_{kmn}\right\rangle_{ABCg} = K_{c_k g}(\theta)\left|\Xi^{(S)}_{kmn}\right\rangle_{aABC}|z\rangle_g \\ = |a_{k\oplus m\oplus 1}\rangle_A \left[(\alpha\,|b_k\rangle - (-1)^{k\oplus m\oplus n}\beta\,|b_{k\oplus 1}\rangle)_B|c_k\rangle_C\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_g \right. \\ \left. + (-1)^k(\alpha\,|b_k\rangle + (-1)^{k\oplus m\oplus n}\beta\,|b_{k\oplus 1}\rangle)_B|c_{k\oplus 1}\rangle_C|z\rangle_g\right]. \tag{19}$$

To disentangle photon B from photon C Charlie measures X-quadrature of the CS. If $|z\rangle_g$ ($\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_g$) is measured she assigns to that event a cbit $s=0$ ($s=1$). Depending on s the state of photons A, B, C turns out to be

$$\left|\Sigma^{(S)}_{kmns}\right\rangle_{ABC} = |a_{k\oplus m\oplus 1}\rangle_A|\psi_{kmns}\rangle_B|c_{k\oplus s\oplus 1}\rangle_C, \tag{20}$$

$$|\psi_{kmns}\rangle_B = (\alpha\,|b_k\rangle + (-1)^{k+m+n+s}\beta\,|b_{k\oplus 1}\rangle)_B. \tag{21}$$

We see that all the photons become separable with respect to S-DOF. Conditioned on all the outcomes *kmns* Bob is always aware of the exact state of photon B and thus infer the proper operator $R^{(S)}_{kmns}$ which he will use in his turn. The dependences of $|\psi_{kmns}\rangle_B$ and $R^{(S)}_{kmns}$ on *kmns* are collected in table 1.

Table 1. The state $|\psi_{kmns}\rangle_B$ of Bob's photon B after Alice's and Charlie's measurements and the operator $R^{(S)}_{kmns}$ with the property $R^{(S)}_{kmns}|\psi_{kmns}\rangle_B = (\alpha\,|b_0\rangle + \beta\,|b_1\rangle)_B$.

$kmns$	$\lvert\psi_{kmns}\rangle_B$	$R^{(S)}_{kmns}$
0000, 0011, 0101, 0110	$(\alpha\,\lvert b_0\rangle + \beta\,\lvert b_1\rangle)_B$	I_S
0001, 0010, 0100, 0111	$(\alpha\,\lvert b_0\rangle - \beta\,\lvert b_1\rangle)_B$	Z_S
1000, 1011, 1101, 1110	$(\alpha\,\lvert b_1\rangle - \beta\,\lvert b_0\rangle)_B$	$Z_S X_S$
1001, 1010, 1100, 1111	$(\alpha\,\lvert b_1\rangle + \beta\,\lvert b_0\rangle)_B$	X_S

In table 1, I_S, X_S and Z_S are operators acting in S-DOF. Explicitly, I_S is the identity operator, $X_S = |b_1\rangle\langle b_0| + |b_0\rangle\langle b_1|$ the path-flip operator which can be realized by exchanging the photon propagation paths and $Z_S = |b_0\rangle\langle b_0| - |b_1\rangle\langle b_1|$ the sign-flip operator which is actually a π-phase-shifter placed on path b_1. Analyzing table 1 we can work out a single formula for the operator $R^{(S)}_{kmns}$ valid for any $k, m, n, s \in \{0, 1\}$:

$$R^{(S)}_{kmns} = Z_S^{k\oplus m\oplus n\oplus s} X_S^k. \tag{22}$$

Having heard the outcomes $kmns$ from Alice's and Charlie's announcements, Bob starts his operations as in figure 3. First, he applies $R^{(S)}_{kmns}$ on photon B to convert its state to $\left|\psi^{(S)}\right\rangle_B = (\alpha\,|b_0\rangle + \beta\,|b_1\rangle)_B$. Then, he implements the operator $U^{(S)}$ in equation (3) on $\left|\psi^{(S)}\right\rangle_B$, bringing photon B to a new state

$$\left|\psi'^{(S)}\right\rangle_B = U^{(S)}\left|\psi^{(S)}\right\rangle_B = (\alpha'\,|b_0\rangle + \beta'\,|b_1\rangle)_B, \tag{23}$$

with α' and β' given by equation (5). Thus, the new state of photons A, B, C reads

$$\left|\Upsilon^{(S)}_{kms}\right\rangle_{ABC} = |a_{k\oplus m\oplus 1}\rangle_A(\alpha'\,|b_0\rangle + \beta'\,|b_1\rangle)_B|c_{k\oplus s\oplus 1}\rangle_C, \tag{24}$$

which is independent of n. Although photons A, B, C in $\left|\Upsilon^{(S)}_{kms}\right\rangle_{ABC}$ are entirely separated in S-DOF, their entanglement in P-DOF remains intact. Dealing with the P-DOF is the job in the second step of the CRIO protocol. Including the P-DOF part the full state is $\left|\Omega^{(SP)}_{kms}\right\rangle_{ABC} = \left|\Upsilon^{(S)}_{kms}\right\rangle_{ABC}\left|Q^{(P)}\right\rangle_{ABC}$, i.e.,

$$\begin{aligned}\left|\Omega^{(SP)}_{kms}\right\rangle_{ABC} &= |a_{k\oplus m\oplus 1}\rangle_A(\alpha'\,|b_0\rangle + \beta'\,|b_1\rangle)_B|c_{k\oplus s\oplus 1}\rangle_C\left(|H\rangle_A|H\rangle_B|H\rangle_C\right.\\ &\quad\left. + |V\rangle_A|V\rangle_B|V\rangle_C\right).\end{aligned} \tag{25}$$

Bob starts the second step by placing a half-wave plate (HWP) on path b_0 to flip the polarization ($|H\rangle_B \leftrightarrows |V\rangle_B$) followed by mixing $|b_0\rangle_B$ and $|b_1\rangle_B$ on a BBS. These operations transform $\left|\Omega^{(SP)}_{kms}\right\rangle_{ABC}$ to

$$\begin{aligned}\left|\Phi^{(SP)}_{kms}\right\rangle_{ABC} &= |a_{k\oplus m\oplus 1}\rangle_A\left[|V, b_1\rangle_B(\alpha'|H\rangle_A|H\rangle_C + \beta'|V\rangle_A|V\rangle_C)\right.\\ &\quad + |V, b_0\rangle_B(\alpha'|H\rangle_A|H\rangle_C - \beta'|V\rangle_A|V\rangle_C)\\ &\quad + |H, b_1\rangle_B(\beta'|H\rangle_A|H\rangle_C + \alpha'|V\rangle_A|V\rangle_C)\\ &\quad \left.- |H, b_0\rangle_B(\beta'|H\rangle_A|H\rangle_C - \alpha'|V\rangle_A|V\rangle_C)\right]|c_{k\oplus s\oplus 1}\rangle_C.\end{aligned} \tag{26}$$

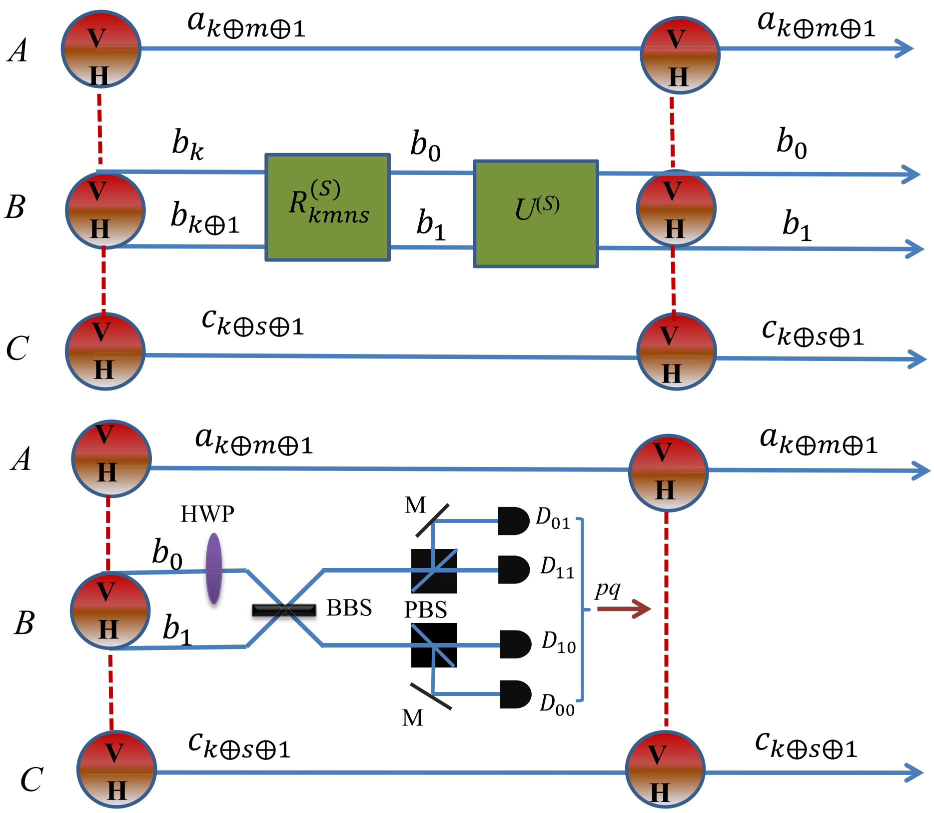

Figure 3. Bob's operations in the (top) first and (bottom) second steps of CRIO on photon state in S-DOF. The operators $R^{(S)}_{kmns}$ and $U^{(S)}$ are defined in equations (22) and (3). HWP is half-wave plate, PBS polarization beam splitter, M mirror and D_{pq} photodetectors.

