

PAPER • OPEN ACCESS

## New phenomena with the $f(R)$ -theory of gravitation in a central gravitational field

To cite this article: Pham Van Ky *et al* 2020 *J. Phys.: Conf. Ser.* **1506** 012001

View the [article online](#) for updates and enhancements.



**IOP | ebooks™**

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection—download the first chapter of every title for free.

# New phenomena with the $f(R)$ -theory of gravitation in a central gravitational field

Pham Van Ky<sup>1</sup>, Nguyen Thi Hong Van<sup>2,3</sup>  
and Nguyen Anh Ky<sup>2</sup>

<sup>1</sup>Graduate university of science and technology, Vietnam academy of science and technology, Hanoi, Viet Nam.

<sup>2</sup>Institute of physics, Vietnam academy of science and technology, Hanoi, Viet Nam.

<sup>3</sup>Institute for interdisciplinary research in science and education, ICISE, Quy Nhon, Viet Nam.

E-mail: phamkyvatly@gmail.com, nhvan@iop.vast.ac.vn, anhky@iop.vast.ac.vn

**Abstract.** The  $f(R)$ -theory (of gravitation) is an extension of Einstein's general theory of relativity (GR) but if a spherically symmetric vacuum solution of the Einstein equation in the GR is always stationary, a spherically symmetric vacuum solution of an  $f(R)$ -theory is not necessary stationary. This may have interesting consequences. In comparison with the GR, a process such as a planet's motion (its orbital precession and parameters) and a gravitational deflection of light now get a correction which is a constant for a static central field and varies with time for a non-static central field even from a source of a constant mass, unlike the corresponding GR value not changing in the same situation. In particular, a spherically symmetric source may radiate gravitational waves. This phenomenon cannot happen in the GR. The present work is an extended version based on a presentation in the 44th Vietnam conference on theoretical physics (Dong Hoi, 29 July - 01 August 2019).

## 1. Introduction

The General theory of Relativity (GR) of A. Einstein is a very successful theory of gravitation [1, 2]. This theory has been verified very precisely, in particular, it was once again confirmed triumphantly by recent detections of gravitational waves [3, 4].

The heart of the GR is Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -kT_{\mu\nu}, \quad (1)$$

$k = \frac{8\pi G}{c^4}$ , derived from the Lagrangian  $\mathcal{L}_G = R$ . However, some cosmological problems require the GR to be extended or modified. One of the modified theories of the GR is called  $f(R)$ -theory of gravitation.

The  $f(R)$ -theory of gravitation is based on the Lagrangian  $\mathcal{L}_G = f(R)$  leading to the equation

$$f'(R)R_{\mu\nu} - g_{\mu\nu}\square f'(R) + \nabla_\mu \nabla_\nu f'(R) - \frac{1}{2}f(R)g_{\mu\nu} = -kT_{\mu\nu}, \quad (2)$$

where  $f(R)$  is a scalar function of the scalar curvature  $R$ , while  $\nabla_\mu$  is a covariant derivative and  $\square = \nabla_\mu \nabla^\mu$ . The function  $f(R)$  could be, for example,  $f(R) = R + \lambda R^2$  or  $f(R) = R - \frac{\lambda}{R^n}$ , etc.



To solve (2) in general is problematic. Here, assuming  $f(R)$  deviating from  $R$  just slightly, we try to solve (2) perturbatively for a central field of a spherically symmetric gravitational source of radius  $R_0$ . More details of the procedure can be found in [5, 6].

## 2. Perturbative spherically symmetric solutions of an $f(R)$ -theory

Starting with

$$f(R) = R + \lambda h(R), \quad \lambda h(R) \ll R,$$

where  $h(R)$  is a scalar function of  $R$  and  $\lambda$  is a parameter (with an appropriate dimension which will be tacit here), we obtain from Eq. (2)

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R + \lambda h'(R)R^\mu{}_\nu - \frac{\lambda}{2}\delta^\mu{}_\nu h(R) - \lambda\delta^\mu{}_\nu \Delta h'(R) + \lambda\nabla^\mu\nabla_\nu h'(R) = -kT^\mu{}_\nu. \quad (3)$$

