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# A Microscopic Model for Superconductivity in Ferromagnetic $UGe_2$

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The phenomenological Ginzburg-Landau model with two order parameters appears in many works of D.V. Shopova and D. I. Uzunov. Here, we develop a microscopic approach that, on the basis of mean-field theory and the functional integral formalism, establishes the two-component Ginzburg-Landau functional to show the correlation between the ferromagnetic and the triplet superconducting order parameters. The meanings of the constants encountered in the phenomenological theory are clarified.

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## I. INTRODUCTION

Magnetism and conventional superconductivity have been known for a long time. Nevertheless, the perfect diamagnetic property of superconductors caused a thought that superconductivity was incompatible with ferromagnetism. However, in 1979, Steglich and *et al.* first discovered superconductivity in the first heavy-fermion material $CeCu_2Si_2$ [1]. In the 1980s, superconductivity was found in U-based heavy-fermion compounds, *e.g.*, $UBe_{13}$ (1983) [2] and $UPt_3$ (1984) [3]. In these heavy fermion compounds which usually contain the rare-earth element cerium (Ce) and uranium (U), superconductivity may coexist with antiferromagnetic (AFM) order, and neighboring electron spins arrange in an antiparallel configuration only under high pressure and at very low temperature. Another breakthrough was the discovery of $UGe_2$ in 2000, and after that, UIr, URhGe, and UCoGe with coexisting ferromagnetism (FM) and superconductivity (SC). In these materials, superconductivity was observed at temperatures below the Curie temperature without expelling the ferromagnetic order. The research on heavy-fermion systems and other exotic materials showed for the first time that competition between magnetic and superconducting orders turns out to be an important characteristic of strongly correlated electron systems.

According to previous theories, magnetism is induced by the spin moments of localized $4f$, $5f$ electrons whereas superconductivity comes from the Cooper pairs formed by conduction electrons. Such a discovery, together with a number of reliable experimental data about the coherence length and the superconducting gap [4–7], leads to the conclusion that $4f$-electrons from Ce-atoms and $5f$-electrons from U-atoms are responsible for both magnetism (AFM or FM) and SC. The Cooper pairs in these metallic compounds belong to a spin-triplet, and magnetic-fluctuation induced pairing is a possible mechanism. In recent years, beside the experimental investigations that examined the dependence of the phase transition on the applied pressure and the magnetic field, other theoretical research concentrated on finding the phase transition mechanism, the natures of phases and the dependences of the temperature of the phase transition and of the spontaneous magnetization moment on the parameters of the materials. Different mechanisms, such as coupled charge density waves and spin density waves [8, 9], magnon exchange [10], electron interaction mediated by ferromagnetically aligned localized moments [11,12], screened phonon interactions [13], d-electron exchange [14], M-trigger [15, 16], the multiband model [17–20], *etc.*, have been proposed. These theoretical works have tackled this important issue and provided invaluable information about the interplay between AFM, FM and conventional SC, and unconventional SC in the coexistence states. Specially, the phenomenological Ginzburg-Landau model proposed by Shopova and *et al.* [15,16, 21–23] has described the coexistence of superconductivity and ferromagnetic orders in U-based compounds well.

The above consideration motivates transforming the

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fermionic field theory to an effective one based on coupling fields that are expressed in terms of order parameter fields for different channels. The main purpose of this paper is the formulation of a microscopic two-component Ginzburg-Landau (G-L) functional that can describe the coexistence of different phases. In our research, through the mean-field theory and the functional integral formalism, we will split microscopic Hamiltonian into possible channels; then, we can get a functional that only depends on the order parameters. Based on the specific problem of ferromagnetic superconductivity for $UGe_2$ systems, we will determine the formal performance of the G-L functional for a two order-parameter system through calculations based on the Green function. Analytical calculations were carried out to show correspondence between Shopova's phenomenological model and our microscopic model.

