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Joint remote implementation of operators

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Abstract
We propose a deterministic protocol for three distant parties to cooperate so that two parties can implement their secret unitary operators on the third party's secret quantum state via shared quantum channel assisted by cross-Kerr nonlinearities. The operators are of general form while the state may be encoded either in spatial degree of freedom or polarization degree of freedom. The quantum channel is served by a three-photon hyperentangled state establishing the minimum of consumed photon number for this type of task. This protocol can be named joint remote implementation of operators which is necessary for distributed quantum tasks throughout a quantum network.

Keywords: joint remote implementation of operators, hyperentanglement, cross-Kerr nonlinearities, X-quadrature measurement

(Some figures may appear in colour only in the online journal)

1. Introduction

'Moore's law' discovered in 1965 was fuelling transistor-based technologies for many years but now turns out redundant because of the emerging revolution in computing innovation based on quantum computers which enable solving problems that are impossible to be solved by any conventional super-computers in a feasible time [1]. A scalable quantum computer should manipulate a large qubit number fault-tolerantly. However, unwanted interactions between the qubits trigger hardly controllable errors in calculations. Although codes exist to correct quantum errors, a nice solution to minimize errors is not to keep all the working qubits in one computer but distribute them among many computers each contains just a relatively small number of qubits but the computation is performable over all the qubits. The set of such remote computers is looked upon as the set of nodes of a quantum network [2, 3] and the computation

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is referred to as 'distributed quantum computation'. Since qubits are in different nodes of the quantum network, in distributed quantum computation necessary remote operations must be done by local operations plus classical communication. Therefore, remote implementation of operators (RIO) plays a very crucial role and in fact is one of the compulsory tasks for quantum networking. The authors of [4] considered the situation when Bob can implement his unitary operator on Alice's quantum state even though the two parties are far apart. A more general version of RIO was dealt with in [5] which is relevant to the issue of quantum secret sharing [6]. Protocols to implement two-qubit operators on two distant qubits was also proposed in [7, 8]. Furthermore, operators whose full set of characteristics is splitted into different subsets [9, 10] and particular unknown operators which are immersed in a lump operator [11] can be remotely implemented as well. Recently, two-photon four-qubit hyperentangled states were also employed for RIO [12]. A more detailed information on RIO can be found in the introduction section of [13] which addresses controlled remote implementation of operators.

Here we solve a novel quantum task named joint remote implementation of operators (JRIO) which will be put forward in the next section, section 2. Section 3 describes specific steps to execute the JRIO task. The final section, section 4, provides some discussion and conclusion.

2. The task of concern

We are interested in a task involving three distant parties Alice, Bob and Charlie. This task uses the same quantum resource as in [13] but its aim is totally different from that of [13]. Alice has a photon X with definite polarization but propagating simultaneously along two distinct spatial paths. Assuming without loss of generality that the photon's polarization is horizontal, its state is of the form

$$|\psi\rangle_X = (\alpha|x_0\rangle_X + \beta|x_1\rangle_X)|H\rangle_X, \tag{1}$$

with $|H\rangle$ denoting state of a photon with horizontal polarization, $|x_0\rangle$ ($|x_1\rangle$) state of a photon traveling along path x_0 (x_1), while α, β the normalization coefficients known only to Alice. As for the other parties, Bob has a unitary operator

$$U_B = \begin{pmatrix} u_B & v_B \\ -v_B^* & u_B^* \end{pmatrix}, \tag{2}$$

which acts on $|\psi\rangle$ as

$$U_B|\psi\rangle = |\psi_B\rangle = (\alpha_B|x_0\rangle + \beta_B|x_1\rangle)|H\rangle, \tag{3}$$

with

$$\alpha_B = \alpha u_B + \beta v_B, \qquad \beta_B = -\alpha v_B^* + \beta u_B^*, \tag{4}$$

while Charlie has another unitary operator

$$U_C = \begin{pmatrix} u_C & v_C \\ -v_C^* & u_C^* \end{pmatrix}, \tag{5}$$

which acts on $|\psi\rangle$ as

$$U_C|\psi\rangle = |\psi_C\rangle = (\alpha_C|x_0\rangle + \beta_C|x_1\rangle)|H\rangle, \tag{6}$$

with

$$\alpha_C = \alpha u_C + \beta v_C, \qquad \beta_C = -\alpha v_C^* + \beta u_C^*. \tag{7}$$

Note that only Bob (Charlie) knows the parameters u_B, $v_B(u_C, v_C)$. The task of our concern is how the three parties should cooperate so that Alice eventually will hold a photon in state

$$|\psi_{BC}\rangle = U_B U_C|\psi\rangle = \big(\alpha_{BC}|x_0\rangle + \beta_{BC}|x_1\rangle\big)|H\rangle, \tag{8}$$

with

$$\alpha_{BC} = \alpha_C u_B + \beta_C v_B, \qquad \beta_{BC} = -\alpha_C v_B^* + \beta_C u_B^*. \tag{9}$$

In essence, this task allows Bob and Charlie to jointly implement their operators on Alice's remote state. Hence, it is named 'joint remote implementation of operators' (JRIO).