The coefficients α', β' have 'hopped' from the state of photon B in S-DOF to the entangled states of photons A and C in P-DOF. To pick up a certain state of photons A and C, Bob measures photon B in the basis $\{|V, b_1\rangle_B, |V, b_0\rangle_B, |H, b_1\rangle_B, \ |H, b_0\rangle_B\}$ by using on each of the two paths b_0 and b_1 a polarization beam splitter (PBS) behind which there are four photodetectors D_{00}, D_{01}, D_{10} and D_{11}. Since PBS transmits horizontally polarized photon and reflects vertically polarized photon, one of the photodetectors may fire. If D_{00}, D_{01}, D_{10} or D_{11} fires, Bob respectively reveals two cbits $pq = 00, 01, 10$ or 11. After a fire of a photodetector photon B disappears but photons A and C survive in a P-DOF entangled state

$$\left|\Phi^{(SP)}_{kmspq}\right\rangle_{AC} = \begin{cases} (\alpha'|H\rangle_A|H\rangle_C + \beta'|V\rangle_A|V\rangle_C)|a_{k\oplus m\oplus 1}\rangle_A|c_{k\oplus s\oplus 1}\rangle_C, \ pq = 00 \\ (\alpha'|H\rangle_A|H\rangle_C - \beta'|V\rangle_A|V\rangle_C)|a_{k\oplus m\oplus 1}\rangle_A|c_{k\oplus s\oplus 1}\rangle_C, \ pq = 01 \\ (\alpha'|V\rangle_A|V\rangle_C + \beta'|H\rangle_A|H\rangle_C)|a_{k\oplus m\oplus 1}\rangle_A|c_{k\oplus s\oplus 1}\rangle_C, \ pq = 10 \\ (\alpha'|V\rangle_A|V\rangle_C - \beta'|H\rangle_A|H\rangle_C)|a_{k\oplus m\oplus 1}\rangle_A|c_{k\oplus s\oplus 1}\rangle_C, \ pq = 11. \end{cases} \tag{27}$$

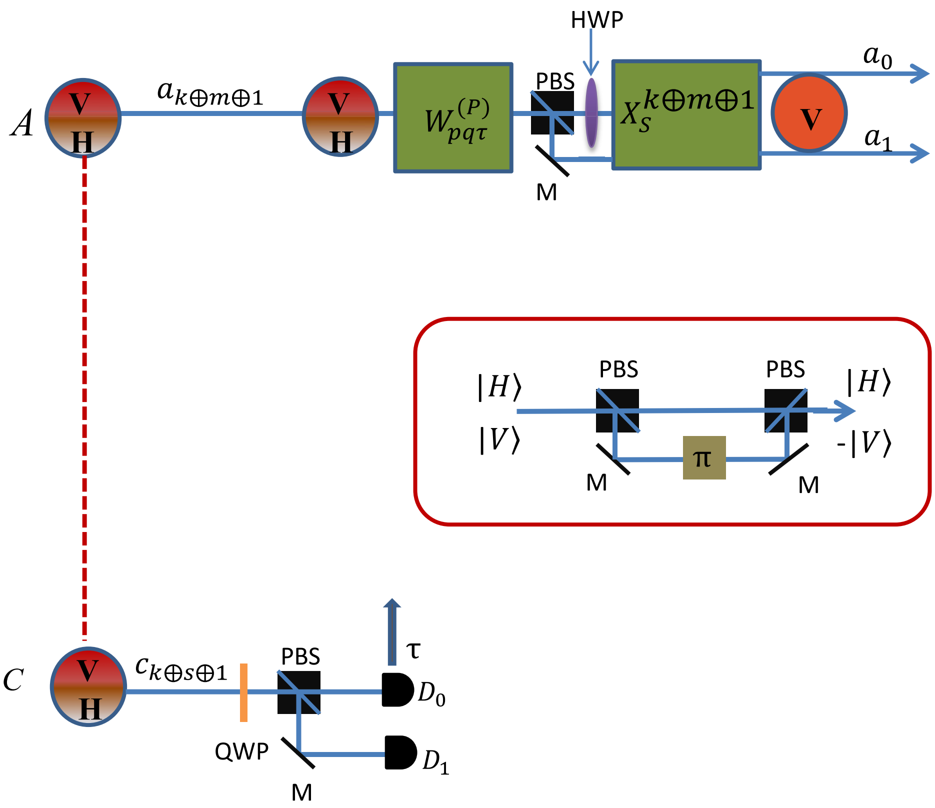

Figure 4. Charlie's and Alice's operations in the second step of CRIO on photon state in S-DOF. QWP is quarter-wave plate and $D_{0(1)}$ photodetector. $W_{pq\tau}^{(P)}$ is defined in equation (30) and X_S path-flip operator. The inset, which contains two PBSs, two Ms and one π-phase-shifter, is the construction of $Z_P = |H\rangle\langle H| - |V\rangle\langle V|$.

In the second step Charlie again demonstrates her power as a controller by executing what is shown in figure 4. She places a quarter-wave plate (QWP) on path $c_{k\oplus s\oplus 1}$ to rotate the polarization state from $|H\rangle_C$ ($|V\rangle_C$) to $|H\rangle_C + |V\rangle_C$ ($|H\rangle_C - |V\rangle_C$), then lets photon C pass through a PBS. By doing so, $\left|\Phi_{kmspq}^{(SP)}\right\rangle_{AC}$ changes to

$$
|a_{k\oplus m\oplus 1}\rangle_A \begin{cases}
\left[\left(\alpha'|H\rangle + \beta'|V\rangle\right)_A |H, c_{k\oplus s\oplus 1}\rangle_C \right. \\
\left. + \left(\alpha'|H\rangle - \beta'|V\rangle\right)_A |V, c_{k\oplus s}\rangle_C\right] & \text{if } pq = 00 \\
\left[\left(\alpha'|H\rangle - \beta'|V\rangle_A\right) |H, c_{k\oplus s\oplus 1}\rangle_C \right. \\
\left. + \left(\alpha'|H\rangle + \beta'|V\rangle\right)_A |V, c_{k\oplus s}\rangle_C\right] & \text{if } pq = 01 \\
\left[\left(\alpha'|V\rangle + \beta'|H\rangle\right)_A |H, c_{k\oplus s\oplus 1}\rangle_C \right. \\
\left. - \left(\alpha'|V\rangle - \beta'|V\rangle\right)_A |V, c_{k\oplus s}\rangle_C\right] & \text{if } pq = 10 \\
\left[\left(\alpha'|V\rangle - \beta'|H\rangle\right)_A |H, c_{k\oplus s\oplus 1}\rangle_C \right. \\
\left. - \left(\alpha'|V\rangle + \beta'|V\rangle\right)_A |V, c_{k\oplus s}\rangle_C\right] & \text{if } pq = 11.
\end{cases} \tag{28}
$$

Behind the PBS two photodetectors D_0 and D_1 are arranged to detect photon C. The event that D_0 (D_1) clicks is revealed by broadcasting a cbit $\tau = 0(\tau = 1)$. When either photodetector clicks, photon C is destroyed while photon A is projected onto

$$\left|\Phi^{(SP)}_{kmspq\tau}\right\rangle_A = |a_{k\oplus m\oplus 1}\rangle_A \begin{cases} (\alpha' |H\rangle + \beta' |V\rangle)_A & \text{for } pq\tau = 000, 011 \\ (\alpha' |H\rangle - \beta' |V\rangle)_A & \text{for } pq\tau = 001, 010 \\ (\alpha' |V\rangle + \beta' |H\rangle)_A & \text{for } pq\tau = 100, 111 \\ (\alpha' |V\rangle - \beta' |H\rangle)_A & \text{for } pq\tau = 101, 110 \end{cases} \tag{29}$$

The last step lies on Alice's shoulder. This is the second time Alice shows up and this time she does the following operations (see also figure 4). As seen from equation (29), knowing $pq\tau$ Alice can apply the operator

$$W^{(P)}_{pq\tau} = Z_P^{q\oplus\tau} X_P^p \tag{30}$$

on photon A to obtain the state $(\alpha' |H\rangle + \beta' |V\rangle)_A |a_{k\oplus m\oplus 1}\rangle_A$. Note that in (30) $X_P = |V\rangle \langle H| + |H\rangle \langle V|$ is the polarization-flip operator which is actually a HWP, while $Z_P = |H\rangle \langle H| - |V\rangle \langle V|$ can be constructed by a combination of two PBSs and one π-phase-shifter arranged as in the inset of figure 4. In [66] $(\alpha' |H\rangle + \beta' |V\rangle)_A |a_{k\oplus m\oplus 1}\rangle_A$ was taken as the target state. However, this is state of a photon propagating along a single path and being superposed of $|H\rangle$ and $|V\rangle$, i.e., encoded in P-DOF, while the desired state $(\alpha' |a_0\rangle + \beta' |a_1\rangle)_A |V\rangle_A$ (see equation (6)) is state of a photon having V-polarization and propagating along two distinct paths $|a_0\rangle$ and $|a_1\rangle$, i.e., encoded in S-DOF. Transparently, $(\alpha' |H\rangle + \beta' |V\rangle)_A |a_{k\oplus m\oplus 1}\rangle_A$ and $(\alpha' |a_0\rangle + \beta' |a_1\rangle)_A |V\rangle_A$ are not identical, so Alice should find a way to convert $(\alpha' |H\rangle + \beta' |V\rangle)_A |a_{k\oplus m\oplus 1}\rangle_A$ to $(\alpha' |a_0\rangle + \beta' |a_1\rangle)_A |V\rangle_A$. This appears possible by Alice using one PBS, one HWP and the operator $X_S^{k\oplus m\oplus 1}$, as illustrated after $W^{(P)}_{pq\tau}$ in figure 4. Actually, after the PBS $(\alpha' |H\rangle + \beta' |V\rangle)_A |a_{k\oplus m\oplus 1}\rangle_A \rightarrow (\alpha' |H, a_{k\oplus m\oplus 1}\rangle + \beta' |V, a_{k\oplus m}\rangle)_A$, then after the HWP, which is placed on path $a_{k\oplus m\oplus 1}$, $(\alpha' |H, a_{k\oplus m\oplus 1}\rangle + \beta' |V, a_{k\oplus m}\rangle)_A \rightarrow (\alpha' |a_{k\oplus m\oplus 1}\rangle + \beta' |a_{k\oplus m}\rangle)_A |V\rangle_A$. And, finally, after $X_S^{k\oplus m\oplus 1}$, $(\alpha' |a_{k\oplus m\oplus 1}\rangle + \beta' |a_{k\oplus m}\rangle)_A |V\rangle_A \rightarrow (\alpha' |a_0\rangle + \beta' |a_1\rangle_A)_A |V\rangle_A \equiv U^{(S)} |\psi\rangle_A$, which is precisely the target state. Therefore, Bob successfully implements a general unknown operator $U^{(S)}$ on Alice's arbitrary unknown state $|\psi\rangle$ under control of Charlie: the CRIO on state in S-DOF has been completed successfully.