Using Einstein's equation in the form  $R = kT$  and  $R^\mu{}_\nu = -k(T^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu T)$  and solving (3) perturbatively, we find the following perturbative solution

$$g_{00}(r, t) = 1 - \frac{kc^2(M - \lambda M_1(r, t) - \lambda M_2(r, t))}{4\pi r},$$

$$g_{11}(r, t) = \frac{-1}{g_{00}(r, t)}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta, \quad (4)$$

where

$$M_1(r, t) = \frac{-2\pi}{kc^2} \int_0^r [h(kT_0^0) + kT_0^0 h'(kT_0^0)] r'^2 dr', \quad M_2(t) = \frac{-4\pi h''(kT_0^0)}{kc^2} \left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t), \quad (5)$$

with

$$\alpha(t) = \frac{3\arcsin[\xi(t)R_0(t)] - \xi(t)R_0(t) \{3 + 2[\xi(t)R_0(t)]^2\} \sqrt{1 - [\xi(t)R_0(t)]^2}}{256\pi^2[\xi(t)]^5 (3k^2c^2)^{-1} (1 - [\xi(t)R_0(t)]^2)^{3/2}}, \quad \xi^2(t) = \frac{kMc^2}{4\pi[R_0(t)]^3}. \quad (6)$$

Here  $R_0(t)$  is the radius of the gravitational source at time  $t$ . Let us apply this result to some special cases.

### 2.1. Model $\mathbf{f(R) = R - 2\lambda}$

In this case  $h(R) = -2$  and, therefore,

$$g_{00}(r, t) = 1 - \frac{kc^2 M}{4\pi r} - \frac{\lambda r^2}{3}, \quad g_{11}(r, t) = \frac{-1}{1 - \frac{kc^2 M}{4\pi r} - \frac{\lambda r^2}{3}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta. \quad (7)$$

This solution coincides with the exact solution with  $\lambda$  being the cosmological constant.

### 2.2. Model $\mathbf{f(R) = R + \lambda R^b}$ , ( $\mathbf{b} > \mathbf{0}$ )

Here  $h(R) = R^b$ , we have

$$g_{00}(r, t) = 1 - \frac{kc^2 M_f(t)}{4\pi r}, \quad g_{11}(r, t) = \frac{-1}{1 - \frac{kc^2 M_f(t)}{4\pi r}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta, \quad (8)$$

with

$$M_f(t) = M - \lambda M_1(t) - \lambda M_2(t), \quad (9)$$

$$M_1(t) = \frac{(b+1)(Mkc^2)^b}{2[\frac{4}{3}\pi R_0^3(t)]^{b-1} kc^2}, \quad M_2(t) = \frac{4\pi b(b-1)(Mkc^2)^{b-2}}{[\frac{4}{3}\pi R_0^3(t)]^{b-2} kc^2} \left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t).$$

2.3. Model  $\mathbf{f}(\mathbf{R}) = \mathbf{R}^{1+\varepsilon}$  with  $\varepsilon$  very small

In this case  $\lambda h(R) = R^{1+\varepsilon} - R$ , we obtain

$$g_{00}(r, t) = 1 - \frac{kc^2 M_f(t)}{4\pi r}, \quad g_{11}(r, t) = \frac{-1}{1 - \frac{kc^2 M_f(t)}{4\pi r}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta \quad (10)$$

where

$$M_f(t) = M - \lambda M_1(t) - \lambda M_2(t), \quad (11)$$

$$\lambda M_1(t) = -M + \frac{(\varepsilon+2)(Mkc^2)^{\varepsilon+1}}{2[\frac{4}{3}\pi R_0^3(t)]^{\varepsilon} kc^2}, \quad \lambda M_2(t) = \frac{4\pi\varepsilon(\varepsilon+1)(Mkc^2)^{\varepsilon-1}}{[\frac{4}{3}\pi R_0^3(t)]^{\varepsilon-1} kc^2} \left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t).$$