3. Protocol for joint remote implementation of operators

What first comes to mind is that the above mentioned JRIO can be fulfilled by performing a four-round quantum state teleportation [14]. In the first round Alice teleports $|\psi\rangle$ to Charlie. In the second round Charlie implements U_C on $|\psi\rangle$ and teleports $|\psi_C\rangle = U_C|\psi\rangle$ back to Alice. In the third round Alice teleports $|\psi_C\rangle$ to Bob. And, in the fourth round Bob implements U_B on $|\psi_C\rangle$ and teleports $|\psi_{BC}\rangle = U_B|\psi_C\rangle = U_B U_C|\psi\rangle$ back to Alice. Such performance is awkward as it actually consists of two independent RIO protocols [5] and requires four EPR pairs [15], which are made by eight photons. A cheaper way consuming six photons in terms of three EPR pairs goes like this. First, Alice teleports $|\psi\rangle$ to Charlie. Second, Charlie implements U_C and teleports $|\psi_C\rangle$ to Bob. Finally, Bob implements U_B then teleports $|\psi_{BC}\rangle$ to Alice.

We shall show that only three photons are consumed in our new protocol, if the photons are exploited at the same time in double degrees of freedom (DOF). Concretely, Alice, Bob and Charlie share ahead of time a three-photon state of the form [16–19]

$$|Q\rangle_{ABC} = \big|Q^{(\mathrm{S})}\big\rangle_{ABC}\big|Q^{(\mathrm{P})}\big\rangle_{ABC}, \tag{10}$$

where

$$\big|Q^{(\mathrm{S})}\big\rangle_{ABC} = |a_0\rangle_A|b_0\rangle_B|c_0\rangle_C + |a_1\rangle_A|b_1\rangle_B|c_1\rangle_C, \tag{11}$$

$$\big|Q^{(\mathrm{P})}\big\rangle_{ABC} = |H\rangle_A|H\rangle_B|H\rangle_C + |V\rangle_A|V\rangle_B|V\rangle_C. \tag{12}$$

In the above formulae the superscript '(S)' implies the spatial degree of freedom (S-DOF) and '(P)' the polarization degree of freedom (P-DOF), $|a_j\rangle_A$ $\big(|b_j\rangle_B, |c_j\rangle_C\big)$, with $j \in \{0, 1\}$, denotes state of photon $A(B, C)$ traveling along spatial path $a_j(b_j, c_j)$, while $|V\rangle$ is state of a photon with vertical polarization. The quantum channel state (10) is tensor product of two GHZ states [20], one in S-DOF, the $\big|Q^{(\mathrm{S})}\big\rangle_{ABC}$, and one in P-DOF, the $\big|Q^{(\mathrm{P})}\big\rangle_{ABC}$. Simultaneous entanglement in more than one DOF is called hyperentanglement [21, 22]. State (10) can thus be called hyperentangled GHZ state. Note that the normalization coefficients in (11) and (12) as well as all other global phase factors that may appear are omitted to simplify the formulation without affecting the total success probability which is 100% in our protocol. Photon $A(B, C)$ of state $|Q\rangle_{ABC}$ must be held by Alice (Bob, Charlie).

The starting total state of the four photons X, A, B and C reads

$$|\psi\rangle_X|Q\rangle_{ABC} = \big|\Phi^{(\mathrm{S})}\big\rangle_{XABC}|H\rangle_X\big|Q^{(\mathrm{P})}\big\rangle_{ABC}, \tag{13}$$

with

$$\left|\Phi^{(S)}\right\rangle_{XABC} = (\alpha|x_0\rangle_X + \beta|x_1\rangle_X)(|a_0\rangle_A|b_0\rangle_B|c_0\rangle_C + |a_1\rangle_A|b_1\rangle_B|c_1\rangle_C). \tag{14}$$

Our JRIO protocol proceeds in several sequential steps as follows.

