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Fermion masses and mixings in a $U(1)_X$ model based on the $\Sigma(18)$ discrete symmetry

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We have built a renormalizable $U(1)_X$ model with a $\Sigma(18) \times Z_4$ symmetry, whose spontaneous breaking yields the observed standard model (SM) fermion masses and fermionic mixing parameters. The tiny masses of the light active neutrinos are produced by the type I seesaw mechanism mediated by very heavy right-handed Majorana neutrinos. To the best of our knowledge, this model is the first implementation of the $\Sigma(18)$ flavor symmetry in a renormalizable $U(1)_X$ model. Our model allows a successful fit for the SM fermion masses, fermionic mixing angles, and CP phases for both quark and lepton sectors. The obtained values for the physical observables of both quark and lepton sectors are in accordance with the experimental data. We obtain an effective neutrino mass parameter of $\langle m_{ee}\rangle = 1.51 \times 10^{-3}$ eV for normal ordering (NO) and $\langle m_{ee}\rangle = 4.88 \times 10^{-2}$ eV for inverted ordering (IO), which are well consistent with the recent experimental limits on neutrinoless double beta decay.

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1. Introduction

In recent years neutrino oscillation experiments have confirmed that the leptonic mixing angles and neutrino mass squared differences measured with high precision require us to extend the standard model (SM) to successfully explain the current pattern of lepton masses and mixing angles. Among the possible extensions of the SM, the versions with an extra $U(1)_X$ gauge symmetry [1–24] are promising scenarios since the simplest possibility is to introduce three right-handed neutrinos that we need to incorporate the neutrino masses in the SM. In this type of model many phenomena are explained, including neutrino masses [10–13], dark matter [13–19], the muon anomalous magnetic moment [20], inflation [21], leptogenesis [22,23], and gravitational wave radiation [24]; however, the most minimal versions of the $U(1)_X$ models do not include a description of SM fermion masses and mixings.

In order to explain the pattern of fermion masses and mixings, many extensions of the SM have been proposed with the inclusion of non-Abelian discrete groups. These have brought many outstanding advantages; see, for instance, S_3 [25–46], T' [47–52], D_4 [53–59], Q_6 [60–68], A_4 [69–88], Q_8 [89], etc. However, there are substantial differences between our present work and others since in most of the previous works the lepton and/or quark masses and mixings are generated (i) by the texture zero mass matrices [25,89], (ii) via non-renormalizable terms [26–30,34–42,46–48,51,54,55,59,

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61,68–73,75–78,80–83,85,86,88], (iii) at loop levels [44,53,72,79,82], and (iv) by combining with other gauge symmetries and/or supplementing other discrete symmetries [29,31–37,43,45,47,49,50, 52,56–58,60,74,77,84,87]. The $U(1)_{B-L}$ extension of the SM based on S_3, D_4, and Q_6 has been studied in Refs. [90–92], in which the fermion masses and mixings are obtained at the first order of perturbation theory.

In this work, we propose a $\mathrm{U}(1)_X$ renormalizable theory based on the $\Sigma(18)$ flavor symmetry, supplemented by the Z_4 discrete group capable of reproducing the SM fermion masses and mixings *at tree level*. We use the $\Sigma(18)$ discrete group, since it is the simplest non-trivial group of the type $\Sigma(2N^2)$ with $N = 3$ which is isomorphic to $(Z_3 \times Z_3') \rtimes Z_2$. The $\Sigma(18)$ discrete group has 18 elements which are divided into nine conjugacy classes and has nine irreducible representations: the six singlets 1_{+0}, 1_{+1}, 1_{+2}, 1_{-0}, 1_{-1}, 1_{-2}, and the three doublets 2_{10}, 2_{20}, and 2_{21}. The mathematical properties of the $\Sigma(18)$ discrete group are discussed in detail in Ref. [93]. However, for convention, we briefly present the tensor products of $\Sigma(18)$ in Appendix A. The reason for adding the auxiliary symmetry $\mathrm{U}(1)_X$ was introduced in Ref. [94] in another, different, multi-Higgs model based on the A_4 discrete symmetry where the global $U(1)_X$ symmetry is softly broken in the scalar potential in order to prevent the appearance of a Goldstone boson; thus, we do not further discuss this issue here. We note that the $\Sigma(18)$ symmetry has not been considered before in this type of model, and to the best of our knowledge the model proposed in this work is the first implementation of the $\Sigma(18)$ flavor symmetry in a renormalizable $U(1)_X$ model.[1]

The layout of the remainder of the paper is as follows. In Sect. 2 we describe our proposed SM extension by adding the $U(1)_X$, $\Sigma(18)$, and Z_4 symmetries and considering an extended scalar sector and right-handed Majorana neutrinos. In Sect. 3 we describe the implications of our model in lepton masses and mixings. Section 4 deals with quark masses and mixings. The implications of our model in K–$\bar{K}$ and B–$\bar{B}$ mixings are discussed in Sect. 5. The consequences of our model in charged lepton flavor violation are analyzed in Sect. 6. We conclude in Sect. 7. A brief description of the Clebsch–Gordan coefficients for the $\Sigma(18)$ group is presented in Appendix A.

2. The model

The electroweak gauge group of the SM is supplemented by a $\Sigma(18) \times Z_4$ discrete symmetry and a global symmetry $\mathrm{U}(1)_X$, where ψ_{iL}, l_{iR} $(i = 1, 2, 3)$ and φ, φ' carry $X = 1$ while all other fields have $X = 0$. In addition to the SM model particle content, three right-handed neutrinos $(\nu_{1R}, \nu_{\alpha R})$ and one $SU(2)_L$ doublet ϕ with $X = 0$, assigned as 2_{10}, two $SU(2)_L$ doublets φ, φ' with $X = 1$, respectively put in 1_{-0} and 2_{20} under $\Sigma(18)$, and two $SU(2)_L$ singlets χ, ρ with $X = 0$, respectively put in 2_{10} and 1_{+1} under $\Sigma(18)$, are introduced. The particle content of the model is summarized in Tables 1 and 2.

The charged lepton masses can arise from the couplings of $\bar{\psi}_{(1,\alpha)L} l_{(1,\alpha)R}$ to scalars, and the neutrino masses are generated by the couplings of $\bar{\psi}_{(1,\alpha)L}\nu_{(1,\alpha)R}$ and $\bar{\nu}^c_{(1,\alpha)R}\nu_{(1,\alpha)R}$ to scalars; quark masses can arise from couplings of $\bar{Q}_{(\beta,3)L} u_{(\beta,3)R}$ and $\bar{Q}_{(\beta,3)L} d_{(\beta,3)R}$ to scalars. Under G symmetry these couplings are summarized in Table 3.

[1] In this model, fermion masses and mixing angles are generated from renormalizable Yukawa interactions. The non-Abelian discrete groups S_3, T', Q_4, D_4, Q_6 contain one- and two-dimensional representations; however, their singlet/doublet components are combined in different ways. Furthermore, $\Sigma(18)$ contains three two-dimensional representations, 2_{10}, 2_{20}, 2_{21}, where $2^*_{10} = 2_{20}$ and $2^*_{20} = 2_{10}$, while 2_{21} is a real representation

Table 1. Fermion assignments under the symmetry $SU(2)_L \times U(1)_Y \times U(1)_X \times \Sigma(18) \times Z_4 \equiv$ G. Here, $\alpha = 2, 3$ and $\beta = 1, 2$.

Fields	ψ_{1L}	$\psi_{\alpha L}$	l_{1R}	$l_{\alpha R}$	ν_{1R}	$\nu_{\alpha R}$	$Q_{\beta L}$	Q_{3L}	$u_{\beta R}$	u_{3R}	$d_{\beta R}$	d_{3R}
$\mathrm{SU(2)}_L$	2	2	1	1	1	1	2	2	1	1	1	1
$\mathrm{U(1)}_Y$	-1	-1	-2	-2	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\mathrm{U(1)}_X$	1	1	1	1	0	0	0	0	0	0	0	0
$\Sigma(18)$	1_{+0}	2_{10}	1_{+1}	2_{21}	1_{+1}	2_{10}	2_{10}	1_{+2}	2_{20}	1_{+1}	2_{21}	1_{+0}
Z_4	i	i	i	i	i	i	$-i$	i	$-i$	i	$-i$	i

Table 2. Scalar assignments under G symmetry.

Fields	H	ϕ	ϕ'	φ	φ'	χ	ρ
$\mathrm{SU(2)}_L$	2	2	2	2	2	1	1
$\mathrm{U(1)}_Y$	1	1	1	-1	-1	0	0
$\mathrm{U(1)}_X$	0	0	0	1	1	0	0
$\Sigma(18)$	1_{+2}	2_{10}	2_{10}	1_{-0}	2_{20}	2_{10}	1_{+1}
Z_4	1	1	-1	1	1	-1	-1

Table 3. List of couplings which can give masses to the fermions.

Couplings	$[SU(2)_L, U(1)_Y, U(1)_X, \Sigma(18), Z_4]$
$\overline{\psi}_{1L} l_{1R}$	$(2, -1, 0, \underline{1}_{+1}, 1)$
$\overline{\psi}_{1L} l_{\alpha R}$	$(2, -1, 0, \underline{2}_{21}, 1)$
$\overline{\psi}_{\alpha L} l_{1R}$	$(2, -1, 0, \underline{2}_{10}, 1)$
$\overline{\psi}_{\alpha L} l_{\alpha R}$	$(2, -1, 0, \underline{1}_{+1} \oplus \underline{1}_{-1} \oplus \underline{2}_{20}, 1)$
$\overline{\psi}_{1L} \nu_{1R}$	$(2, 1, -1, \underline{1}_{+1}, 1)$
$\overline{\psi}_{1L} \nu_{\alpha R}$	$(2, 1, -1, \underline{2}_{10}, 1)$
$\overline{\psi}_{\alpha L} \nu_{1R}$	$(2, 1, -1, \underline{2}_{10}, 1)$
$\overline{\psi}_{\alpha L} \nu_{\alpha R}$	$(2, 1, -1, \underline{1}_{+0} \oplus \underline{1}_{-0} \oplus \underline{2}_{21}, 1)$
$\overline{\nu}^c_{1R} \nu_{1R}$	$(1, 0, 0, \underline{1}_{+2}, -1)$
$\overline{\nu}^c_{1R} \nu_{\alpha R}$	$(1, 0, 0, \underline{2}_{21}, -1)$
$\overline{\nu}^c_{\alpha R} \nu_{1R}$	$(1, 0, 0, \underline{2}_{21}, -1)$
$\overline{\nu}^c_{\alpha R} \nu_{\alpha R}$	$(1, 0, 0, \underline{1}_{+1} \oplus \underline{1}_{-1} \oplus \underline{2}_{20}, -1)$
$\overline{Q}_{\beta L} u_{\beta R}$	$(2, 1, 0, \underline{1}_{+2} + \underline{1}_{-2} + \underline{2}_{10}, 1)$
$\overline{Q}_{\beta L} u_{3R}$	$(2, 1, 0, \underline{2}_{10}, -1)$
$\overline{Q}_{3L} u_{\beta R}$	$(2, 1, 0, \underline{2}_{10}, -1)$
$\overline{Q}_{3L} u_{3R}$	$(2, 1, 0, \underline{1}_{+2}, 1)$
$\overline{Q}_{\beta L} d_{\beta R}$	$(2, -1, 0, \underline{1}_{+1} + \underline{1}_{-1} + \underline{2}_{20}, 1)$
$\overline{Q}_{\beta L} d_{3R}$	$(2, -1, 0, \underline{2}_{20}, -1)$
$\overline{Q}_{3L} d_{\beta R}$	$(2, -1, 0, \underline{2}_{20}, -1)$
$\overline{Q}_{3L} d_{3R}$	$(2, -1, 0, \underline{1}_{+1}, 1)$

together with its tensor products presented in Appendix A, giving the $\Sigma(18)$ group some advantages compared to the other discrete groups. Our proposed model is completely different from previous works.

In order to generate all SM fermion masses, we introduce seven scalars as shown in Table 2, where H, ϕ, and ϕ' give the charged lepton and quark masses, whereas φ, φ' are responsible for generating the Dirac mass terms and χ, ρ yield the Majorana mass terms. The Yukawa interactions for leptons and quarks invariant under all the symmetries of the model are:[2]

$$\begin{aligned}
-\mathcal{L}_Y^l &= h_1\bar{\psi}_{1L}Hl_{1R} + h_2\left(\bar{\psi}_{\alpha L}l_{\alpha R}\right)_{1_{+1}}H + h_3\left(\bar{\psi}_{\alpha L}l_{\alpha R}\right)_{2_{20}}\phi \\
&+ \frac{x_1}{2}\left(\bar{\psi}_{\alpha L}\nu_{\alpha R}\right)_{\underline{1}_{-0}}\varphi + \frac{x_2}{2}\bar{\psi}_{1L}\left(\varphi'\nu_{\alpha R}\right)_{\underline{1}_{+0}} + \frac{x_3}{2}\left(\bar{\psi}_{\alpha L}\varphi'\right)_{\underline{1}_{+2}}\nu_{1R} \\
&+ \frac{y_1}{2}(\bar{\nu}^c_{1R}\nu_{1R})_{\underline{1}_{+2}}\rho + \frac{y_2}{2}(\bar{\nu}^c_{\alpha R}\nu_{\alpha R})_{\underline{1}_{+1}}\rho^* + \frac{y_3}{2}(\bar{\nu}^c_{\alpha R}\nu_{\alpha R})_{\underline{2}_{20}}\chi + \text{H.c.}, \qquad (1)
\end{aligned}$$

$$\begin{aligned}
-\mathcal{L}_q &= h_{1u}(\bar{Q}_{\beta L}u_{\beta R})_{\underline{1}_{+2}}\widetilde{H} + h_{2u}(\bar{Q}_{3L}\widetilde{H})_{\underline{1}_{+0}}u_{3R} + h_{3u}(\bar{Q}_{\beta L}u_{\beta R})_{\underline{2}_{10}}\widetilde{\phi} \\
&+ h_{4u}(\bar{Q}_{\beta L}u_{3R})_{\underline{2}_{10}}\widetilde{\phi}' + h_{5u}(\bar{Q}_{3L}u_{\beta R})_{\underline{2}_{10}}\widetilde{\phi}' \\
&+ h_{1d}(\bar{Q}_{\beta L}d_{\beta R})_{\underline{1}_{+1}}H + h_{2d}(\bar{Q}_{3L}H)_{\underline{1}_{+0}}d_{3R} + h_{3d}(\bar{Q}_{\beta L}d_{\beta R})_{\underline{2}_{20}}\phi \\
&+ h_{4d}(\bar{Q}_{\beta L}d_{3R})_{\underline{2}_{20}}\phi' + h_{5d}(\bar{Q}_{3L}d_{\beta R})_{\underline{2}_{20}}\phi' + \text{H.c.} \qquad (2)
\end{aligned}$$

It is important to note that the $U(1)_X$ and $\Sigma(18)$ symmetries forbid some Yukawa interactions, thus giving rise to the desired textures for the lepton and quark sectors as shown in Eqs. (32), (34), and (75), and this is an interesting feature of these symmetries. For instance, for the known scalars in Table 2, in the charged lepton sector the interactions $(\overline{\psi}_{1L}l_{1R})\phi$, $(\overline{\psi}_{1L}l_{\alpha R})\phi$, $(\overline{\psi}_{\alpha L}l_{1R})\phi$, $(\overline{\psi}_{1L}l_{\alpha R})H$, and $(\overline{\psi}_{\alpha L}l_{1R})H$ are forbidden by the $\Sigma(18)$ symmetry; in the neutrino sector the interactions $(\overline{\psi}_{1L}\nu_{1R})\varphi$, $(\overline{\psi}_{1L}\nu_{1R})\varphi'$, $(\overline{\psi}_{1L}\nu_{\alpha R})\varphi$, $(\overline{\psi}_{\alpha L}\nu_{1R})\varphi$, $(\overline{\psi}_{\alpha L}\nu_{\alpha R})\varphi'$, $(\nu^c_{1R}\nu_{1R})\chi$, $(\nu^c_{1R}\nu_{\alpha R})\chi$, $(\nu^c_{\alpha R}\nu_{1R})\chi$, $(\nu^c_{1R}\nu_{\alpha R})\rho$, and $(\nu^c_{\alpha R}\nu_{1R})\rho$ are forbidden by the $\Sigma(18)$ symmetry; and in the quark sector $(\bar{Q}_{\beta L}u_{3R})\widetilde{H}$, $(\bar{Q}_{3L}u_{\beta R})\widetilde{H}$, $(\bar{Q}_{\beta L}u_{3R})\widetilde{\phi}$, $(\bar{Q}_{3L}u_{\beta R})\widetilde{\phi}$, $(\bar{Q}_{\beta L}d_{3R})H$, $(\bar{Q}_{3L}d_{\beta R})H$, $(\bar{Q}_{\beta L}d_{3R})\phi$, and $(\bar{Q}_{3L}d_{\beta R})\phi$ are prevented by the $\Sigma(18)$ symmetry, whereas the interactions $(\overline{\psi}_{\alpha L}l_{\alpha R})\widetilde{\varphi}'$, $(\overline{\psi}_{\alpha L}\nu_{\alpha R})\widetilde{\phi}$, $(\overline{\psi}_{1L}\nu_{\alpha R})\widetilde{\phi}$, $(\overline{\psi}_{\alpha L}\nu_{1R})\widetilde{\phi}$, $(\bar{Q}_{\beta L}u_{3R})\varphi'$, and $(\bar{Q}_{\beta L}d_{3R})\widetilde{\varphi}'$ are prevented by the $U(1)_X$ symmetry.