The paper is structured as follows: Sec. II reviews the phenomenological Ginzburg-Landau model with two magnetization and superconducting order parameters when both $i\mathbf{M}.(\mathbf{d}\times\mathbf{d}^*)$-linear and $|\mathbf{d}|^2.|\mathbf{M}|^2$-quadratic couplings of the magnetization to the superconducting order parameter are taken into account, and the anisotropies are ignored. The uniform phases and the phase diagram in this case were investigated in Ref. 15. Here, we summarize the main results in order to make a clear comparison with the new results presented in Sec. III, which discusses the competition between the AFM, FM and SC orders on phenomenological grounds. We present our microscopic model and derive the Ginzburg-Landau functional in which the coefficients that couple the FM and the SC order parameters are clarified. Finally, Sec. IV summarize the results and provides a general discussion. The two appendices contain additional technical details of the derivations discussed in the main text. Some of the results reported in this work have been presented in a shorter publication [24].

## II. THE PHENOMENOLOGICAL GINZBURG - LANDAU MODEL

In this section, we briefly review the main experimental results for $UGe_2$ and consider the phenomenological Ginzburg-Landau model with two order parameters, which was proposed by Shopova and Uzunov [15]. The main results for the Meissner phases in spin-triplet ferromagnetic superconductors are summarized in order to make clear the role of the phenomenological Ginzburg-Landau model for describing the coexistence of superconductivity and ferromagnetic orders in $UGe_2$.

### 1. Experiments

The experimental research on $UGe_2$ [4,5,25,26] shows that at ambient pressure, $UGe_2$ is an itinerant ferromagnet whose Curie temperature is $T_c = 52$ K, and the spontaneous moment is $\mu_s = 1.4\mu B/\mathrm{U-atom}$. The easy axis is the $a$-axis in the orthorhombic crystal [8]. With increasing pressure, the system passes through two successive quantum phase transitions, one from the ferromagnetism phase to the FM-SC phase at $P = 1$ GPa, and the other at higher pressure $P_c = 1.6$ GPa to the paramagnetism phase. The superconducting phase exists entirely within the ferromagnetic domain at low temperatures and pressure intervals between 1.0 and 1.6 GPa with a maximum superconducting transition temperature $T_{sc} = 0.8$ K near 1.2 GPa. Two distinct ferromagnetic phases, usually denoted by FM2 and FM1 with different magnitudes of the magnetic moments, exist. With increasing pressure, the magnetic ground state switches from a highly polarized phase (FM2, $\mu = 1.5\mu B$) to a weakly polarized phase (FM1, $\mu = 0.9\mu B$) at a pressure of $P_x = 1.2$ GPa; the phase line ends where $T_x = T_{sc}$ has its maximal value. When the pressure is increased, the order of the transition from FM1 to the paramagnetic phase changes from second order to first order at the tricritical point $T_{cr}$ on the $T(P)$ diagram. With further increases in $P$, both $T_{FM}$ and $T_{FS}$ decrease and disappear almost simultaneously at a pressure around $P \sim 1.7$ GPa.

### 2. Meissner phases in spin-triplet ferromagnetic superconductors

If we follow Shopova and Uzunov [15], and ignoring the anisotropy of both Cooper pairs and crystals, the phenomenological Ginzburg-Landau energy functional of the triplet ferromagnetic superconductor is as follows:

$$\begin{aligned} f_{\mathrm{GL}}(\mathbf{d},\mathbf{M}) &= a_s|\mathbf{d}|^2 + \frac{b_s}{2}|\mathbf{d}|^4 + a_f|\mathbf{M}|^2 + \frac{b_f}{2}|\mathbf{M}|^4 \\ &+ i\gamma_0\mathbf{M}.(\mathbf{d}\times\mathbf{d}^*) + \delta_0|\mathbf{d}|^2|\mathbf{M}|^2 . \end{aligned} \quad (1)$$