In step 1 (see figure 1), Alice prepares an auxiliary coherent state (CS) $|z\rangle = \exp(-|z|^2/2)\sum_{n=0}^{\infty}(z^n/\sqrt{n!})|n\rangle$ ($|n\rangle$ a Fock state containing n photons) and lets it first interact with photon X on path x_0 and then with photon A on path a_0 by means of cross-Kerr nonlinearities (see, e.g., [23–25]) $K_{x_0}(\theta)$ and $K_{a_0}(-\theta)$, with $|\theta|$ a dimensionless measure of cross-Kerr nonlinearity strength. Such nonlinear interactions cause the transitions $|z\rangle|x_0\rangle_X \to K_{x_0}(\theta)|z\rangle|x_0\rangle_X = \left|z\mathrm{e}^{\mathrm{i}\theta}\right\rangle|x_0\rangle_X$ and $|z\rangle|a_0\rangle_A \to K_{a_0}(-\theta)|z\rangle|a_0\rangle_A = \left|z\mathrm{e}^{-\mathrm{i}\theta}\right\rangle|a_0\rangle_A$, but keep $|z\rangle|x_1\rangle_X$ and $|z\rangle|a_1\rangle_A$ as they are. After the interactions Alice measures X-quadrature of the CS by means of homodyne detection technique to find out whether the CS is in $|z\rangle$ or in $\left|z\mathrm{e}^{\pm\mathrm{i}\theta}\right\rangle$ ($\left|z\mathrm{e}^{+\mathrm{i}\theta}\right\rangle$ and $\left|z\mathrm{e}^{-\mathrm{i}\theta}\right\rangle$ are indistinguishable by this kind of measurement). If $|z\rangle$ ($\left|z\mathrm{e}^{\pm\mathrm{i}\theta}\right\rangle$) is detected she publishes one bit $k = 0 (k = 1)$ in which event photon X gets entangled with photons A, B and C with respect to S-DOF. Depending on the measurement outcome k of X-quadrature measurement after the specified cross-Kerr interactions the photon path changes accordingly. Namely, if $k = 0$ the resulting state is $[\alpha|x_0\rangle_X|a_0\rangle_A|b_0\rangle_B|c_0\rangle_C + \beta|x_1\rangle_X|a_1\rangle_A|b_1\rangle_B|c_1\rangle_C]|H\rangle_X\left|Q^{(P)}\right\rangle_{ABC}$, but if $k = 1$ the state turns out to be $[\alpha|x_0\rangle_X|a_1\rangle_A|b_1\rangle_B|c_1\rangle_C + \beta|x_1\rangle_X|a_0\rangle_A|b_0\rangle_B|c_0\rangle_C]|H\rangle_X\left|Q^{(P)}\right\rangle_{ABC}$. So, to avoids writing formulae for each outcome separately that consumes paper space, we shall write the resulting state so that it is valid for both $k = 0$ and $k = 1$ as

$$[\alpha|x_0\rangle_X|a_k\rangle_A|b_k\rangle_B|c_k\rangle_C + \beta|x_1\rangle_X|a_{k\oplus1}\rangle_A|b_{k\oplus1}\rangle_B|c_{k\oplus1}\rangle_C]|H\rangle_X\left|Q^{(P)}\right\rangle_{ABC}, \tag{15}$$

with $\oplus$ an addition mod 2. Note that all the other formulae that follow are also written in such a fashion so that they are valid for any possible measurement outcome. Next, Alice mixes $|x_0\rangle_X$ ($|a_k\rangle_A$) with $|x_1\rangle_X$ ($|a_{k\oplus1}\rangle_A$) on a beam-splitter (BS), which obeys the following rule of transformation $|d_k\rangle_D \to (-1)^k|d_k\rangle_D + |d_{k\oplus1}\rangle_D$ for $d = \{x, a\}$, $D = \{X, A\}$ with $k = \{0, 1\}$. After passing the BS, $|x_0\rangle_X$ ($|a_k\rangle_A$) is let to interact with another CS via $K_{x_0}(\theta)$ ($K_{a_k}(2\theta)$), followed by X-quadrature measurement of the outgoing CS. The four possible measurement outcomes labeled by $mn = 00$, 01, 10 or 11 correspond respectively to finding the CS in $|z\rangle$, $\left|z\mathrm{e}^{\mathrm{i}\theta}\right\rangle$, $\left|z\mathrm{e}^{2\mathrm{i}\theta}\right\rangle$ or $\left|z\mathrm{e}^{3\mathrm{i}\theta}\right\rangle$. Conditioned on mn, state (15) changes to

$$[|x_{n\oplus1}\rangle_X|a_{k\oplus m\oplus1}\rangle_A\left(\alpha|b_k\rangle_B|c_k\rangle_C + (-1)^{k+m+n}\beta|b_{k\oplus1}\rangle_B|c_{k\oplus1}\rangle_C\right)]|H\rangle_X\left|Q^{(P)}\right\rangle_{ABC}. \tag{16}$$

We see that photon X is entirely disentangled from the system and no more 'carries' the coefficients α, β. We shall thus ignore it from this moment. Photon A is also disentangled from photons B and C with respect to S-DOF, but it should not be ignored because of their untouched P-DOF entanglement. A worthy point is that the coefficients α, β are now shared between photons B and C whose S-DOF entanglement still survives.

The purpose of step 2 (see figure 2) is for Bob to help Charlie to implement U_C on her photon C. Bob mixes $|b_k\rangle_B$ and $|b_{k\oplus1}\rangle_B$ on a BS, turns on cross-Kerr interaction $K_{b_k}(\theta)$ between $|b_k\rangle_B$ and a CS $|z\rangle$, then carries out X-quadrature measurement of the resulting CS. Depending on the measurement outcome $l = 0$ or $l = 1$ (corresponding to finding $|z\rangle$ or $\left|z\mathrm{e}^{\mathrm{i}\theta}\right\rangle$), state of photons A, B, C collapses to

$$|a_{k\oplus m\oplus1}\rangle_A|b_{k\oplus l\oplus1}\rangle_B(\alpha|c_k\rangle - (-1)^{m\oplus n\oplus l}\beta|c_{k\oplus1}\rangle)_C\left|Q^{(P)}\right\rangle_{ABC}. \tag{17}$$

