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Heat fluxes in a two-qubit cascaded system due to coherences of a non-thermal bath

Zhong-Xiao Man^{1,*} , Yun-Jie Xia¹ and Nguyen Ba An^{2,3} 

¹ Shandong Provincial Key Laboratory of Laser Polarization and Information Technology, Department of Physics, Qufu Normal University, Qufu 273165, People's Republic of China

² Thang Long Institute of Mathematics and Applied Sciences, Thang Long University, Nghiem Xuan Yem, Hoang Mai, Hanoi, Vietnam

³ Center for Theoretical Physics, Institute of Physics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam

E-mail: zxman@qfnu.edu.cn, yjxia@qfnu.edu.cn and nban@iop.vast.ac.vn

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Abstract

We study a cascaded system of two subsystems repeatedly interacting with a bath prepared in various non-thermal states. Effects of different types of bath coherence on dynamics of the system heat flux are explored. For a single-qubit bath with the displacement coherence, heat flux of each subsystem is divided into dissipative and coherent parts. In the stationary regime, the dissipative heat fluxes are the same for the two subsystems, but the coherent parts are not due to the cascaded feature. In the transient regime, heat fluxes occur even when the system and the bath have the same temperature and, more surprisingly, heat flow reversal appears when temperatures of the system and the bath differ. For a two-qubit bath we consider two types of coherence: the squeezing and the heat-exchange ones. In the stationary regime, the squeezing coherence establishes correlations between the subsystems, each of which is evolving to its thermal state with a temperature identical to the bath's. The squeezing coherence slightly influences only the dynamics of the subsystem that interacts with the bath later. As for the heat-exchange coherence, it forces the total system to thermalize to a state with the temperature related to both magnitude and relative phase of the coherence. Particularly, depending on the relative phases, the heat-exchange coherence can amplify or suppress the heat flux during the time evolution.

Keywords: quantum thermodynamics, coherence, heat flux

(Some figures may appear in colour only in the online journal)

1. Introduction

The effort to explore the laws of classical thermodynamics in the quantum domain results in the growing field of quantum thermodynamics (QT) [1]. Of particular interest in this quest is to clarify how the quantum effects, such as quantum coherence and quantum correlation identified as resources in quantum technology, influence the thermodynamics process. In a pioneering work [2], Scully *et al* studied a quantum Carnot engine where the randomly-injected atomic beam is treated

as an effective bath for the cavity field mode. It was found out that, by exploiting the coherence of the three-level atoms and properly tuning the relative phase therein, the field can be thermalized to a temperature larger than that of the atoms without coherence [2]. As a result, the quantum efficiency can be improved to outperform the classical Carnot efficiency [2]. It was shown that the coherence and correlation can be exploited as resources to improve the performance and/or the power of thermal machines [3–17], to enhance work extraction from a quantum system [18–21] and to increase the system's thermalization temperature [22–24]. On the other hand, ones also design various thermal machines to produce and/or amplify

* Author to whom any correspondence should be addressed.

the quantum coherence and correlation [25–27] embodying their close interplay. Although the quantum coherence and correlation of non-thermal baths have been studied extensively in different contexts [3–24], their roles in manipulating heat fluxes, particularly the different manners and components of the heat fluxes, need further studies. We address this issue in this work by focusing on the cascaded model since it involves two manners of heat exchanges, namely, the local and nonlocal ones. With the specific type of bath coherence, the heat flux can be divided into different components, such as the coherent and incoherent ones, whose detailed behaviors in both transient and stationary regimes are to be compared as well.

Heat exchange between quantum systems is the essence of QT and much work has been devoted to the study on its characterization [28–35]. In addition to the fundamental relevance, on-demand manipulation of heat flux is the prerequisite for its exploitation, particularly in designing various quantum thermal machines and devices [36–46]. In this connection, how the quantum coherence and correlation affect the heat flux attracts a lot of interest [47–58]. It was shown that the natural heat flux from a hot quantum system to a cold one can be reverted in the presence of initial quantum correlations between the two systems [53, 54]. This anomalous phenomenon has recently been realized in a nuclear magnetic resonance setup [55] and explained from different points of view, such as the identification of the nonclassical contribution [56] and the construction of the concept of apparent temperature [58]. The relation between quantum coherence and heat flux is a subtle issue since the coherence includes different types and the heat exchange can also manifest various manners. In this work, by classifying the bath coherences into different types, we investigate and compare their effects on different manners of heat fluxes.

According to whether they come into being from the degenerate or the non-degenerate energy levels of the bath, the coherences are classified into non-energetic and energetic coherences [57] or internal and external coherences [59]. The non-energetic (internal) coherence contributes to the excitation (decay) rate of the system being coupled to the bath and determines the temperature [58] to which the system arrives in the stationary regime. The non-energetic coherence is also called heat-exchange coherence [4, 5] since the non-thermal bath in their presence can still be identified as an effective thermal one for the system. The energetic (external) coherence is associated with the energy levels connected by transition of different numbers of excitation. The coherence between the levels with one-excitation transition is termed displacement coherence [4, 5] since it exerts the coherent drive on the system akin to quantum mechanical work [51]. The levels connected by two-excitation transition lead to the so-called squeezing coherence [4, 5], under which the bath can be thought of as an effective squeezed one. Therefore, the displacement and squeezing coherences belong to the energetic coherences. We shall examine the effects of these three types of coherences on the heat fluxes.

In practical situations, the subsystems of a composite quantum system may be affected either by independent baths or by a

common one. When the subsystems are locally coupled to their own independent baths, the heat flux of a subsystem is related to its state evolution induced by the local dissipation of its bath. In this case, only local heat flux appears for each individual subsystem. By contrast, if two or more subsystems are coupled to a common bath, the subsystems can be seen as a whole by the bath even when there are no direct interactions among them [60–63]. In this case, besides the local dissipation for each subsystem due to the bath, there appears a collective dissipation for the subsystems resulting in nonlocal heat fluxes. In addition to those two configurations, the cascaded model [52, 64–66] depicts a situation when subsystems, say, S_1 and S_2 , interact successively with a common bath, namely, the interaction of S_1 with the bath occurs first, which is then followed by that of S_2 with the bath leading to the unidirectional influence of S_1 to S_2 . The cascaded model has an intrinsic time structure allowing to depict the sequence of interactions between subsystems and the environment. For instance, the system of a linear array of QED cavities can exchange information (energy) unidirectionally via the injected atoms passing through the cavities one by one. From the perspective of thermodynamics, such a cascaded manner of interaction does not influence the local heat flux of S_1 on the one hand, which is fully consistent with that in the absence of S_2 , but, on the other hand, it induces the nonlocal heat flux that is completely due to S_2 . Hence, among other fundamental and practical relevances, the cascaded model allows one to deal with both local and nonlocal heat fluxes, which is the reason for us to consider the cascaded model.

In this work, focusing on a bipartite system S_1 and S_2 interacting with a non-thermal bath in the cascaded manner, we study the impacts of three types of coherences of the bath, namely, the displacement, the squeezing and the heat-exchange ones, on both the local and nonlocal heat fluxes. The dynamics of the total system is described by a master equation derived within the framework of the repeated interaction model [22–24, 50–52, 65–70]. We first consider the bath constituted by a single qubit whose coherence is completely attributed to the displacement coherence. In this case, besides the usual dissipative ones, both the heat fluxes regarding S_1 and S_2 contain the contributions from coherent dynamics induced by the displacement coherence, namely, the coherent ingredients of the heat fluxes. We discuss the features of all these heat fluxes, i.e., the local versus nonlocal ones as well as the dissipative versus coherent parts, in both stationary and transient regimes, respectively. Then, we turn to the two-qubit bath that possesses the squeezing or heat-exchange coherences. For the bath with squeezing coherence, S_1 and S_2 are thermalized to the same thermal states and meanwhile possess nonzero global correlations between them in the stationary regime. The dynamics of heat flux of S_1 is shown to be independent to the squeezing coherence, which can however slightly influences that regarding S_2 . For the bath with heat-exchange coherence, the total system reaches the thermal state in the stationary regime with the temperature being related to the magnitude and relative phase of the heat-exchange coherence. The heat fluxes can be amplified or suppressed in the time-evolution depending on the relative phase of the coherence.

2. The model and master equation

The non-thermal bath R in our model is assumed as a cluster of N identical spin-1/2 particles (qubits). The system of interest consists of two identical subsystems in form of two qubits S_1 and S_2 , which randomly and repeatedly interact with an N -qubit cluster in a *cascaded* manner. The cascaded manner means that the interaction of S_1 - R occurs first, which is then followed by that of S_2 - R inducing an indirect coupling between S_1 and S_2 and an unidirectional influence of the dynamics of S_1 on that of S_2 .