In order to generate the observed pattern of SM fermion masses and mixing angles from the potential minimization condition, we consider the following vacuum expectation value (VEV) configuration for the scalar fields:

$$\begin{aligned}
&\langle H\rangle = \begin{pmatrix} 0 \\ v_H \end{pmatrix}, \quad \langle\phi\rangle = (\langle\phi_1\rangle\ \ \langle\phi_2\rangle), \quad \langle\phi_i\rangle = \begin{pmatrix} 0 \\ v_i \end{pmatrix} \quad (i = 1, 2), \\
&\langle\phi'\rangle = (\langle\phi'_1\rangle\ \ \langle\phi'_1\rangle), \quad \langle\phi'_1\rangle = \begin{pmatrix} 0 \\ v' \end{pmatrix}, \quad \langle\varphi\rangle = \begin{pmatrix} v_\varphi \\ 0 \end{pmatrix}, \\
&\langle\varphi'\rangle = (\langle\varphi'_1\rangle\ \ \langle\varphi'_1\rangle), \quad \langle\varphi'_1\rangle = \begin{pmatrix} v_{\varphi'} \\ 0 \end{pmatrix}, \quad \langle\chi\rangle = (0\ \ \langle\chi_2\rangle), \quad \langle\chi_2\rangle = v_\chi, \quad \langle\rho\rangle = v_\rho. \qquad (3)
\end{aligned}$$

In order to prove that the scalar fields with the VEV alignments as chosen in Eq. (3) are obtained from the minimization condition of $\mathcal{V}_{\text{total}}$ in Appendix B, we set

$$v_{\phi'_2} = v_{\phi'_1} = v', \quad v_{\varphi'_2} = v_{\varphi'_1} = v_{\varphi'}, \quad v_{\chi_1} = 0, \quad v_{\chi_2} = v_\chi, \qquad (4)$$

$$v^*_H = v_H, \ v^*_1 = v_1, \ v^*_2 = v_2, \ v'^* = v', \ v^*_\varphi = v_\varphi, \ v^*_{\varphi'} = v_{\varphi'}, \ v^*_\chi = v_\chi, \ v^*_\rho = v_\rho, \qquad (5)$$

[2] Here, $\widetilde{\phi}$, $\widetilde{\phi}'$, and $\widetilde{H}$ are respectively the complex conjugate fields of ϕ, ϕ', and H, i.e. $\widetilde{\phi} = i\sigma_2\phi^* = (\phi^0_2\ \ -\phi^-_1)^{\mathrm{T}} \sim [2, -1, 0, \underline{2}_{20}, 1]$, $\widetilde{\phi}' \sim [2, -1, 0, \underline{2}_{20}, -1]$, $\widetilde{H} \sim [2, -1, 0, \underline{1}_{+1}, 1]$.

which leads to

$$\frac{\partial \mathcal{V}_{\rm total}}{\partial v_j^*} = \frac{\partial \mathcal{V}_{\rm total}}{\partial v_j}, \qquad \frac{\partial^2 \mathcal{V}_{\rm total}}{\partial v_j^{*2}} = \frac{\partial^2 \mathcal{V}_{\rm total}}{\partial v_j^2} \quad (v_j = v_H, v_1, v_2, v', v_\varphi, v_{\varphi'}, v_\chi, v_\rho), \tag{6}$$

and the minimization conditions of $\mathcal{V}_{\rm total}$ become

$$\frac{\partial \mathcal{V}_{\rm total}}{\partial v_j} = 0, \qquad \frac{\partial^2 \mathcal{V}_{\rm total}}{\partial v_j^2} > 0 \qquad (v_j = v_H, v_1, v_2, v', v_\varphi, v_{\varphi'}, v_\chi, v_\rho). \tag{7}$$

Furthermore, for simplicity and without loss of generality, we consider the following benchmark point of the Yukawa couplings:

$$\lambda_1^\phi = \lambda_2^\phi = \lambda_3^\phi = \lambda^\phi, \quad \lambda_1^{\phi'} = \lambda_2^{\phi'} = \lambda_3^{\phi'} = \lambda^{\phi'}, \quad \lambda_1^{\varphi'} = \lambda_2^{\varphi'} = \lambda_3^{\varphi'} = \lambda^{\varphi'}, \tag{8}$$
$$\lambda_1^\chi = \lambda_2^\chi = \lambda_3^\chi = \lambda^\chi, \quad \lambda_1^{H\phi} = \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda_4^{H\phi} = \lambda^{H\phi}, \tag{9}$$
$$\lambda_1^{H\phi'} = \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda_4^{H\phi'} = \lambda^{H\phi'}, \quad \lambda_1^{H\varphi} = \lambda_2^{H\varphi} = \lambda^{H\varphi}, \tag{10}$$
$$\lambda_1^{H\varphi'} = \lambda_2^{H\varphi'} = \lambda^{H\varphi'}, \quad \lambda_1^{H\chi} = \lambda_2^{H\chi} = \lambda^{H\chi}, \quad \lambda_1^{H\rho} = \lambda_2^{H\rho} = \lambda^{H\rho}, \tag{11}$$
$$\lambda_1^{\phi\phi'} = \lambda_2^{\phi\phi'} = \lambda_3^{\phi\phi'} = \lambda_4^{\phi\phi'} = \lambda_5^{\phi\phi'} = \lambda_6^{\phi\phi'} = \lambda^{\phi\phi'}, \quad \lambda_1^{\phi\varphi} = \lambda_2^{\phi\varphi} = \lambda^{\phi\varphi}, \tag{12}$$
$$\lambda_1^{\phi\varphi'} = \lambda_2^{\phi\varphi'} = \lambda_3^{\phi\varphi'} = \lambda_4^{\phi\varphi'} = \lambda_5^{\phi\varphi'} = \lambda_6^{\phi\varphi'} = \lambda^{\phi\varphi'}, \tag{13}$$
$$\lambda_1^{\phi\chi} = \lambda_2^{\phi\chi} = \lambda_3^{\phi\chi} = \lambda_4^{\phi\chi} = \lambda_5^{\phi\chi} = \lambda_6^{\phi\chi} = \lambda^{\phi\chi}, \quad \lambda_1^{\phi\rho} = \lambda_2^{\phi\rho} = \lambda^{\phi\rho}, \tag{14}$$
$$\lambda_1^{\phi'\varphi} = \lambda_2^{\phi'\varphi} = \lambda^{\phi'\varphi}, \quad \lambda_1^{\phi'\varphi'} = \lambda_2^{\phi'\varphi'} = \lambda_3^{\phi'\varphi'} = \lambda_4^{\phi'\varphi'} = \lambda_5^{\phi'\varphi'} = \lambda_6^{\phi'\varphi'} = \lambda^{\phi'\varphi'}, \tag{15}$$
$$\lambda_1^{\phi'\chi} = \lambda_2^{\phi'\chi} = \lambda_3^{\phi'\chi} = \lambda_4^{\phi'\chi} = \lambda_5^{\phi'\chi} = \lambda_6^{\phi'\chi} = \lambda^{\phi'\chi}, \quad \lambda_1^{\phi'\rho} = \lambda_2^{\phi'\rho} = \lambda^{\phi'\rho}, \tag{16}$$
$$\lambda_1^{\varphi\varphi'} = \lambda_2^{\varphi\varphi'} = \lambda^{\varphi\varphi'}, \quad \lambda_1^{\varphi\chi} = \lambda_2^{\varphi\chi} = \lambda^{\varphi\chi}, \quad \lambda_1^{\varphi\rho} = \lambda_2^{\varphi\rho} = \lambda^{\varphi\rho}, \tag{17}$$
$$\lambda_1^{\varphi'\chi} = \lambda_2^{\varphi'\chi} = \lambda_3^{\varphi'\chi} = \lambda_4^{\varphi'\chi} = \lambda_5^{\varphi'\chi} = \lambda_6^{\varphi'\chi} = \lambda^{\varphi'\chi}, \tag{18}$$
$$\lambda_1^{\varphi'\rho}\lambda_2^{\varphi'\rho} = \lambda^{\varphi'\rho}, \quad \lambda_1^{\chi\rho} = \lambda_2^{\chi\rho} = \lambda^{\chi\rho}, \quad \lambda_1^{H\phi\phi'} = \lambda_2^{H\phi\phi'} = \lambda^{H\phi\phi'}, \tag{19}$$
$$\lambda_1^{H\phi\varphi'} = \lambda_2^{H\phi\varphi'} = \lambda^{H\phi\varphi'}, \quad \lambda_1^{H\phi\chi} = \lambda_2^{H\phi\chi} = \lambda^{H\phi\chi}, \quad \lambda_1^{\varphi\varphi'\chi\rho} = \lambda_2^{\varphi\varphi'\chi\rho} = \lambda^{\varphi\varphi'\chi\rho}. \tag{20}$$

The expressions of the scalar potential minimum equations in Eq. (7) thus reduce to the expressions in Appendix C, in which the system of Eqs. (C.1)–(C.8) always have the solution

$$\lambda^H = \frac{\beta_H}{2(v_1 - v_2)(2v_1^2 + v_1 v_2 + 2v_2^2)v_H^4}, \quad \lambda^\phi = \frac{\beta_\phi}{2(v_1 - v_2)(2v_1^2 + v_1 v_2 + 2v_2^2)}, \tag{21}$$
$$\lambda^{\phi'} = -\frac{\beta_{\phi'}}{12v'^2}, \quad \lambda^\varphi = -\frac{\beta_\varphi}{2v_\varphi^3}, \quad \lambda^{\varphi'} = -\frac{\beta_{\varphi'}}{24v_{\varphi'}^3}, \quad \lambda^\chi = -\frac{\beta_\chi}{4v_\chi^3}, \quad \lambda^\rho = -\frac{\beta_\rho}{v_\rho^3}, \tag{22}$$
$$\lambda^{H\phi} = \frac{\beta_{H\phi}}{2(2v_1^4 + v_1^3 v_2 - v_1 v_2^3 - 2v_2^4)v_H^2}, \tag{23}$$

where β_Φ ($\Phi = H, \phi, \phi', \varphi, \varphi', \chi, \rho$) and $\beta_{H\phi}$ are defined in Appendix D.

We will show that, with λ_Φ and $\lambda_{H\phi}$ in Eqs. (21)–(23), there exist Yukawa couplings such that the expressions in Eq. (7) always give the solution as chosen in Eq. (3). For instance, for the benchmark

$$v_1 = 10^{10}\,\mathrm{eV}, \qquad v_\chi = v_\rho = 1.725\times 10^{11}\,\mathrm{eV}, \tag{24}$$

$$v_H = v' = v_\varphi = v_{\varphi'} = v_2 = 1.5\times 10^{10}\,\mathrm{eV}, \tag{25}$$

$$\mu_H = \mu_\phi = \mu_{\phi'} = \mu_\varphi = \mu_{\varphi'} = \mu_\chi = \mu_\rho = 10^8\,\mathrm{eV}, \tag{26}$$

$$\begin{aligned}\lambda^{H\phi'} &= \lambda^{H\varphi} = \lambda^{H\varphi'} = \lambda^{\phi\phi'} = \lambda^{\phi\varphi} = \lambda^{\phi\varphi'} = \lambda^{\varphi\varphi'}\\ &= \lambda^{\phi'\varphi'} = \lambda^{\phi'\varphi} = \lambda^{H\chi} = \lambda^{H\rho} = \lambda^{x}, \qquad \lambda^{\varphi\varphi'\chi\rho} = \lambda^{H\phi\varphi'} = \lambda^{z},\end{aligned} \tag{27}$$

$$\lambda^{\phi\chi} = \lambda^{\phi\rho} = \lambda^{\varphi\chi} = \lambda^{\varphi'\chi} = \lambda^{\varphi'\rho} = \lambda^{\varphi\rho} = \lambda^{\phi'\chi} = \lambda^{\phi'\rho} = \lambda^{\chi\rho} = \lambda^{y}, \tag{28}$$

the expressions in Eq. (7) are always satisfied in the case of $\lambda^x < 0$, $\lambda^y < 0$, and $\lambda^z < 0$, for example $\lambda^{x,y,z} \in (-10^{-3}, -10^{-5})$ as shown in Fig. 1.[3] Therefore, the VEV alignments in Eq. (3) are the natural solution of the potential minimum condition.

In models with more than one $SU(2)_L$ Higgs doublet, as in the present model, flavor-changing neutral current (FCNC) processes exist; however, they can be suppressed by adding discrete symmetries as presented in Refs. [90,95–101]. In addition, the large amount of parametric freedom allows us to find a suitable region of parameter space where these FCNC processes can be suppressed. A numerical analysis of the FCNC, along with other phenomenological aspects in a multi-Higgs doublet model with the D_4 discrete symmetry is presented in Ref. [59]. The implications of our model for the FCNC interactions are discussed in Sect. 4.

3. Lepton masses and mixings

From the lepton Yukawa terms given by Eq. (1) and the tensor product of $\Sigma(18)$ in Appendix A, we can rewrite the Yukawa interactions in the lepton sector:

$$\begin{aligned}-\mathcal{L}^{l}_{Y} &= h_1\bar{\psi}_{1L}Hl_{1R} + h_2\left(\bar{\psi}_{2L}Hl_{2R} + \bar{\psi}_{3L}Hl_{3R}\right) + h_3\left(\bar{\psi}_{2L}\phi_2 l_{3R} + \bar{\psi}_{3L}\phi_1 l_{2R}\right)\\ &+ \frac{x_1}{2}\left(\bar{\psi}_{2L}\nu_{2R} - \bar{\psi}_{3L}\nu_{3R}\right)\varphi + \frac{x_2}{2}\left(\bar{\psi}_{1L}\varphi'_1\nu_{2R} + \bar{\psi}_{1L}\varphi'_2\nu_{3R}\right)\\ &+ \frac{x_3}{2}\left(\bar{\psi}_{2L}\varphi'_2\nu_{1R} + \bar{\psi}_{3L}\varphi'_1\nu_{1R}\right)\\ &+ \frac{y_1}{2}(\bar{\nu}^c_{1R}\nu_{1R})\rho + \frac{y_2}{2}(\bar{\nu}^c_{2R}\nu_{3R} + \bar{\nu}^c_{3R}\nu_{2R})\rho^* + \frac{y_3}{2}(\bar{\nu}^c_{2R}\chi_1\nu_{2R} + \bar{\nu}^c_{3R}\chi_2\nu_{3R}) + H.c.\end{aligned} \tag{29}$$

With the help of Eq. (3), we get the mass terms for leptons as follows:

$$\begin{aligned}-\mathcal{L}^{\mathrm{mass}}_{\mathrm{lep}} &= h_1 v_H \bar{l}_{1L} l_{1R} + h_2 v_H(\bar{l}_{2L}l_{2R} + \bar{l}_{3L}l_{3R}) + h_3(v_2\bar{l}_{2L}l_{3R} + v_1\bar{l}_{3L}l_{2R})\\ &+ \frac{x_1 v_\varphi}{2}\bar{\nu}_{2L}\nu_{2R} - \frac{x_1 v_\varphi}{2}\bar{\nu}_{3L}\nu_{3R} + \frac{x_2 v_{\varphi'}}{2}\bar{\nu}_{1L}\nu_{2R} + \frac{x_2 v_{\varphi'}}{2}\bar{\nu}_{1L}\nu_{3R}\\ &+ \frac{x_3 v_{\varphi'}}{2}\bar{\nu}_{2L}\nu_{1R} + \frac{x_3 v_{\varphi'}}{2}\bar{\nu}_{3L}\nu_{1R} + \frac{y_1 v_\rho}{2}\bar{\nu}^c_{1R}\nu_{1R}\end{aligned}$$

[3] Here we have used the notations $\delta^2_{v_H} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v_H^2}$, $\delta^2_{v_1} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v_1^2}$, $\delta^2_{v_2} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v_2^2}$, $\delta^2_{v'} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v'^2}$, $\delta^2_{v_\varphi} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v_\varphi^2}$, $\delta^2_{v_{\varphi'}} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v_{\varphi'}^2}$, $\delta^2_{v_\chi} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v_\chi^2}$, $\delta^2_{v_\rho} \equiv \frac{\partial^2\mathcal{V}_{\mathrm{total}}}{\partial v_\rho^2}$.