As seen from (17), if Charlie applies $R = \mathcal{Z}^{m\oplus n\oplus l\oplus1}\mathcal{X}^k$ on photon C, with $\mathcal{X} = |c_0\rangle\langle c_1| + |c_1\rangle\langle c_0|$ and $\mathcal{Z} = |c_0\rangle\langle c_0| - |c_1\rangle\langle c_1|$, the S-DOF state of photon C is transformed to

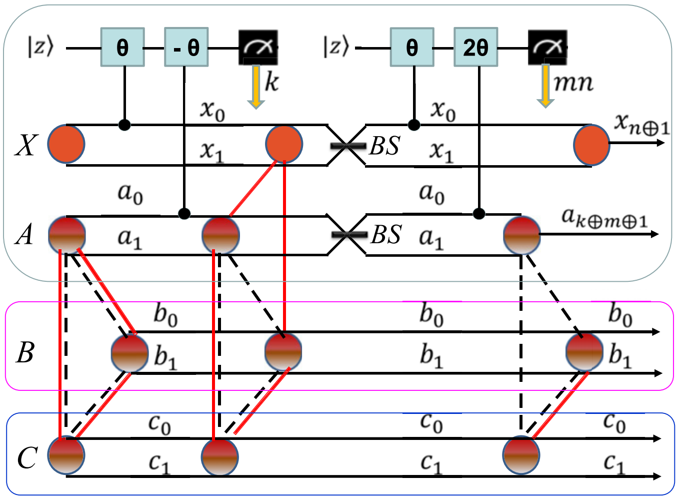

Figure 1. The operations in step 1. A circle filled by one color (two colors) represents a photon with horizontal polarization (in a superposition of horizontal and vertical polarizations). A circle attached with one (two) tail(s) represents a photon propagating along only one spatial path (two spatial paths simultaneously). Photons entangled in S-DOF (P-DOF) are connected by solid (dashed) lines. $|z\rangle$ is a high-intensity coherent state, θ is a dimensionless parameter quantifying the strength of cross-Kerr nonlinearity, bold arrows indicate X-quadrature measurements outcomes coded by classical bits to be openly announced and BS is short for balanced beam-splitter. At the beginning photon X is not entangled with the other photons. After the first X-quadrature measurement with outcome $k \in \{0, 1\}$ photon X gets entangled in S-DOF with photons A, B and C, and the type of S-DOF entanglement between the four photons is selected by the measurement outcome k as in the expression (15). After the second X-quadrature measurement with outcome $mn \in \{00, 01, 10, 11\}$ photons X and A become separated and each propagates only along a single path, while photons B and C are still entangled in S-DOF. The path of photons X, A and the type of S-DOF entanglement between photons B and C are determined by the measurement outcomes kmn as in the expression (16).

$(\alpha|c_0\rangle + \beta|c_1\rangle)_C$, so that Charlie is able to locally implement her operator U_C. As a result of such Charlie's operation, state (17) turns out to be

$$|a_{k\oplus m\oplus 1}\rangle_A |b_{k\oplus l\oplus 1}\rangle_B (\alpha_C|c_0\rangle + \beta_C|c_1\rangle)_C \left|Q^{(\mathrm{P})}\right\rangle_{ABC}, \tag{18}$$

with α_C, β_C given in (7). The S-DOF entanglement died totally, but the P-DOF entanglement remained intact that will be exploited in the next steps.

Step 3 (see figure 3) serves as a preliminary step for Bob to implement U_B on his photon B. It proceeds as follows. Bob first sends photon B in state $|b_{k\oplus l\oplus 1}\rangle_B$ through a BS which triggers a new path $|b_{k\oplus l}\rangle_B$. Behind the BS, he takes a CS $|z\rangle$ and lets it interact with $|b_{k\oplus l\oplus 1}\rangle_B$ via $K_{b_{k\oplus l\oplus 1}}(\theta)$. After that he forwards the CS to Charlie. Charlie turns on cross-Kerr interaction $K_{c_0}(-\theta)$ then measures X-quadrature of the obtained CS. Charlie's measurement outcome may be $r = 0$ or $r = 1$ (i.e., $|z\rangle$ or $\left|z\mathrm{e}^{\pm \mathrm{i}\theta}\right\rangle$ is detected), that projects the photons' state onto

$$|a_{k\oplus m\oplus 1}\rangle_A [(-1)^{k\oplus l\oplus 1}\alpha_C |b_{k\oplus l\oplus r\oplus 1}\rangle_B |c_0\rangle_C + \beta_C |b_{k\oplus l\oplus r}\rangle_B |c_1\rangle_C] \left|Q^{(\mathrm{P})}\right\rangle_{ABC}. \tag{19}$$