The free Hamiltonians H_S and H_R of the system and the N bath particles are respectively given as ($\hbar = 1$)

$$H_S = \sum_{j=1}^2 H_{S_j} = \sum_{j=1}^2 \frac{\omega_0}{2} \sigma_{S_j}^z, \quad H_R = \frac{\omega_b}{2} \sum_{k=1}^N \sigma_k^z, \quad (1)$$

with $\sigma_X^z = |1_X\rangle\langle 1_X| - |0_X\rangle\langle 0_X|$ the Pauli z matrix for the qubit X . In the following, we assume $\omega_0 = \omega_b = \omega$, i.e., the resonant interactions of S_j - R . We adopt the central spin model for the interaction of S_j - R where the subsystem S_j interacts at random times with the cluster of N qubits of the bath. The interaction Hamiltonian of S_j - R can be expressed as

$$H_{\text{int}}^{(j)} = g_j \sum_{k=1}^N \left(\sigma_{S_j}^- \sigma_k^+ + \sigma_{S_j}^+ \sigma_k^- \right) = g_j \left(\sigma_{S_j}^- A^+ + \sigma_{S_j}^+ A^- \right), \quad (2)$$

where g_j is the coupling constant between S_j and all the bath qubits, $\sigma_X^- = |0_X\rangle\langle 1_X|$ and $\sigma_X^+ = |1_X\rangle\langle 0_X|$ are the lowering and raising operators for qubit X , while $A^\pm = \sum_{k=1}^N \sigma_k^\pm$ denote the collective lowering and raising operators of the bath qubits. For simplicity, we let $g_1 = g_2 = g$.

The interaction of S_j - R for a short time τ can be described by the unitary propagator $U_j(\tau) = \exp(-iH_{\text{int}}^{(j)}\tau)$ which is approximated up to the second order in τ as

$$\begin{aligned} U_j(\tau) &\approx I - i\tau H_{\text{int}}^{(j)} - \frac{\tau^2}{2} \left(H_{\text{int}}^{(j)} \right)^2 \\ &= I - ig\tau \left(\sigma_{S_j}^- A^+ + \sigma_{S_j}^+ A^- \right) \\ &\quad - \frac{(g\tau)^2}{2} \left(\sigma_{S_j}^- \sigma_{S_j}^+ A^+ A^- + \sigma_{S_j}^+ \sigma_{S_j}^- A^- A^+ \right). \end{aligned} \quad (3)$$

After the sequential interactions of S_1 and S_2 with R under the governments of $U_1(\tau)$ and $U_2(\tau)$, respectively, the state $\rho(t)$ of the system evolves to

$$\begin{aligned} \rho(t + 2\tau) &= \text{Tr}_R \left[U_2(\tau) U_1(\tau) \rho(t) \otimes \rho_R U_1^\dagger(\tau) U_2^\dagger(\tau) \right] \\ &\equiv \mathcal{M}(2\tau) \rho(t), \end{aligned} \quad (4)$$

where ρ_R is the initial state of the bath R . According to the framework of repeated interaction model, we assume that after each round of system-bath interactions, the bath state is reinitiated to ρ_R or equivalently, is replaced by an identical bath with the same state.

Similarly to the micromaser model [4], we introduce a rate p to depict the random action of system-bath

interaction. In the given time interval $[t, t + \delta t]$, the system-bath interaction governed by $\mathcal{M}(2\tau)\rho(t)$, equation (4), takes place with a probability $p\delta t$. During the time δt we also take into account the free evolution of the system being described by the unitary propagator $U_S(\delta t) = \exp[-iH_S\delta t]$. Thus, the system's state after the evolution time δt takes the form $\rho(t + \delta t) = U_S(\delta t)\mathcal{M}(2\tau)\rho(t)U_S^\dagger(\delta t) = \mathcal{M}(2\tau)\rho(t) - i\delta t[H_S, \rho(t)]$, in which we have approximated $U_S(\delta t)$ to the first order in δt . Otherwise, with a probability $1 - p\delta t$, the system-bath interaction does not occur and the system only freely evolves under the unitary propagator $U_S(\delta t)$. Putting these two aspects together, the state of the system at time $t + \delta t$ can be expressed as

$$\begin{aligned} \rho(t + \delta t) &= p\delta t(\mathcal{M}(2\tau)\rho(t) - i\delta t[H_S, \rho(t)]) \\ &\quad + (1 - p\delta t)(\rho(t) - i\delta t[H_S, \rho(t)]). \end{aligned} \quad (5)$$

Taking the limit of $\delta t \rightarrow 0$, we obtain the following master equation for the dynamics of the system

$$\begin{aligned} \dot{\rho}(t) &\equiv \lim_{\delta t \rightarrow 0} \frac{\rho(t + \delta t) - \rho(t)}{\delta t} \\ &= p(\mathcal{M}(2\tau) - 1)\rho(t) - i[H_S, \rho(t)] \\ &= -i[H_S, \rho(t)] - i \left[\sum_{j=1}^2 H_{\text{eff}}^{(j)}, \rho(t) \right] \\ &\quad + \sum_{j=1}^2 \left(\mathcal{L}_s^{(j)}\rho(t) + \mathcal{L}_h^{(j)}\rho(t) \right) + \mathcal{L}_s^{(12)}\rho(t) + \mathcal{L}_h^{(12)}\rho(t), \end{aligned} \quad (6)$$

in which

$$H_{\text{eff}}^{(j)} = p\tilde{g} \left(d\sigma_{S_j}^+ + d^* \sigma_{S_j}^- \right), \quad (7)$$

$$\mathcal{L}_s^{(j)}\rho(t) = p\tilde{g}^2 \left(s\sigma_{S_j}^+\rho(t)\sigma_{S_j}^+ + s^* \sigma_{S_j}^-\rho(t)\sigma_{S_j}^- \right), \quad (8)$$

$$\begin{aligned} \mathcal{L}_h^{(j)}\rho(t) &= p\tilde{g}^2 h_g \left(\sigma_{S_j}^-\rho(t)\sigma_{S_j}^+ - \frac{1}{2} \left[\sigma_{S_j}^+ \sigma_{S_j}^-, \rho(t) \right]_+ \right) \\ &\quad + p\tilde{g}^2 h_e \left(\sigma_{S_j}^+\rho(t)\sigma_{S_j}^- - \frac{1}{2} \left[\sigma_{S_j}^- \sigma_{S_j}^+, \rho(t) \right]_+ \right), \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{L}_s^{(12)}\rho(t) &= p\tilde{g}^2 s \left([\sigma_{S_1}^+\rho(t), \sigma_{S_2}^+] + [\sigma_{S_2}^+, \rho(t)\sigma_{S_1}^+] \right) \\ &\quad + p\tilde{g}^2 s^* \left([\sigma_{S_1}^-\rho(t), \sigma_{S_2}^-] + [\sigma_{S_2}^-, \rho(t)\sigma_{S_1}^-] \right), \end{aligned} \quad (10)$$

$$\begin{aligned} \mathcal{L}_h^{(12)}\rho(t) &= p\tilde{g}^2 h_g \left([\sigma_{S_1}^-\rho(t), \sigma_{S_2}^+] + [\sigma_{S_2}^-, \rho(t)\sigma_{S_1}^+] \right) \\ &\quad + p\tilde{g}^2 h_e \left([\sigma_{S_1}^+\rho(t), \sigma_{S_2}^-] + [\sigma_{S_2}^+, \rho(t)\sigma_{S_1}^-] \right), \end{aligned} \quad (11)$$

and $\tilde{g} = g\tau$, $d = \langle A^- \rangle$, $d^* = \langle A^+ \rangle$, $s = \langle (A^-)^2 \rangle$, $s^* = \langle (A^+)^2 \rangle$, $h_e = \langle A^+ A^- \rangle$, $h_g = \langle A^- A^+ \rangle$ with $\langle \dots \rangle = \text{Tr} \{ (\dots) \rho_R \}$ and $[\dots, \dots]_+$ the anticommutator. The

master equation (6) contains two types of actions of the bath on the system: the terms given by equations (7)–(9) represent the local actions on S_j coinciding with the case when only one subsystem S_j exists in the absence of the other subsystem, while those given by equations (10) and (11) represent the collective actions on the two subsystems due to the cascaded interaction of the system with the bath [52, 64–66]. In the master equation (6), there are three types of coherences of the initial bath state characterized by the coefficients $d(d^*)$, $s(s^*)$ and $h_e(h_g)$, respectively. Note that we treat the correlations of the constituent qubits of the bath as the coherences by regarding them as a single quantum system. The displacement and squeezing coherences determine the coefficients $d(d^*)$ and $s(s^*)$ appearing in the effective Hamiltonian (7) and the Liouvillians (8) and (10), respectively. The heat-exchange coherence contributes to the excitation (decay) rate in terms of $h_e(h_g)$ and appears in the Liouvillians (9) and (11).

In the following, we assume that the subsystem S_j is prepared in the thermal state of the form

$$\rho_{S_j} = [(1 + q_j)/2] |0\rangle\langle 0| + [(1 - q_j)/2] |1\rangle\langle 1|, \quad (12)$$

where $q_j = \tanh(\beta_{S_j}\omega/2)$ with $\beta_{S_j} = 1/T_{S_j}$ the inverse temperature of S_j (we set Boltzmann constant $k_B = 1$). We shall consider $T_{S_1} = T_{S_2} = T_S$ throughout the paper. As for the bath, we shall consider two scenarios: the first one consists of a single qubit with the displacement coherence and the second one consists of two qubits with the squeezing or the heat-exchange coherence.