3. CRIO on photon state in P-DOF

In this section we are concerned with Alice having a photon a which propagates along only one path (path x) but is encoded in P-DOF, i.e., her state is of the form

$$|\phi\rangle_a = \left|\psi^{(P)}\right\rangle_a |x\rangle_a, \tag{31}$$

$$\left|\psi^{(P)}\right\rangle_a = (\gamma |H\rangle + \delta |V\rangle)_a, \tag{32}$$

with $|\gamma|^2 + |\delta|^2 = 1$. The apparatus of Bob is now capable of executing a general unitary operator

$$U^{(P)} = \begin{pmatrix} \zeta & \eta \\ -\eta^* & \zeta^* \end{pmatrix} \tag{33}$$

on $\left|\psi^{(P)}\right\rangle$ as

$$U^{(P)}\left|\psi^{(P)}\right\rangle=\gamma'\left|H\right\rangle+\delta'\left|V\right\rangle=\left|\psi'^{(P)}\right\rangle, \tag{34}$$

$$\gamma'=\gamma\zeta-\delta\eta^{*}, \delta'=\gamma\eta+\delta\zeta^{*}. \tag{35}$$

The second task of our concern is to design a protocol for CRIO on P-DOF state: Alice and Bob, under Charlie's control, should cooperate in order that upon completion of the protocol Alice will have at hand the state

$$U^{(P)}\left|\phi\right\rangle=\left|\psi'^{(P)}\right\rangle\left|x\right\rangle. \tag{36}$$

This task can also be fulfilled in two steps using the same hyperentangled GHZ state (7) assisted by cross-Kerr nonlinearities. The starting total state is

$$|\phi\rangle_a\left|Q^{(SP)}\right\rangle_{ABC}=\left|\Phi^{(P)}\right\rangle_{aABC}|x\rangle_a\left|Q^{(S)}\right\rangle_{ABC}, \tag{37}$$

$$\begin{aligned}\left|\Phi^{(P)}\right\rangle_{aABC}=\big(&\gamma|H\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C+\gamma|H\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C\\&+\delta|V\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C+\delta|V\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C\big).\end{aligned} \tag{38}$$

The first step manipulates only the P-DOF part $\left|\Phi^{(P)}\right\rangle_{aABC}$ so we put the S-DOF aside for a while. Alice starts by performing the operations sketched in figure 5. She prepares a CS $|z\rangle_d$ and turns on the cross-Kerr interactions $K_{xd}^{(H)}(-\theta)$, $K_{a_0d}^{(H)}(\theta)$ and $K_{a_1d}^{(H)}(\theta)$. The label (H) in $K_{xd}^{(H)}$, $K_{a_0d}^{(H)}$ and $K_{a_1d}^{(H)}$ indicates that only the $|H\rangle$-component of the photon participates in the nonlinear interaction (see the inset of figure 6). The interactions $K_{xd}^{(H)}(-\theta)$, $K_{a_0d}^{(H)}(\theta)$ and $K_{a_1d}^{(H)}(\theta)$ bring $\left|\Phi^{(P)}\right\rangle_{aABC}|z\rangle_d$ to

$$\begin{aligned}&(\gamma|H\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C+\delta|V\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C)|z\rangle_d\\&\quad+\gamma|H\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C\left|z\,\mathrm{e}^{-\mathrm{i}\theta}\right\rangle_d+\delta|V\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_d.\end{aligned} \tag{39}$$

The X-quadrature of the CS is then measured. When an outcome $k=0$ or $k=1$ corresponding to finding $|z\rangle_d$ or $\left|z\mathrm{e}^{\pm\mathrm{i}\theta}\right\rangle_d$ occurs photon a gets entangled with respect to P-DOF with the others and their state reads

$$\left|\Gamma_k^{(P)}\right\rangle_{aABC}=\begin{cases}\gamma|H\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C+\delta|V\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C & \text{if } k=0\\ \gamma|H\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C+\delta|V\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C & \text{if } k=1\end{cases} \tag{40}$$

Alice goes on by putting a QWP on each of path x of photon a and paths a_0 and a_1 of photons A, then, after the two photons have passed through the QWPs, she lets their $|H\rangle$-components interact with a CS $|z\rangle_e$ via the cross-Kerr interactions $K_{xe}^{(H)}(\mu)$, $K_{a_0e}^{(H)}(\nu)$ and $K_{a_1e}^{(H)}(\nu)$, thus casting $\left|\Gamma_k^{(P)}\right\rangle_{aABC}|z\rangle_e$ to

$$\begin{aligned}\left|\Theta_0^{(P)}\right\rangle_{aABCe}=&|H\rangle_a|H\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}(\mu+\nu)}\right\rangle_e(\gamma|H\rangle_B|H\rangle_C+\delta|V\rangle_B|V\rangle_C)\\&+|H\rangle_a|V\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}\mu}\right\rangle_e(\gamma|H\rangle_B|H\rangle_C-\delta|V\rangle_B|V\rangle_C)\\&+|V\rangle_a|H\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}\nu}\right\rangle_e(\gamma|H\rangle_B|H\rangle_C-\delta|V\rangle_B|V\rangle_C)\\&+|V\rangle_a|V\rangle_A|z\rangle_e(\gamma|H\rangle_B|H\rangle_C+\delta|V\rangle_B|V\rangle_C)\end{aligned} \tag{41}$$

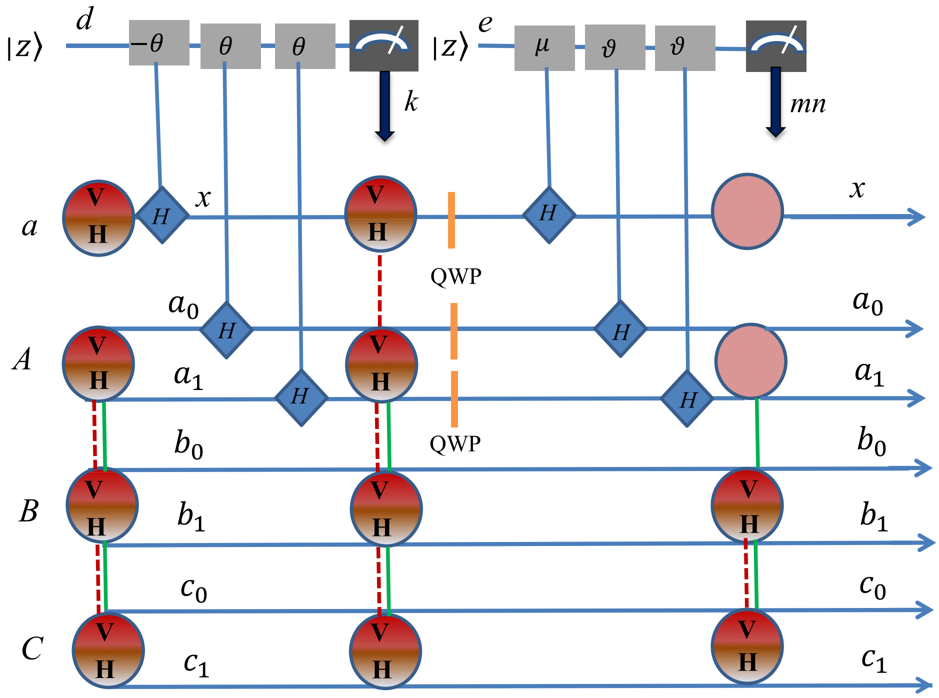

Figure 5. Alice's operations in the first step of CRIO on photon state in P-DOF. A circle without any letters inside represents a photon that is in either $|H\rangle$ or $|V\rangle$. The cross-Kerr interaction between an $|H\rangle$-part of the photon and a CS is detailed in the inset of figure 6.

for $k = 0$ and

$$\begin{aligned}\left|\Theta_1^{(P)}\right\rangle_{aABCe} = &|H\rangle_a|H\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}(\mu+\nu)}\right\rangle_e(\gamma|V\rangle_B|V\rangle_C + \delta|H\rangle_B|H\rangle_C)\\ &- |H\rangle_a|V\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}\mu}\right\rangle_e(\gamma|V\rangle_B|V\rangle_C - \delta|H\rangle_B|H\rangle_C)\\ &+ |V\rangle_a|H\rangle_A\left|z\,\mathrm{e}^{\mathrm{i}\nu}\right\rangle_e(\gamma|V\rangle_B|V\rangle_C - \delta|H\rangle_B|H\rangle_C)\\ &- |V\rangle_a|V\rangle_A|z\rangle_e(\gamma|V\rangle_B|V\rangle_C + \delta|H\rangle_B|H\rangle_C)\end{aligned} \tag{42}$$

for $k = 1$. The X-quadrature measurement of the CS yields four possible outcomes $mn = 00$, $01, 10$ or 11 if $\left|z\,\mathrm{e}^{\mathrm{i}(\mu+\nu)}\right\rangle_e$, $\left|z\,\mathrm{e}^{\mathrm{i}\mu}\right\rangle_e$, $\left|z\,\mathrm{e}^{\mathrm{i}\nu}\right\rangle_e$ or $|z\rangle_e$ is found, respectively. The collapsed state $\left|\Lambda_{kmn}^{(P)}\right\rangle_{aABC}$ depends on both k and mn as presented in table 2.

As noticed from table 2, after Alice's operations, photon a is no longer in a superposition of $|H\rangle_a$ and $|V\rangle_a$ and appears in either $|H\rangle_a$ or $|V\rangle_a$. That is, it is separable and ceases its role from now on. As for photon A, it also becomes either horizontally or vertically polarized, i.e., no longer entangled in P-DOF with photons B and C. The P-DOF entanglement survives only between photons B and C, while the S-DOF entanglement between photons A, B and C remains as before.