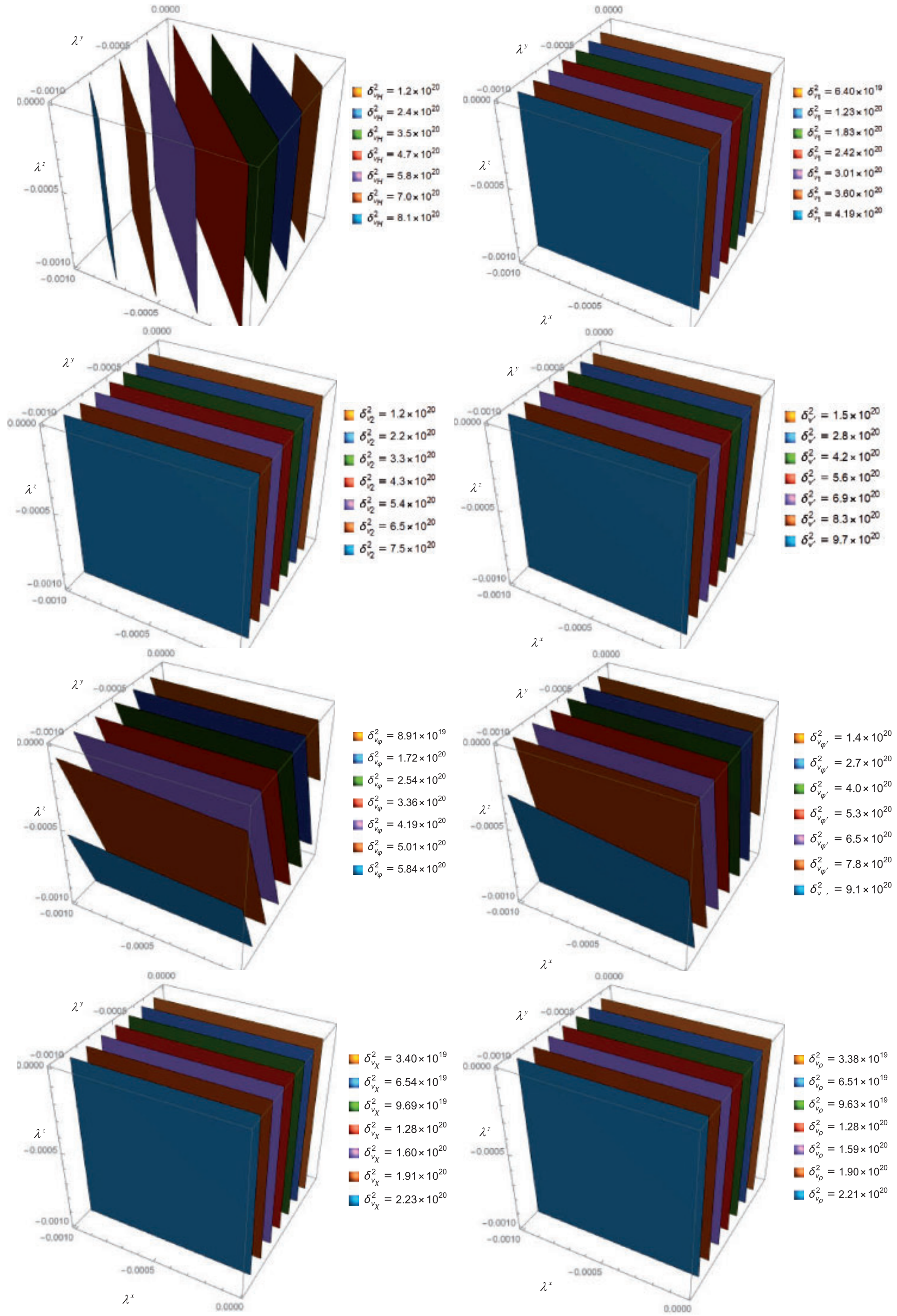

Fig. 1. $\delta^2_{v_h}, \delta^2_{v_1}, \delta^2_{v_2}, \delta^2_{v'}, \delta^2_{v_\varphi}, \delta^2_{v_{\varphi'}}, \delta^2_{v_\chi}$, and $\delta^2_{v_\rho}$ vs. λ^x, λ^y, and λ^z with $\lambda^x \in (-10^{-3}, -10^{-5})$, $\lambda^y \in (-10^{-3}, -10^{-5})$, and $\lambda^z \in (-10^{-3}, -10^{-5})$.

$$+\frac{y_2 v_\rho^*}{2}\left(\bar{\nu}^c_{2R}\nu_{3R} + \bar{\nu}^c_{3R}\nu_{2R}\right) + \frac{y_3 v_\chi}{2}\bar{\nu}^c_{3R}\nu_{3R} + H.c., \tag{30}$$

which can be written in the matrix form

$$-\mathcal{L}^{\text{mass}}_{\text{lep}} = \bar{l}_L \mathcal{M}_{cl} l_R + \frac{1}{2}\bar{n}^c_L \mathcal{M}_\nu n_L + H.c., \tag{31}$$

where

$$l_l=(l_{1L},l_{2L},l_{3L})^{\mathrm{T}},\qquad l_R=(l_{1R},l_{2R},l_{3R})^{\mathrm{T}},\qquad \mathcal{M}_{cl}=\begin{pmatrix} a_1 & 0 & 0\\ 0 & a_2 & b_2\\ 0 & b_1 & a_2\end{pmatrix},\tag{32}$$

$$n_L=\left(\nu_L^c,\nu_R\right)^{\mathrm{T}},\qquad \mathcal{M}_\nu=\begin{pmatrix} 0 & \mathcal{M}_D\\ \mathcal{M}_D^{\mathrm{T}} & \mathcal{M}_R\end{pmatrix},\tag{33}$$

with $\nu_L^c=(\nu_{1L}^c,\nu_{2L}^c,\nu_{3L}^c)^{\mathrm{T}}$, $\nu_R=(\nu_{1R},\nu_{2R},\nu_{3R})^{\mathrm{T}}$; $\mathcal{M}_D$ and $\mathcal{M}_R$ are respectively the Dirac and Majorana neutrino mass matrices,

$$\mathcal{M}_D=\begin{pmatrix} 0 & b_D & b_D\\ c_D & a_D & 0\\ c_D & 0 & -a_D\end{pmatrix},\qquad \mathcal{M}_R=\begin{pmatrix} a_R & 0 & 0\\ 0 & 0 & b_R\\ 0 & b_R & c_R\end{pmatrix},\tag{34}$$

and

$$a_i=h_iv_H,\qquad b_i=h_3v_i\quad (i=1,2),\tag{35}$$

$$a_D=x_1v_\varphi,\qquad b_D=x_2v_{\varphi'},\qquad c_D=x_3v_{\varphi'},$$

$$a_R=y_1v_\rho,\qquad b_R=y_2v_\rho^*,\qquad c_R=y_3v_\chi.\tag{36}$$

Let us define a Hermitian matrix as follows:

$$\mathcal{M}_l=\mathcal{M}_{cl}\mathcal{M}_{cl}^+=\begin{pmatrix} |a_1|^2 & 0 & 0\\ 0 & |a_2|^2+|b_2|^2 & a_2b_1^*+a_2^*b_2\\ 0 & (a_2b_1^*+a_2^*b_2)^* & |a_2|^2+|b_1|^2\end{pmatrix},\tag{37}$$

which can be diagonalized by $\mathcal{U}_{L,R}$ satisfying $\mathcal{U}_L^+\mathcal{M}_l\mathcal{U}_R=\mathrm{diag}(m_e^2,m_\mu^2,m_\tau^2)$, where

$$\mathcal{U}_L=\mathcal{U}_R=\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\theta_l & -\sin\theta_l.e^{-i\alpha}\\ 0 & \sin\theta_l.e^{i\alpha} & \cos\theta_l\end{pmatrix},\tag{38}$$

$$m_e^2=|a_1|^2,\qquad m_{\mu,\tau}^2=\frac{1}{2}\left(\gamma_1\mp\gamma_2\right),\tag{39}$$

with

$$\gamma_2=\sqrt{(|b_1|^2-|b_2|^2)^2+4|a_2|^2\left(|b_1|^2+|b_2|^2\right)+8|a_2|^2|b_1||b_2|\cos(2\alpha_2-\beta_1-\beta_2)},$$

$$\gamma_1=2|a_2|^2+|b_1|^2+|b_2|^2,\qquad \alpha_2=\arg(a_2),\qquad \beta_i=\arg(b_i)\quad (i=1,2),\tag{40}$$

$$\alpha=\frac{i}{2}\log\left[\frac{a_2b_1^*+a_2^*b_2}{(a_2b_1^*+a_2^*b_2)^*}\right],\qquad \theta_l=\arctan\left[\frac{(a_2^*b_1+a_2b_2^*)e^{-i\alpha}}{|a_2|^2+|b_2|^2-m_\tau^2}\right].\tag{41}$$

Comparing the result in Eq. (39) with the experimental values of the charged lepton masses given in Ref. [102], $m_e\simeq 0.51999\,\mathrm{MeV}$, $m_\mu\simeq 105.65837\,\mathrm{MeV}$, $m_\tau=1776.86\,\mathrm{MeV}$, we obtain:

$$|a_1|=0.510999\times 10^6\,\mathrm{eV},\quad \gamma_1=3.1684\times 10^{18}\,\mathrm{eV}^2,\quad \gamma_2=3.14607\times 10^{18}\,\mathrm{eV}^2.\tag{42}$$

In the case $\alpha_2 = \beta_1 = \beta_2$ and $|v_1| \sim |v_2|$, we get:

$$|h_1| \sim \frac{5\times 10^5}{|v_H|}, \qquad |h_2| \sim \frac{8\times 10^8}{|v_H|}, \qquad |h_3| \sim \frac{9\times 10^8}{|v_2|}. \tag{43}$$

As we will see below, since the charged lepton mixing matrix $\mathcal{U}_L$ is non-trivial in our model it can contribute to the final leptonic mixing matrix, defined by $U = \mathcal{U}_L^{+}\mathcal{U}_\nu$, where $\mathcal{U}_L$ refers to the left-handed charged-lepton mixing matrix and $\mathcal{U}_\nu$ is the neutrino mixing matrix.

Regarding the neutrino sector, from Eq. (34), the light active neutrino mass matrix arises from the type-I seesaw mechanism as follows:

$$\mathcal{M}_\nu = -\mathcal{M}_D\mathcal{M}_R^{-1}\mathcal{M}_D^{\mathrm{T}} = \begin{pmatrix} \frac{b_D^2(c_R-2b_R)}{b_R^2} & \frac{a_Db_D(c_R-b_R)}{b_R^2} & \frac{a_Db_D}{b_R} \\ \frac{a_Db_D(c_R-b_R)}{b_R^2} & \frac{a_D^2c_R}{b_R^2}-\frac{c_D^2}{a_R} & \frac{a_D^2}{b_R}-\frac{c_D^2}{aR} \\ \frac{a_Db_D}{b_R} & \frac{a_D^2}{b_R}-\frac{c_D^2}{aR} & -\frac{c_D^2}{a_R} \end{pmatrix}, \tag{44}$$

which has three exact eigenvalues,

$$m_1 = 0, \qquad m_{2,3} = \kappa_1 \mp \kappa_2, \tag{45}$$

where

$$\kappa_1 = \frac{(a_D^2+b_D^2)c_R}{2b_R^2} - \frac{b_D^2}{a_R} - \frac{c_D^2}{a_R}, \qquad \kappa_2 = \frac{\sqrt{\Bbbk}}{2a_Rb_R^4}, \tag{46}$$
$$\Bbbk = b_R^4\Big\{4b_R^2\left[a_R(a_D^2+b_D^2)-b_Rc_D^2\right]^2 - 4a_Rb_D^2b_R\left[a_R(a_D^2+b_D^2)-b_Rc_D^2\right]c_R + a_R^2(a_D^2+b_D^2)^2cR^2\Big\},$$

and the corresponding mixing matrix is

$$\mathbb{R} = \begin{pmatrix} \frac{\mathbb{K}}{\sqrt{\mathbb{K}^2+2}} & \frac{\mathbb{K}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{\mathbb{K}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}} \\ -\frac{1}{\sqrt{\mathbb{K}^2+2}} & \frac{\mathbb{N}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{\mathbb{N}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}} \\ \frac{1}{\sqrt{\mathbb{K}^2+2}} & \frac{1}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{1}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}} \end{pmatrix}.\mathbb{P}, \tag{47}$$

where $\mathbb{P} = \mathrm{diag}(1,\ 1,\ i)$ and $\mathbb{K}$, $\mathbb{K}_\mp$, and $\mathbb{N}_\mp$ are defined as

$$\mathbb{K} = \frac{a_D}{b_D}, \qquad \mathbb{K}_\mp = \kappa_{11} \mp \kappa_{12}, \qquad \mathbb{N}_\mp = \epsilon_{11} \mp \epsilon_{12}, \tag{48}$$

with

$$\kappa_{11} = \frac{b_D}{2a_D}\left\{\frac{(a_D^2+b_D^2)a_Rc_R}{b_R\left[(a_D^2+b_D^2)a_R-b_Rc_D^2\right]} - 2\right\}, \quad \kappa_{12} = \frac{b_D\sqrt{\Bbbk}}{2a_Db_R^3\left[(a_D^2+b_D^2)a_R-c_D^2b_R\right]},$$
$$\epsilon_{11} = \frac{(a_D^2+b_D^2)a_Rc_R}{2b_R\left[a_R(a_D^2+b_D^2)-c_D^2b_R\right]}, \qquad \epsilon_{12} = \frac{\sqrt{\Bbbk}}{2b_R^3\left[(a_D^2+b_D^2)a_R-c_D^2b_R\right]}, \tag{49}$$

and a_D, b_D, c_D, a_R, b_R, and c_R as given in Eq. (36).