3. Single-qubit bath with displacement coherence

We first deal with a situation in which the bath comprises only one qubit (i.e., $N = 1$). Instead of the thermal state, such as that considered in reference [52], we assume the qubit being prepared in non-thermal state with a finite coherence. The coherence of the qubit is bound to the energetic coherence, or rather, the displacement coherence, therefore the master equation (6) of the system is reduced to

$$\begin{aligned} \dot{\rho}(t) = & -i[H_S, \rho(t)] - i \left[\sum_{j=1}^2 H_{\text{eff}}^{(j)}, \rho(t) \right] \\ & + \sum_{j=1}^2 \mathcal{L}_h^{(j)} \rho(t) + \mathcal{L}_h^{(12)} \rho(t). \end{aligned} \quad (13)$$

The energy evolution of the total system, namely, the total heat flux $J^{\text{tot}}(t)$, can be evaluated as

$$J^{\text{tot}}(t) \equiv \text{Tr} \{ \dot{\rho}(t) H_S \} = J_1^{\text{tot}}(t) + J_2^{\text{tot}}(t), \quad (14)$$

where $J_1^{\text{tot}}(t) = J_1(t) + W_1(t)$ and $J_2^{\text{tot}}(t) = J_2(t) + W_2(t) + J_{12}(t)$ represent the total heat fluxes of S_1 and S_2 , respectively, with

$$W_j(t) \equiv -i \text{Tr} \left\{ \left[H_{\text{eff}}^{(j)}, \rho(t) \right] H_S \right\} = -i \text{Tr} \left\{ \rho(t) \left[H_S, H_{\text{eff}}^{(j)} \right] \right\}, \quad (15)$$

$$J_j(t) \equiv \text{Tr} \left\{ \mathcal{L}_h^{(j)} \rho(t) H_S \right\} = \text{Tr} \left\{ \mathcal{L}_h^{(j)} \rho(t) H_{S_j} \right\}, \quad (16)$$

$$J_{12}(t) \equiv \text{Tr} \left\{ \mathcal{L}_h^{(12)} \rho(t) H_S \right\} = \text{Tr} \left\{ \mathcal{L}_h^{(12)} \rho(t) H_{S_2} \right\}. \quad (17)$$

From equations (14)–(17), we note that the total heat flux $J^{\text{tot}}(t)$ contains both coherent and incoherent contributions due to different physical processes. The terms $W_1(t)$ and $W_2(t)$ stand for the heat fluxes taking the form of coherent work [12], which is dependent on and driven by the displacement coherence of the bath. Therefore, the displacement coherence of non-thermal bath can be regarded as a thermodynamics resource to transform the disordered energy to the ordered one [12]. The other terms of $J^{\text{tot}}(t)$ embody the usual heat with $J_1(t)$ and $J_2(t)$ stemming from the local dissipators $\mathcal{L}_h^{(1)}$ and $\mathcal{L}_h^{(2)}$, respectively, while $J_{12}(t)$ from the nonlocal one $\mathcal{L}_h^{(12)}$.

Suppose the bath qubit is prepared in the state

$$\rho_R = \rho_R^{\text{th}} + \lambda \rho_R^{\text{coh}} \quad (18)$$

where $\rho_R^{\text{th}} = [(1 + \xi)/2] |0\rangle\langle 0| + [(1 - \xi)/2] |1\rangle\langle 1|$ is a thermal state of the bath at the inverse temperature $\beta_R = 1/T_R$ with $\xi = \tanh(\beta_R\omega/2)$, and $\rho_R^{\text{coh}} = \sqrt{(1 - \xi^2)/4} (e^{-i\phi} |0\rangle\langle 1| + e^{i\phi} |1\rangle\langle 0|)$ denotes the coherent part of the bath state with $\lambda \in [0, 1]$ controlling the value of coherence and $\phi \in [0, 2\pi]$. Though we use T_R to measure the temperature of bath qubit, it is actually an effective one when the qubit possesses nonzero coherence (i.e., $\lambda \neq 0$). For this state, we have $d = \lambda\sqrt{(1 - \xi^2)/4} e^{i\phi}$, $h_e = (1 - \xi)/2$ and $h_g = (1 + \xi)/2$. By means of equations (15)–(17), we then derive the explicit expressions of all the contributions to the total heat flux $J^{\text{tot}}(t)$ as

$$J_j(t) = p\tilde{g}^2\omega \left[h_e \rho_{S_j}^g(t) - h_g \rho_{S_j}^e(t) \right], \quad (19)$$

$$W_j(t) = 2p\tilde{g}\omega |d| \text{Im} \left[e^{i\phi} \rho_{S_j}^{ge}(t) \right], \quad (20)$$

$$J_{12}(t) = 2p\tilde{g}^2\omega (h_e - h_g) \text{Re}[\rho_{23}(t)], \quad (21)$$

where $\rho_{S_j}^g(t) = \langle 0 | \rho_{S_j}(t) | 0 \rangle$, $\rho_{S_j}^e(t) = \langle 1 | \rho_{S_j}(t) | 1 \rangle$ and $\rho_{S_j}^{ge}(t) = \langle 0 | \rho_{S_j}(t) | 1 \rangle$ with $\rho_{S_j}(t)$ the reduced state of S_j at time t , while $\rho_{kl}(t) = \langle \tilde{k} | \rho(t) | \tilde{l} \rangle$ with $\{ |\tilde{1}\rangle = |00\rangle_{S_1 S_2}, |\tilde{2}\rangle = |01\rangle_{S_1 S_2}, |\tilde{3}\rangle = |10\rangle_{S_1 S_2}, |\tilde{4}\rangle = |11\rangle_{S_1 S_2} \}$ the ordered basis of the subsystems S_1 and S_2 .

3.1. The stationary regime

By solving $\dot{\rho}(t) = 0$ in equation (13), we can derive the stationary state of the system and the stationary heat fluxes. The heat fluxes regarding the subsystem S_1 , i.e., $\mathcal{J}_1 \equiv J_1(t \rightarrow \infty)$ and $\mathcal{W}_1 \equiv W_1(t \rightarrow \infty)$, can be expressed as

$$\mathcal{J}_1 = -\mathcal{W}_1 = \frac{4p^3 |d|^2 \tilde{g}^4 \omega (h_e - h_g)}{8p^2 \tilde{g}^2 |d|^2 + (h_e + h_g)^2 p^2 \tilde{g}^4 + 4\omega^2}. \quad (22)$$

The expression (22) indicates that for $d \neq 0$, namely, when the bath has nonzero coherence (with $\lambda \neq 0$), the nonzero \mathcal{J}_1 and \mathcal{W}_1 can always be achieved, which are also true for the heat fluxes regarding S_2 , i.e., $\mathcal{J}_2 \equiv J_2(t \rightarrow \infty)$, $\mathcal{W}_2 \equiv W_2(t \rightarrow \infty)$

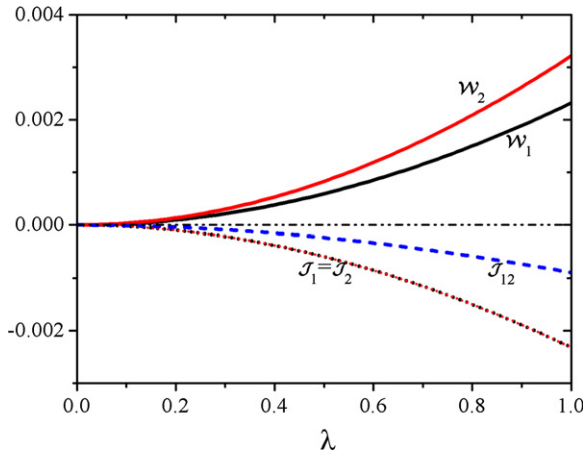


Figure 1. The stationary heat fluxes \mathcal{J}_1 , \mathcal{W}_1 , \mathcal{J}_2 , \mathcal{W}_2 and \mathcal{J}_{12} versus the magnitude of coherence of the bath in terms of λ . The other parameters are chosen as $T_R/\omega = 1$, $p = 0.3$, $\bar{g}/\omega = 1$ and $\phi = 0$.

and $\mathcal{J}_{12} \equiv J_{12}(t \rightarrow \infty)$. The analytic expressions of \mathcal{W}_2 and \mathcal{J}_{12} are provided in the [appendix](#), i.e., equations (A.1)–(A.4), while $\mathcal{J}_2 = \mathcal{J}_1$. As can be seen from equation (20), $\mathcal{W}_j \neq 0$ implies that the subsystems S_1 and S_2 possess nonzero coherences in the stationary regime, which means that the stationary states of S_1 and S_2 are non-thermal. These nonzero stationary coherences are induced by the bath's displacement coherence which can be seen as the classical drive. For the dissipative qubit driven by real field, its quantum non-equilibrium steady state can also carry coherences [71], which is a general fact for non-adiabatically driven quantum open systems [72]. Therefore, it appears to be that the steady coherence of the system can always arise irrespective of the real driving field and the drive provided by the bath coherence.