To control the task in the first step, Charlie puts a QWP on path c_0 and another one on path c_1 of photon C, then turns on the interactions $K_{c_0g}^{(H)}(\theta)$ and $K_{c_1g}^{(H)}(\theta)$ between photon C and a CS $|z\rangle_g$ and measures X-quadrature of the CS to see whether $\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_g$ or $|z\rangle_g$ is found. In

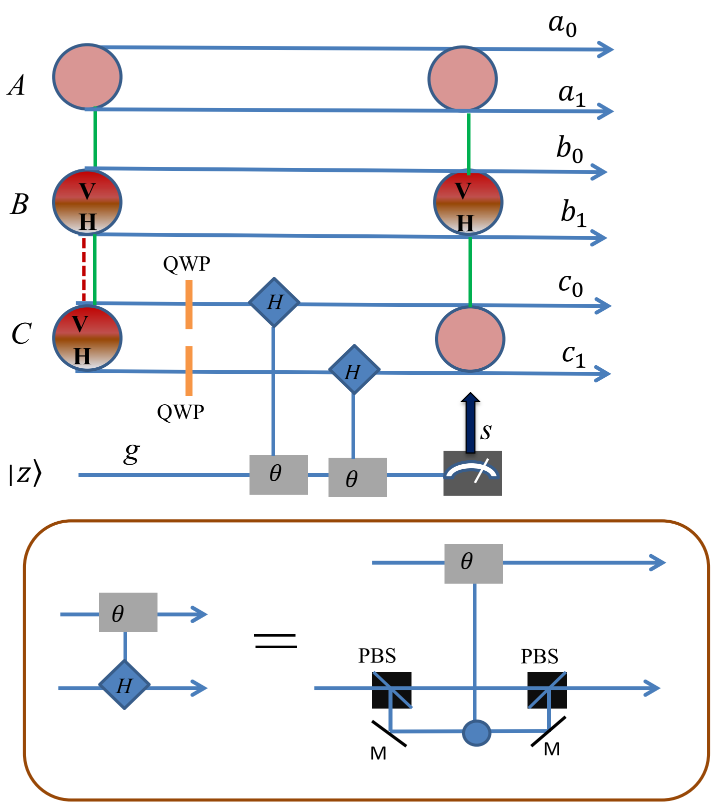

Figure 6. Charlie's operations in the first step of CRIO on photon state in P-DOF. The inset details the cross-Kerr interaction between an $|H\rangle$-part of a photon and a coherent state.

Table 2. The collapsed state $\left|\Lambda_{kmn}^{(P)}\right\rangle_{aABC}$ conditioned on Alice's measurement outcomes *kmn*.

kmn	$\left\|\Lambda_{kmn}^{(P)}\right\rangle_{aABC}$
000	$\|H\rangle_a\|H\rangle_A(\gamma\|H\rangle_B\|H\rangle_C+\delta\|V\rangle_B\|V\rangle_C)$
001	$\|H\rangle_a\|V\rangle_A(\gamma\|H\rangle_B\|H\rangle_C-\delta\|V\rangle_B\|V\rangle_C)$
010	$\|V\rangle_a\|H\rangle_A(\gamma\|H\rangle_B\|H\rangle_C-\delta\|V\rangle_B\|V\rangle_C)]$
011	$\|V\rangle_a\|V\rangle_A(\gamma\|H\rangle_B\|H\rangle_C+\delta\|V\rangle_B\|V\rangle_C)$
100	$\|H\rangle_a\|H\rangle_A(\gamma\|V\rangle_B\|V\rangle_C+\delta\|H\rangle_B\|H\rangle_C)$
101	$\|H\rangle_a\|V\rangle_A(\gamma\|V\rangle_B\|V\rangle_C-\delta\|H\rangle_B\|H\rangle_C)$
110	$\|V\rangle_a\|H\rangle_A(\gamma\|V\rangle_B\|V\rangle_C-\delta\|H\rangle_B\|H\rangle_C)$
111	$\|V\rangle_a\|V\rangle_A(\gamma\|V\rangle_B\|V\rangle_C+\delta\|H\rangle_B\|H\rangle_C)$

the former event she announces a cbit $s=0$ but in the latter event $s=1$. For whatever the outcome s the P-DOF entanglement between photons B and C dies and photon C turns out to be either horizontally or vertically polarized while photon B catches the coefficients γ, δ being in a polarization-superposed state. The resulting state $\left|\Sigma_{kmns}^{(P)}\right\rangle_{ABC}$ exhibits only S-DOF

Table 3. States $\left|\Phi^{(P)}_{kmns}\right\rangle_{ABC}$ of photons A, B and C versus Alice's and Charlie's measurement outcomes $kmns$.

$kmns$	$\left\|\Sigma^{(P)}_{kmns}\right\rangle_{ABC}$	$kmns$	$\left\|\Sigma^{(P)}_{kmns}\right\rangle_{ABC}$
0000	$\|H\rangle_A(\gamma\|H\rangle+\delta\|V\rangle)_B\|H\rangle_C$	1000	$\|H\rangle_A(\gamma\|V\rangle+\delta\|H\rangle)_B\|H\rangle_C$
0001	$\|H\rangle_A(\gamma\|H\rangle-\delta\|V\rangle)_B\|V\rangle_C$	1001	$\|H\rangle_A(\gamma\|V\rangle-\delta\|H\rangle)_B\|V\rangle_C$
0010	$\|V\rangle_A(\gamma\|H\rangle-\delta\|V\rangle)_B\|H\rangle_C$	1010	$\|V\rangle_A(\gamma\|V\rangle-\delta\|H\rangle)_B\|H\rangle_C$
0011	$\|V\rangle_A(\gamma\|H\rangle+\delta\|V\rangle)_B)\|V\rangle_C$	1011	$\|V\rangle_A(\gamma\|V\rangle+\delta\|H\rangle)_B\|V\rangle_C$
0100	$\|H\rangle_A(\gamma\|H\rangle-\delta\|V\rangle)_B\|H\rangle_C$	1100	$\|H\rangle_A(\gamma\|V\rangle-\delta\|H\rangle)_B\|H\rangle_C$
0101	$\|H\rangle_A(\gamma\|H\rangle+\delta\|V\rangle)_B\|V\rangle_C$	1101	$\|H\rangle_A(\gamma\|V\rangle+\delta\|H\rangle)_B\|V\rangle_C$
0110	$\|V\rangle_A(\gamma\|H\rangle+\delta\|V\rangle)_B\|H\rangle_C$	1110	$\|V\rangle_A(\gamma\|V\rangle+\delta\|H\rangle)_B\|H\rangle_C$
0111	$\|V\rangle_A(\gamma\|H\rangle-\delta\|V\rangle)_B\|V\rangle_C$	1111	$\|V\rangle_A(\gamma\|V\rangle-\delta\|H\rangle)_B\|V\rangle_C$

entanglement between the three photons. Table 3 lists $\left|\Sigma^{(P)}_{kmns}\right\rangle_{ABC}$ subjected to kmn (Alice's measurement outcomes) and s (Charlie's measurement outcome).

Now the second step starts and the S-DOF part $\left|Q^{(S)}\right\rangle_{ABC}$ should be supplemented to $\left|\Sigma^{(P)}_{kmns}\right\rangle_{ABC}$ to form the full state $\left|\Pi^{(SP)}_{kmns}\right\rangle_{ABC} = \left|\Sigma^{(P)}_{kmns}\right\rangle_{ABC}\left|Q^{(S)}\right\rangle_{ABC}$. Bob's operations in this step are displayed in figure 7. The content of table 3 signals that Bob can transform the state of photon B in $\left|\Pi^{(SP)}_{kmns}\right\rangle_{ABC}$ to $(\gamma\,|H\rangle+\delta\,|V\rangle)_B$ by applying on it the operator

$$R^{(P)}_{kmns} = Z_P^{m\oplus n\oplus s}X_P^k. \tag{43}$$

After doing so he uses his apparatus to implement $U^{(P)}$ in equation (33) on photon B to obtain

$$\left|\psi'^{(P)}\right\rangle_B = (\gamma'\,|H\rangle+\delta'\,|V\rangle)_B, \tag{44}$$

with γ' and δ' given in equation (35). Next, he makes a path-exchanging operation (PE) as constructed in the inset of figure 7 which exchanges the path of photon B when it is in horizontal polarization state, i.e.,

$$PE\left\{\begin{array}{l}|H,b_j\rangle_B\\ |V,b_j\rangle_B\end{array}\right. = \left\{\begin{array}{l}|H,b_{j\oplus 1}\rangle_B\\ |V,b_j\rangle_B.\end{array}\right. \tag{45}$$

The state of photons A, B, C now becomes dependent only on n, s and has the form

$$\begin{aligned}\left|\Upsilon^{(SP)}_{ns}\right\rangle_{ABC} = |f_n(H,V)\rangle_A\,\big[&|a_0\rangle_A(\gamma'\,|H,b_1\rangle+\delta'\,|V,b_0\rangle)_B|c_0\rangle_C\\ &+|a_1\rangle_A(\gamma'\,|H,b_0\rangle+\delta'\,|V,b_1\rangle)_B|c_1\rangle_C\big]\,|f_s(H,V)\rangle_C,\end{aligned} \tag{46}$$

$$|f_n(H,V)\rangle = \left|H^{n\oplus 1}V^n\right\rangle, \tag{47}$$

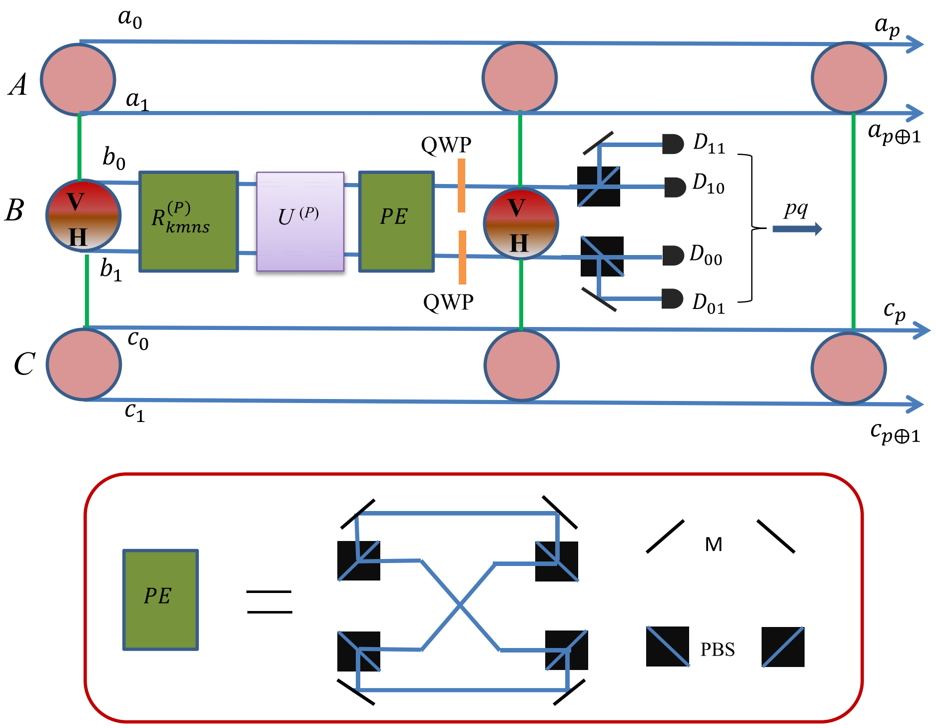

Figure 7. Bob's operations for CRIO on photon state in P-DOF. The operators $R^{(P)}_{kmns}$ and $U^{(P)}$ are defined in equations (43) and (33), while PE is the path-exchanger constructed as in the inset.