From the explicit expressions of $m_{2,3}$, $\mathbb{K}$, $\mathbb{K}_{\mp}$, and $\mathbb{N}_{\mp}$ in Eqs. (45), (46), (48), and (49), the following relations hold:

$$1+\mathbb{K}\mathbb{K}_{-}-\mathbb{N}_{-}=0, \qquad 1+\mathbb{K}\mathbb{K}_{+}-\mathbb{N}_{+}=0, \qquad 1+\mathbb{K}_{-}\mathbb{K}_{+}+\mathbb{N}_{-}\mathbb{N}_{+}=0, \tag{50}$$

$$\begin{aligned}
a_D &= \frac{(\mathbb{N}_{-}+\mathbb{N}_{+}-2)b_D}{\mathbb{K}_{-}+\mathbb{K}_{+}},\\
a_R &= \frac{(\Lambda_{\mathrm{N}}+2)\Lambda_{\mathrm{K}}^2 b_R c_D^2}{\left[\Lambda_{\mathrm{K}}^2+\Lambda_{\mathrm{N}}(\Lambda_{\mathrm{N}}-2)\right](\Lambda_{\mathrm{N}}-2)b_D^2-(m_2+m_3)\Lambda_{\mathrm{K}}^2 b_R},\\
c_R &= \frac{\left\{2[2\Lambda_{\mathrm{K}}^2+(\Lambda_{\mathrm{N}}-2)^2]b_D^2+\Lambda_{\mathrm{K}}^2(m_2+m_3)b_R\right\}\Lambda_{\mathrm{N}} b_R}{\left[\Lambda_{\mathrm{K}}^2+(\Lambda_{\mathrm{N}}-2)^2\right](\Lambda_{\mathrm{N}}+2)b_D^2},
\end{aligned} \tag{51}$$

with

$$\Lambda_{\mathrm{K}}=\mathbb{K}_{-}+\mathbb{K}_{+}, \qquad \Lambda_{\mathrm{N}}=\mathbb{N}_{-}+\mathbb{N}_{+}. \tag{52}$$

The effective neutrino mass matrix $\mathcal{M}_\nu$ in Eq. (44) is diagonalized as

$$\mathcal{U}_\nu^\dagger \mathcal{M}_\nu \mathcal{U}_\nu = \begin{cases}
\begin{pmatrix} 0 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3 \end{pmatrix}, \ \mathcal{U}_\nu = \begin{pmatrix}
\frac{\mathbb{K}}{\sqrt{\mathbb{K}^2+2}} & \frac{\mathbb{K}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{i\mathbb{K}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}}\\
-\frac{1}{\sqrt{\mathbb{K}^2+2}} & \frac{\mathbb{N}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{i\mathbb{N}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}}\\
\frac{1}{\sqrt{\mathbb{K}^2+2}} & \frac{1}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{i}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}}
\end{pmatrix} & \text{for NO},\\[2ex]
\begin{pmatrix} m_2 & 0 & 0\\ 0 & m_3 & 0\\ 0 & 0 & 0 \end{pmatrix}, \ \mathcal{U}_\nu = \begin{pmatrix}
\frac{\mathbb{K}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{\mathbb{K}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}} & \frac{i\mathbb{K}}{\sqrt{\mathbb{K}^2+2}}\\
\frac{\mathbb{N}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{\mathbb{N}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}} & -\frac{i}{\sqrt{\mathbb{K}^2+2}}\\
\frac{1}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{1}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}} & \frac{i}{\sqrt{\mathbb{K}^2+2}}
\end{pmatrix} & \text{for IO},
\end{cases} \tag{53}$$

where $m_{2,3}$ and $\mathbb{K}$, $\mathbb{K}_{\mp}$, and $\mathbb{N}_{\mp}$ are respectively given in Eqs. (45) and (48).

The final leptonic mixing matrix then reads:

$$U=\mathcal{U}_L^{+}\mathcal{U}_\nu = \begin{cases}
\begin{pmatrix}
\frac{\mathbb{K}}{\sqrt{\mathbb{K}^2+2}} & \frac{\mathbb{K}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{i\mathbb{K}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}}\\
\frac{e^{-i\alpha}\sin\theta_l-\cos\theta_l}{\sqrt{\mathbb{K}^2+2}} & \frac{\cos\theta_l\mathbb{N}_-+e^{-i\alpha}\sin\theta_l}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{i(\cos\theta_l\mathbb{N}_++e^{-i\alpha}\sin\theta)}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}}\\
\frac{e^{i\alpha}\sin\theta_l+\cos\theta_l}{\sqrt{\mathbb{K}^2+2}} & \frac{\cos\theta_l-e^{i\alpha}\sin\theta_l\mathbb{N}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{N}_-^2+1}} & \frac{i(\cos\theta_l-e^{i\alpha}\sin\theta_l\mathbb{N}_+)}{\sqrt{\mathbb{K}_+^2+\mathbb{N}_+^2+1}}
\end{pmatrix} & \text{for NO},\\[2ex]
\begin{pmatrix}
\frac{\mathbb{K}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{K}_-^2+1}} & \frac{\mathbb{K}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{K}_+^2+1}} & \frac{i\mathbb{K}}{\sqrt{\mathbb{K}^2+2}}\\
\frac{\cos\theta_l\mathbb{N}_-+e^{-i\alpha}\sin\theta_l}{\sqrt{\mathbb{K}_-^2+\mathbb{K}_-^2+1}} & \frac{\cos\theta_l\mathbb{N}_++e^{-i\alpha}\sin\theta_l}{\sqrt{\mathbb{K}_+^2+\mathbb{K}_+^2+1}} & \frac{i(e^{-i\alpha}\sin\theta_l-\cos\theta_l)}{\sqrt{\mathbb{K}^2+2}}\\
\frac{\cos\theta_l-e^{i\alpha}\sin\theta_l\mathbb{N}_-}{\sqrt{\mathbb{K}_-^2+\mathbb{K}_-^2+1}} & \frac{\cos\theta_l-e^{i\alpha}\sin\theta_l\mathbb{N}_+}{\sqrt{\mathbb{K}_+^2+\mathbb{K}_+^2+1}} & \frac{i(e^{i\alpha}\sin\theta_l+\cos\theta_l)}{\sqrt{\mathbb{K}^2+2}}
\end{pmatrix} & \text{for IO}.
\end{cases} \tag{54}$$

In the three-neutrino oscillation picture, the lepton mixing matrix can be parametrized as [102][4]

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.P, \tag{55}$$

whereby θ_{12}, θ_{23}, and θ_{13} can be defined via the elements of the leptonic mixing matrix:

$$s_{13}^2 = |U_{13}|^2, \qquad t_{12}^2 = \left|\frac{U_{12}}{U_{11}}\right|^2, \qquad t_{23}^2 = \left|\frac{U_{23}}{U_{33}}\right|^2. \tag{56}$$

The neutrino mass spectrum is currently unknown, and it can be NO or IO depending on the sign of Δm_{32}^2 [102], as presented in the next section.

3.1. Normal spectrum

In NO, the Jarlskog invariant $J_{\rm CP}$ which determines the magnitude of CP violation in neutrino oscillations [102], determined from Eqs. (50) and (54), takes the form

$$J_{\rm CP}^{\rm N} = {\rm Im}(U_{23}U_{13}^*U_{12}U_{22}^*) = \frac{\mathbb{K}_+^2(1-\mathbb{N}_+^2)\cos\theta_l\sin\theta_l\sin\alpha}{\left[2\mathbb{K}_+^2 + (\mathbb{N}_+ - 1)^2\right](1+\mathbb{K}_+^2+\mathbb{N}_+^2)}. \tag{57}$$

Comparing Eq. (57) with its corresponding expression in the standard parametrization of the neutrino mixing matrix given in Ref. [102], $J_{\rm CP} = s_{13}c_{13}^2s_{12}c_{12}s_{23}c_{23}\sin\delta$, we get:

$$\sin\delta^{\rm N} = \frac{\mathbb{K}_+^2(1-\mathbb{N}_+^2)}{\left[2\mathbb{K}_+^2 + (\mathbb{N}_+ - 1)^2\right](1+\mathbb{K}_+^2+\mathbb{N}_+^2)}\frac{\cos\theta_l\sin\theta_l\sin\alpha}{s_{13}c_{13}^2s_{12}c_{12}s_{23}c_{23}}. \tag{58}$$

Furthermore, from Eqs. (54) and (56), for NO we get:

$$\begin{aligned} s_{13}^2 &= \frac{\mathbb{K}_+^2}{1+\mathbb{K}_+^2+\mathbb{N}_+^2}, \qquad t_{12}^2 = \frac{\mathbb{K}_+^2(\mathbb{N}_+ + 1)^2}{(\mathbb{N}_+ - 1)^2(1+\mathbb{K}_+^2+\mathbb{N}_+^2)}, \\ t_{23}^2 &= \frac{\sin^2\theta_l + \cos^2\theta_l\mathbb{N}_+^2 + \sin(2\theta_l)\cos\alpha.\mathbb{N}_+}{\cos^2\theta_l + \sin^2\theta_l\mathbb{N}_+^2 - \sin(2\theta_l)\cos\alpha.\mathbb{N}_+}. \end{aligned} \tag{59}$$

Combining Eqs. (50) and (59) yields

$$\mathbb{K}_+ = \frac{\sqrt{2}s_{13}\sqrt{s_{13}^2+t_{12}^2}}{c_{13}(t_{12}-s_{13})}, \qquad \mathbb{N}_+ = 1 + \frac{2s_{13}}{t_{12}-s_{13}}, \tag{60}$$

$$\cos\alpha = \frac{(1-t_{23}^2)(s_{13}^2+t_{12}^2) + 2(1-2\sin^2\theta_l)(1+t_{23}^2)s_{13}t_{12}}{(s_{13}^2-t_{12}^2)(1+t_{23}^2)\sin(2\theta_l)}. \tag{61}$$

Next, substituting Eq. (60) into Eq. (58) yields

$$\sin\delta = -\frac{\sin(2\theta_l)\sin\alpha}{\sin(2\theta_{23})}. \tag{62}$$

[4] Here, δ is the Dirac CP-violating phase and $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with θ_{12}, θ_{23}, and θ_{13} being the solar, atmospheric, and reactor angles, respectively. P contains two Majorana phases (α_{21}, α_{31}) which play no role in neutrino oscillations, $P = {\rm diag}\big(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}}\big)$, and thus will be ignored.

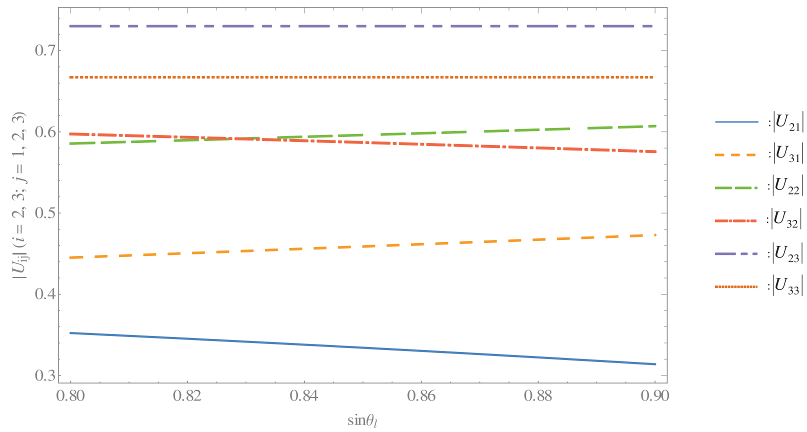

Fig. 2. $|U^{\rm N}_{ij}|$ $(i = 2, 3; j = 1, 2, 3)$ as functions of $\sin\theta_l$ with $\sin\theta_l \in (0.8, 0.9)$ for NO.

Table 4. The model parameters in the case $\delta = 1.36\pi$ [102] for NO.

Parameters	Derived values
$\mathbb{K}$	2.05
$\mathbb{K}_-$	−0.734
$\mathbb{K}_+$	0.278
$\mathbb{N}_-$	−0.507
$\mathbb{N}_+$	1.57
θ_l	55.1°
α	73.9°

We note that the elements U_{1i} $(i = 1, 2, 3)$ depend only on θ_{12} and θ_{13}, while U_{2i} and U_{3i} $(i = 1, 2, 3)$ depend on all the lepton mixing angles θ_{ij} $(ij = 12, 23, 13)$ and θ_l.

For NO, by taking the best-fit values of the leptonic mixing angles θ_{ij} $(i, j = 1, 2, 3)$, $\sin^2\theta_{12} = 0.307$, $\sin^2\theta_{13} = 2.18 \times 10^{-2}$, and $\sin^2\theta_{23} = 0.545$ [102], we obtain $U^{\rm N}_{11} = 0.823$, $U^{\rm N}_{12} = -0.548$, and $U^{\rm N}_{13} = 0.148i$, and $U^{\rm N}_{ij}$ $(i = 2, 3; j = 1, 2, 3)$ depend only on $\sin\theta_l$ as plotted in Fig. 2 with $\sin\theta_l \in (0.8, 0.9)$. In the case $\delta = 1.36\pi$ [102], from Eq. (62) we obtain the model parameters shown in Table 4.

The lepton mixing matrix in Eq. (54) then takes the form

$$U^{\rm N} = \begin{pmatrix} 0.823 & -0.548 & 0.148i \\ -0.138 - 0.316i & -0.046 - 0.588i & 0.419 + 0.598i \\ 0.321 + 0.316i & 0.513 + 0.298i & 0.657 + 0.113i \end{pmatrix}, \tag{63}$$

which is unitary and consistent with the constraint on the absolute values of the entries of the lepton mixing matrix given in Ref. [117].

Now, by using the recent best-fit values for the squared neutrino mass differences, $\Delta m^2_{21} = 7.53 \times 10^{-5}\,\text{eV}^2$ and $\Delta m^2_{32} = 2.453 \times 10^{-3}\,\text{eV}^2$ for NO [102], we get a solution

$$\kappa_1 = 2.95 \times 10^{-2}, \qquad \kappa_2 = 2.08 \times 10^{-2},$$
$$m_1 = 0\,\text{eV}, \qquad m_2 = 8.68 \times 10^{-3}\,\text{eV}, \qquad m_3 = 5.03 \times 10^{-2}\,\text{eV}. \tag{64}$$

The absolute neutrino mass, defined as the sum of the masses of the three neutrino mass eigenstates, is found to be $\sum_{i=1}^{3} m^{\mathrm{N}}_{\nu_i} = 5.90 \times 10^{-2}$ eV. At present, the sum of the three neutrino masses has not been precisely determined; however, the result obtained from our model is well consistent with the strongest bound from cosmology, $\sum m_\nu < 0.078$ eV [103].

Next, substituting explicit expressions for b_D, b_D, c_D, a_R, b_R, and c_R from Eq. (36), the obtained values of $\mathbb{K}_\mp$ and $\mathbb{N}_\mp$ in Table 4, and $m_{2,3}$ in Eq. (64) into Eq. (51), we get the following relations:

$$x_1 = 2.05\left(\frac{v_{\varphi'}}{v_\varphi}\right)x_2, \qquad y_1 = \frac{1}{1.16\left(\frac{v_\rho}{v^*_\rho}\right)\frac{x_2^2}{x_3^2 y_2} - 0.0192\left(\frac{v_\rho}{v^2_{\varphi'}}\right)\frac{1}{x_3^2}},$$

$$y_3 = \frac{0.827 v^*_\rho y_2}{v_\chi} + \frac{0.00393 (v^*_\rho)^2 y_2^2}{v_\chi v^2_{\varphi'} x_2^2}. \tag{65}$$

3.2. *Inverted spectrum*

Similar to the NO, from Eqs. (54) and (56) for IO we get a solution

$$\mathbb{K}_+ = -\frac{\sqrt{2} c_{13} t_{12}}{1 + s_{13} t_{12}}, \qquad \mathbb{N}_+ = \frac{2}{1 + s_{13} t_{12}} - 1, \qquad \cos\alpha = \frac{1}{\sin(2\theta)}\frac{1 - t^2_{23}}{1 + t^2_{23}}, \tag{66}$$

and the Jarlskog invariant J_{CP} determined from Eq. (54) and $\sin\delta$ take the form

$$J^{\mathrm{I}}_{\mathrm{CP}} = -\frac{\mathbb{K}\mathbb{K}_+(1 + \mathbb{N}_+)\cos\theta_l \sin\theta_l \sin\alpha}{(\mathbb{K}^2 + 2)(\mathbb{K}^2_+ + \mathbb{N}^2_+ + 1)}, \tag{67}$$

$$\sin\delta^{\mathrm{I}} = \frac{\mathbb{K}\mathbb{K}_+(1 + \mathbb{N}_+)}{(\mathbb{K}^2_+ + 2)(1 + \mathbb{K}^2_+ + \mathbb{N}^2_+)}\frac{\cos\theta_l \sin\theta_l \sin\alpha}{s_{13} c^2_{13} s_{12} c_{12} s_{23} c_{23}}. \tag{68}$$

With the help of Eq. (50), it is easy to show that $J^{\mathrm{N}}_{\mathrm{CP}} = J^{\mathrm{I}}_{\mathrm{CP}}$ and $\sin\delta^{\mathrm{N}} = \sin\delta^{\mathrm{I}}$; thus, the relations in Eqs. (57) and (58) are satisfied for both normal and inverted orderings, and the differences start from Eqs. (60) and (66).

Next, by taking the best-fit values of the leptonic mixing angles θ_{ij} $(i, j = 1, 2, 3)$ for IO, $s^2_{12} = 0.307$, $s^2_{23} = 0.547$, and $s^2_{13} = 2.18 \times 10^{-2}$ [102], we get $U^{\mathrm{I}}_{11} = 0.823$, $U^{\mathrm{I}}_{12} = -0.548$, and $U^{\mathrm{I}}_{13} = 0.148i$, and the other elements of U depend only on $\sin\theta_l$ as plotted in Fig. 3.