Though both \mathcal{J}_1 and \mathcal{W}_1 can retain nonzero, the expression in equation (22) also indicates that their sum becomes zero, namely, the heat transferred in terms of coherent work \mathcal{W}_1 between the bath and S_1 is completely transferred back via the dissipative heat \mathcal{J}_1 . For S_2 , we also obtain that $\mathcal{J}_2 + \mathcal{J}_{12} = -\mathcal{W}_2$, which implies that the heat exchange in terms of \mathcal{W}_2 between the bath and S_2 can be totally offset via the sum of the local and nonlocal dissipative heats \mathcal{J}_2 and \mathcal{J}_{12} . This result further demonstrates that, although \mathcal{J}_{12} is induced by the non-local dissipator, it totally contributes to S_2 (but not at all to S_1) due to the cascaded manner of the interaction.

It is known that with the thermal bath both the two subsystems S_1 and S_2 will be thermalized to the same temperature T_R of the bath and there are no heat exchanges of any forms between the system and the bath. Here, in the presence of non-thermal bath with $\lambda \neq 0$, it can be verified that S_1 and S_2 can still be thermalized to the same effective temperature which, however, is not equal to that of the bath, i.e., $\tilde{T}_{S_1} = \tilde{T}_{S_2} \neq T_R$ with \tilde{T}_{S_j} the effective temperature of S_j in the stationary regime. This can also be derived from equation (19) that $\mathcal{J}_j \neq 0$ leads to $\tilde{T}_{S_j} = \frac{\omega}{\ln(\frac{\rho_{S_j}^g}{\rho_{S_j}^e})} \neq T_R$ with $\tilde{\rho}_{S_j}^{g(e)} = \rho_{S_j}^{g(e)}(t \rightarrow \infty)$. Correspondingly, we have $\mathcal{J}_1 = \mathcal{J}_2$ even though S_1 and S_2 interact with the bath successively. The cascaded feature of the interaction actually embodies in the discrepancy between \mathcal{W}_1

and \mathcal{W}_2 , i.e., the coherent parties of the exchanged heat. That is, by virtue of $\mathcal{J}_2 + \mathcal{J}_{12} = -\mathcal{W}_2$ and $\mathcal{J}_1 = \mathcal{J}_2 = -\mathcal{W}_1$, we have $\mathcal{J}_{12} = \mathcal{W}_1 - \mathcal{W}_2 \neq 0$, namely, $\mathcal{W}_1 \neq \mathcal{W}_2$. These observations can be visualized in figure 1, where we exhibit the relations of all the stationary heat fluxes, i.e., \mathcal{J}_1 , \mathcal{W}_1 , \mathcal{J}_2 , \mathcal{W}_2 and \mathcal{J}_{12} , with the coherence of the bath in terms of λ .

3.2. The dynamics of heat fluxes

In this subsection, we further study the effects of the displacement coherence of the bath on the dynamics of heat fluxes before reaching stationary regime. Of particular interest is the fact that when $T_S = T_R$ there are no heat fluxes between the system and the bath if the bath is prepared in a thermal state with $\lambda = 0$. By contrast, as shown in figure 2, when $\lambda \neq 0$ all the constituent heat fluxes regarding S_1 (cf figure 2(a)), i.e., $J_1(t)$ and $W_1(t)$, as well as that regarding S_2 (cf figures 2(b) and (c)), i.e., $J_2(t)$, $W_2(t)$ and $J_{12}(t)$, become oscillatory with nonzero amplitudes which are amplified by increasing λ .

As given in its explicit expression (19), the emergence of dissipative heat flux $J_j(t)$ requires $\rho_{S_j}^g(t)/\rho_{S_j}^e(t) \neq h_g/h_e = \exp(\omega/T_R)$, otherwise it does not occur. At time $t = 0$, we have $\rho_{S_j}^g(0)/\rho_{S_j}^e(0) = (1 + q_j)/(1 - q_j) = \exp(\omega/T_S)$ (cf the initial state (12) of S_1). If $T_S = T_R$ and meanwhile the bath is in the thermal state, the relation $\rho_{S_j}^g(0)/\rho_{S_j}^e(0) = h_g/h_e$ will always hold for $t \geq 0$ implying $J_j(t) = 0$. By contrast, even with $T_S = T_R$ at $t = 0$, the effective Hamiltonian $H_{\text{eff}}^{(j)}$ of the coherent bath with $\lambda \neq 0$ can trigger the time-dependence of $\rho_{S_j}^g(t)$ and $\rho_{S_j}^e(t)$ leading to $\rho_{S_j}^g(t)/\rho_{S_j}^e(t) \neq h_g/h_e$ and thus $J_j(t) \neq 0$ in this case. In the other words, the dynamics of the system induced by $H_{\text{eff}}^{(j)}$ of the non-thermal bath leads to the deviations of the temperature of S_j from its initial value, namely, $T_S(t > 0) \neq T_S = T_R$, so that the temperature gradient is formed and we can still observe nonzero $J_j(t)$. On the other hand, if at $t = 0$ the temperature gradient already exists, namely, $T_S \neq T_R$, the nonzero $J_j(t)$ can always appear but with a fixed sign (direction) in the presence of the thermal bath. Nevertheless, for the non-thermal bath with $\lambda \neq 0$, the reversal of the direction of $J_j(t)$ can be achieved due to the temperature reversal of S_j and R by the action of $H_{\text{eff}}^{(j)}$. In figure 3(a), we show the dynamics of $J_1(t)$ and $J_2(t)$ for $T_S \neq T_R$ in the non-thermal bath with $\lambda = 1$. One can see that for $T_S < T_R$ (e.g., $T_S = 0.1T_R, 0.5T_R$), both $J_1(t)$ and $J_2(t)$ experience a transition from positive to negative in the dynamical process implying the transformation from heat absorption to heat release of the system. Here, we also observe a cascaded feature of the model, namely, $J_2(t)$ takes more time than $J_1(t)$ to experience such a direction reversal (for $T_S < T_R$) and also to reach the stationary regime, as shown in figure 3(a).

As is shown in equation (20), the dynamics of $W_j(t)$ is related to both the magnitude and relative phase of the coherence of S_j that is formed in the time evolution for the given coherence of the bath. This is confirmed in figure 3(b) where generally $|W_j(t)|$ are inversely proportional to the temperature T_S of the system since the lower the temperature, the purer the state of the system and thus the more coherence it can acquire via the interaction with the coherent bath. Due to the variations of the relative phases of the system's coherence in the