i.e., $|f_0(H,V)\rangle = |H\rangle$ and $|f_1(H,V)\rangle = |V\rangle$. Bob then places two QWPs, one on path b_0 and the other on path b_1 bringing $\left|\Upsilon^{(SP)}_{ns}\right\rangle_{ABC}$ to

$$\begin{aligned}\left|\Omega^{(SP)}_{ns}\right\rangle_{ABC} = |f_n(H,V)\rangle_A \big[&|H,b_1\rangle_B \left(\gamma'|a_0\rangle_A|c_0\rangle_C + \delta'|a_1\rangle_A|c_1\rangle_C\right)\\ &+ |V,b_1\rangle_B \left(\gamma'|a_0\rangle_A|c_0\rangle_C - \delta'|a_1\rangle_A|c_1\rangle_C\right)\\ &+ |H,b_0\rangle_B \left(\gamma'|a_1\rangle_A|c_1\rangle_C + \delta'|a_0\rangle_A|c_0\rangle_C\right)\\ &+ |V,b_0\rangle_B \left(\gamma'|a_1\rangle_A|c_1\rangle_C - \delta'|a_0\rangle_A|c_0\rangle_C\right)\big]\,|f_s(H,V)\rangle_C. \qquad (48)\end{aligned}$$

The last thing for Bob to do is to detect photon B by four photodetectors D_{00}, D_{01}, D_{10} and D_{11} arranged after two PBSs as in figure 7. If D_{00} (D_{01}, D_{10} or D_{11}) clicks, this means that state $|H,b_1\rangle_B$ ($|V,b_1\rangle_B$, $|H,b_0\rangle_B$ or $|V,b_0\rangle_B$) is found and that event is labeled by two cbits $pq = 00$ (01, 10 or 11). Photon B is destroyed while photons A and C remain entangled in S-DOF as

$$\begin{aligned}\left|\Psi^{(SP)}_{nspq}\right\rangle_{AC} = &|f_n(H,V)\rangle_A[\gamma'|a_p\rangle_A|c_p\rangle_C + (-1)^q\delta'|a_{p\oplus1}\rangle_A|c_{p\oplus1}\rangle_C]\\ &\times|f_s(H,V)\rangle_C. \qquad (49)\end{aligned}$$

Now Charlie joins to control the task the second time. She mixes $|c_p\rangle_C$ and $|c_{p\oplus1}\rangle_C$ on a BBS then detects photon C by two photodetectors D_0 and D_1. A click of D_0 or D_1, labeled by

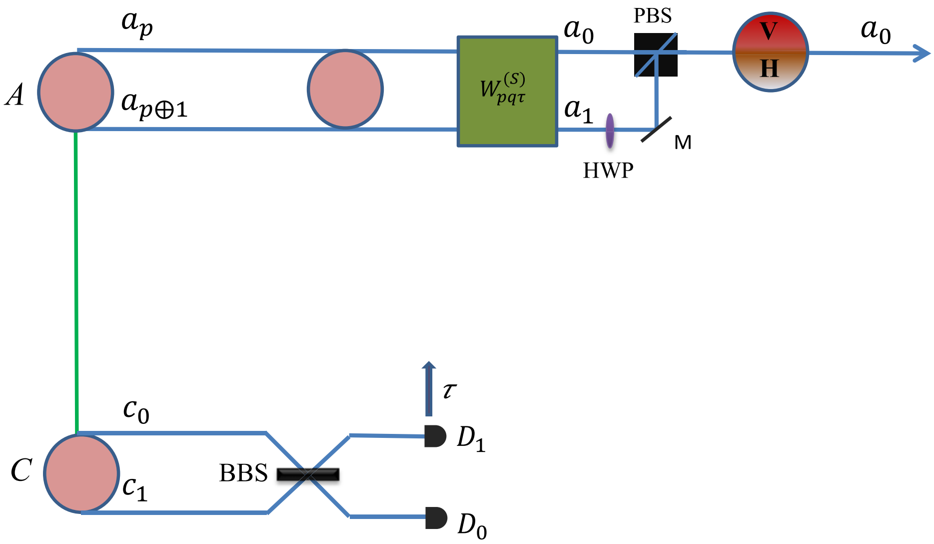

Figure 8. Charlie's and Alice's operations in the second step of CRIO on photon state in P-DOF. $W^{(S)}_{pq\tau}$ is the operator defined in equation (51). This figure is plotted for the case $n = 0$ for which the HWP is placed on path a_1 (otherwise, the HWP should be on path a_0).

a cbit $\tau = 0$ or $\tau = 1$, destroys photon C and projects photon A onto

$$\left|\Psi^{(SP)}_{npq\tau}\right\rangle_A = |f_n(H,V)\rangle_A(\gamma'\,|a_p\rangle + (-1)^{p\oplus q\oplus\tau}\delta'\,|a_{p\oplus1}\rangle)_A. \tag{50}$$

Finally, depending on the outcomes $pq\tau$, Alice applies on photon A the operator

$$W^{(S)}_{pq\tau} = Z_S^{p\oplus q\oplus\tau}X_S^p \tag{51}$$

to have it in state $|f_n(H,V)\rangle_A(\gamma'\,|a_0\rangle + \delta'\,|a_1\rangle)_A$, which has not yet been the desired state $(\gamma'\,|H\rangle + \delta'\,|V\rangle)_A|a_0\rangle_A$. To transform $|f_n(H,V)\rangle_A(\gamma'\,|a_0\rangle + \delta'\,|a_1\rangle)_A$ to $(\gamma'\,|H\rangle + \delta'\,|V\rangle)_A|a_0\rangle_A$ Alice puts a HWP on path $a_{n\oplus1}$, then combines the two paths into a PBS. Photon A then goes out from the PBS in state $(\gamma'\,|H,a_0\rangle + \delta'\,|V,a_0\rangle)_A = U^{(P)}|\phi\rangle_A$, which is the target state now traveling along path a_0. The above-described operations of Charlie and Alice are shown in figure 8 (for $n = 0$).

4. CRISO on photon state in both S-DOF and P-DOF

CRIO on photon states encoded either in S-DOF or in P-DOF was studied in two preceding sections, each employs one hyperentangled GHZ state. This section poses a third task which is controlled remote implementation of a subset of operators (CRISO) on photon state encoded at the same time in both S-DOF and P-DOF. Suppose Alice has photon a in state of the form

$$\left|\psi^{(SP)}\right\rangle_a = \left|\psi^{(S)}\right\rangle_a\left|\psi^{(P)}\right\rangle_a = (\alpha|x_0\rangle_a + \beta|x_1\rangle_a)(\gamma|H\rangle_a + \delta|V\rangle_a), \tag{52}$$

with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$. Such state, though being single-photon, is in fact worth two qubits because it exists at the same time in two orthogonal polarization states and also at the same time propagates along two distinct spatial paths. Can Bob, under Charlie's control, implement general operators $U^{(S)}$ acting in S-DOF and $U^{(P)}$ acting in P-DOF on Alice's state $\left|\psi^{(SP)}\right\rangle$ so that Alice's target state will be of the form $(U^{(S)}\left|\psi^{(S)}\right\rangle)(U^{(P)}\left|\psi^{(P)}\right\rangle) = \left|\psi'^{(S)}\right\rangle\left|\psi'^{(P)}\right\rangle$ with $\left|\psi'^{(S)}\right\rangle$ and $\left|\psi'^{(P)}\right\rangle$ given in equations (4) and (34)? A straightforward way to do that is to sequentially resort to the protocol in section 2 to implement $U^{(S)}$ and to the protocol in section 3 to implement $U^{(P)}$. Such method deems awkward since it is composed of two independent protocols consuming two separate hyperentangled GHZ states. Hence, a question arises: can both $U^{(S)}$ and $U^{(P)}$ be controllably and parallelly implemented on the same remote state (52) using just one hyperentangled GHZ state? The answer is that this is impossible if $U^{(S)}$ and $U^{(P)}$ are of the general forms as in equations (3) and (33). However, if the to-be-implemented operators belong to some particular subset of operators, then the above-said task would be possible. We shall propose a protocol for CRISO in which the operators belong to the following subset of unitary operators $U_m^{(S)} \in \{U_0^{(S)}, U_1^{(S)}\}$ and $U_n^{(P)} \in \{U_0^{(P)}, U_1^{(P)}\}$ with

$$U_0^{(S)} = \begin{pmatrix} s_0 & 0 \\ 0 & s_0^* \end{pmatrix}, U_1^{(S)} = \begin{pmatrix} 0 & s_1 \\ -s_1^* & 0 \end{pmatrix}, \tag{53}$$

$$U_0^{(P)} = \begin{pmatrix} p_0 & 0 \\ 0 & p_0^* \end{pmatrix}, U_1^{(P)} = \begin{pmatrix} 0 & p_1 \\ -p_1^* & 0 \end{pmatrix}. \tag{54}$$

The specific property of the above subset of operators is that their action on one of the two basis states (i.e., $|x_0\rangle$ or $|x_1\rangle$ in S-DOF and $|H\rangle$ or $|V\rangle$ in P-DOF) does not superimpose the two. In mathematical language,

$$U_m^{(S)}|x_0\rangle = s_m|x_m\rangle, \; U_m^{(S)}|x_1\rangle = (-1)^m s_m^*|x_{m\oplus 1}\rangle, \tag{55}$$

$$U_n^{(P)}|H\rangle = p_n|f_n(H,V)\rangle, \; U_n^{(P)}|V\rangle = (-1)^n p_n^*|f_{n\oplus 1}(H,V)\rangle, \tag{56}$$

with $|f_n(H,V)\rangle$ defined in equation (47). It is the properties (55) and (56) that make possible the CRISO with $U_m^{(S)}$ and $U_n^{(P)}$ on $\left|\psi^{(SP)}\right\rangle$ consuming only one hyperentangled GHZ state $\left|Q^{(SP)}\right\rangle_{ABC}$.

The CRISO protocol begins with the total state

$$\left|\psi^{(SP)}\right\rangle_a\left|Q^{(SP)}\right\rangle_{ABC} = \left|\Phi^{(S)}\right\rangle_{aABC}\left|\Phi^{(P)}\right\rangle_{aABC}, \tag{57}$$

with $\left|\psi^{(SP)}\right\rangle_a$, $\left|Q^{(SP)}\right\rangle_{ABC}$, $\left|\Phi^{(S)}\right\rangle_{aABC}$ and $\left|\Phi^{(P)}\right\rangle_{aABC}$ given in equations (52), (7), (11) and (38). The state Alice wishes to have is

$$\left|\psi'^{(SP)}_{mn}\right\rangle_A = \left|\psi'^{(S)}_m\right\rangle_A\left|\psi'^{(P)}_n\right\rangle_A, \tag{58}$$

with $\left|\psi'^{(S)}_m\right\rangle = U_m^{(S)}\left|\psi^{(S)}\right\rangle$ and $\left|\psi'^{(P)}_n\right\rangle = U_n^{(P)}\left|\psi^{(P)}\right\rangle$ defined in equations (6) and (36).