### 3. Motion in a central field of the $f(R)$ -theory

Applying metrics  $g_{\mu\nu}$  obtained to the Hamilton-Jacobi equation

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = m^2 c^2,$$

we get a general equation of motion in a central field for an  $f(R)$ -theory, and, then a planet's orbital precession [6]

$$\Delta\varphi(n) = \frac{6\pi m^2 G^2 [M - \lambda M_1(t_n) - \lambda M_2(t_n)]^2}{c^2 L^2},$$

where  $L$  is the planet's angular momentum. Using [2]

$$\frac{L^2}{m^2} = a(1 - e^2)GM_f(t) \quad (12)$$

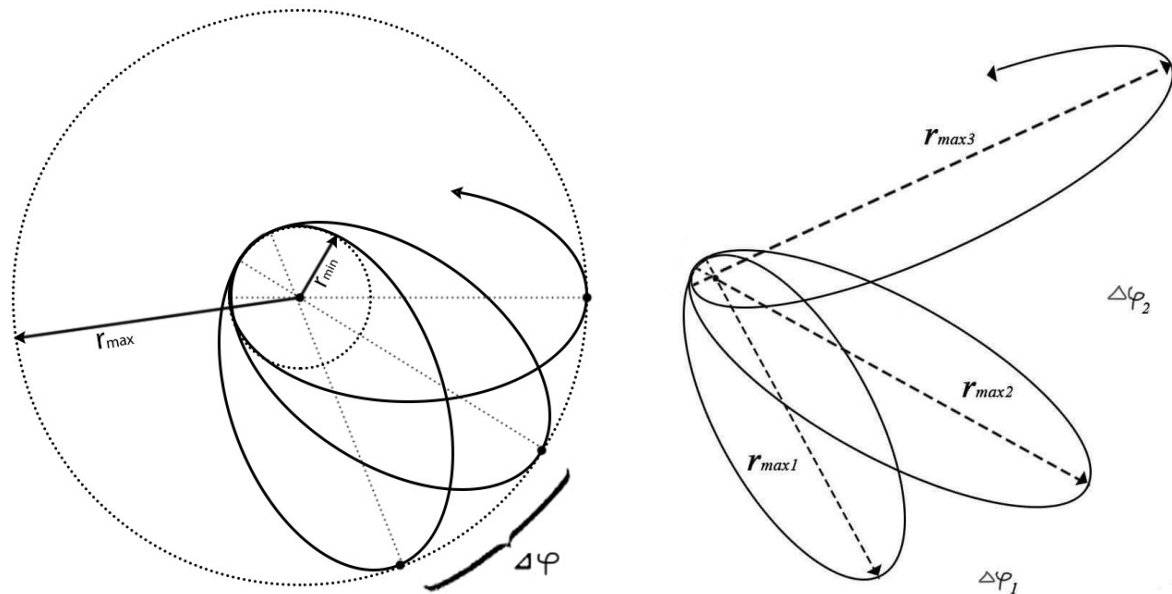
we get

$$\Delta\varphi_{f(R)} = \frac{6\pi GM_f(t)}{c^2 a(1 - e^2)}. \quad (13)$$

(for the deflection angle of light, see [5]).

### 4. Examination of the $f(R)$ -theory

We will examine the  $f(R)$ -theory for  $f(R) = R + \lambda R^2$  and  $f(R) = R + \frac{\lambda'}{R}$  in two cases: in a static central (gravitational) field and in a non-static central field. For this goal, let us apply the theory to some real gravitational systems, for example, Sun-Mercury (small system) and Sgr A\*-S2 (big system).



**Figure 1.** Illustrations of the orbital precession in the GR and in the  $f(R)$ -theory [5, 6]

#### 4.1. Static central field

Let us take Starobinsky's model [8]  $f(R) = R + \lambda R^2$  as an example (for other models, see [5, 6]). Using the data [7]

$$\begin{aligned}
 c &= 299792458 \text{ m/s}; \\
 G &= 6.67259 \times 10^{-11} \text{ kg}^{-1}\text{m}^3\text{s}^{-2}; \\
 k &= \frac{8\pi G}{c^4} = 2.0761154 \times 10^{-43} \text{ kg}^{-1}\text{m}^{-1}\text{s}^2; \\
 M &\equiv M_{\odot} = 1.988919 \times 10^{30} \text{ kg}; \\
 \frac{2GM}{c^2} &= 2.95325008 \times 10^3 \text{ m}; \\
 a &= 5.7909175 \times 10^{10} \text{ m}; \\
 e &= 0.20563069; \\
 \Delta\varphi_{obs} &= 2\pi(7.98734 \pm 0,00037) \times 10^{-8} \text{ radian/revolution},
 \end{aligned} \tag{14}$$

$$\frac{6\pi G}{c^2 a(1-e^2)} = 2.523307 \times 10^{-37}. \tag{15}$$

we obtain the orbital precession of Mercury orbiting around the Sun (the Sun-Mercury system) as follows