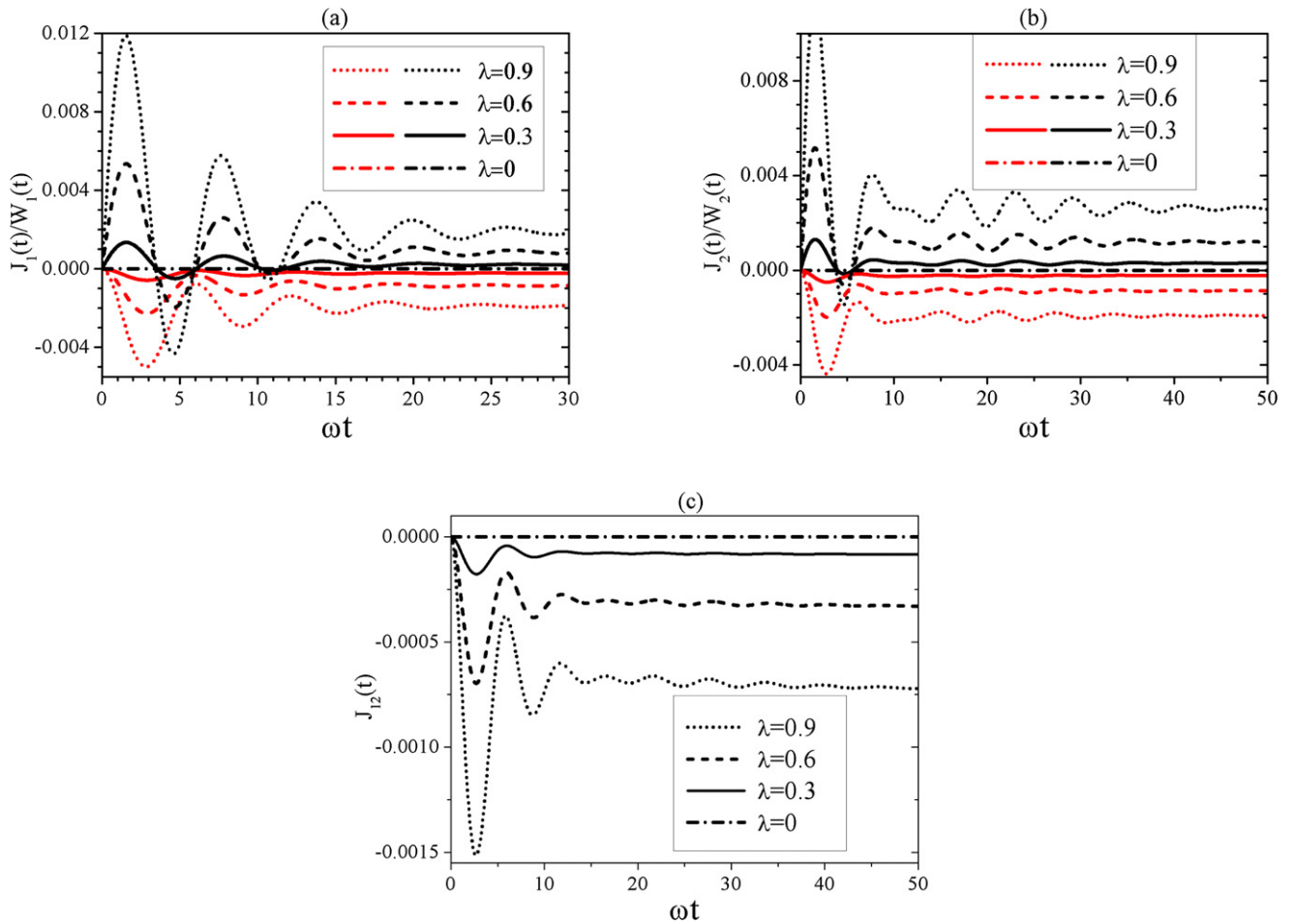


Figure 2. The dynamics of heat fluxes $J_1(t)$ (red curves) and $W_1(t)$ (black curves) regarding S_1 (a), $J_2(t)$ (red curves) and $W_2(t)$ (black curves) regarding S_2 (b), and $J_{12}(t)$ (c) for different magnitudes of displacement coherences of the bath in terms of λ and for the identical effective temperatures of the system and the bath with $T_S/\omega = T_R/\omega = 1$. The other parameters are the same as those given in figure 1.

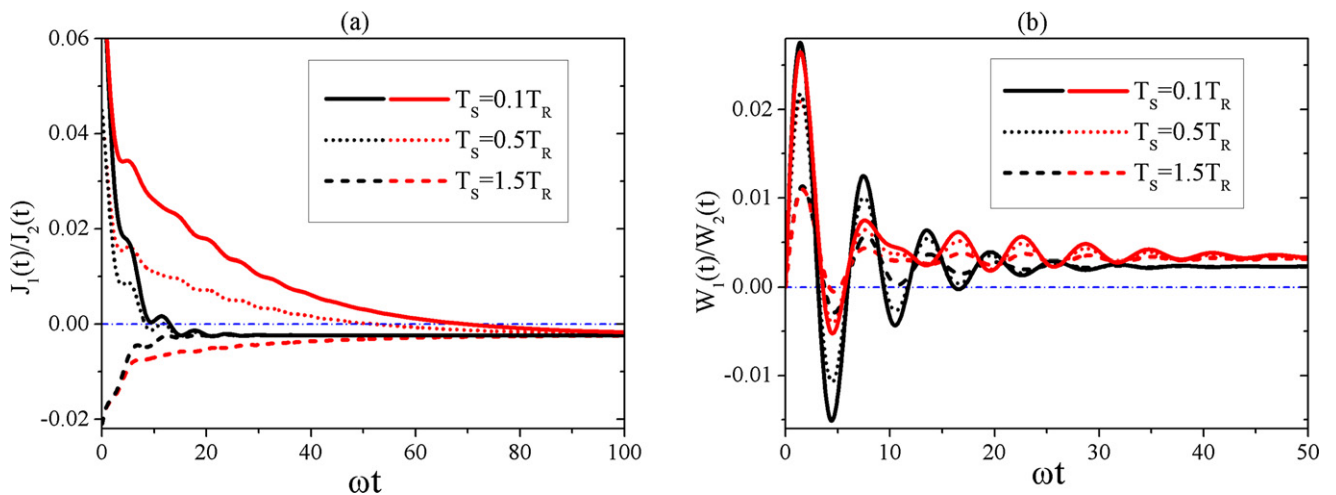


Figure 3. The dynamics of heat fluxes $J_1(t)$ (black curves) and $J_2(t)$ (red curves) in (a) as well as $W_1(t)$ (black curves) and $W_2(t)$ (red curves) in (b) for different T_S . We set $\lambda = 1$ for the bath. The other parameters are the same as those given in figure 1.

time evolution, $W_j(t)$ manifest alternations between positive and negative values before reaching stationary values. From figures 3(b) and 2, we also observe the clear discrepancies for the dynamics of $W_1(t)$ and $W_2(t)$, namely, the oscillations of

$W_1(t)$ are more severe, while $W_2(t)$ take more time to reach the stationary regime. From both figures 3(a) and (b), it shows that the smaller the ratio T_S/T_R the more pronounced the fluxes' direction change.

At last, we point out that the dynamics of $J_{12}(t)$, as given in equation (21), is determined by the correlation of S_1 and S_2 in terms of $\rho_{23}(t)$ constructed in the time evolution. The possible change of direction of this nonlocal heat flux is related to the variation of the phase of $\rho_{23}(t)$ resembling the case of $W_j(t)$.

4. Two-qubit bath with squeezing and heat-exchange coherences

In this section, we proceed to the situation in which the bath comprises two qubits (i.e., $N = 2$) being prepared in the state $\rho_R = \sum_{m,n=1}^4 c_{mn} |\tilde{m}\rangle \langle \tilde{n}|$, with $c_{mn} = \langle \tilde{m} | \rho_R | \tilde{n} \rangle$ the matrix elements of ρ_R in the ordered basis $\{|\tilde{1}\rangle = |00\rangle, |\tilde{2}\rangle = |01\rangle, |\tilde{3}\rangle = |10\rangle, |\tilde{4}\rangle = |11\rangle\}$ of the two bath qubits. In this case, we obtain coefficients of the master equation (6) as $d = c_{21} + c_{31} + c_{42} + c_{43}$, $s = 2c_{41}$, $h_e = c_{22} + c_{33} + 2c_{44} + 2\text{Re}[c_{23}]$ and $h_g = 2c_{11} + c_{22} + c_{33} + 2\text{Re}[c_{23}]$ with $\text{Re}[c_{mn}]$ the real part of c_{mn} . We can see that $N = 2$ is the minimal number of the qubits in the bath that could include all the three types of coherences. Since we have explored the effect of displacement coherence in single-qubit bath on the heat flux dynamics, in the following we concentrate on two specific states of the bath containing squeezing and heat-exchange coherences, respectively.

4.1. The effect of squeezing coherence

We first consider that the two qubits in the bath are prepared in the state

$$\rho_R = \frac{(1 + \xi)^2}{4} |00\rangle \langle 00| + \frac{1 - \xi^2}{4} |01\rangle \langle 01| + \frac{1 - \xi^2}{4} |10\rangle \langle 10| + \frac{(1 - \xi)^2}{4} |11\rangle \langle 11| + c_{14} |00\rangle \langle 11| + c_{41} |11\rangle \langle 00|, \quad (23)$$

where $|c_{14}| \leq \frac{1}{4}(1 - \xi^2)$ to ensure ρ_R to be positive and $\xi = \tanh(\beta_R \omega / 2)$. This state contains the squeezing coherence and results in $s = 2c_{41}$, $d = 0$, $h_e = 1 - \xi$ and $h_g = 1 + \xi$ for the coefficients in the master equation (6). As a result, the master equation (6) is reduced to

$$\dot{\rho}(t) = -i[H_S, \rho] + \sum_{j=1}^2 \left(\mathcal{L}_s^{(j)} \rho + \mathcal{L}_h^{(j)} \rho \right) + \mathcal{L}_s^{(12)} \rho + \mathcal{L}_h^{(12)} \rho. \quad (24)$$

We find that in the stationary regime the total system acquires nonzero squeezing coherence in terms of $\tilde{\rho}_{14} = \tilde{\rho}_{41}^* = \rho_{14}(t \rightarrow \infty)$ as

$$\tilde{\rho}_{14} = \tilde{\rho}_{41}^* = -\frac{s\mu\xi^2(\mu - i\omega)}{2(\mu^2 - |s|^2\mu^2 + \omega^2)} \quad (25)$$

with $\mu = pg^2$, which can be thought of as the transfer of coherence from the bath to the system since it is zero when $s = 0$. Therefore, in the stationary regime the total system is not thermalized to the thermal state when the bath contains nonzero

squeezing coherence. However, we also note that the two individual subsystems S_1 and S_2 are thermalized to the thermal state with the same effective temperature T_R of the bath being of the form

$$\tilde{\rho}_{S_j} = \rho_{S_j}(t \rightarrow \infty) = \frac{1 + \xi}{2} |0\rangle \langle 0| + \frac{1 - \xi}{2} |1\rangle \langle 1|. \quad (26)$$

This observation implies that in the stationary regime there are no heat exchanges between the system and the bath in presence of the bath's squeezing coherence.