In Eq. (1), the first two terms describe the superconductivity for $M = H = 0$, the next two terms describe the free energy of a ferromagnetic phase which is considered as having uniaxial anisotropy of the Ising type, and last two terms describe the interaction between the ferromagnetic order parameter $\mathbf{M}$ and the superconducting order parameter $\mathbf{d}$. Where $b_s, b_f > 0$ and $a_s = \alpha_s(T - T_s)$, the parameter $a_f$ is slightly modified by choosing $a_f = \alpha_f\left[T^n - T_f^n(P)\right]$ in which $n = 1$ gives the standard form of $a_f$ and $n = 2$ applies for other models. $\alpha_s, \alpha_f$ are positive material parameters, $T_s$ is the critical temperature of a standard second-order phase transition which may take place at $M = H = 0$, and the temperature $T_f$ is the critical temperature of the ferromagnet. The parameter $\gamma_0$ for ferromagnetic superconductors may take both positive and negative values. In general, the values of the material parameters $(T_s, T_f, a_s, b_s, a_f, b_f, \gamma_0)$ depend on the choice of the concrete substance and on thermodynamic parameters such as the temperature $T$ and the pressure $P$.

Next, Shopova and Uzunov redefined, for convenience, the free energy in Eq. (1) in a dimensionless form as $f = f_{\rm GL}/\left(b_f M_0^4\right)$, where $M_0 = [\alpha_f T_{f0}/b_f]^{1/2}$ is the value of the magnetization $M$ corresponding to the pure magnetic subsystem $\boldsymbol{\psi} = 0$ at $T = P = 0$ and $T_{f0} = T_f(0)$. The order parameters assume the scaling $m = M/M_0$ and $\boldsymbol{\varphi} = \boldsymbol{d}/\left[(b_f/b)^{1/4} M_0\right]$, and as a result, the free energy becomes

$$f = r\phi^2 + \frac{1}{2}\phi^4 + tm^2 + \frac{1}{2}m^4 + 2\gamma\phi_1\phi_2 \sin\theta m + \delta\phi^2 m^2, \quad (2)$$

where $\phi = |\boldsymbol{d}|$, $\phi_j = |d_j|$, and $\theta = \theta_1 - \theta_2$ is the phase angle between the complex $d_1 = \phi_1 e^{i\theta_1}$ and $d_2 = \phi_2 e^{i\theta_2}$. The dimensionless parameters $t, r, \gamma$ and $\delta$ in Eq. (2) are given by

$$\begin{aligned} r &= \kappa\left(\tilde{T} - \tilde{T}_s(P)\right); \\ t &= \tilde{T} - \tilde{T}_f(P) = \tilde{T} - 1 + \tilde{P}; \\ \gamma &= \gamma_0/\left[\alpha_f T_{f0}/b\right]^{1/2}; \\ \delta &= \delta_0/\left(b b_f\right)^{1/2}, \end{aligned} \quad (3)$$

where $\kappa = \alpha_s b_f^{1/2}/\alpha_f b^{1/2}$ and $\tilde{P} = P/P_{\rm c}$. The reduced temperatures are $\tilde{T} = T/T_{f0}$, $\tilde{T}_f(P) = T_f(P)/T_{f0}$ and $\tilde{T}_s = T_s(P)/T_{f0}$. The analysis involves making simple assumptions for the $P$ dependences of the $t, r, \gamma$ and $\delta$ parameters in Eq. (2). In particular, only $T_f$ has a significant $P$ dependence, described by

$$\tilde{T}_f(P) = (1 - \tilde{P})^{1/n},$$

where $\tilde{P} = P/P_0$ and $P_0$ is a characteristic pressure deduced later. In $UGe_2$ the $P_0$ values are very close to the critical pressure $P_{\rm c}$ at which both the ferromagnetic and the superconducting orders vanish, but in other systems, this is not necessarily the case. The linear dependence of $\tilde{T}_f$ on $\tilde{P}$ ($n = 1$) agrees well with the experimentally measured pressure dependence for $UGe_2$.