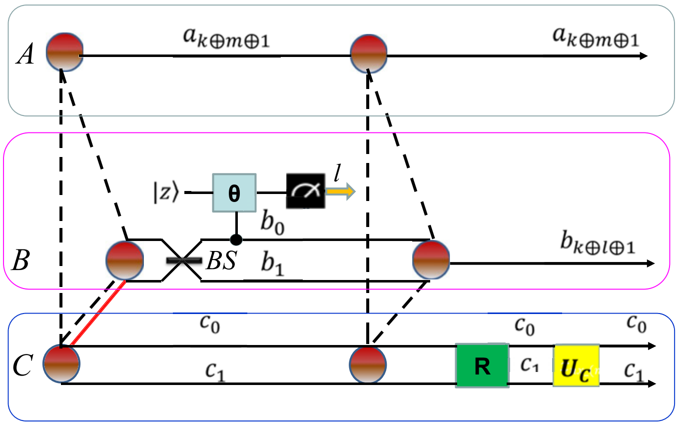

Figure 2. The operations in step 2. In this step, after the X-quadrature measurement with outcome $l \in \{0, 1\}$ photon B appears disentangled in S-DOF from photon C and propagates only along a single path. The path of photon B and the form in S-DOF of photon C depend on the measurement outcomes $kmnl$ as in the expression (17). After applications of operators $R = Z^{m\oplus n\oplus l\oplus 1}X^k$ and U_C, equation (5), photon C turns out to be of the form $(\alpha_C|c_0\rangle + \beta_C|c_1\rangle)_C$, with α_C, β_C given in (7).

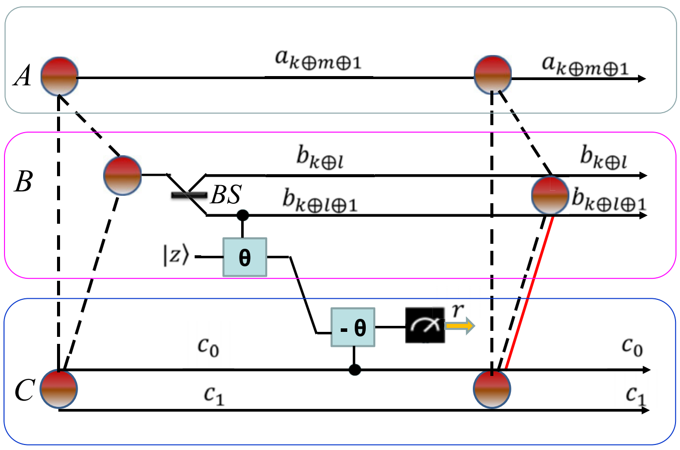

Figure 3. The operations in step 3. After this step, S-DOF entanglement between photons B and C revives but their entangled state is characterized by α_C, β_C which differs from that at the end of step 1 where the characteristic coefficients were α, β. The type of S-DOF entanglement between photons B and C is conditioned not only on the outcome $r \in \{0, 1\}$ of the X-quadrature measurement in this step but also on the outcomes k and l of the X-quadrature measurements in the previous steps, as seen from the expression (19).

That is, step 3 restores entanglement in S-DOF between photons B and C, but there is an important difference between the S-DOF entangled states (16) and (19): in (16) the coefficients are α, β while in (19) they are α_C, β_C.

As α_C, β_C are shared between Bob and Charlie, Charlie can 'convey' those coefficients to Bob exclusively, as is shown in figure 4 in step 4. Explicitly, in step 4 Charlie superimposes

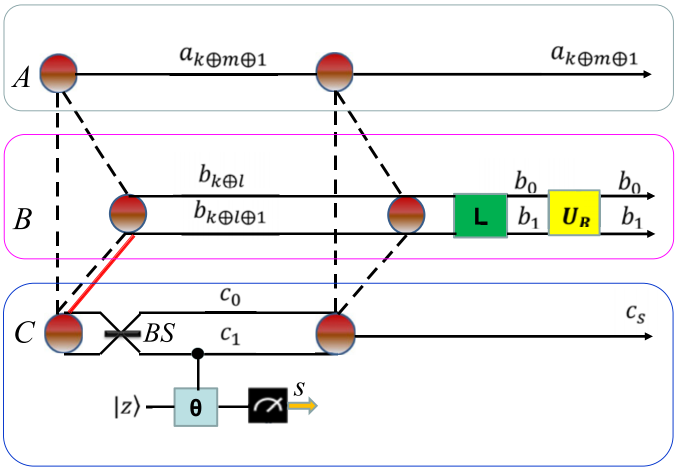

Figure 4. The operations in step 4. In this step, after Charlie performs her X-quadrature measurement with outcome $s \in \{0, 1\}$ photons B and C cease to be entangled in S-DOF. The state in S-DOF of photons B is decided by the outcomes $klrs$ obtained in both the previous (klr) and the present (s) X-quadrature measurements as is in the expression (20), which upon applications of $L = Z^{k\oplus l\oplus s\oplus 1}X^{k\oplus l\oplus r\oplus 1}$ and the operator U_B, equation (2), becomes $(\alpha_{BC}|b_0\rangle + \beta_{BC}|b_1\rangle)_B = U_B U_C(\alpha|b_0\rangle + \beta|b_1\rangle)_B$. As for photon C, it propagates along a single path which is solely determined by the Charlie's measurement outcomes s.