This CRISO protocol is also two-step. In the first step only $\left|\Phi^{(S)}\right\rangle_{aABC}$ plays a role so $\left|\Phi^{(P)}\right\rangle_{aABC}$ is out of consideration until the second step. In the first step Alice performs the same operations with photons a and A as in section 2 to obtain the state (13). But here, for the CRISO, Bob joins right after Alice announced the outcome k of her X-quadrature measurement by applying X_S^k on photon B to modify (13) to

$$\left|G_k^{(S)}\right\rangle_{aABC} = \alpha|x_0\rangle_a|a_k\rangle_A|b_0\rangle_B|c_k\rangle_C + \beta|x_1\rangle_a|a_{k+1}\rangle_A|b_1\rangle_B|c_{k+1}\rangle_C. \tag{59}$$

Then, Bob implements $U_m^{(S)}$ with $m \in \{0, 1\}$ on photon B to transform $\left|G_k^{(S)}\right\rangle_{aABC}$ to

$$\left|G_{mk}^{(S)}\right\rangle_{aABC} = |x_0\rangle_a(\alpha X_S^{m\oplus k} U_m^{(S)}|a_0\rangle_A)|b_m\rangle_B|c_k\rangle_C + |x_1\rangle_a(\beta X_S^{m\oplus k} U_m^{(S)}|a_1\rangle_A)|b_{m\oplus 1}\rangle_B|c_{k\oplus 1}\rangle_C, \tag{60}$$

by virtue of the properties (55). Next, Alice and Bob proceed as follows. Alice mixes $|x_0\rangle_a$ and $|x_1\rangle_a$ on a BBS then switches on cross-Kerr interaction $K_{x_0 e}(\theta)$ between $|x_0\rangle_a$ and a CS $|z\rangle_e$, while Bob mixes $|b_m\rangle_B$ and $|b_{m\oplus 1}\rangle_B$ on another BBS then switches on cross-Kerr interaction $K_{b_{m\oplus 1} g}(\theta)$. Those operations realize the transition

$$\begin{aligned}
&\left|G_{mk}^{(S)}\right\rangle_{aABC}|z\rangle_e|z\rangle_g \\
&\to (-1)^m|x_0\rangle_a\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_e|b_{m\oplus 1}\rangle_B\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_g\left[(X_S^{m\oplus k}U_m^{(S)}\alpha|a_0\rangle_A)|c_k\rangle_C\right.\\
&\left.+(-1)^m(X_S^{m\oplus k}U_m^{(S)}\beta|a_1\rangle_A)|c_{k\oplus 1}\rangle_C\right]\\
&+|x_0\rangle_a\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_e|b_m\rangle_B|z\rangle_g\left[(X_S^{m\oplus k}U_m^{(S)}\alpha|a_0\rangle_A)\right.\\
&\left.-(-1)^m(X_S^{m\oplus k}U_m^{(S)}\beta|a_1\rangle_A)|c_{k\oplus 1}\rangle_C\right]\\
&+(-1)^m|x_1\rangle_a|z\rangle_e|b_{m\oplus 1}\rangle_B\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_g\left[(X_S^{m\oplus k}U_m^{(S)}\alpha|a_0\rangle_A)|c_k\rangle_C\right.\\
&\left.-(-1)^m(X_S^{m\oplus k}U_m^{(S)}\beta|a_1\rangle_A)|c_{k\oplus 1}\rangle_C\right]\\
&+|x_1\rangle_a|z\rangle_e|b_m\rangle_B|z\rangle_g\left[(X_S^{m\oplus k}U_m^{(S)}\alpha|a_0\rangle_A)\right.\\
&\left.+(-1)^m(X_S^{m\oplus k}U_m^{(S)}\beta|a_1\rangle_A)|c_{k\oplus 1}\rangle_C\right].
\end{aligned} \tag{61}$$

After the cross-Kerr interaction Alice measures X-quadrature of her CS, while Bob does the same of his CS. The outcomes will be published as two cbits $rs = 00, 01, 10$ or 11, that correspond to Alice and Bob finding $|z\rangle_e|z\rangle_g$, $|z\rangle_e\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_g$, $\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_e|z\rangle_g$ or $\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_e\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_g$, respectively. For an outcome rs, state (61) collapses into

$$\left|G_{mkrs}^{(S)}\right\rangle_{aABC} = |x_{r\oplus 1}\rangle_a|b_{m\oplus s}\rangle_B\left[(\alpha X_S^{m\oplus k}U_m^{(S)}|a_0\rangle_A)|c_k\rangle_C + (-1)^{m\oplus r\oplus s}(\beta X_S^{m\oplus k}U_m^{(S)}|a_1\rangle_A)|c_{k\oplus 1}\rangle_C\right], \tag{62}$$

unveiling that photon a just propagates along one path, path $x_{r\oplus 1}$, and it is not at all entangled with the other photons neither in S-DOF nor in P-DOF, but unlike in the CRIO on state in S-DOF in section 2 where it gets out of the game, here photon a must be kept to help the implementation of $U_n^{(P)}$ in the next step when P-DOF will plays its role. As for photon B, it also propagates only along one path, path $b_{m\oplus s}$, eliminating its entanglement in S-DOF with photons A and C. Yet, the latter are still entangled with each other in S-DOF. Figure 9 depicts the above-described actions of Alice and Bob.

The first step of the CRISO is continued by Charlie who uses a BBS to superimpose $|c_k\rangle_C$ and $|c_{k\oplus 1}\rangle_C$, followed by letting $|c_{k\oplus 1}\rangle_C$ and CS $|z\rangle_h$ interact via $K_{c_{k\oplus 1}h}(\theta)$ and measuring the CS's X-quadrature to see whether $|z\rangle_h$ or $\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_h$ is found. If it is $\left|z\,\mathrm{e}^{\mathrm{i}\theta}\right\rangle_h$ ($|z\rangle_h$), Charlie

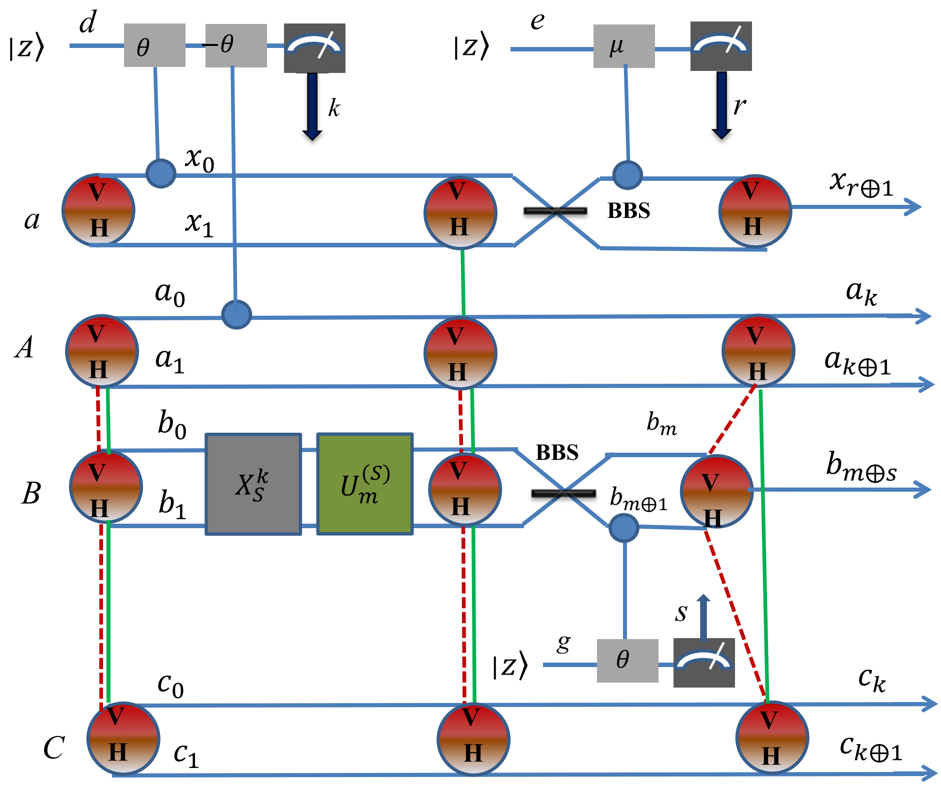

Figure 9. Alice's and Bob's operations in the first step of CRISO on photon state in both S-DOF and P-DOF. X_S is the path-flip operator and $U_m^{(S)}$ the subset of operators defined in equation (53).

broadcasts one cbit $\tau = 0(\tau = 1)$ in which case $\left|G_{mkrs}^{(S)}\right\rangle_{aABC}$ is projected onto

$$\left|G_{mkrs\tau}^{(S)}\right\rangle_{aABC} = |x_{r\oplus 1}\rangle_a |b_{m\oplus s}\rangle_B (X_S^{m\oplus k} Z_S^{m\oplus k\oplus r\oplus s\oplus \tau} U_m^{(S)} \left|\psi^{(S)}\right\rangle_A) |c_{k\oplus\tau\oplus 1}\rangle_C. \tag{63}$$

Obviously, entanglement of photons A and C in S-DOF disappears and Alice can apply on photon A the operator

$$F_{mkrs\tau}^{(S)} = Z_S^{m\oplus k\oplus r\oplus s\oplus \tau} X_S^{m\oplus k} \tag{64}$$

to have

$$\left|L_{mkrs\tau}^{(S)}\right\rangle_{aABC} = |x_{r\oplus 1}\rangle_a |b_{m\oplus s}\rangle_B (U_m^{(S)} \left|\psi^{(S)}\right\rangle_A) |c_{k\oplus\tau\oplus 1}\rangle_C. \tag{65}$$

The above Charlie's and Alice's operations are as in figure 10. Although the operator $U_m^{(S)}$ has successfully been implemented on $\left|\psi^{(S)}\right\rangle_A$, Bob still needs implementing $U_n^{(P)}$. The second step will do that via the P-DOF part $\left|\Phi^{(P)}\right\rangle_{aABC}$ in equation (38). Therefore, the total state from this