The transient expressions of heat fluxes $J_j(t)$ and $J_{12}(t)$ can be formally expressed by equations (19) and (21) with h_e and h_g given above. We note that the squeezing coherences do not have any influence on the heat flux $J_1(t)$ of S_1 . To verify this result, we obtain the explicit expression of $J_1(t)$ as

$$J_1(t) = [q_1 h_e - (1 - q_1) h_g] \mu \omega e^{-2\mu t}, \quad (27)$$

with $q_1 = \tanh(\beta_S \omega / 2)$, which is obviously independent of the squeezing coherence in terms of s and consistent with the situation of the thermal bath. In contrast, through modifying the construction of correlations between S_1 and S_2 , the squeezing coherence of the bath slightly influences the dynamics of heat flux of S_2 .

4.2. The effect of heat-exchange coherence

We turn to consider that the two bath qubits possess heat-exchange coherence being of the form

$$\rho_R = \frac{(1 + \xi)^2}{4} |00\rangle \langle 00| + \frac{1 - \xi^2}{4} |01\rangle \langle 01| + \frac{1 - \xi^2}{4} |10\rangle \langle 10| + \frac{(1 - \xi)^2}{4} |11\rangle \langle 11| + c_{23} |01\rangle \langle 10| + c_{32} |10\rangle \langle 01|, \quad (28)$$

where $|c_{23}| \leq \frac{1}{4}(1 - \xi^2)$ to ensure ρ_R to be positive. With the bath state (28), the coefficients in the master equation (6) take the values $d = 0$, $s = 0$, $h_e = 1 - \xi + 2\text{Re}[c_{23}]$ and $h_g = 1 + \xi + 2\text{Re}[c_{23}]$. It is worth noting that both the populations and the heat-exchange coherence of the bath state make contributions to the coefficients h_e and h_g . In this case, the master equation (6) becomes

$$\dot{\rho}(t) = -i[H_S, \rho] + \sum_{j=1}^2 \mathcal{L}_h^{(j)} \rho + \mathcal{L}_h^{(12)} \rho. \quad (29)$$

We find that in the stationary regime the total system is thermalized to a thermal state $\tilde{\rho} \equiv \rho(t \rightarrow \infty) = \frac{e^{-H_S/\mathcal{T}_S}}{\text{tr}[e^{-H_S/\mathcal{T}_S}]}$ with \mathcal{T}_S the system's thermalization temperature given as

$$\mathcal{T}_S = \omega \left(\ln \frac{h_g}{h_e} \right)^{-1} = \omega \left(\ln \frac{1 + \xi + 2\text{Re}[c_{23}]}{1 - \xi + 2\text{Re}[c_{23}]} \right)^{-1}, \quad (30)$$

which is obviously related to both the magnitude and relative phase of the heat-exchange coherence of the bath. Only when $\text{Re}[c_{23}] = 0$ we have $\mathcal{T}_S = T_R$, otherwise the system is thermalized to the so-called apparent temperature \mathcal{T}_R [58] of the bath, namely, $\mathcal{T}_S = \mathcal{T}_R \neq T_R$. Therefore, in the stationary

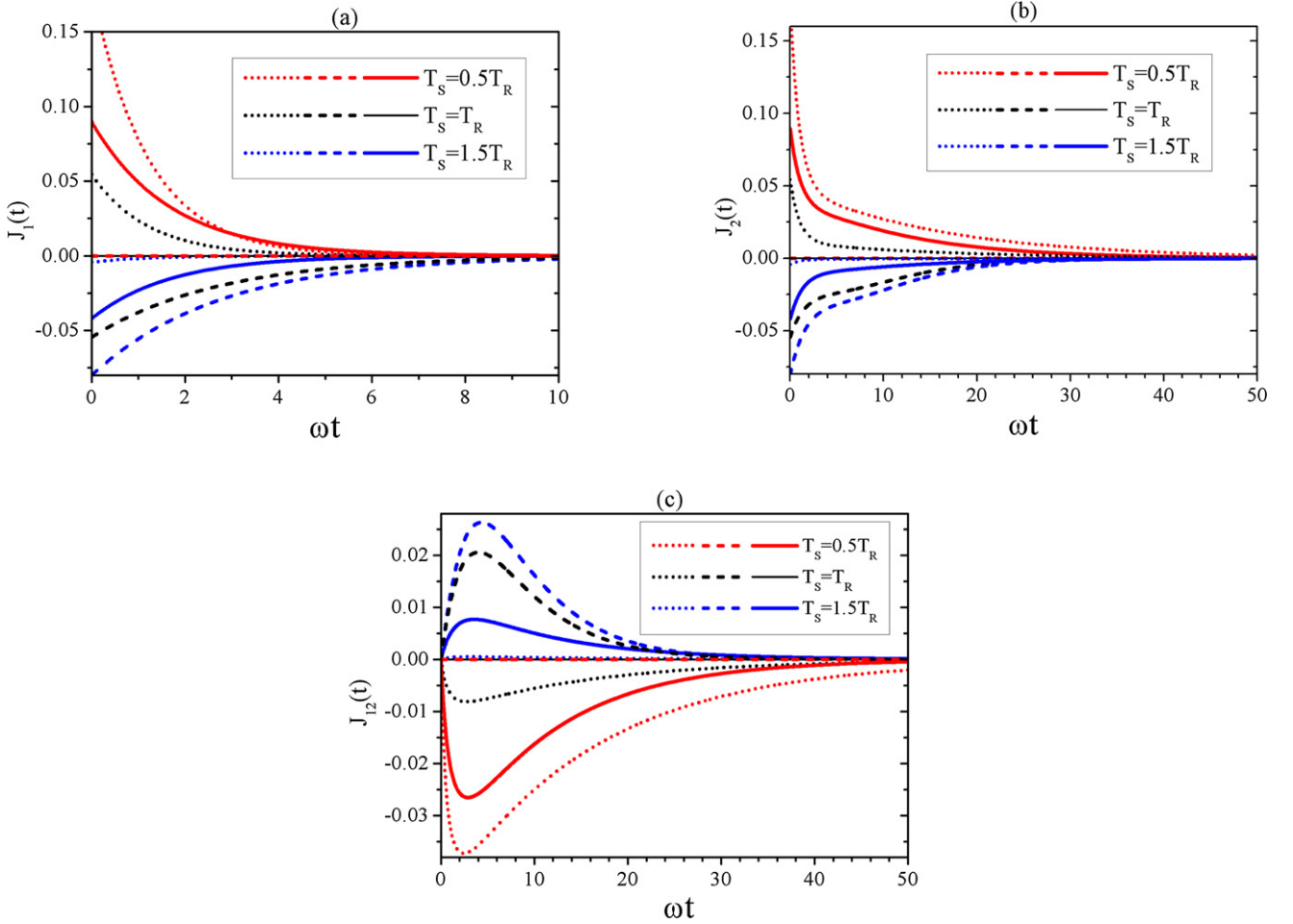


Figure 4. The time evolution of $J_1(t)$, $J_2(t)$ and $J_{12}(t)$ for different T_S in the presence of two-qubit bath initially prepared in the state ρ_R (28) for $\phi = 0$ (dotted lines) and $\phi = \pi$ (dashed lines). The corresponding dynamics under the thermal state with $c_{23} = 0$ (solid lines) are also presented for comparison. The other parameters are chosen as $T_R/\omega = 1$, $p = 0.3$ and $\tilde{g}/\omega = 1$.

regime, the system and the bath reach thermal equilibrium and no heat exchanges occur.

Next, we explore the dynamics of the heat fluxes $J_j(t)$ and $J_{12}(t)$ with the expressions formally given in equations (19) and (21). In figure 4, we demonstrate the heat fluxes dynamics by setting $c_{23} = \frac{1}{4}(1 - \xi^2)e^{i\phi}$ with the different relative phases (e.g., $\phi = 0$ and $\phi = \pi$) and compare their behaviors with the case of $c_{23}=0$, i.e., the thermal bath state. For the chosen $T_R/\omega = 1$ used in figure 4, we have the bath apparent temperatures $\mathcal{T}_R/\omega \approx 1.45$ for $\phi = 0$ and $\mathcal{T}_R/\omega = 0.5$ for $\phi = \pi$, which are the *real* temperatures seen from the system and thermalized temperatures the system will reach in the stationary regime. As a result, we observe in figure 4(a) for the dynamics of $J_1(t)$, that for $T_S/\omega = T_R/\omega = 1$ (black lines) under which $J_1(t) = 0$ when $c_{23} = 0$, the heat is flowed into S_1 from the bath with $J_1(t) > 0$ when $\phi = 0$ ($\mathcal{T}_R > T_S$), whereas the heat is taken away from S_1 to the bath with $J_1(t) < 0$ when $\phi = \pi$ ($\mathcal{T}_R < T_S$). For $T_S/\omega = 0.5T_R/\omega = 0.5$ (red lines), $\phi = 0$ makes an amplification of $J_1(t)$ in the initial stage of the time evolution, while reduces it to zero if $\phi = \pi$ is chosen as $T_S = \mathcal{T}_R$ in this case. For $T_S/\omega = 1.5T_R/\omega = 1.5$ (blue lines), $|J_1(t)|$ is amplified for $\phi = \pi$, while reduced almost to zero for $\phi = 0$. Therefore, the heat-exchange coherence, in

particular the phase in it, can be identified as an efficient resource to modify the heat fluxes. The influences of heat-exchange coherence on the dynamics of $J_2(t)$ and $J_{12}(t)$ are displayed in figures 4(b) and (c), respectively. In addition to some similar features as shown in figure 4(a), we also observe that the heat fluxes $J_2(t)$ and $J_{12}(t)$ of S_2 take more time to reach the stationary regime which means that S_2 is thermalized to the thermal state later than S_1 .