- Einstein's value:  
 $\Delta\varphi_E = 1.59748694\pi \times 10^{-7} \text{ radian/rev.}$
- Observed value:  
 $\Delta\varphi_{obs} = 1.597468\pi \times 10^{-7} \text{ radian/rev.}$

- Deviation:

$$\delta\varphi^{Mer} = \Delta\varphi_{obs} - \Delta\varphi_E = -0.1906\pi \times 10^{-11} \text{ radian/rev.}$$

Roughly, if the measurement's error is smaller than the latter,  $\lambda$  should take the value

$$\lambda = 0.296631 \times 10^{18} \quad (16)$$

in order to explain the deviation

$$\Delta\varphi_{f(R)} = \Delta\varphi_{obs} = 1.597468\pi \times 10^{-7} \text{ radian/rev.} \quad (17)$$

and

$$\delta\varphi^{Mer} = \Delta\varphi_{f(R)} - \Delta\varphi_E = \Delta\varphi_{obs} - \Delta\varphi_E = -0.1906\pi \times 10^{-11} \text{ radian/rev.} \quad (18)$$

It is worth noting that the value of  $\lambda$  in (16) satisfies the perturbation condition  $\lambda h(R) \ll R$  (see [6]), that is

$$\lambda h(kT_0^0) \ll \frac{6GM}{c^2[R_0]^3}, \quad (19)$$

or, equivalently,

$$\lambda \ll \frac{c^2[R_0]^3}{6GM} = 0.380053 \times 10^{23}. \quad (20)$$

Applying  $\lambda$  given above to a stronger gravitational system, e.g., the system Sgr A\*-S2 (with mass  $M = 4.31 \times 10^6 M_\odot$ ) we obtain

- $\Delta\varphi_{f(R)}^{S2} = 1.149305\pi \times 10^{-3} \text{ radian/rev.}$
- $\Delta\varphi_E^{S2} = 1.15114\pi \times 10^{-3} \text{ radian/rev.}$
- $\delta\varphi^{S2} = \Delta\varphi_{f(R)}^{S2} - \Delta\varphi_E^{S2} = -1.835\pi \times 10^{-6} \text{ radian/rev.}$

We do not consider  $f(R) = R + \lambda'/R$  for this case as it, compared with the GR, does not give a new correction (to the orbital precession). To check the theory it is necessary to work in a non-static field.

#### 4.2. Non-static central field

The theory  $f(R) = R + \lambda R^2$  is examined above in a static central field and its examination can be repeated straightforwardly for a non-static case. In this case we will examine one more theory, namely, the theory  $f(R) = R + \lambda'/R$ . As the value of  $\lambda$ , assumed to be universal and, therefore, applicable to the present case, is already estimated in (16), now we estimate the value of  $\lambda'$ . The perturbation condition  $\lambda h(R) \ll R$  applied to the theory  $f(R) = R + \lambda'/R$  gives

$$\lambda' \ll \frac{9}{[R_0]^6} \left( \frac{2GM}{c^2} \right)^2 = 6.923265 \times 10^{-46}. \quad (21)$$

Next, we estimate the correction to the orbital parameters (eccentricity and axes), compared with their classical values, of a planet moving around a collapsing star with the beginning radius  $R_0$  (before the collapse) and the final radius  $R_0(t)$  in a moment  $t$  (during the collapsing process or at the end of the collapse). Here, we consider a collapsing star as example but it is possible to consider an expanding (exploding) star. In order to have data for reference we will take a

Sun-similar star and its Mercury. Below, we will see how an orbit of a planet (Mercury) would change under the star (Sun) contraction keeping its spherical form. In this situation the GR gives no effect unlike the  $f(R)$  theory predicting new interesting phenomena (such as corrections to a planet's orbital eccentricity and axes, gravitational waves, etc.) which can be used for testing an  $f(R)$  theory.