her photon states $|c_0\rangle_C$ and $|c_1\rangle_C$ on a BS, then cross-Kerr interaction $K_{c_1}(\theta)$ is switched on between $|c_1\rangle_C$ and a CS $|z\rangle$. After that she homodyne detects the CS to find whether it is in $|z\rangle$ or $|ze^{i\theta}\rangle$ (corresponding to the measurement outcome $s = 0$ or $s = 1$). If the outcome is $s \in \{0, 1\}$, the S-DOF entanglement between photons B and C disappears bringing state (19) to

$$|a_{k\oplus m\oplus 1}\rangle_A(\alpha_C|b_{k\oplus l\oplus r\oplus 1}\rangle_B + (-1)^{k\oplus l\oplus s\oplus 1}\beta_C|b_{k\oplus l\oplus r}\rangle_B)|c_s\rangle_C|Q^{(P)}\rangle_{ABC}. \tag{20}$$

Having heart s from Charlie's announcement, Bob applies $L = \mathcal{Z}^{k\oplus l\oplus s\oplus 1}\mathcal{X}^{k\oplus l\oplus r\oplus 1}$ then implements U_B on photon B so that state (20) becomes

$$|a_{k\oplus m\oplus 1}\rangle_A(\alpha_{BC}|b_0\rangle_B + \beta_{BC}|b_1\rangle_B)|c_s\rangle_C|Q^{(P)}\rangle_{ABC}, \tag{21}$$

where $(\alpha_{BC}|b_0\rangle + \beta_{BC}|b_1\rangle)_B = U_B U_C(\alpha|b_0\rangle + \beta|b_1\rangle)_B$.

As seen from (21), the S-DOF entanglement is completely demolished, but the P-DOF entanglement (i.e., $|Q^{(P)}\rangle_{ABC}$) is fully reserved that will be used to fulfill the JRIO task in step 5 (see figure 5). Taking into account of the explicit form of $|Q^{(P)}\rangle_{ABC}$ from (12), state (21) is written in an expanded form as

$$\begin{aligned}&|a_{k\oplus m\oplus 1}\rangle_A\big[\alpha_{BC}|H\rangle_A|H, b_0\rangle_B|H, c_s\rangle_C + \alpha_{BC}|V\rangle_A|V, b_0\rangle_B|V, c_s\rangle_C\\ &\quad + \beta_{BC}|H\rangle_A|H, b_1\rangle_B|H, c_s\rangle_C + \beta_{BC}|V\rangle_A|V, b_1\rangle_B|V, c_s\rangle_C\big],\end{aligned} \tag{22}$$

where $|H, b_0\rangle$ denotes state of a horizontally polarized photon propagating along path b_0 and similarly for $|V, b_0\rangle$, $|H, b_1\rangle_B$, $|V, b_1\rangle$, $|H, c_s\rangle_C$ and $|V, c_s\rangle_C$. Bob and Charlie independently execute their local operations as follows. Bob puts a half-wave plate (HWP) on path b_1 to

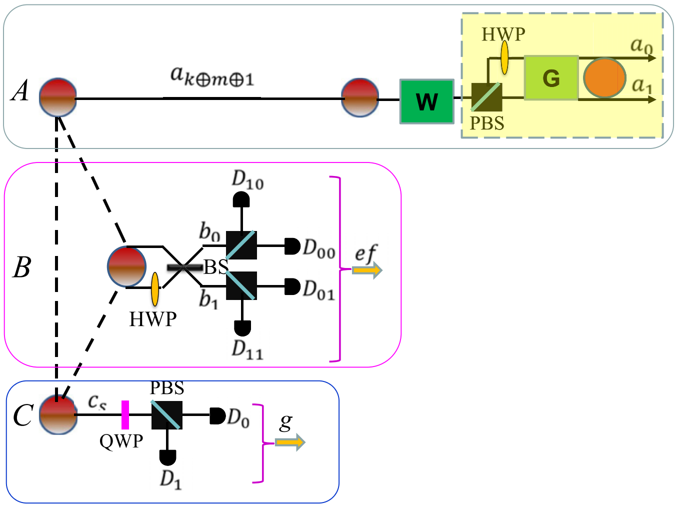

Figure 5. The operations in steps 5 and 6 (the highlighted box is for step 6). Here HWP is half-wave plate, QWP quarter-wave plate and PBS polarization beam-splitter (PBS transmits horizontal polarization but reflects vertical polarization). The measurements of photon B and photon C are realized by a set of HWP, QWP, BS, PBS combined with photodetectors D_{ef} and D_g arranged as in the figure. $D_{00}(D_{01}, D_{10}, D_{11})$ clicks signals that $|H, b_0\rangle_B$ $(|H, b_1\rangle_B, |V, b_0\rangle_B, |V, b_1\rangle_B\}$ is measured, while $|H, c_s\rangle_C$ $(|V, c_s\rangle_C)$ triggers a click of $D_0(D_1)$. If Bob's and Charlie's measurement outcomes are respectively $ef \in \{00, 01, 10, 11)$ and $g \in \{0, 1\}$, then Alice's photon A is projected onto state (24) which can be transformed to (25) by the operator $W = Z^{f\oplus g}\mathrm{HWP}^e$. In the final step, step 6, Alice uses a PBS, a HWP and the operator $G = X^{k\oplus m\oplus 1}$ to achieve the desired target state.