5. Conclusion

We have studied the effects of different types of coherences in a non-thermal bath on the heat exchanges between the bath and the system which may take place in various manners, such as through local/global dissipation and dissipative/coherent dynamics. Specifically, we consider a model that consists of two cascaded subsystems S_1 and S_2 sequentially interacting with a bath modeled as a cluster of N qubits: the interaction of S_2-R occurs after that of S_1-R . Therefore, in addition to the local heat flux regarding each subsystem, the nonlocal heat flux appears due to the collective interaction of the two subsystems with the bath. As a typical character of the cascaded

interaction model, the nonlocal heat flux is completely caused by the subsystem S_2 . Within the framework of repeated interaction model, we have constructed a master equation to describe the dynamics of the total system. The master equation (6) is valid for a bath of any N qubits, but we explicitly study only the cases with $N = 1$ and $N = 2$ because they are sufficient to have all the three types of bath coherences. We first consider the single-qubit bath with the so-called displacement coherence, i.e., the coherence between the energy levels connected by single-excitation transition. In this case, the heat fluxes of both subsystems include dissipative and coherent ingredients. In the stationary regime, though all the ingredients retain nonzero, the heat exchanges in the dissipative manners completely flow back in the coherent ways. We also show that the two subsystems possess the same locally dissipative heat fluxes but different coherent ones owing to the cascaded feature of the model. The displacement coherence brings about additional effective Hamiltonians for the subsystems, which make the dynamics of the system deviate from the situation when the bath is prepared in a thermal state. In the transient regime, we thus observe nonzero heat fluxes even when the system has the same temperature with the bath and heat flow reversal when they have different temperatures. We then turn to the two-qubit bath that is prepared in the non-thermal states with squeezing coherence and heat-exchange coherence, respectively. For the bath with squeezing coherence, i.e., the coherence between levels connected by two-excitation transition, nonzero correlation exists between the two subsystems in the stationary regime, though each individual subsystem is thermalized to its thermal state with the temperature of the bath. The squeezing coherence does not influence on the dynamics of the subsystem S_1 , but slightly affects the dynamics of S_2 . For the bath with heat-exchange coherence, i.e., the coherence between degenerate levels, the total system is thermalized to a thermal state with the temperature related to both the magnitude and relative phase of the bath coherence. During the transient dynamical process the heat-exchange coherence, particularly the relative phases therein, can greatly amplify or suppress the heat fluxes. Our obtained results would provide an instance of exploiting types of bath coherence and ways of system-bath interaction to manage heat fluxes. In the future work, we shall explore the QT of the cascaded model with additional driving field by means of the framework provided in reference [71].

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Appendix

In the following, we present the analytical expressions of $\mathcal{W}_2 \equiv P/Q$ and $\mathcal{J}_{12} \equiv X/Y$ with P , Q , X and Y are given as

$$\begin{aligned}
P = & 4|d|^2\tilde{g}^2h_-p^2\mu\omega (h_+^6 (9h_e^2 - 38h_e h_g + 9h_g^2) (9h_e^2 + 2h_e h_g) \\
& + 9h_g^2) \mu^8 - 64h_+^4 (3h_e^2 - 2h_e h_g + 3h_g^2) (3h_e^2 + 13h_e h_g \\
& + 3h_g^2) \mu^6\omega^2 - 32h_+^2 (139h_e^4 + 172h_e^3 h_g + 2h_e^2 h_g^2 \\
& + 172h_e h_g^3 + 139h_g^4) \mu^4\omega^4 - 512 (7h_e^4 + 18h_e^3 h_g + 6h_e^2 h_g^2 \\
& + 18h_e h_g^3 + 7h_g^4) \mu^2\omega^6 - 256 (3h_e^2 - 2h_e h_g + 3h_g^2) \omega^8 \\
& + 1024|d|^6g^6h_+^2p^6 ((h_e^2 - 6h_e h_g + h_g^2) \mu^2 - 4\omega^2) \\
& + 64|d|^4\tilde{g}^4p^4 (h_+^2 (29h_e^4 - 132h_e^3 h_g - 66h_e^2 h_g^2 - 132h_e h_g^3) \\
& + 29h_g^4) \mu^4 - 16 (h_e^2 + h_g^2) (7h_e^2 + 30h_e h_g + 7h_g^2) \mu^2\omega^2) \\
& - 16 (9h_e^2 + 2h_e h_g + 9h_g^2) \omega^4 + 32|d|^2\tilde{g}^2p^2 \\
& \times (h_+^4 (27h_e^4 - 112h_e^3 h_g + 10h_e^2 h_g^2 - 112h_e h_g^3 + 27h_g^4) \mu^6 \\
& - 2h_+^2 (57h_e^4 + 248h_e^3 h_g + 62h_e^2 h_g^2 + 248h_e h_g^3 + 57h_g^4) \mu^4\omega^2 \\
& - 32h_+^2 (13h_e^2 - 2h_e h_g + 13h_g^2) \mu^2\omega^4 - 32(5h_e^2 - 2h_e h_g \\
& + 5h_g^2) \omega^6), \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
Q = & h_+^2 (h_+ \mu - 2i\omega) (h_+ \mu + 2i\omega) (8|d|^2\tilde{g}^2p^2 + h_+^2\mu^2 + 4\omega^2)^2 \\
& \times (128|d|^4\tilde{g}^4p^4 + (9h_e^2 + 2h_e h_g + 9h_g^2) \mu^4 \\
& + 8(9h_e^2 + 34h_e h_g + 9h_g^2) \mu^2\omega^2 + 16\omega^4 \\
& + 24|d|^2\tilde{g}^2p^2((9h_e^2 + 2h_e h_g + 9h_g^2) \mu^2 + 4\omega^2)), \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
X = & -h_- \mu \omega (8|d|^2\tilde{g}^2(h_e - h_g)^2p^2 (1024|d|^6\tilde{g}^6(h_e + h_g)^2p^6\mu^2 \\
& + 9h_+^6(9h_e^2 + 2h_e h_g + 9h_g^2) \mu^8 + 8h_+^4(9h_e^2 - 70h_e h_g + 9h_g^2) \\
& \times \mu^6\omega^2 - 256h_+^2(5h_e^2 + 13h_e h_g + 5h_g^2) \mu^4\omega^4 - 128(9h_e^2 \\
& + 26h_e h_g + 9h_g^2) \mu^2\omega^6 - 256\omega^8 + 64|d|^4\tilde{g}^4p^4 (h_+^2(29h_e^2 \\
& + 26h_e h_g + 29h_g^2) \mu^4 + 4(h_e - 3h_g)(3h_e - h_g) \mu^2\omega^2 - 32\omega^4) \\
& + 32|d|^2\tilde{g}^2p^2 (h_+^4 (27h_e^2 + 10h_e h_g + 27h_g^2) \mu^6 + 7(h_e - 3h_g) \\
& \times (3h_e - h_g)h_+^2\mu^4\omega^2 - 104h_+^2\mu^2\omega^4 - 48\omega^6))) \tag{A.3}
\end{aligned}$$

and

$$\begin{aligned}
Y = & h_+^2 (h_+ \mu - 2i\omega)(h_+ \mu + 2i\omega)(8|d|^2\tilde{g}^2p^2 + h_+^2\mu^2 + 4\omega^2)^2 \\
& \times (128|d|^4\tilde{g}^4p^4 + (9h_e^2 + 2h_e h_g + 9h_g^2) \mu^4 \\
& + 8(9h_e^2 + 34h_e h_g + 9h_g^2) \mu^2\omega^2 + 16\omega^4 \\
& + 24|d|^2\tilde{g}^2p^2((9h_e^2 + 2h_e h_g + 9h_g^2) \mu^2 + 4\omega^2)), \tag{A.4}
\end{aligned}$$

in which $h_{\pm} = h_e \pm h_g$ and $\mu = pg^2$.