A star of the Sun's size having a radius of the order

$$R_0 \approx 6.957 \times 10^8 \text{ m}, \quad (22)$$

would collapse to a white dwarf <sup>1</sup> of the Earth's size (or smaller) with a radius of the order

$$R_0(t) \approx 6.371 \times 10^6 \text{ m}. \quad (23)$$

The radius change

$$\Delta R_0(t) = R_0(t) - R_0 = -689329000 \text{ m} \quad (24)$$

would happen for the free falling (assumed) time interval [9]

$$\begin{aligned} \Delta t &= - \left( \frac{8\pi G \rho_0}{3} \right)^{-1/2} \int_1^{\frac{R_0(t)}{R_0}} \left( \frac{\zeta}{1-\zeta} \right)^{1/2} d\zeta \\ &= \left( \frac{3\pi}{32G\rho_0} \right)^{1/2} (1 + 3.45 \times 10^{-4}) \\ &= \left( \frac{4\pi^2 (R_0)^3}{32GM} \right)^{1/2} (1 + 3.45 \times 10^{-4}) \\ &= 1769.83 \text{ s}, \end{aligned} \quad (25)$$

with  $\frac{R_0(t)}{R_0} = 9.158 \times 10^{-3}$ . From here we can estimate the average free falling speed as

$$\frac{|\Delta R_0(t)|}{\Delta t} = 389488.82 \text{ m/s}. \quad (26)$$

Now, let us calculate some parameters of a planet's (Mercury's) orbit in an  $f(R)$  theory, where, the star's (Sun's) mass  $M$  is replaced by an effective mass  $M_f$  [5]. This mass would change with a value  $\Delta M_f$  during the star contraction until its total collapse. The semi-major axis  $a$  and the eccentricity  $e$  of the planet's elliptical orbit in an  $f(R)$  theory are calculated by the formulas

$$a = \frac{GmM_f(t)}{2|E|}, \quad (27)$$

$$e \approx \sqrt{1 - \frac{2|E|L^2}{G^2m^3M_f^2}}, \quad (28)$$

where  $L$  is the angular momentum which is a conserved quantity but the energy  $E$  is not conserved for a non-static field. To calculate the energy change during the star contraction (until the total collapse) we use the approximation  $E \approx \frac{-m(GmM_f)^2}{2L^2}$  (for a circular orbit), thus,

$$\Delta|E| = \frac{2|E|}{M_f} \Delta M_f. \quad (29)$$

<sup>1</sup> More precisely, according to the standard theory of the star evolution, a Sun-type star would first become a red giant before collapsing to a white dwarf.

Therefore, the planet's orbital semi-major axis and eccentricity would change by the quantities

$$\Delta a = \frac{Gm}{2|E|} \left( \Delta M_f - \frac{M_f \Delta |E|}{|E|} \right) = -\frac{Gm}{2|E|} \Delta M_f = -\frac{L^2 \Delta M_f}{Gm^2 M_f^2}, \quad (30)$$

$$\Delta e = -\frac{L^2}{G^2 m^3 M_f^2} \frac{\left( \Delta |E| - \frac{2|E| \Delta M_f}{M_f} \right)}{e} = 0, \quad (31)$$

or with (12) taken into account these changes become

$$\Delta a(t) = a(t) - a_0 = -a_0(1 - e^2) \frac{\Delta M_f(t)}{M_f}, \quad (32)$$

$$\Delta e = 0, \quad (33)$$

where  $a_0$  and  $a(t)$  are the semi-major axis before the contraction and after the collapse of the star, respectively.

#### 4.2.1. Model $f(R) = R + \lambda R^2$ :

Assuming that the star's mass  $M$  remains unchanged during the collapse process (in reality, some gravitational radiation and other matter loss may be possible), we calculate the change of the components  $M_1$  and  $M_2$  of  $M_f$  during the collapsing (in this  $f(R)$ -theory  $M_1$  and  $M_2$  have dimension of  $[mass/length^2]$  which will be tacit below).

Let us start with calculating the change  $\Delta M_1$  of  $M_1$ . It is easy to see that

$$\Delta M_1(t) = M_1(t) - M_1(0) = 1.02213108 \times 10^{14} \quad (34)$$

where

$$M_1(0) = \frac{9M^2 kc^2}{8\pi[R_0]^3} = 78498929.12, \quad (35)$$

is the value of  $M_1$  before the collapse starting ( $t = 0$ ), and

$$M_1(t) = \frac{9M^2 kc^2}{8\pi[R_0(t)]^3} = 1.02213186 \times 10^{14}, \quad (36)$$

is the value of  $M_1$  immediately before the end of the collapse. In order to calculate the change  $\Delta M_2$  of  $M_2$  we must first calculate  $\xi(t)$  and  $\alpha(t)$ . Using (6), (14) and (23) we get