exchange the photon polarization $|H, b_1\rangle_B \leftrightarrows |V, b_1\rangle_B$, then uses a BS to mix the two paths $|b_0\rangle_B$ and $|b_1\rangle_B$. As for Charlie, she puts a quarter-wave plate (QWP) on path c_s to transform $|H, c_s\rangle_C$ to $(|H, c_s\rangle_C + |V, c_s\rangle_C)$ and $|V, c_s\rangle_C$ to $(|H, c_s\rangle_C - |V, c_s\rangle_C)$. Those local operations of Bob and Charlie result in a quite cumbersome-in-form state that can be expressed as

$$\begin{aligned}
|a_{k\oplus m\oplus 1}\rangle_A \big[&|H, b_0\rangle_B|H, c_s\rangle_C(\alpha_{BC}|H\rangle_A + \beta_{BC}|V\rangle_A) \\
&+ |H, b_0\rangle_B|V, c_s\rangle_C(\alpha_{BC}|H\rangle_A - \beta_{BC}|V\rangle_A) \\
&+ |H, b_1\rangle_B|H, c_s\rangle_C(\alpha_{BC}|H\rangle_A - \beta_{BC}|V\rangle_A) \\
&+ |H, b_1\rangle_B|V, c_s\rangle_C(\alpha_{BC}|H\rangle_A + \beta_{BC}|V\rangle_A) \\
&+ |V, b_0\rangle_B|H, c_s\rangle_C(\alpha_{BC}|V\rangle_A + \beta_{BC}|H\rangle_A) \\
&- |V, b_0\rangle_B|V, c_s\rangle_C(\alpha_{BC}|V\rangle_A - \beta_{BC}|H\rangle_A) \\
&+ |V, b_1\rangle_B|H, c_s\rangle_C(\alpha_{BC}|V\rangle_A - \beta_{BC}|H\rangle_A) \\
&- |V, b_1\rangle_B|V, c_s\rangle_C(\alpha_{BC}|V\rangle_A + \beta_{BC}|H\rangle_A)\big].
\end{aligned} \tag{23}$$

Looking closer at (23) indicates that if Bob measures photon B in the basis $\{|H, b_0\rangle_B, |H, b_1\rangle_B, |V, b_0\rangle_B, |V, b_1\rangle_B\}$, while Charlie measures photon C in the basis $\{|H, c_s\rangle_C, |V, c_s\rangle_C\}$, then photon A will be projected onto a state which Alice can locally transform to the target state $|\psi_{BC}\rangle_A = (\alpha_{BC}|a_0\rangle + \beta_{BC}|a_1\rangle)_A|H\rangle_A = U_BU_C|\psi\rangle_A$. Indeed, let Bob's measurement outcome be $ef = 00$ (01, 10 or 11) corresponding to detecting photon B in state $|H, b_0\rangle_B$ ($|H, b_1\rangle_B, |V, b_0\rangle_B$ or $|V, b_1\rangle_B$) and Charlie's measurement outcome be $g = 0(1)$ corresponding to finding photon C in state $|H, c_s\rangle_C$ ($|V, c_s\rangle_C$), then for any of the eight possible sets of measurement outcomes efg the state of photon A becomes

$$(\alpha_{BC}\left|H^{e\oplus 1}V^e\right\rangle_A + (-1)^{f\oplus g}\beta_{BC}\left|H^e V^{e\oplus 1}\right\rangle_A)|a_{k\oplus m\oplus 1}\rangle_A, \tag{24}$$

where $H^{e\oplus 1}V^e = H$ and $H^e V^{e\oplus 1} = V$ for $e = 0$, while $H^{e\oplus 1}V^e = V$ and $H^e V^{e\oplus 1} = H$ for $e = 1$. When Bob and Charlie publicly reveal their measurement outcomes, Alice is in the position to convert her photon to the desired state. Namely, if $e = 0$ she just applies $Z^{f\oplus g}$ with $Z = |H\rangle\langle H| - |V\rangle\langle V|$. Otherwise, if $e = 1$ she puts a HWP on path $a_{k\oplus m\oplus 1}$ before applying $Z^{f\oplus g}$. That means that Alice's operator to be applied on photon A has the form $W = Z^{f\oplus g}\mathrm{HWP}^e$, which transforms state (24) to

$$(\alpha_{BC}|H\rangle + \beta_{BC}|V\rangle)_A|a_{k\oplus m\oplus 1}\rangle_A. \tag{25}$$

In the final step, step 6 (see the highlighted box in figure 5), Alice arranges on path $a_{k\oplus m\oplus 1}$ a polarization beam-splitter (PBS) that transmits horizontal polarization but reflects vertical polarization. The PBS changes state (25) to the form $(\alpha_{BC}|H, a_{k\oplus m\oplus 1}\rangle + \beta_{BC}|V, a_{k\oplus m}\rangle)_A$. Next, a HWP is put on path $a_{k\oplus m}$ followed by application of $G = X^{k\oplus m\oplus 1}$ on both paths. The resulting state will be $(\alpha_{BC}|a_0\rangle + \beta_{BC}|a_1\rangle)_A|H\rangle_A$ which is nothing else but the target state $|\psi_{BC}\rangle_A = U_B U_C|\psi\rangle_A$ defined by (8).