ORCID iDs

Zhong-Xiao Man  <https://orcid.org/0000-0003-1906-5923>
 Nguyen Ba An  <https://orcid.org/0000-0002-6802-0388>

References

- [1] Gemma G, Michel M and Mahler G 2004 *Quantum Thermodynamics* (Berlin: Springer)
- [2] Scully M O, Zubairy M S, Agarwal G S and Walther H 2003 *Science* **299** 862
- [3] Quan H T, Zhang P and Sun C P 2006 *Phys. Rev. E* **73** 036122
- [4] Dağ C B, Niedenzu W, Müstecaplıoğlu O E and Kurizki G 2016 *Entropy* **18** 244
- [5] Türkcençe D and Müstecaplıoğlu O E 2016 *Phys. Rev. E* **93** 012145
- [6] Shi Y-H, Shi H-L, Wang X-H, Hu M-L, Liu S-Y, Yang W-L and Fan H 2020 *J. Phys. A: Math. Theor.* **53** 085301
- [7] Mehta V and Johal R S 2017 *Phys. Rev. E* **96** 032110
- [8] Dillenschneider R and Lutz E 2009 *Europhys. Lett.* **88** 50003
- [9] Scully M O, Chapin K R, Dorfman K E, Kim M B and Svidzinsky A 2011 *Proc. Natl Acad. Sci.* **108** 15097
- [10] Brandner K, Bauer M, Schmid M T and Seifert U 2015 *New J. Phys.* **17** 065006
- [11] Uzdin R, Levy A and Kosloff R 2015 *Phys. Rev. X* **5** 031044
- [12] Brunner N, Huber M, Linden N, Popescu S, Silva R and Skrzypczyk P 2014 *Phys. Rev. E* **89** 032115
- [13] Niedenzu W, Gelbwaser-Klimovsky D, Kofman A G and Kurizki G 2016 *New J. Phys.* **18** 083012
- [14] Niedenzu W, Gelbwaser-Klimovsky D and Kurizki G 2015 *Phys. Rev. E* **92** 042123
- [15] Leggio B, Bellomo B and Antezza M 2015 *Phys. Rev. A* **91** 012117
- [16] Uzdin R 2016 *Phys. Rev. A* **6** 024004
- [17] Doyeux P, Leggio B, Messina R and Antezza M 2016 *Phys. Rev. E* **93** 022134
- [18] Skrzypczyk P, Short A J and Popescu S 2014 *Nat. Commun.* **5** 4185
- [19] Åberg J 2014 *Phys. Rev. Lett.* **113** 150402
- [20] Korzekwa K, Lostaglio M, Oppenheim J and Jennings D 2016 *New J. Phys.* **18** 023045
- [21] Li H, Zou J, Yu W L, Xu B M, Li J G and Shao B 2014 *Phys. Rev. E* **89** 052132
- [22] Liao J Q, Dong H and Sun C P 2010 *Phys. Rev. A* **81** 052121
- [23] Manatuly A, Niedenzu W, Román-Ancheyta R, Çakmak B, Müstecaplıoğlu O E and Kurizki G 2019 *Phys. Rev. E* **99** 042145
- [24] Dağ C B, Niedenzu W, Ozaydin F, Müstecaplıoğlu O E and Kurizki G 2019 *J. Phys. Chem. C* **123** 4035
- [25] Brask J B, Haack G, Brunner N and Huber M 2015 *New J. Phys.* **17** 113029
- [26] Çakmak B, Manatuly A and Müstecaplıoğlu O E 2017 *Phys. Rev. A* **96** 032117
- [27] Manzano G, Silva R and Parrondo J M R 2019 *Phys. Rev. E* **99** 042135
- [28] Wichterich H, Henrich M J, Breuer H P, Gemmer J and Michel M 2007 *Phys. Rev. E* **76** 031115
- [29] Manzano D, Tiersch M, Asadian A and Briegel H J 2012 *Phys. Rev. E* **86** 061118
- [30] Levy A and Kosloff R 2014 *Europhys. Lett.* **107** 20004
- [31] De Chiara G, Landi G, Hewgill A, Reid B, Ferraro A, Roncaglia A J and Antezza M 2018 *New J. Phys.* **20** 113024
- [32] Werlang T and Valente D 2015 *Phys. Rev. E* **91** 012143
- [33] Ronzani A, Karimi B, Senior J, Chang Y-C, Peltonen J T, Chen C and Pekola J P 2018 *Nat. Phys.* **14** 991
- [34] Brange F, Menczel P and Flindt C 2019 *Phys. Rev. B* **99** 085418
- [35] Man Z X, An N B and Xia Y J 2016 *Phys. Rev. E* **94** 042135
- [36] Rossnagel J, Dawkins S T, Tolazzi K N, Abah O, Lutz E, Schmidt-Kaler F and Singer K 2016 *Science* **352** 325
- [37] Malabarba A S L, Short A J and Kammerlander P 2015 *New J. Phys.* **17** 045027
- [38] Alicki R and Gelbwaser-Klimovsky D 2015 *New J. Phys.* **17** 115012
- [39] Leggio B and Antezza M 2016 *Phys. Rev. E* **93** 022122
- [40] Guo B Q, Liu T and Yu C S 2019 *Phys. Rev. E* **99** 032112
- [41] Guo B Q, Liu T and Yu C S 2018 *Phys. Rev. E* **98** 022118
- [42] Kargı C, Naseem M T, Opatrný T, Müstecaplıoğlu O E and Kurizki G 2019 *Phys. Rev. E* **99** 042121
- [43] Wang C, Xu D, Liu H and Gao X L 2019 *Phys. Rev. E* **99** 042102
- [44] Ordóñez-Miranda J, Ezzahri Y and Joulain K 2017 *Phys. Rev. E* **95** 022128
- [45] Werlang T, Marchiori M A, Cornelio M F and Valente D 2014 *Phys. Rev. E* **89** 062109
- [46] Joulain K, Drevillon J, Ezzahri Y and Ordóñez-Miranda J 2016 *Phys. Rev. Lett.* **116** 200601
- [47] Rahav S, Harbola U and Mukamel S 2012 *Phys. Rev. A* **86** 043843
- [48] Mitchison M T 2015 *New J. Phys.* **17** 115013
- [49] Huber Woods B, Messina R and Antezza M 2015 *Europhys. Lett.* **110** 40002
- [50] Li L, Zou J, Li H, Xu B M, Wang Y M and Shao B 2018 *Phys. Rev. E* **97** 022111
- [51] Rodrigues F L S, De Chiara G, Paternostro M and Landi G T 2019 *Phys. Rev. Lett.* **123** 140601
- [52] Lorenzo S, Farace A, Ciccarello F, Palma G M and Giovannetti V 2015 *Phys. Rev. A* **91** 022121
- [53] Partovi M H 2008 *Phys. Rev. E* **77** 021110
- [54] Jennings D and Rudolph T 2010 *Phys. Rev. E* **81** 061130
- [55] Micadei K, Peterson J P S, Souza A M, Sarthour R S, Oliveira I S, Landi G T, Batalhao T B, Serra R M and Lutz E 2019 *Nat. Commun.* **10** 2456
- [56] Henao I and Serra R M 2018 *Phys. Rev. E* **97** 062105
- [57] Latune C L, Sinayskiy I and Petruccione F 2019 *Phys. Rev. Res.* **1** 033097
- [58] Latune C L, Sinayskiy I and Petruccione F 2019 *Quantum Sci. Technol.* **4** 025005
- [59] Kwon H, Jeong H, Jennings D, Yadin B and Kim M S 2018 *Phys. Rev. Lett.* **120** 150602
- [60] Liao J Q, Huang J F and Kuang L M 2011 *Phys. Rev. A* **83** 052110
- [61] Man Z-X, Tavakoli A, Brask J B and Xia Y-J 2019 *Phys. Scr.* **94** 075101
- [62] Karimi B and Pekola J P 2017 *Phys. Rev. B* **96** 115408
- [63] Manzano G, Giorgi G-L, Fazio R and Zambrini R 2019 *New J. Phys.* **21** 123026
- [64] Giovannetti V and Palma G M 2012 *Phys. Rev. Lett.* **108** 040401
- [65] Giovannetti V and Palma G M 2012 *J. Phys. B: At. Mol. Opt. Phys.* **45** 154003
- [66] Lorenzo S, McCloskey R, Ciccarello F, Paternostro M and Palma G M 2015 *Phys. Rev. Lett.* **115** 120403
- [67] Scarani V, Ziman M, Stelmachovic P, Gisin N and Bužek V 2002 *Phys. Rev. Lett.* **88** 097905
- [68] Man Z X, Xia Y J and Lo Franco R 2019 *Phys. Rev. A* **99** 042106
- [69] Pezzutto M, Paternostro M and Omar Y 2016 *New J. Phys.* **18** 123018
- [70] Strasberg P, Schaller G, Brandes T and Esposito M 2017 *Phys. Rev. X* **7** 021003
- [71] Elouard C, Herrera-Martí D, Esposito M and Auffèves A 2020 arXiv:2001.08033
- [72] Dann R, Levy A and Kosloff R 2018 arXiv:1805.10689