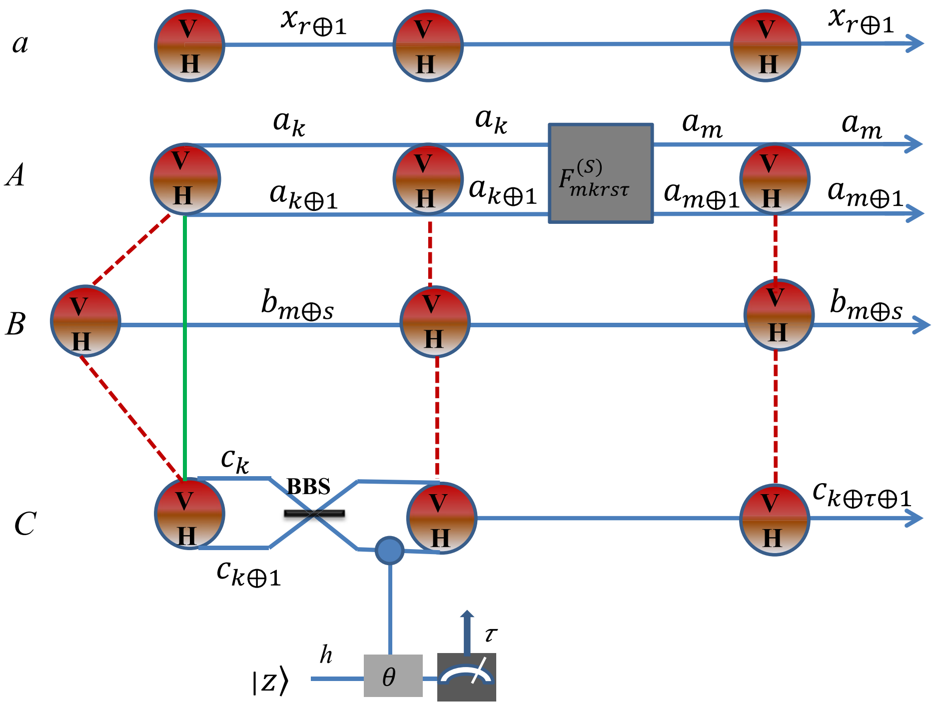

Figure 10. Charlie's and Alice's operations in the first step of CRISO on photon state in both S-DOF and P-DOF. The operator $F^{(S)}_{mkrs\tau}$ is defined in equation (64).

moment is $\left|L^{(SP)}_{mkrs\tau}\right\rangle_{aABC} = \left|L^{(S)}_{mkrs\tau}\right\rangle_{aABC}\left|\Phi^{(P)}\right\rangle_{aABC}$ which explicitly reads

$$\begin{aligned}\left|L^{(SP)}_{mkrs\tau}\right\rangle_{aABC} = \left|L^{(S)}_{mkrs\tau}\right\rangle_{aABC}&\left[\gamma|H\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C\right.\\ &+\gamma|H\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C+\delta|V\rangle_a|H\rangle_A|H\rangle_B|H\rangle_C\\ &\left.+\delta|V\rangle_a|V\rangle_A|V\rangle_B|V\rangle_C\right].\end{aligned}\tag{66}$$

Alice starts the second step by the operations shown in figure 11. Namely, she executes the cross-Kerr interactions $K^{(H)}_{x_{r\oplus 1}d}(-\theta)$, $K^{(H)}_{a_md}(\theta)$ and $K^{(H)}_{a_{m\oplus 1}d}(\theta)$ followed by measuring X-quadrature of the CS. If she finds $|z\rangle_d$ ($\left|z\,\mathrm{e}^{\pm i\theta}\right\rangle_d$) she publishes a cbit $k'=0(k'=1)$ for Bob to apply $X_P^{k'}$ and then $U_n^{(P)}$ on photon B. Thanks to the properties (56) the resulting state is

$$\begin{aligned}\left|L^{(SP)}_{mkrs\tau nk'}\right\rangle_{aABC} = \left|L^{(S)}_{mkrs\tau}\right\rangle_{aABC}&\left[\gamma p_n|H\rangle_a|f_{k'}(H,V)\rangle_A|f_n(H,V)\rangle_B\right.\\ &\times|f_{k'}(H,V)\rangle_C+\delta(-1)^n p_n^*|V\rangle_a|f_{k'\oplus 1}(H,V)\rangle_A\\ &\left.\times|f_{n\oplus 1}(H,V)\rangle_B|f_{k'\oplus 1}(H,V)\rangle_C\right].\end{aligned}\tag{67}$$

Next, Alice sends photon a through a QWP and a PBS behind which there are two photodetectors D_{A0} and D_{A1}. Similarly, Bob sends photon B through a QWP and a PBS behind which there are two photodetectors D_{B0} and D_{B1}. The event that D_{A0} (D_{A1}) clicks is announced as a cbit $r'=0(r'=1)$, while the event that D_{B0} (D_{B1}) clicks as a cbit $s'=0(s'=1)$. For any

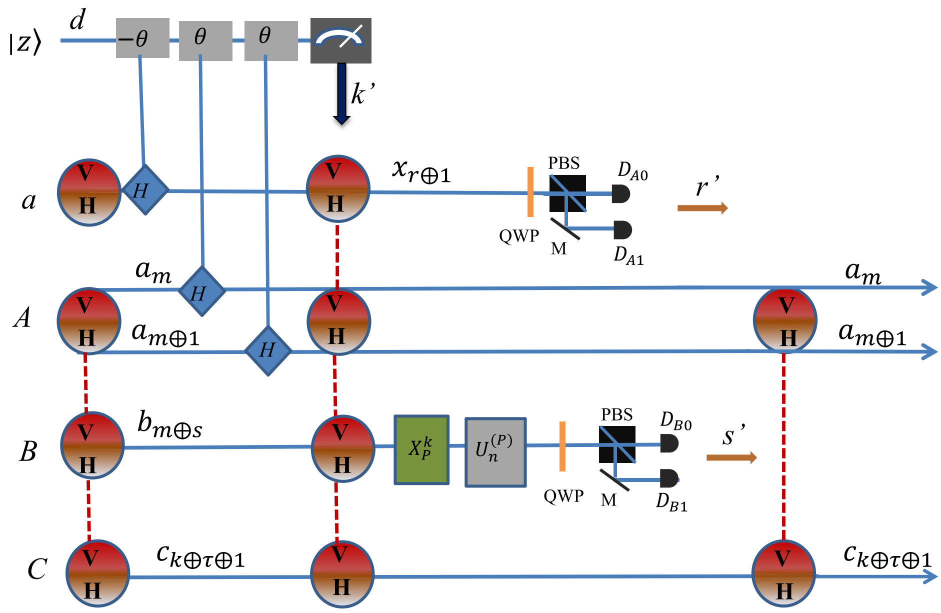

Figure 11. Alice's and Bob's operations in the second step of CRISO on photon state in both S-DOF and P-DOF. X_P is the polarization-flip operator and $U_n^{(P)}$ is defined in equation (54).

possible events $r's'$ photons a and B are annihilated and $\left|L^{(SP)}_{mkrs\tau nk'}\right\rangle_{aABC}$ collapses into

$$\left|\Psi^{(SP)}_{mk\tau nk'r's'}\right\rangle_{AC} = (U_m^{(S)}|\psi^{(S)}\rangle_A)|c_{k\oplus\tau\oplus 1}\rangle_C \times \Big[(X_P^{n\oplus k'}Z_P^{r'\oplus s'}\gamma U_n^{(P)}|H\rangle_A)|f_{k'}(H,V)\rangle_C + (X_P^{n\oplus k'}Z_P^{r'\oplus s'}\delta U_n^{(P)}|V\rangle_A)|f_{k'\oplus 1}(H,V)\rangle_C\Big], \tag{68}$$

again thanks to the properties (56) together with the equalities $X_P^m|f_n(H,V)\rangle = |f_{n\oplus m}(H,V)\rangle$ and $Z_P^m|f_n(H,V)\rangle = (-1)^{mn}|f_n(H,V)\rangle\ \forall\ m,n.$

As seen from equation (68), photons A and C are left entangled (in P-DOF) that gives room for Charlie to control the CRISO by detecting photon C polarization state. Conditioned on the detected state, photon A will be projected onto a state which Alice is able to manipulate towards the target state. As usual, Charlie passes photon C through a QWP and a PBS and then looks at photodetectors D_0 and D_1 arranged as in figure 12. A click of D_0 (D_1), which corresponds to the outcome labeled by a cbit $\tau' = 0$ ($\tau' = 1$), signals that photon A becomes in the state

$$\left|\Psi^{(SP)}_{mnk'r's'\tau'}\right\rangle_A = (U_m^{(S)}|\psi^{(S)}\rangle_A)(X_P^{n\oplus k'}Z_P^{r'\oplus s'\oplus\tau'}U_n^{(P)}|\psi^{(P)}\rangle_A). \tag{69}$$

Finally, Alice applies the operator

$$V^{(P)}_{nk'r's'\tau'} = Z_P^{r'\oplus s'\oplus\tau'}X_P^{n\oplus k'} \tag{70}$$

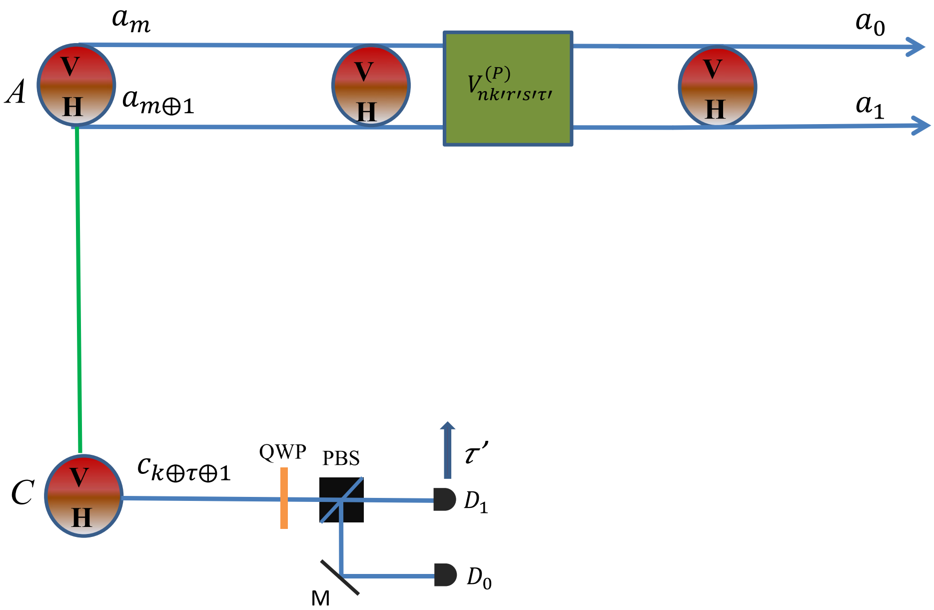

Figure 12. Charlie's and Alice's operations in the second step of CRISO on photon state in both S-DOF and P-DOF. The operator $V^{(P)}_{nk'r's'\tau'}$ is defined in equation (70).

on state $\left|\Psi^{(SP)}_{mnk'r's'\tau'}\right\rangle_A$ to obtain $(U^{(S)}_m\left|\psi^{(S)}\right\rangle_A)(U^{(P)}_n\left|\psi^{(P)}\right\rangle_A)$ which is the state $\left|\psi'^{(SP)}_{mn}\right\rangle_A$ in equation (58) that she wants.