$$\xi(t) = \sqrt{\frac{kMc^2}{4\pi[R_0(t)]^3}} = 3.37939 \times 10^{-9} \quad (37)$$

and

$$\alpha(t) = 4.5029183 \times 10^{-36}. \quad (38)$$

With the approximation  $\frac{\partial}{\partial t} R_0(t) \approx \frac{\Delta R_0(t)}{\Delta t}$ ,

$$\left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t) = \left( \frac{-3M}{[R_0(t)]^4} \frac{\Delta R_0(t)}{\Delta t} \right)^2 \alpha(t) = 8.959833 \times 10^{-18}, \quad (39)$$

$M_2$  takes the value

$$M_2 = \frac{8\pi}{kc^2} \left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t) = 1.206832 \times 10^{10}. \quad (40)$$



Combining (36) and (40) we have

$$M_f(t) = M(t) - \lambda M_1(t) - \lambda M_2(t) = -2.833426 \times 10^{31} \text{ kg}, \quad (41)$$

$$\Delta M_f = M_f(t) - M_f = -3.03231558 \times 10^{31} \text{ kg}. \quad (42)$$

Inserting this result in (30) we get

$$\Delta a(t) = a(t) - a = 0.4166264 \times 10^{10} \text{ m}. \quad (43)$$

In comparison with the GR, i.e., comparing (43) with (14), we see that the semi-major axis  $a(t)$  of the Mercury-like planet in the  $f(R)$ -theory would increase with 7.19% after the collapse of the Sun-like star. The sign minus in (41) and (42) shows something like an “anti-gravitational” effect which could be a strong argument for verifying the present model.

#### 4.2.2. Model $f(R) = R + \frac{\lambda'}{R}$ :

In this model, following similar calculations as above it is not difficult to get for a Sun-Mercury-like system

$$\Delta M_f(t) \simeq 0, \quad (44)$$

that is, there is no sensitive correction to the GR.

## 5. Conclusion

The  $f(R)$ -theory is a modified theory of gravitation which makes correction to the general theory of relativity and may replace the latter in explaining new cosmological observations.

We have shown that the  $f(R)$ -theory allows a non-static spherically symmetric solution and predicts (non-static in general) corrections to cosmological observations (such as orbital precessions and deformations, deflections of light, etc.). Another prediction which is about gravitational radiations of a (non-static) spherically symmetric source, a phenomenon not possible in the general theory of relativity, can be also considered, but it is a subject of a later work being in progress. A testing measurement (observation) may not be easy at the present technical level but we hope it can be done in a not very far future.

## Acknowledgments

This work is funded by the Vietnam academy of science and technology (VAST) under the grant NVCC05.09/20-20.

## References

- [1] Weinberg S 1972 *Gravitation and cosmology: Principles and applications of the general theory of relativity* (John Wiley & Son, New York)
- [2] Landau L D and Lifshitz E M 1994 *The classical theory of fields* vol. 2 (Elsevier Oxford)
- [3] Abbott B P *et al* [LIGO scientific and Virgo collaborations] 2016 *Phys. Rev. Lett.* **116** 061102 (*Preprint* arXiv:1602.03837 [gr-qc])
- [4] Abbott B P *et al* [LIGO scientific and Virgo collaborations] 2017 *Phys. Rev. Lett.* **119**, 161101 (*Preprint* arXiv:1710.05832 [gr-qc])
- [5] Nguyen Anh Ky, Pham Van Ky and Nguyen Thi Hong Van 2018 *Eur. Phys. J. C* **78** 539; Erratum: 2018 *Eur. Phys. J. C* **78** 664 (*Preprint* arXiv:1807.04628 [gr-qc])
- [6] Nguyen Anh Ky, Pham Van Ky and Nguyen Thi Hong Van 2019 *Commun. Phys.* **29** 35 (*Preprint* arXiv:1904.04013 [physics.gen-ph])

- [7] Majumder B *Preprint* arXiv: 1105.2428
- [8] Starobinsky A A 1980 *Phys. Lett. B* **91** 99 (1987 *Adv. Ser. Astrophys. Cosmol.* **3** 130)
- [9] Kippenhahn R and Weigert A and Weiss A 2012 *Stellar structure and evolution* (Springer-Verlag)