In the above consideration the initial state of Alice's photon (1) was superposed of two states propagating along two distinct paths, i.e., it was encoded in S-DOF. We now turn to the situation when Alice's initial photon is encoded in P-DOF, i.e., it is superposed of two states having different polarizations but traveling only along one path which is of the form

$$|\phi\rangle_X = (\alpha|H\rangle + \beta|V\rangle)_X|x_0\rangle_X. \tag{26}$$

Another difference between states (1) and (26) is that the photon in state (1) has a certain polarization while the photon in state (26) propagates along a certain path. The photon state that Alice now wishes to obtain is

$$|\phi_{BC}\rangle = (\alpha_{BC}|H\rangle + \beta_{BC}|V\rangle)|x_0\rangle, \tag{27}$$

with α_{BC}, β_{BC} given in (9). The state $|\phi_{BC}\rangle$ can be interpreted as $U_B U_C|\phi\rangle$ where U_B is Bob's operator (2) acting on $|\phi\rangle$ as

$$U_B|\phi\rangle = |\phi_B\rangle = (\alpha_B|H\rangle + \beta_B|V\rangle)|x_0\rangle \tag{28}$$

and U_C Charlie's operator (5) acting on $|\phi\rangle$ as

$$U_C|\phi\rangle = |\phi_C\rangle = (\alpha_C|H\rangle + \beta_C|V\rangle)|x_0\rangle, \tag{29}$$

with α_B, β_B and α_C, β_C given in (4) and (7), respectively. This task can also be fulfilled via the same three-photon hyperentangled GHZ state $|Q\rangle_{ABC}$ in (10) by first exploiting the P-DOF entanglement $\left|Q^{(\mathrm{P})}\right\rangle_{ABC}$, with the rules (28) and (29) to be applied, and then its S-DOF entanglement $\left|Q^{(\mathrm{S})}\right\rangle_{ABC}$. Here we show an alternative way for Alice to have the desired state (27) by precisely following the JRIO protocol described in detail above, with the rules (3) and (6) to be applied. However, Alice has to do something before starting the protocol. Namely, she puts a PBS on path x_0 of $|\phi\rangle_X$ so that $|\phi\rangle_X \to (\alpha|H, x_0\rangle + \beta|V, x_1\rangle)_X$. Behind the PBS, a HWP is inserted on path x_1 to transform $(\alpha|H, x_0\rangle + \beta|V, x_1\rangle)_X$ to $(\alpha|x_0\rangle + \beta|x_1\rangle)_X|H\rangle_X$ which is exactly the state $|\psi\rangle_X$ in (1). Therefore, performing the five first steps of the above JRIO protocol will readily give, at the end of step 5 (step 6 needs not to be done), the state (25), which with a formal relabeling of the path $a_{k\oplus m\oplus 1} \to x_0$ is nothing else but the state $|\phi_{BC}\rangle$ in (27) that Alice wished.

4. Discussion and conclusion

Although there are measurements with probabilistic outcomes our proposed JRIO protocol always succeeds because any possible set of outcomes can be well handled by appropriate local operators. The important technique used here is X-quadrature measurements of CSs after their cross-Kerr nonlinear interactions with a photon. Since practical cross-Kerr nonlinearity is very weak, i.e., $|\theta| \ll 1$, X-quadrature measurements might be not helpful. The technique would not be acceptable in our protocol if the X-quadrature measurement outcomes associated with states $|z\rangle$, $|ze^{i\theta}\rangle$, $|ze^{2i\theta}\rangle$ and $|ze^{3i\theta}\rangle$ could not be resolved. Fortunately, the measurement resolution is determined not solely by $|\theta|$. In fact, the X-quadrature value of state $|ze^{in\theta}\rangle$, with n an integer, is given by $X_n = \sqrt{2}z\cos(n\theta)$. Then, for cross-Kerr nonlinearity with $|\theta| \ll 1$, the resolutions between states $|ze^{mi\theta}\rangle$ and $|ze^{ni\theta}\rangle$ amount to $|X_m - X_n| \sim |(m^2 - n^2)z\theta^2|$, which can be made sufficiently large by using CSs with $|z|$ big enough. In other words, a countermeasure for weak cross-Kerr nonlinearities is use of high-intensity CSs [23–25].

As hyperentangled photon system is an important resource in this protocol, the foremost step is to generate the relevant hyperentanglement. Description of detailed processes for production of various two- and multi-photon hyperentangled states, including the state (10), is beyond the scope of this paper, but can be found in the literature (see, e.g., references [18, 21, 22, 26–32]).

In conclusion, we have proposed a protocol for JRIO involving three parties in three different locations. In this protocol two parties can implement their secret operators on the third party's secret quantum state via shared quantum channel. Here we have used a three-photon hyperentangled GHZ state as the quantum channel so the required number of photons is just three which is much smaller than that in protocols using conventional entanglements. It is worth noting that the same task of JRIO can also be achieved by running the remote implementation of operator developed in [12] two times, the first time is between Alice and Charlie and the second time is between Alice and Bob. Since each time uses one two-photon hyperentangled EPR state [12], the total number of photons consumed for the quantum channels is four. Therefore, the number of photons needed for the task under consideration is minimum in our protocol.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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