5. Conclusion

We have put forward three tasks of deterministic controlled implementation of operators on remote photon states. The first task deals with general operators and photon states in S-DOF. For the second task the operator is also general but the photon state is in P-DOF. The third task however concerns particular subsets of operators with photon states being in both S-DOF and P-DOF. Each of the three tasks can be fulfilled in two steps via one hyperentangled GHZ state serving as the shared quantum channel. In the first step of the first task the S-DOF part of the quantum channel is exploited to transfer the state in S-DOF from Alice to Bob so that Bob can locally implement his general operator on the transferred state. Then, the P-DOF part of the quantum channel is exploited in the second step for Bob to transfer his state in S-DOF to Alice but the state Alice obtains is not in S-DOF but in P-DOF. Finally, Alice applies some technique to transform her state in P-DOF to the desired target state in S-DOF. The protocol for the second task is quite different. The first step of it exploits the P-DOF part of the quantum channel to transfer the state in P-DOF from Alice to Bob and the second step exploits the S-DOF part of the quantum channel to transfer the state in P-DOF at Bob's to a state in S-DOF at Alice's and Alice also needs some other technique to turn the state she obtains in S-DOF to the target state in P-DOF. The third task, like the first one, exploits the P-DOF part of the quantum channel in the first step and the P-DOF part of it in the second step. But, the third task just deals with limited subsets of operators with peculiar properties, so in the first step the desired photon state in S-DOF is readily obtained, leaving the desired P-DOF photon state to the second step.

The role of the controller is played in each step of each task. All the tasks are assisted by cross-Kerr nonlinearities and X-quadrature measurements of the coherent states which are a kind of quantum nondemolition measurement necessary for achieving the goal. Although the cross-Kerr nonlinearities are very weak in practice, the weakness of such nonlinear interaction strength could be compensated by using coherent states with high enough intensity. The tasks considered here are beneficial to distributed quantum computation.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Harrow A W and Montanaro A 2017 *Nature* **549** 203
[2] Cirac J I, Ekert A K, Huelga S F and Macchiavello C 1999 *Phys. Rev.* A **59** 4249
[3] DiVincenzo D P 2000 *Fortschr. Phys.* **48** 771
[4] Pompili M *et al* 2021 *Science* **372** 259
[5] Wehner S, Elkouss D and Hanson R 2018 *Science* **362** 6412
[6] Long G L, Pan D, Sheng Y B, Xue Q, Lu J and Hanzo L 2022 arXiv:2202.03619 [quant-ph]
[7] Britt B C 2020 *Quantum Eng.* **2** e29
[8] Zhang Y and Ni Q 2021 *Quantum Eng.* **3** e75
[9] Wei S, Chen Y, Zhou Z and Long G 2022 *AAPPS Bull.* **32** 2
[10] Cirac J I, Zoller P, Kimble H J and Mabuchi H 1997 *Phys. Rev. Lett.* **78** 3221
[11] Duan L-M, Blinov B B, Moehring D L and Monroe C 2004 *Quantum Inf. Comput.* **4** 165
[12] Nielsen M A and Chuang I L 1997 *Phys. Rev. Lett.* **79** 321
[13] Sørensen A and Mølmer K 1998 *Phys. Rev.* A **58** 2745
[14] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[15] Huelga S F, Vaccaro J A, Chefles A and Plenio M B 2001 *Phys. Rev.* A **63** 042303
[16] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[17] Huelga S F, Plenio M B and Vaccaro J A 2002 *Phys. Rev.* A **65** 042316
[18] Xiang G Y, Li J and Guo G C 2005 *Phys. Rev.* A **71** 044304
[19] Yang C-P and Gea-Banacloche J 2001 *J. Opt. B: Quantum Semiclass. Opt.* **3** 407
[20] Greenberger D M, Horne M A and Zeilinger A 1989 *Bells Theorem, Quantum Theory, and Conception of the Universe* ed M Kafatos (Dordrecht: Kluwer) p 69
[21] Hillery M, Bužek V and Berthiaume A 1998 *Phys. Rev.* A **59** 1829
[22] Collins D, Linden N and Popescu S 2001 *Phys. Rev.* A **64** 032302
[23] Huang Y-F, Ren X-F, Zhang Y-S, Duan L-M and Guo G-C 2004 *Phys. Rev. Lett.* **93** 240501

[24] Reznik B, Aharonov Y and Groisman B 2002 *Phys. Rev.* A **65** 032312
[25] Groisman B and Reznik B 2005 *Phys. Rev.* A **71** 032322
[26] Huelga S F, Plenio M B, Xiang G-Y, Li J and Guo G-C 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** S384
[27] Wang A M 2006 *Phys. Rev.* A **74** 032317
[28] Wang A M 2007 *Phys. Rev.* A **75** 062323
[29] Bo F Q and Dong L D 2008 *Sci. China* G **51** 1661
[30] Zhao N B and Wang A M 2007 *Phys. Rev.* **76** 062317
[31] An N B 2007 *Phys. Lett.* A **364** 198
[32] Zhong H S *et al* 2020 *Science* **370** 1460
[33] Knill E, Laflamme R and Milburn G J 2001 *Nature* **409** 46
[34] Nemoto K and Munro W J 2004 *Phys. Rev. Lett.* **93** 250502
[35] Munro W J, Nemoto K and Spiller T P 2005 *New J. Phys.* **7** 137
[36] An N B, Kim K and Kim J 2011 *Quantum Inf. Comput.* **11** 0124
[37] Luo M-X, Li H-R and Lai H 2016 *Sci. Rep.* **6** 29939
[38] Li X H and Ghose S 2017 *Phys. Rev.* A **93** 022302
[39] Zhou P and Lv L 2019 *Sci. Rep.* **9** 15939
[40] Zeng Z and Zhu K-D 2020 *New J. Phys.* **22** 083051
[41] Zheng Y-B, Zhou X-J, Wei H-R, Du F-F and Song G-Z 2021 *Quantum Inf. Process.* **20** 273
[42] Paul G and Kwiat J 1997 *Mod. Opt.* **44** 2173
[43] Deng F-G, Ren B-C and Li X-H 2017 *Sci. Bull.* **62** 46
[44] Sheng Y B, Deng F G and Long G L 2010 *Phys. Rev.* A **82** 032318
[45] Li X H and Ghose S 2017 *Phys. Rev.* A **96** 020303
[46] Gao C Y, Ren B C, Zhang Y X, Ai Q and Deng F G 2020 *Appl. Phys. Express* **13** 027004
[47] Sheng Y B and Deng F G 2010 *Phys. Rev.* A **81** 032307
[48] Wang G Y, Liu Q and Deng F G 2016 *Phys. Rev.* A **94** 032319
[49] Li C Y and Sheng Y 2019 *Opt. Express* **27** 13127
[50] Ren B-C, Wei H-R and Deng F-G 2013 *Laser Phys. Lett.* **10** 095202
[51] Kwek L-C, Cao L, Luo W, Wang Y, Sun S, Wang X and Liu A Q 2021 *AAPPS Bull.* **31** 15
[52] Wang X-L, Cai X-D, Su Z-E, Chen M-C, Wu D, Li L, Liu N-L, Lu C-Y and Pan J-W 2015 *Nature* **518** 516
[53] Graham T M, Bernstein H J, Wei T-C, Junge M and Kwiat P G 2015 *Nat. Commun.* **6** 7185
[54] Luo M X, Li H R, Lai H and Wang X 2016 *Phys. Rev.* A **93** 012332
[55] Walborn S P 2008 *Nat. Phys.* **4** 268
[56] Barreiro J T, Wei T-C and Kwiat P G 2008 *Nat. Phys.* **4** 282
[57] Williams B P, Sadlier R J and Humble T S 2017 *Phys. Rev. Lett.* **118** 050501
[58] Nawaz M, ul-Islam R and Ikram M 2018 *J. Phys. B: At. Mol. Opt. Phys.* **51** 075501
[59] Zhou P, Jiao X-F and Lv S-X 2018 *Quantum Inf. Process.* **17** 298
[60] Jiao X-F, Zhou P, Lv S-X and Wang Z-Y 2019 *Sci. Rep.* **9** 4663
[61] Zhou P and Lv L 2020 *Quantum Inf. Process.* **19** 283
[62] Wu X-D, Zhou L, Zhong W and Sheng Y-B 2020 *Quantum Inf. Process.* **19** 354
[63] Gao C Y, Guo P L and Ren B C 2021 *Quantum Eng.* **3** e83
[64] Sheng Y-B, Zhou L and Long G-L 2022 *Sci. Bull.* **67** 367
[65] Han K-Q, Zhou L, Zhong W and Sheng Y-B 2021 *Quantum Inf. Process.* **20** 280
[66] Jiao X-F, Zhou P and Lv S-X 2019 *J. Opt. Soc. Am.* B **36** 867
[67] Karlsson A and Bourennane M 1998 *Phys. Rev.* A **58** 4394
[68] An N B 2003 *Phys. Rev.* A **68** 022321
[69] Wang X L *et al* 2018 *Phys. Rev. Lett.* **120** 260502
[70] Xia Y, Chen Q-Q, Song J and Song H-S 2012 *J. Opt. Soc. Am.* B **29** 1029
[71] Ding D and Yan F-L 2013 *Phys. Lett.* A **377** 1088
[72] Liu A-P, Han X, Cheng L-Y, Guo Q, Su S-L, Wang H-F and Zhang S 2019 *Eur. Phys. J.* D **73** 118
[73] Gianni J and Qu Z 2021 *J. Quantum Comput.* **3** 46