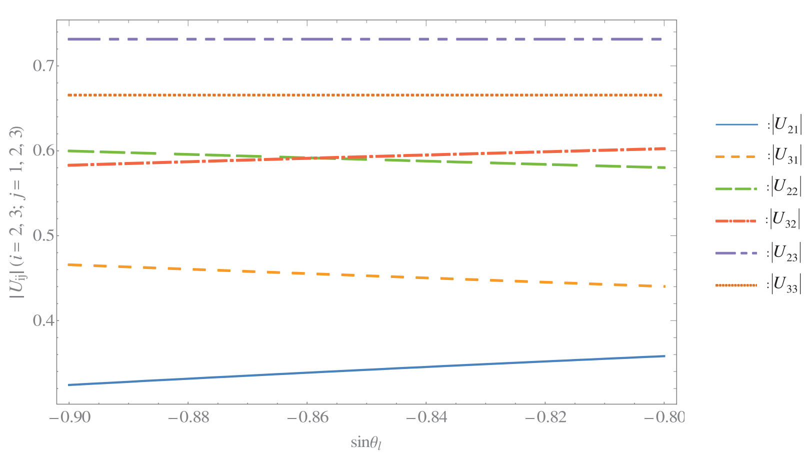

Fig. 3. $|U^{\mathrm{I}}_{ij}|$ $(i = 2, 3; j = 1, 2, 3)$ as functions of $\sin\theta_l$ with $\sin\theta_l \in (-0.9, -0.8)$ for IO.

Table 5. The model parameters in the case $\delta = 1.36\pi$ [102] for IO.

Parameters	Derived values
$\mathbb{K}$	0.211
$\mathbb{K}_-$	2.7
$\mathbb{K}_+$	−0.848
$\mathbb{N}_-$	1.57
$\mathbb{N}_+$	0.821
α	84.0°

In the case where the CP-violating phase takes the best-fit values, $\delta = 1.36\pi$ [102], we find $\sin\theta_l = -0.537$ ($\theta_l = 327.5°$); the other model parameters are explicitly given in Table 5.

The PMNS leptonic mixing matrix of Eq. (54) takes the form

$$U^{\rm I} = \begin{pmatrix} 0.823 & -0.548 & 0.148i \\ 0.387 + 0.163i & 0.412 + 0.345i & -0.373 - 0.629i \\ 0.284 + 0.256i & 0.575 + 0.283i & 0.373 + 0.551i \end{pmatrix}, \tag{69}$$

which is unitary and consistent with the constraint on the absolute values of the entries of the lepton mixing matrix given in Ref. [117].

Now, by using the recent best-fit values for the squared neutrino mass differences [102], $\Delta m^2_{21} = 7.53 \times 10^{-5}\,\mathrm{eV}^2$ and $\Delta m^2_{32} = -2.546 \times 10^{-3}\,\mathrm{eV}^2$ for IO, we get a solution

$$\begin{aligned} &\kappa_1 = 5.01 \times 10^{-2}, \qquad \kappa_2 = 3.76 \times 10^{-4}, \\ &m_1 = 4.97 \times 10^{-2}\,\mathrm{eV}, \qquad m_2 = 5.05 \times 10^{-2}\,\mathrm{eV}, \qquad m_3 = 0\,\mathrm{eV}. \end{aligned} \tag{70}$$

Using our best-fit results given above, we find that the sum of three light neutrino masses is given by $\sum_{i=1}^{3} m^{\rm I}_{\nu_i} = 0.1\,\mathrm{eV}$. Currently, the cosmological data set limits on the sum of light active neutrino masses [103] are $\sum m_\nu < 0.152\,\mathrm{eV}$ in the minimal ΛCDM + $\sum m_\nu$ model, $\sum m_\nu < 0.118\,\mathrm{eV}$ in in the minimal ΛCDM + $\sum m_\nu$ model with high-l polarization data, $\sum m_\nu < 0.305\,\mathrm{eV}$ in the DDE model with TT + BAO + PAN + $\tau 0p055$, $\sum m_\nu < 0.305\,\mathrm{eV}$ in the DDE model with TT + BAO + PAN + $\tau 0p055$, $\sum m_\nu < 0.247\,\mathrm{eV}$ in the DDE model with TTTEEE + BAO + PAN + R16 + $\tau 0p055$, and $\sum m_\nu < 0.101\,\mathrm{eV}$ in the NPDDE model with TTTEEE + BAO + PAN + $\tau 0p055$. Thus, our obtained value for the sum of the light active neutrino masses is well consistent with the aforementioned bounds arising from cosmology.

By substituting explicit expressions for b_D, b_D, c_D, a_R, b_R, and c_R from Eq. (36), the obtained values of $\mathbb{K}_\mp$ and $\mathbb{N}_\mp$ in Table 5, and $m_{1,2}$ in Eq. (70) into Eq. (51), we get the relations

$$x_1 = \frac{0.211 v_{\varphi'} x_2}{v_\varphi}, \qquad y_1 = \frac{1}{\frac{0.113 v_\rho x_2^2}{v_\rho^* x_3^2 y_2} - \frac{0.0228 v_\rho}{v_{\varphi'}^2 x_3^2}}, \qquad y_3 = \frac{2.13 v_\rho^* y_2}{v_\chi} + \frac{0.052 (v_\rho^*)^2 y_2^2}{v_\chi v_{\varphi'}^2 x_2^2}. \tag{71}$$

3.3. *Effective neutrino mass parameters*

The effective neutrino mass parameters governing beta decay and neutrinoless double beta decay are defined as $m_\beta = \sqrt{\sum_{k=1}^{3} |U_{ek}|^2 m_k^2}$ and $\langle m_{ee}\rangle = \left|\sum_{k=1}^{3} U_{ek}^2 m_k\right|$, where U_{ek} ($k = 1, 2, 3$) are the leptonic mixing matrix elements and m_k correspond to the masses of the three light neutrinos. Using the model parameters obtained in Sects. 3.1 and 3.2, we find the following numerical values for the abovementioned mass parameters:

$$\langle m_{ee}\rangle = \begin{cases} 1.51\times 10^{-3}\,\mathrm{eV} & \text{for NO,}\\ 4.88\times 10^{-2}\,\mathrm{eV} & \text{for IO;}\end{cases} \tag{72}$$

$$m_\beta = \begin{cases} 8.82\times 10^{-3}\,\mathrm{eV} & \text{for NO,}\\ 4.94\times 10^{-2}\,\mathrm{eV} & \text{for IO.}\end{cases} \tag{73}$$

The resulting effective neutrino mass parameters in Eqs. (72) and (73), for both normal and inverted orderings, are below all the upper bounds arising from present $0\nu\beta\beta$ decay experiments such as KamLAND-Zen $\langle m_{ee}\rangle < 0.05\div 0.16\,\mathrm{eV}$ [104], GERDA $\langle m_{ee}\rangle < 0.12\div 0.26\,\mathrm{eV}$ [105], MAJORANA $\langle m_{ee}\rangle < 0.24\div 0.53\,\mathrm{eV}$ [106], EXO $\langle m_{ee}\rangle < 0.17\div 0.49\,\mathrm{eV}$ [107–109], and CUORE $\langle m_{ee}\rangle < 0.11\div 0.5\,\mathrm{eV}$ [110,111]. Hence, our obtained effective neutrino mass parameter is beyond the reach of the present and forthcoming $0\nu\beta\beta$ decay experiments.

4. Quark masses and mixings

In this section we show that our model is able to successfully reproduce the observed pattern of SM quark masses and mixing parameters. Indeed, from the quark Yukawa terms given by Eq. (2) and the tensor product of $\Sigma(18)$ in Appendix A, we can rewrite the Yukawa interactions in the quark sector in the form

$$\begin{aligned}-\mathcal{L}_{\mathrm{q}} &= h_{1u}(\bar{Q}_{1L}\widetilde{H}u_{2R}+\bar{Q}_{2L}\widetilde{H}u_{1R})+h_{2u}\bar{Q}_{3L}\widetilde{H}u_{3R}+h_{3u}(\bar{Q}_{1L}\widetilde{\phi}_1u_{1R}+\bar{Q}_{2L}\widetilde{\phi}_2u_{2R})\\ &\quad+h_{4u}(\bar{Q}_{1L}\widetilde{\phi}'_2u_{3R}+\bar{Q}_{2L}\widetilde{\phi}'_1u_{3R})+h_{5u}(\bar{Q}_{3L}\widetilde{\phi}'_2u_{1R}+\bar{Q}_{3L}\widetilde{\phi}'_1u_{2R})\\ &\quad+h_{1d}(\bar{Q}_{1L}Hd_{1R}+\bar{Q}_{2L}Hd_{2R})+h_{2d}\bar{Q}_{3L}Hd_{3R}+h_{3d}(\bar{Q}_{1L}\phi_2d_{2R}+\bar{Q}_{2L}\phi_1d_{1R})\\ &\quad+h_{4d}(\bar{Q}_{1L}\phi'_1d_{3R}+\bar{Q}_{2L}\phi'_2d_{3R})+h_{5d}(\bar{Q}_{3L}\phi'_2d_{1R}+\bar{Q}_{3L}\phi'_1d_{2R})+\mathrm{H.c.}\end{aligned} \tag{74}$$

With the VEV alignments of H and ϕ as chosen in Eq. (3), the mass Lagrangian of quarks reads

$$\begin{aligned}-\mathcal{L}^{\mathrm{mass}}_{\mathrm{q}} &= h_{1u}v_H(\bar{u}_{1L}u_{2R}+\bar{u}_{2L}u_{1R})+h_{2u}v_H\bar{u}_{3L}u_{3R}+h_{3u}(v_1^*\bar{u}_{1L}u_{1R}+v_2^*\bar{u}_{2L}u_{2R})\\ &\quad+h_{4u}(v_2^{'*}\bar{u}_{1L}u_{3R}+v_1^{'*}\bar{u}_{2L}u_{3R})+h_{5u}(v_2^{'*}\bar{u}_{3L}u_{1R}+v_1^{'*}\bar{u}_{3L}u_{2R})\\ &\quad+h_{1d}v_H(\bar{d}_{1L}d_{1R}+\bar{d}_{2L}d_{2R})+h_{2d}v_H\bar{d}_{3L}d_{3R}+h_{3d}(v_2\bar{d}_{1L}d_{2R}+v_1\bar{d}_{2L}d_{1R})\\ &\quad+h_{4d}(v'_1\bar{d}_{1L}d_{3R}+v'_2\bar{d}_{2L}d_{3R})+h_{5d}(v'_2\bar{d}_{3L}d_{1R}+v'_1\bar{d}_{3L}d_{2R})+\mathrm{H.c.}\\ &\equiv(\bar{u}_{1L},\bar{u}_{2L},\bar{u}_{3L})M_u(u_{1R},u_{2R},u_{3R})^{\mathrm{T}}+(\bar{d}_{1L},\bar{d}_{2L},\bar{d}_{3L})M_d(d_{1R},d_{2R},d_{3R})^{\mathrm{T}}+\mathrm{H.c.},\end{aligned}$$

where the up- and down-quark mass matrices are

$$M_u=\begin{pmatrix} b_{1u} & a_{1u} & c_{2u}\\ a_{1u} & b_{2u} & c_{1u}\\ g_{2u} & g_{1u} & a_{2u}\end{pmatrix},\qquad M_d=\begin{pmatrix} a_{1d} & b_{1d} & c_{1d}\\ b_{2d} & a_{1d} & c_{2d}\\ g_{2d} & g_{1d} & a_{2d}\end{pmatrix}, \tag{75}$$

with

$$\begin{aligned} &a_{(1,2)u}=h_{(1,2)u}v_H,\quad b_{(1,2)u}=h_{3u}v^*_{(1,2)},\quad c_{(1,2)u}=h_{4u}v^{'*}_{(1,2)},\quad g_{(1,2)u}=h_{5u}v^{'*}_{(1,2)},\\ &a_{(1,2)d}=h_{(1,2)d}v_H,\quad b_{(1,2)d}=h_{3d}v_{(2,1)},\quad c_{(1,2)d}=h_{4d}v'_{(1,2)},\quad g_{(1,2)d}=h_{5d}v'_{(1,2)}.\end{aligned} \tag{76}$$

Now we turn our attention to the experimental values for the SM quark masses and Cabibbo–Kobayashi–Maskawa (CKM) parameters [112,113]:

$$
\begin{aligned}
& m_u = 1.24 \pm 0.22\,\text{MeV}, \quad m_d = 2.69 \pm 0.19\,\text{MeV}, \quad m_s = 53.5 \pm 4.6\,\text{MeV}, \\
& m_c = 0.63 \pm 0.02\,\text{GeV}, \quad m_t = 172.9 \pm 0.4\,\text{GeV}, \quad m_b = 2.86 \pm 0.03\,\text{GeV}, \\
& \sin\theta_{12} = 0.2245 \pm 0.00044, \quad \sin\theta_{23} = 0.0421 \pm 0.00076, \quad \sin\theta_{13} = 0.00365 \pm 0.00012, \\
& J = (3.18 \pm 0.15) \times 10^{-5}.
\end{aligned} \tag{77}
$$

We look for the eigenvalue problem solutions reproducing the experimental values of the quark masses and the CKM parameters given by Eq. (77), finding the following:

$$
\begin{aligned}
M_u &= \begin{pmatrix} -27.6375 - 62.7392i & -31.3158 - 66.0758i & 24.5526 - 34.1468i \\ -31.3158 - 66.0758i & -36.0052 - 69.6769i & 25.5349 - 37.6027i \\ 24.5526 - 34.1468i & 25.5349 - 37.6027i & 25.9351 + 2.79953i \end{pmatrix} \text{GeV}, \\
M_d &= \begin{pmatrix} 1.24842 & 1.22041 - 0.00504449i & 0.37787 - 0.563654i \\ 1.22041 + 0.00504449i & 1.24842 & 0.368746 - 0.599054i \\ 0.37787 + 0.563654i & 0.368746 + 0.599054i & 0.419347 \end{pmatrix} \text{GeV}.
\end{aligned} \tag{78}
$$

This shows that our model is consistent with and successfully accommodates the experimental values of the physical observables of the quark sector: the six quark masses, the quark mixing angles, and the CP-violating phase in the quark sector.

5. K–$\bar{K}$ and B–$\bar{B}$ mixings

In this section we discuss the implications of our model in the FCNC interactions in the down-type quark sector. The FCNC Yukawa interactions in the down-type quark sector give rise to meson oscillations. Here we focus on the K–$\bar{K}$ mixing, whose corresponding ΔM_K parameter arises from the following effective Hamiltonian:

$$
\mathcal{H}_{\text{eff}}^{(\Delta S=2)} = \frac{G_F^2 m_W^2}{16\pi^2} \sum_i C_i(\mu)\, O_i(\mu). \tag{79}
$$

In our analysis of the K–$\bar{K}$ mixing we follow the approach of Refs. [114,115]: the K–$\bar{K}$ mixing in our model mainly arises from the tree-level exchange of neutral CP-even and CP-odd scalars, thus giving rise to the following operators:

$$
O_1^{LL} = (\bar{s}P_L d)(\bar{s}P_L d), \qquad O_1^{RR} = (\bar{s}P_R d)(\bar{s}P_R d), \qquad O_1^{LR} = (\bar{s}P_L d)(\bar{s}P_R d), \tag{80}
$$

where the corresponding Wilson coefficients are given by

$$
C_1^{LL} = \frac{16\pi^2}{G_F^2 m_W^2}\left(\sum_{i=1}^{N} \frac{y^2_{H_i^0 \bar{s}_R d_L}}{m^2_{H_i^0}} - \sum_{i=1}^{N-4} \frac{y^2_{A_i^0 \bar{s}_R d_L}}{m^2_{H_i^0}}\right) = \frac{16\pi^2}{G_F^2 m_W^2}\widetilde{C}_1^{LL}, \tag{81}
$$

$$
C_1^{RR} = \frac{16\pi^2}{G_F^2 m_W^2}\left(\sum_{i=1}^{N} \frac{y^2_{H_i^0 \bar{s}_L d_R}}{m^2_{H_i^0}} - \sum_{i=1}^{N-4} \frac{y^2_{A_i^0 \bar{s}_L d_R}}{m^2_{H_i^0}}\right) = \frac{16\pi^2}{G_F^2 m_W^2}\widetilde{C}_1^{RR}, \tag{82}
$$

$$C_2^{LR} = \frac{16\pi^2}{G_F^2 m_W^2}\left(\sum_{i=1}^{N} \frac{y_{H_i^0 \bar{s}_R d_L} y_{H_i^0 \bar{s}_L d_R}}{m^2_{H_i^0}} - \sum_{i=1}^{N-4} \frac{y_{A_i^0 \bar{s}_R d_L} y_{A_i^0 \bar{s}_L d_R}}{m^2_{H_i^0}}\right) = \frac{16\pi^2}{G_F^2 m_W^2}\widetilde{C}_2^{LR}, \tag{83}$$

with

$$\widetilde{C}_1^{LL} = \sum_{i=1}^{N} \frac{y^2_{H_i^0 \bar{s}_R d_L}}{m^2_{H_i^0}} - \sum_{i=1}^{N-4} \frac{y^2_{A_i^0 \bar{s}_R d_L}}{m^2_{A_i^0}}, \tag{84}$$

$$\widetilde{C}_1^{RR} = \sum_{i=1}^{N} \frac{y^2_{H_i^0 \bar{s}_L d_R}}{m^2_{H_i^0}} - \sum_{i=1}^{N-4} \frac{y^2_{A_i^0 \bar{s}_L d_R}}{m^2_{A_i^0}}, \tag{85}$$

$$\widetilde{C}_2^{LR} = \sum_{i=1}^{N} \frac{y_{H_i^0 \bar{s}_R d_L} y_{H_i^0 \bar{s}_L d_R}}{m^2_{H_i^0}} - \sum_{i=1}^{N-4} \frac{y_{A_i^0 \bar{s}_R d_L} y_{A_i^0 \bar{s}_L d_R}}{m^2_{A_i^0}}. \tag{86}$$

Here, $N = 11$ is the number of CP-even scalars of our model, whereas $N - 4 = 7$ is the number of CP-odd scalars. We note that our model is an extended 8HDM where the scalar sector is enlarged by the inclusion of three real gauge singlet scalars.

On the other hand, the K–$\bar{K}$ mass splitting has the form

$$\begin{aligned}\Delta m_K &= 2\mathrm{Re}\left\langle \overline{K^0} \left| \mathcal{H}_{\mathrm{eff}}^{(\Delta S=2)} \right| K^0 \right\rangle \\ &= \frac{G_F^2 m_W^2}{12\pi^2} m_K f_K^2 \eta_K B_K \left[P_2^{LR} C_2^{LR} + P_1^{LL}\left(C_1^{LL} + C_1^{RR}\right)\right] \\ &= \frac{4}{3} m_K f_K^2 \eta_K B_K \left[P_2^{LR} \widetilde{C}_2^{LR} + P_1^{LL}\left(\widetilde{C}_1^{LL} + \widetilde{C}_1^{RR}\right)\right].\end{aligned} \tag{87}$$

Then, it follows that

$$M_{12} = \frac{\Delta m_K}{m_K} = \frac{4}{3} f_K^2 \eta_K B_K \left[P_2^{LR} \widetilde{C}_2^{LR} + P_1^{LL}\left(\widetilde{C}_1^{LL} + \widetilde{C}_1^{RR}\right)\right]. \tag{88}$$

Using the parameters [114,115]

$$\begin{aligned}&\Delta m_K = 3.483 \times 10^{-12}\,\mathrm{MeV}, \qquad m_K = 497.614\,\mathrm{MeV}, \qquad f_K = 160\,\mathrm{MeV}, \\ &B_K = 0.85 \pm 0.15, \qquad \sqrt{B_K} f_K = 135\,\mathrm{MeV}, \qquad \eta_K = 0.57, \qquad P_1^{LL} = -9.3, \\ &P_2^{LR} = 30.6, \qquad M_{12} = \frac{\Delta m_K}{m_K} = 7.2948 \times 10^{-15},\end{aligned} \tag{89}$$

we get the following constraint arising from K–$\bar{K}$ mixing:

$$P_2^{LR} \widetilde{C}_2^{LR} + P_1^{LL}\left(\widetilde{C}_1^{LL} + \widetilde{C}_1^{RR}\right) \le 4.41 \times 10^{-19}\,\mathrm{MeV}^{-2}. \tag{90}$$

Given the large amount of parametric freedom in both the fermion and scalar sectors of our model, such a constraint can be fulfilled. To show explicitly that the above constraint resulting from K–$\bar{K}$ mixing is successfully fulfilled, and given the large number of parameters in our model, we consider a simplified benchmark scenario:

$$y_{H_1^0 \bar{s}_R d_L} = y_{H_1^0 \bar{s}_L d_R} = y_{H_j^0 \bar{s}_R d_L} = y_{H_j^0 \bar{s}_L d_R} = y_{A_i^0 \bar{s}_R d_L} = y_{A_i^0 \bar{s}_L d_R} = y, \tag{91}$$

$$m_{H_1^0} = m_h = 126\,\mathrm{GeV}, \qquad m_{H_j^0} = m_H, \tag{92}$$

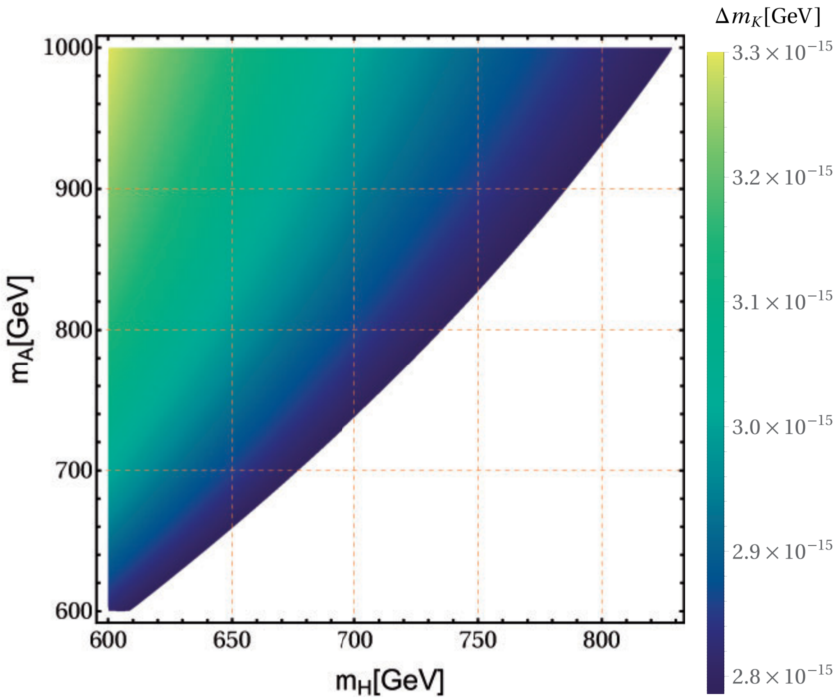

Fig. 4. Allowed region in the m_H–m_A plane consistent with the constraint arising from K–$\bar{K}$ mixing.

$$m_{A_j^0} = m_A, \qquad j = 2, 3, \ldots, 11, \; i = 1, 2, \ldots, 7. \tag{93}$$

Here we identified H_1^0 with a 126 GeV SM-like Higgs boson. We plot in Fig. 4 the allowed parameter space in the m_H–m_A plane consistent with the constraint arising from K–$\bar{K}$ mixing in the aforementioned simplified benchmark scenario of our model. Here, for the sake of simplicity we have set $y = 2\times10^{-5}$. The figure shows that our model can successfully accommodate the constraint arising from K–$\bar{K}$ mixing in a large region of parameter space.

It is worth mentioning that we are considering a scenario where the down-type quark Yukawa couplings have been taken to be real, which implies that the CP violation in the quark sector only arises from the up-type quark sector. Consequently, in that scenario the stringent constraints that are usually imposed on any possible new contribution to the K–$\bar{K}$ mixing from CP-violating processes are not relevant in our case. Furthermore, as regards the B_d–$\bar{B}_d$ and B_s–$\bar{B}_s$ mixings, we have numerically checked that in the aforementioned simplified benchmark scenario and the region of parameter space described above with a flavor-violating Yukawa coupling of the order of 10^{-5}, the obtained values for the Δm_{B_d} and Δm_{B_s} parameters are about two and four orders of magnitude, respectively, below their corresponding experimental values of $\Delta m_{B_d} = 3.337\times10^{-10}$ MeV and $\Delta m_{B_s} = 1.042\times10^{-8}$ MeV. On the other hand, in the simplified benchmark scenario, when the couplings of the flavor-changing neutral Yukawa interactions responsible for the B_d–$\bar{B}_d$ and B_s–$\bar{B}_s$ mixings take values of about 2×10^{-4} and 10^{-3}, respectively, the B_d–$\bar{B}_d$ and B_s–$\bar{B}_s$ mixings arising from these interactions reach values close to their experimental upper bounds, thus giving rise to the allowed regions in the m_H–m_A plane consistent with these constraints and displayed in Fig. 5.

6. Charged lepton flavor violation

In this section we analyze the implications of our model in charged lepton flavor violation. From the charged lepton Yukawa interactions it follows that the $\mu \to e\gamma$ and $\tau \to e\gamma$ decays are absent in our

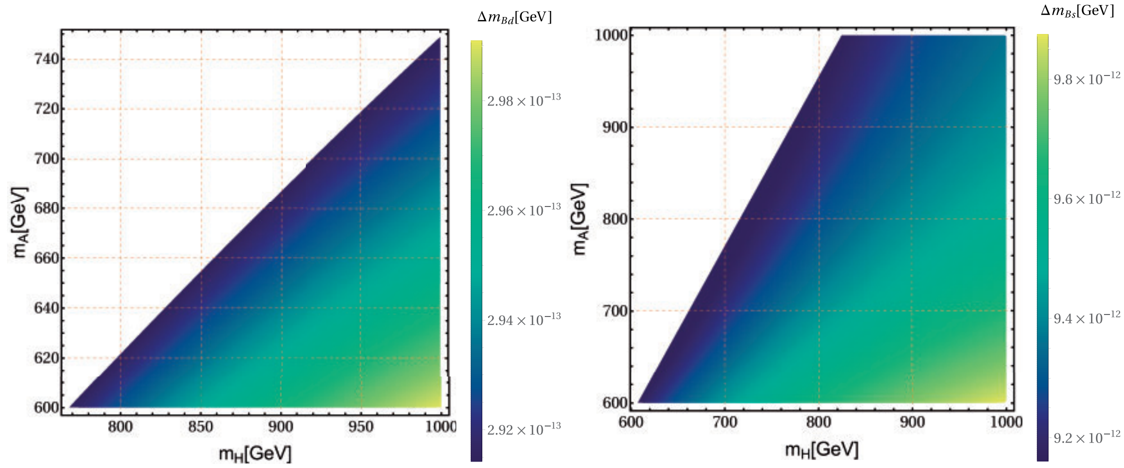

Fig. 5. Allowed region in the m_H–m_A plane consistent with the constraint arising from B_d–$\bar{B}_d$ (left) and B_s–$\bar{B}_s$ (right) mixings. The couplings of the flavor-changing neutral Yukawa interactions have been set to be equal to 2×10^{-4} (left) and 10^{-3} (right).

model since there are no flavor-changing neutral (FCN) scalar interactions involving the first family of SM charged leptons with the remaining ones. However, there are flavor-changing neutral scalar interactions involving the tau and the muon that give rise to the $\tau\to\mu\gamma$ decay. The $\tau\to\mu\gamma$ decay appears at one-loop level and involves the exchange of electrically neutral CP-even and CP-odd scalars and the tau and muon leptons running in the internal lines of the loop. Its branching ratio is given by [116]

$$Br(\tau\to\mu\gamma)\simeq\frac{3(4\pi)^3\alpha_{EM}}{4G_F^2}\left(\frac{1}{16\pi^2}\right)^2\left|\sum_{l=\mu,\tau}\sum_{i=1}^{N}\frac{y_{H_i^0\tau\bar{l}}\,y_{H_i^0\bar{l}\mu}}{m^2_{H_i^0}}\left\{\frac{1}{6}-\frac{m_l}{m_\mu}\left[\frac{3}{2}+\ln\left(\frac{m_l^2}{m^2_{H_i^0}}\right)\right]\right\}\right.$$
$$\left.+\sum_{l=e,\mu}\sum_{i=1}^{N-4}\frac{y_{A_i^0\tau\bar{l}}\,y_{A_i^0\bar{l}\mu}}{m^2_{A_i^0}}\left\{\frac{1}{6}+\frac{m_l}{m_\mu}\left[\frac{3}{2}+\ln\left(\frac{m_l^2}{m^2_{A_i^0}}\right)\right]\right\}\right|^2. \quad (94)$$

To simplify our analysis we choose the benchmark scenario described in Sect. 4. We display in Fig. 6 the allowed parameter space in the m_H–m_A plane consistent with the existing $\tau\to\mu\gamma$ experimental constraints. Here we set the Yukawa couplings of the FCN leptonic Yukawa interactions equal to 10^{-2} and 5×10^{-3} for the left and right plots, respectively. Consequently, our model is highly consistent with the constraints arising from lepton flavor-violating decays for a large region of the parameter space. Furthermore, this allows for charged lepton flavor-violating (CLFV) processes within the reach of future experimental sensitivity.

7. Conclusions

We have built a renormalizable theory where the SM gauge symmetry is extended by the inclusion of the global $U(1)_X$ symmetry and the $\Sigma(18)\times Z_4$ discrete group, which leads to a successful fit of SM fermion masses and mixings. The right-handed neutrinos are responsible for the generation of the tiny active neutrino masses through a type-I seesaw mechanism mediated by heavy right-handed Majorana neutrinos. The resulting physical parameters are in accordance with the recent experimental data. We find values for the effective neutrino mass parameters $\langle m_{ee}\rangle=1.51\times10^{-3}$ eV

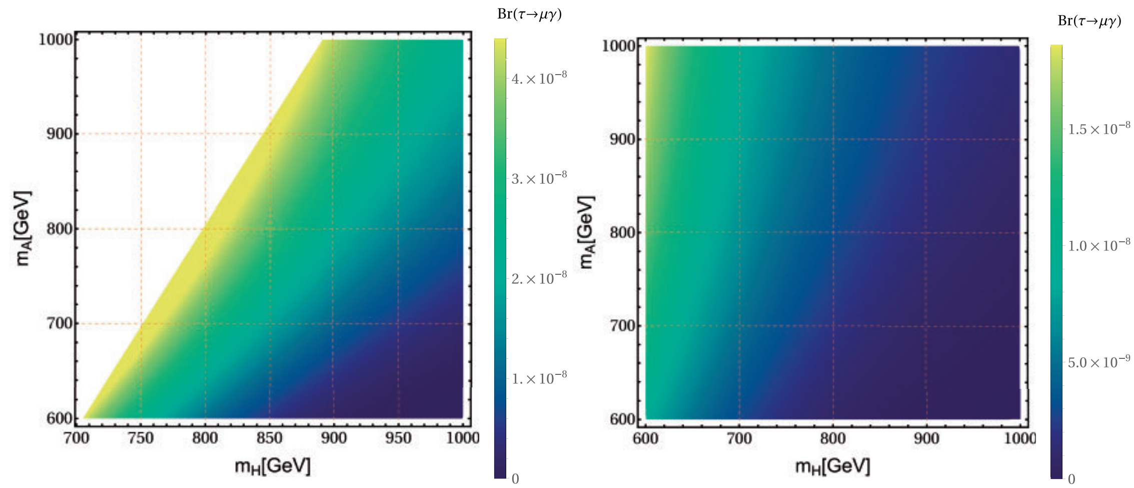

Fig. 6. Allowed region in the m_H–m_A plane consistent with the charged lepton flavor-violating constraints. The Yukawa couplings of the FCN leptonic Yukawa interactions have been set to be equal to 10^{-2} and 5×10^{-3} for the left and right plots, respectively.

for normal ordering and $\langle m_{ee}\rangle = 4.88 \times 10^{-2}$ eV for inverted ordering, which are well in accordance with the recent experimental limits on neutrinoless double beta decay. The proposed model also successfully accommodates the recent experimental values of the physical observables of the quark sector, including the six quark masses, the quark mixing angles, and the CP-violating phase in the quark sector. Furthermore, our model can also accommodate the constraints arising from K–$\bar{K}$, B_d–$\bar{B}_d$, and B_s–$\bar{B}_s$ mixings as well as the constraints arising from charged lepton flavor violation.

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Appendix A. The Clebsch–Gordan coefficients of the $\Sigma(18)$ group

$\Sigma(18)$ is the simplest non-trivial group of $\Sigma(2N^2)$ with $N = 3$ which is isomorphic to $(Z_3 \times Z_3') \rtimes Z_2$. It has 18 elements, $b^k a^m a'^n$ for $k = 0, 1$ and $m, n = 0, 1, 2$, where a, a', and b satisfy $a^3 = a'^3 = e$, $b^2 = e$, $aa' = a'a$, and $bab = a'$. The elements of $\Sigma(18)$ are divided into nine conjugacy classes with 1_{+0}, 1_{+1}, 1_{+2}, 1_{-0}, 1_{-1}, 1_{-2}, 2_{10}, 2_{20}, and 2_{21} as the nine irreducible representations. The tensor products between doublets of $\Sigma(18)$ are given by [93]:

$$2_{10}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes 2_{10}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1_{+1}(x_1y_2 + x_2y_1) \oplus 1_{-1}(-x_1y_2 + x_2y_1) \oplus 2_{20}\begin{pmatrix} x_1y_1 \\ x_2y_2 \end{pmatrix},$$

$$2_{20}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes 2_{20}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1_{+2}(x_1y_2 + x_2y_1) \oplus 1_{-2}(-x_1y_2 + x_2y_1) \oplus 2_{10}\begin{pmatrix} x_1y_1 \\ x_2y_2 \end{pmatrix},$$

$$2_{21}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes 2_{21}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1_{+0}(x_1y_2 + x_2y_1) \oplus 1_{-0}(-x_1y_2 + x_2y_1) \oplus 2_{21}\begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix},$$

$$2_{20}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes 2_{10}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1_{+0}(x_1y_1 + x_2y_2) \oplus 1_{-0}(x_1y_1 - x_2y_2) \oplus 2_{21}\begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix},$$
$$2_{21}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes 2_{10}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1_{+2}(x_1y_2 + x_2y_1) \oplus 1_{-2}(-x_1y_2 + x_2y_1) \oplus 2_{10}\begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix},$$
$$2_{21}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes 2_{20}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 1_{+1}(x_1y_1 + x_2y_2) \oplus 1_{-1}(x_1y_1 - x_2y_2) \oplus 2_{20}\begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix}, \quad \text{(A.1)}$$

where x_i and y_i $(i = 1, 2)$ are the components of two different representations.

The tensor products between singlets and doublets of $\Sigma(18)$ are [93]:

$$1_{\pm0}(x) \otimes 2_{21}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{21}\begin{pmatrix} xy_1 \\ xy_2 \end{pmatrix}, \qquad 1_{\pm1}(x) \otimes 2_{21}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{20}\begin{pmatrix} xy_2 \\ xy_1 \end{pmatrix},$$
$$1_{\pm2}(x) \otimes 2_{21}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{10}\begin{pmatrix} xy_1 \\ xy_2 \end{pmatrix}, \qquad 1_{\pm0}(x) \otimes 2_{20}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{20}\begin{pmatrix} xy_1 \\ xy_2 \end{pmatrix},$$
$$1_{\pm1}(x) \otimes 2_{20}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{10}\begin{pmatrix} xy_2 \\ xy_1 \end{pmatrix}, \qquad 1_{\pm2}(x) \otimes 2_{20}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{21}\begin{pmatrix} xy_2 \\ xy_1 \end{pmatrix},$$
$$1_{\pm0}(x) \otimes 2_{10}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{10}\begin{pmatrix} xy_1 \\ xy_2 \end{pmatrix}, \qquad 1_{\pm1}(x) \otimes 2_{10}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{21}\begin{pmatrix} xy_1 \\ xy_2 \end{pmatrix},$$
$$1_{\pm2}(x) \otimes 2_{10}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2_{20}\begin{pmatrix} xy_2 \\ xy_1 \end{pmatrix}. \quad \text{(A.2)}$$

The tensor products between singlets of $\Sigma(18)$ are [93]:

$$1_{\pm0}(x) \otimes 1_{\pm0}(y) = 1_{+0}(xy), \qquad 1_{\pm1}(x) \otimes 1_{\pm1}(y) = 1_{+2}(xy),$$
$$1_{\pm2}(x) \otimes 1_{\pm2}(y) = 1_{+1}(xy), \qquad 1_{\pm1}(x) \otimes 1_{\pm0}(y) = 1_{+1}(xy),$$
$$1_{\pm2}(x) \otimes 1_{\pm0}(y) = 1_{+2}(xy), \qquad 1_{\pm2}(x) \otimes 1_{\pm1}(y) = 1_{+0}(xy),$$
$$1_{\pm0}(x) \otimes 1_{\mp0}(y) = 1_{-0}(xy), \qquad 1_{\pm1}(x) \otimes 1_{\mp1}(y) = 1_{-2}(xy),$$
$$1_{\pm2}(x) \otimes 1_{\mp2}(y) = 1_{-1}(xy), \qquad 1_{\pm1}(x) \otimes 1_{\mp0}(y) = 1_{-1}(xy),$$
$$1_{\pm2}(x) \otimes 1_{\mp0}(y) = 1_{-2}(xy), \qquad 1_{\pm2}(x) \otimes 1_{\mp1}(y) = 1_{-0}(xy). \quad \text{(A.3)}$$

The rules to conjugate all the representations of $\Sigma(18)$ are:

$$1^*_{+0}(x^*) = 1_{+0}(x^*), \qquad 1^*_{+1}(x^*) = 1_{+2}(x^*), \qquad 1^*_{+2}(x^*) = 1_{+1}(x^*),$$
$$1^*_{-0}(x^*) = 1_{-0}(x^*), \qquad 1^*_{-1}(x^*) = 1_{-2}(x^*), \qquad 1^*_{-2}(x^*) = 1_{-1}(x^*), \quad \text{(A.4)}$$

$$2^*_{10}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = 2_{20}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}, \qquad 2^*_{20}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = 2_{10}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix},$$
$$2^*_{21}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = 2_{21}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}. \quad \text{(A.5)}$$

Appendix B. Renormalizable Higgs potential invariant under G symmetry

The general renormalizable potential invariant under G symmetry is a sum of the following components:[5]

$$\begin{aligned}\mathcal{V}_{\text{total}} = {} & \mathcal{V}(H) + \mathcal{V}(\phi) + \mathcal{V}(\phi') + \mathcal{V}(\varphi) + \mathcal{V}(\varphi') + \mathcal{V}(\chi) + \mathcal{V}(\rho) + \mathcal{V}(H,\phi) + \mathcal{V}(H,\phi') \\ & + \mathcal{V}(H,\varphi) + \mathcal{V}(H,\varphi') + \mathcal{V}(H,\chi) + \mathcal{V}(H,\rho) + \mathcal{V}(\phi,\phi') + \mathcal{V}(\phi,\varphi) + \mathcal{V}(\phi,\varphi') \\ & + \mathcal{V}(\phi,\chi) + \mathcal{V}(\phi,\rho) + \mathcal{V}(\phi',\varphi) + \mathcal{V}(\phi',\varphi') + \mathcal{V}(\phi',\chi) + \mathcal{V}(\phi',\rho) + \mathcal{V}(\varphi,\varphi') \\ & + \mathcal{V}(\varphi,\chi) + \mathcal{V}(\varphi,\rho) + \mathcal{V}(\varphi',\chi) + \mathcal{V}(\varphi',\rho) + \mathcal{V}(\chi,\rho) + \mathcal{V}_{\text{three}} + \mathcal{V}_{\text{four}},\end{aligned} \tag{B.1}$$

where

$$\mathcal{V}(H) = \mu_H^2 H^\dagger H + \lambda^H (H^\dagger H)_{1_{+0}} (H^\dagger H)_{1_{+0}}, \tag{B.2}$$

$$\mathcal{V}(\phi) = \mu_\phi^2 \phi^\dagger \phi + \lambda_1^\phi (\phi^\dagger\phi)_{1_{+0}}(\phi^\dagger\phi)_{1_{+0}} + \lambda_2^\phi (\phi^\dagger\phi)_{1_{-0}}(\phi^\dagger\phi)_{1_{-0}} + \lambda_3^\phi (\phi^\dagger\phi)_{2_{21}}(\phi^\dagger\phi)_{2_{21}}, \tag{B.3}$$

$$\mathcal{V}(\phi') = \mathcal{V}(\phi \to \phi'), \quad \mathcal{V}(\varphi) = \mathcal{V}(H \to \varphi), \quad \mathcal{V}(\varphi') = \mathcal{V}(\phi \to \varphi'), \quad \mathcal{V}(\chi) = \mathcal{V}(\phi \to \chi), \tag{B.4}$$

$$\mathcal{V}(\rho) = \mathcal{V}(H \to \rho), \tag{B.5}$$

$$\begin{aligned}\mathcal{V}(H,\phi) = {} & \lambda_1^{H\phi}(H^\dagger H)_{1_{+0}}(\phi^\dagger\phi)_{1_{+0}} + \lambda_2^{H\phi}(H^\dagger\phi)_{2_{21}}(\phi^\dagger H)_{2_{21}} \\ & + \lambda_3^{H\phi}(H^\dagger\phi)_{2_{21}}(H^\dagger\phi)_{2_{21}} + \lambda_4^{H\phi}(\phi^\dagger H)_{2_{21}}(\phi^\dagger H)_{2_{21}}, \quad \mathcal{V}(H,\phi') = \mathcal{V}(H,\phi\to\phi'),\end{aligned} \tag{B.6}$$

$$\mathcal{V}(H,\varphi) = \lambda_1^{H\varphi}(H^\dagger H)_{1_{+0}}(\varphi^\dagger\varphi)_{1_{+0}} + \lambda_2^{H\varphi}(H^\dagger\varphi)_{1_{-1}}(\varphi^\dagger H)_{1_{-2}}, \tag{B.7}$$

$$\mathcal{V}(H,\varphi') = \lambda_1^{H\varphi'}(H^\dagger H)_{1_{+0}}(\varphi'^\dagger\varphi')_{1_{+0}} + \lambda_2^{H\varphi'}(H^\dagger\varphi')_{2_{10}}(\varphi'^\dagger H)_{2_{20}}, \tag{B.8}$$

$$\mathcal{V}(H,\chi) = \lambda_1^{H\chi}(H^\dagger H)_{1_{+0}}(\chi^\dagger\chi)_{1_{+0}} + \lambda_2^{H\chi}(H^\dagger\chi)_{2_{21}}(\chi^\dagger H)_{2_{21}}, \tag{B.9}$$

$$\mathcal{V}(H,\rho) = \lambda_1^{H\rho}(H^\dagger H)_{1_{+0}}(\rho^\dagger\rho)_{1_{+0}} + \lambda_2^{H\rho}(H^\dagger\rho)_{1_{+2}}(\rho^\dagger H)_{1_{+1}}, \tag{B.10}$$

$$\begin{aligned}\mathcal{V}(\phi,\phi') = {} & \lambda_1^{\phi\phi'}(\phi^\dagger\phi)_{1_{+0}}(\phi'^\dagger\phi')_{1_{+0}} + \lambda_2^{\phi\phi'}(\phi^\dagger\phi)_{1_{-0}}(\phi'^\dagger\phi')_{1_{-0}} + \lambda_3^{\phi\phi'}(\phi^\dagger\phi)_{2_{21}}(\phi'^\dagger\phi')_{2_{21}} \\ & + \lambda_4^{\phi\phi'}(\phi^\dagger\phi')_{1_{+0}}(\phi'^\dagger\phi)_{1_{+0}} + \lambda_5^{\phi\phi'}(\phi^\dagger\phi')_{1_{-0}}(\phi'^\dagger\phi)_{1_{-0}} + \lambda_6^{\phi\phi'}(\phi^\dagger\phi')_{2_{21}}(\phi'^\dagger\phi)_{2_{21}},\end{aligned} \tag{B.11}$$

$$\mathcal{V}(\phi,\varphi) = \lambda_1^{\phi\varphi}(\phi^\dagger\phi)_{1_{+0}}(\varphi^\dagger\varphi)_{1_{+0}} + \lambda_2^{\phi\varphi}(\phi^\dagger\varphi)_{2_{20}}(\varphi^\dagger\phi)_{2_{10}}, \tag{B.12}$$

$$\begin{aligned}\mathcal{V}(\phi,\varphi') = {} & \lambda_1^{\phi\varphi'}(\phi^\dagger\phi)_{1_{+0}}(\varphi'^\dagger\varphi')_{1_{+0}} + \lambda_2^{\phi\varphi'}(\phi^\dagger\phi)_{1_{-0}}(\varphi'^\dagger\varphi')_{1_{-0}} + \lambda_3^{\phi\varphi'}(\phi^\dagger\phi)_{2_{21}}(\varphi'^\dagger\varphi')_{2_{21}} \\ & + \lambda_4^{\phi\varphi'}(\phi^\dagger\varphi')_{1_{+2}}(\varphi'^\dagger\phi)_{1_{+1}} + \lambda_5^{\phi\varphi'}(\phi^\dagger\varphi')_{1_{-2}}(\varphi'^\dagger\phi)_{1_{-1}} + \lambda_6^{\phi\varphi'}(\phi^\dagger\varphi')_{2_{10}}(\varphi'^\dagger\phi)_{2_{20}},\end{aligned} \tag{B.13}$$

$$\mathcal{V}(\phi,\chi) = \mathcal{V}(\phi,\phi' \to \chi), \quad \mathcal{V}(\phi,\rho) = \mathcal{V}(H \to \phi, \varphi' \to \rho), \quad \mathcal{V}(\phi',\varphi) = \mathcal{V}(\phi \to \phi', \varphi), \tag{B.14}$$

[5] Here, we have used the notation $\mathcal{V}(a_1 \to a_2, b_1 \to b_2, \ldots) \equiv \mathcal{V}(a_1, b_1, \ldots)|_{\{a_1=a_2, b_1=b_2, \ldots\}}$.

$$\mathcal{V}(\phi',\varphi')=\mathcal{V}(\phi\to\phi',\varphi'),\quad \mathcal{V}(\phi',\chi)=\mathcal{V}(\phi\to\chi,\phi'),\quad \mathcal{V}(\phi',\rho)=\mathcal{V}(H\to\phi',\varphi'\to\rho), \tag{B.15}$$

$$\mathcal{V}(\varphi,\varphi')=\mathcal{V}(\phi\to\varphi,\varphi\to\varphi'),\quad \mathcal{V}(\varphi,\chi)=\mathcal{V}(H\to\varphi,\varphi'\to\chi),$$
$$\mathcal{V}(\varphi,\rho)=\mathcal{V}(H\to\varphi,\varphi\to\rho), \tag{B.16}$$

$$\mathcal{V}(\varphi',\chi)=\mathcal{V}(\phi\to\chi,\varphi'),\quad \mathcal{V}(\varphi',\rho)=\mathcal{V}(H\to\varphi',\chi\to\rho),\quad \mathcal{V}(\chi,\rho)=\mathcal{V}(H\to\chi,\varphi'\to\rho), \tag{B.17}$$

$$\begin{aligned}\mathcal{V}_{\text{three}}=&\ \lambda_1^{H\phi\phi'}(H^\dagger\phi)_{2_{21}}(\phi'^\dagger\phi')_{2_{21}}+\lambda_2^{H\phi\phi'}(\phi^\dagger H)_{2_{21}}(\phi'^\dagger\phi')_{2_{21}}+\lambda_1^{H\phi\varphi'}(H^\dagger\phi)_{2_{21}}(\varphi'^\dagger\varphi')_{2_{21}}\\&+\lambda_2^{H\phi\varphi'}(\phi^\dagger H)_{2_{21}}(\varphi'^\dagger\varphi')_{2_{21}}+\lambda_1^{H\phi\chi}(H^\dagger\phi)_{2_{21}}(\chi^\dagger\chi)_{2_{21}}+\lambda_2^{H\phi\chi}(\phi^\dagger H)_{2_{21}}(\chi^\dagger\chi)_{2_{21}},\end{aligned} \tag{B.18}$$

$$V_{\text{four}}=\lambda_1^{\varphi\varphi'\chi\rho}(\varphi^\dagger\varphi')_{\underline{2}_{20}}(\chi^\dagger\rho)_{\underline{2}_{10}}+\lambda_2^{\varphi\varphi'\chi\rho}(\varphi'^\dagger\varphi)_{\underline{2}_{10}}(\rho^\dagger\chi)_{\underline{2}_{20}}. \tag{B.19}$$

Note that all the other renormalizable three- and four-scalar interactions are forbidden by one/some of the model symmetries.

Appendix C. The potential minimum condition

$$\begin{aligned}&\mu_H^2 v_H+\lambda^{H\phi\varphi}(v_1+v_2)v_{\varphi'}^2+2\lambda^{H\phi}(v_1+v_2)^2v_H+\lambda^{H\phi\phi'}(v_1+v_2)v'^2\\&\quad+2v_H(\lambda^{H\varphi}v_\varphi^2+2\lambda^{H\varphi'}v_{\varphi'}^2+\lambda^{H\chi}v_\chi^2+\lambda^H v_H^2+4\lambda^{H\phi'}v'^2+\lambda^{H\rho}v_\rho^2)=0,\end{aligned} \tag{C.1}$$

$$\begin{aligned}&\mu_\phi^2 v_1+2\lambda^\phi v_1(2v_1^2+v_2^2)+2\lambda^{\phi\varphi}v_1v_\varphi^2+3\lambda^{\phi\varphi'}v_1v_\varphi'^2+3\lambda^{\phi\varphi'}v_2v_{\varphi'}^2\\&\quad+\lambda^{\phi\chi}v_1v_\chi^2+\lambda^{H\phi\varphi'}v_{\varphi'}^2v_H+2\lambda^{H\phi}v_1v_H^2+2\lambda^{H\phi}v_2v_H^2+5\lambda^{\phi\phi'}v_1v'^2\\&\quad+\lambda^{\phi\phi'}v_2v'^2+\lambda^{H\phi\phi'}v_Hv'^2+2\lambda^{\phi\rho}v_1v_\rho^2=0,\end{aligned} \tag{C.2}$$

$$\begin{aligned}&\mu_\phi^2 v_2+2\lambda^\phi v_2(2v_2^2+v_1^2)+2\lambda^{\phi\varphi}v_2v_\varphi^2+3\lambda^{\phi\varphi'}v_1v_{\varphi'}^2+3\lambda^{\phi\varphi'}v_2v_{\varphi'}^2\\&\quad+\lambda^{\phi\chi}v_2v_\chi^2+\lambda^{H\phi\varphi'}v_{\varphi'}^2v_H+2\lambda^{H\phi}v_1v_H^2+2\lambda^{H\phi}v_2v_H^2+5\lambda^{\phi\phi'}v_2v'^2\\&\quad+\lambda^{\phi\phi'}v_1v'^2+\lambda^{H\phi\phi'}v_Hv'^2+2\lambda^{\phi\rho}v_2v_\rho^2=0,\end{aligned} \tag{C.3}$$

$$\begin{aligned}&2\mu_{\phi'}^2+\lambda^{\phi\phi'}(5v_1^2+2v_1v_2+5v_2^2)+4\lambda^{\phi'\varphi}v_\varphi^2+12\lambda^{\phi'\varphi'}v_{\varphi'}^2+5\lambda^{\phi'\chi}v_\chi^2\\&\quad+2v_H\left[\lambda^{H\phi\phi'}(v_1+v_2)+4\lambda^{H\phi'}v_H\right]+12\lambda^{\phi'}v'^2+4\lambda^{\phi'\rho}v_\rho^2=0,\end{aligned} \tag{C.4}$$

$$\begin{aligned}&2\mu_\varphi^2 v_\varphi+4v_\varphi\left[\lambda^{\phi\varphi}(v_1^2+v_2^2)+\lambda^\varphi v_\varphi^2+2\lambda^{\varphi\varphi'}v_\varphi^2+\lambda^{\varphi\chi}v_\chi^2+\lambda^{H\varphi}v_H^2+2\lambda^{\phi'\varphi}v'^2\right]\\&\quad+2\lambda^{\varphi\varphi'\chi\rho}v_{\varphi'}v_\chi v_\rho+4\lambda^{\varphi\rho}v_\varphi v_\rho^2=0,\end{aligned} \tag{C.5}$$

$$\begin{aligned}&2\mu_{\varphi'}^2 v_{\varphi'}+v_{\varphi'}\Big\{3\lambda^{\phi\varphi'}(v_1+v_2)^2+4\lambda^{\varphi\varphi'}v_\varphi^2+12\lambda^{\varphi'}v_{\varphi'}^2+3\lambda^{\varphi'\chi}v_\chi^2+12\lambda^{\phi'\varphi'}v'^2\\&\quad+2v_H\left[\lambda^{H\phi\varphi'}(v_1+v_2)+2\lambda^{H\varphi'}v_H\right]\Big\}+\lambda^{\varphi\varphi'\chi\rho}v_\varphi v_\chi v_\rho+4\lambda^{\varphi'\rho}v_{\varphi'}v_\rho^2=0,\end{aligned} \tag{C.6}$$

$$2v_\chi \left[\lambda^{\phi\chi}(v_1^2+4v_2^2)+2\lambda^{\varphi\chi}v_\varphi^2+3\lambda^{\varphi'\chi}v_{\varphi'}^2+4\lambda^{\chi}v_\chi^2+2\lambda^{H\chi}v_H^2+5\lambda^{\phi\phi'\chi}v'^2\right] + 2v_\chi\mu_\chi^2+2\lambda^{\varphi\varphi'\chi\rho}v_\varphi v_{\varphi'}v_\rho+4\lambda^{\chi\rho}v_\chi v_\rho^2=0, \quad \text{(C.7)}$$

$$2v_\rho \left[\lambda^{\phi\rho}(v_1^2+v_2^2)+\lambda^{\varphi\rho}v_\varphi^2+2\lambda^{\varphi'\rho}v_{\varphi'}^2+\lambda^{\chi\rho}v_\chi^2+\lambda^{H\rho}v_H^2+2\lambda^{\phi\phi'\rho}v'^2+\lambda^{\rho}v_\rho^2\right] + v_\rho\mu_\rho^2+\lambda^{\varphi\varphi'\chi\rho}v_\varphi v_{\varphi'}v_\chi=0, \quad \text{(C.8)}$$

$$\mu_H^2+2\left[\lambda^{H\phi}(v_1+v_2)^2+\lambda^{H\varphi}v_\varphi^2+2\lambda^{H\varphi'}v_{\varphi'}^2+\lambda^{H\chi}v_\chi^2+3\lambda^{H}v_H^2+4\lambda^{H\phi'}v'^2+\lambda^{H\rho}v_\rho^2\right]>0, \quad \text{(C.9)}$$

$$\mu_\phi^2+2\lambda^{\phi}(6v_1^2+v_2^2)+2\lambda^{\phi\varphi}v_\varphi^2+3\lambda^{\phi\varphi'}v_{\varphi'}^2+\lambda^{\phi\chi}v_\chi^2+2\lambda^{H\phi}v_H^2+5\lambda^{\phi\phi'}v'^2+2\lambda^{\phi\rho}v_\rho^2>0, \quad \text{(C.10)}$$

$$\mu_\phi^2+2\lambda^{\phi}(v_1^2+6v_2^2)+2\lambda^{\phi\varphi}v_\varphi^2+3\lambda^{\phi\varphi'}v_{\varphi'}^2+\lambda^{\phi\chi}v_\chi^2+2\lambda^{H\phi}v_H^2+5\lambda^{\phi\phi'}v'^2+2\lambda^{\phi\rho}v_\rho^2>0, \quad \text{(C.11)}$$

$$2\mu_{\phi'}^2+\lambda^{\phi\phi'}(5v_1^2+2v_1v_2+5v_2^2)+4\lambda^{\phi'\varphi}v_\varphi^2+12\lambda^{\phi'\varphi'}v_{\varphi'}^2+5\lambda^{\phi'\chi}v_\chi^2+36\lambda^{\phi'}v'^2+4\lambda^{\phi'\rho}v_\rho^2 + 2v_H\left[\lambda^{H\phi\phi'}(v_1+v_2)+4\lambda^{H\phi'}v_H\right]>0, \quad \text{(C.12)}$$

$$\mu_\varphi^2+2\left[\lambda^{\phi\varphi}(v_1^2+v_2^2)+3\lambda^{\varphi}v_\varphi^2+2\lambda^{\varphi\varphi'}v_{\varphi'}^2+\lambda^{\varphi\chi}v_\chi^2+\lambda^{H\varphi}v_H^2+2\lambda^{\phi'\varphi}v_H^2+\lambda^{\varphi\rho}v_\rho^2\right]>0, \quad \text{(C.13)}$$

$$2\mu_{\varphi'}^2+3\lambda^{\phi\varphi'}(v_1+v_2)^2+4\lambda^{\varphi\varphi'}v_\varphi^2+36\lambda^{\varphi'}v_{\varphi'}^2+3\lambda^{\varphi'\chi}v_\chi^2+12\lambda^{\phi\phi'\varphi'}v'^2+4\lambda^{\varphi'\rho}v_\rho^2 + 2v_H\left[\lambda^{H\phi\varphi'}(v_1+v_2)+2\lambda^{H\varphi'}v_H\right]>0, \quad \text{(C.14)}$$

$$\mu_\chi^2+\lambda^{\phi\chi}(v_1^2+4v_2^2)+2\lambda^{\varphi\chi}v_\varphi^2+3\lambda^{\varphi'\chi}v_{\varphi'}^2+12\lambda^{\chi}v_\chi^2+2\lambda^{H\chi}v_H^2+5\lambda^{\phi\phi'\chi}v'^2+2\lambda^{\chi\rho}v_\rho^2>0, \quad \text{(C.15)}$$

$$\mu_\rho^2+2\left[\lambda^{\phi\rho}(v_1^2+v_2^2)+\lambda^{\varphi\rho}v_\varphi^2+2\lambda^{\varphi'\rho}v_{\varphi'}^2+\lambda^{\chi\rho}v_\chi^2+\lambda^{H\rho}v_H^2+2\lambda^{\phi\phi'\rho}v'^2+3\lambda^{\rho}v_\rho^2\right]>0. \quad \text{(C.16)}$$

Appendix D. Explicit expressions for $\boldsymbol{\beta_H}$, $\boldsymbol{\beta_\phi}$, $\boldsymbol{\beta_{\phi'}}$, $\boldsymbol{\beta_\varphi}$, $\boldsymbol{\beta_{\varphi'}}$, $\boldsymbol{\beta_\chi}$, $\boldsymbol{\beta_\rho}$, and $\boldsymbol{\beta_{H\phi}}$

$$\begin{aligned}\beta_H =& \left(\mu_\phi^2+2\lambda^{\phi\varphi}v_\varphi^2\right)v_1v_2(v_1-v_2)(v_1+v_2)^2\\
&+\lambda^{\phi\varphi'}\left(6v_1^5+9v_1^4v_2+3v_1^3v_2^2-3v_1^2v_2^3-9v_1v_2^4-6v_2^5\right)v_{\varphi'}^2\\
&-\left(\mu_H^2+2\lambda^{H\varphi}+4\lambda^{H\varphi'}+2\lambda^{H\chi}\right)\left(2v_1^3-v_1^2v_2+v_1v_2^2-2v_2^3\right)v_H^2\\
&+\lambda^{\phi\phi'}\left(2v_1^5+7v_1^4v_2+5v_1^3v_2^2-5v_1^2v_2^3-7v_1v_2^4-2v_2^5\right)v'^2\\
&+\lambda^{\phi\chi}\left(7v_1^3+7v_1^2v_2+2v_1v_2^2+2v_2^3\right)v_1v_2v_\chi^2\\
&-8\lambda^{H\phi'}\left(2v_1^3-v_1^2v_2+v_1v_2^2-2v_2^3\right)v_H^2v'^2\\
&+2(v_1-v_2)\left[\lambda^{\phi\rho}v_1v_2(v_1+v_2)^2-\lambda^{H\rho}(2v_1^2+v_1v_2+2v_2^2)v_H^2\right]v_\rho^2,\end{aligned} \quad \text{(D.1)}$$

$$\beta_\phi = \left(\mu_\phi^2+2\lambda^{\phi\varphi}v_\varphi^2+4\lambda^{\phi\phi'}v'^2+2\lambda^{\phi\rho}v_\rho^2\right)(v_2-v_1)+\lambda^{\phi\chi}(4v_2-v_1)v_\chi^2, \quad \text{(D.2)}$$

$$\beta_{\phi'} = 2\mu_{\phi'}^2+\lambda^{\phi\phi'}(5v_1^2+2v_1v_2+5v_2^2)+4\lambda^{\phi'\varphi}v_\varphi^2+12\lambda^{\phi'\varphi'}v_{\varphi'}^2$$

$$+\ 4\lambda^{\phi'\rho}v_\rho^2+5\lambda^{\phi'\chi}v_\chi^2+2v_H\left[\lambda^{H\phi\phi'}(v_1+v_2)+4\lambda^{H\phi'}v_H\right], \tag{D.3}$$

$$\beta_\varphi = v_\varphi\left\{\mu_\varphi^2+2\left[\lambda^{\phi\varphi}(v_1^2+v_2^2)+2\lambda^{\varphi\varphi'}v_{\varphi'}^2+\lambda^{\varphi\chi}v_\chi^2+\lambda^{H\varphi}v_H^2+2\lambda^{\phi'\varphi}v'^2\right]\right\}$$
$$+\ \lambda^{\varphi\varphi'\chi\rho}v_\varphi v_\chi v_\rho+2\lambda^{\varphi\rho}v_\varphi v_\rho^2, \tag{D.4}$$

$$\beta_{\varphi'} = 4\mu_{\varphi'}^2v_{\varphi'}+2\lambda^{\varphi\varphi'\chi\rho}v_\varphi v_\chi v_\rho+8\lambda^{\varphi'\rho}v_{\varphi'}v_\rho^2+2v_{\varphi'}\left\{3\lambda^{\phi\varphi'}(v_1+v_2)^2\right.$$
$$\left.+\ 4\lambda^{\varphi\varphi'}v_\varphi^2+3\lambda^{\varphi'\chi}v_\chi^2+2v_H\left[\lambda^{H\phi\varphi'}(v_1+v_2)+2\lambda^{H\varphi'}v_H\right]+12\lambda^{\phi'\varphi'}v'^2\right\}, \tag{D.5}$$

$$\beta_\chi = v_\chi\left[\mu_\chi^2+\lambda^{\phi\chi}(v_1^2+4v_2^2)+2\lambda^{\varphi\chi}v_\varphi^2+3\lambda^{\varphi'\chi}v_{\varphi'}^2+2\lambda^{H\chi}v_H^2+5\lambda^{\phi\phi'\chi}v'^2\right]$$
$$+\ \lambda^{\varphi\varphi'\chi\rho}v_\varphi v_{\varphi'}v_\rho+2\lambda^{\chi\rho}v_\chi v_\rho^2, \tag{D.6}$$

$$\beta_\rho = \left\{\mu_\rho^2+2\left[\lambda^{\phi\rho}(v_1^2+v_2^2)+\lambda^{\varphi\rho}v_\varphi^2+2\lambda^{\varphi'\rho}v_{\varphi'}^2+\lambda^{\chi\rho}v_\chi^2+\lambda^{H\rho}v_H^2+2\lambda^{\phi'\rho}v'^2\right]\right\}v_\rho$$
$$+\ \lambda^{\varphi\varphi'\chi\rho}v_\varphi v_{\varphi'}v_\chi, \tag{D.7}$$

$$\beta_{H\phi} = (\mu_\phi^2+2\lambda^{\phi\varphi}v_\varphi^2)v_1v_2(v_2^2-v_1^2)-3\lambda^{\phi\varphi'}\left(2v_1^4+v_1^3v_2-v_1v_2^3-2v_2^4\right)v_{\varphi'}^2$$
$$-\ \lambda^{\phi\chi}(7v_1^2+2v_2^2)v_1v_2v_\chi^2-\lambda^{H\phi\varphi'}(2v_1^3-v_1^2v_2+v_1v_2^2-2v_2^3)v_{\varphi'}^2v_H$$
$$-\ \lambda^{\phi\phi'}(2v_1^4+5v_1^3v_2-5v_1v_2^3-2v_2^4)v'^2-\lambda^{H\phi\phi'}(2v_1^3-v_1^2v_2+v_1v_2^2-2v_2^3)v_Hv'^2$$
$$+\ 2\lambda^{\phi\rho}v_1v_2(v_2^2-v_1^2)v_\rho^2. \tag{D.8}$$

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