Finally, the conditions of equilibrium and the stable phases for the $UGe_2$ system which has the free energy given by Eq. (2) are used to outline the $(t, r)$ phase diagram, which indicates the domains of stability for the N, FM and FS phases. The results are presented in an analytical form; only a small part of phase diagram is calculated numerically.

The phase diagram for concrete parameters of $\gamma$ and $\delta$ is shown in Fig. 1. The domains of stability of the N, FM and FS phases are indicated. DCB is the line of demarcation between two domains of stability of the FM and FS phases, where the curve DC (the dashed line, to the left of point C) is the second-order phase transition and the line CB is the first-order phase transition. The phase

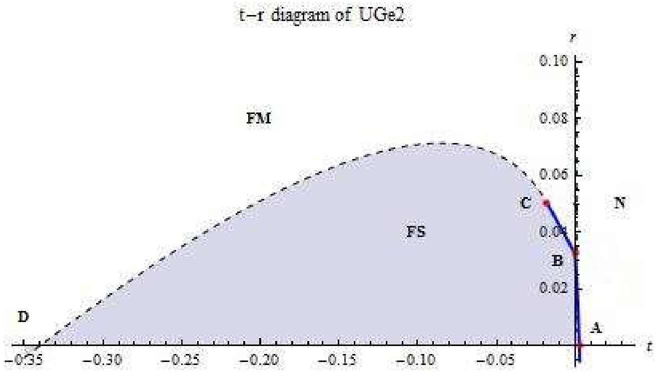


Fig. 1. (Color online) Phase diagram in the $(t, r)$ plane for $\gamma = 0.49, \delta = 0.84$

transition between the N and the FS phases is first-order and goes along the equilibrium line BA. The vertical dashed line coinciding with the $r$-axis above B, which is the line of demarcation between two domains of stability for the FM and N phases, indicates the N-FM phase transition to be of second order. A and C are tricritical points of the phase transitions; B is the triple point. An analytical calculation also show that the superconducting critical temperature $T_m$ and the corresponding pressure $P_m$ for $UGe_2$ when $\gamma = 0.49, \delta = 0.84, \kappa = 5, T_{f0} = 52$, and $P_{\rm c} = 1.6$, together with $T_s = 0$, are $T_m = 0.743$ K and $P_m = 1.44$ GPa, respectively. These values are in relative agreement with the main experimental findings.

The phenomenological Ginzburg-Landau model, Eq. (1), not only successfully describes the domains of stability of the N, FM and FS phases but also shows the dependence of phase transition temperature $T$ on the pressure $P$. The model and its extended models used in many works of Shopova and *et al.* [15, 16, 21–23] have provided invaluable information about the interplay between the FM and the triplet SC phases in the coexistence states of U-based compounds.

In the next section, we develop a microscopic approach on the basic of the mean-field theory and the functional integral formalism, we will establish the two-component Ginzburg-Landau functional to show the correlation between the ferromagnetic and triplet superconducting order parameters. Moreover the meanings of the constants encountered in the phenomenological theory will be clarified.

## III. MICROSCOPIC DERIVATION OF THE GINZBURG-LANDAU FUNCTIONAL FOR A FERROMAGNETIC SUPERCONDUCTOR

The competition between the superconductivity and the magnetism orders has been one of the central issues in heavy fermion systems. In particular, one usually is interested in materials showing the coexistence of SC and ferromagnetic (FM) or antiferromagnetic (AF) order in uranium-based heavy fermions intermetallic compounds. These materials contain a periodic array of uranium ions. The strong interactions of the localized $5f$ moments with the light conduction electrons of the other constituent metal atoms give rise to the formation of heavy quasiparticles with an effective mass $m^*$ up to several hundred times larger than that of a usual metal. When these materials are cooled from high temperatures, localized moments seem to decrease progressively due to local spin fluctuations. When a characteristic energy associated with the spin fluctuations is dominated by the indirect Ruderman-Kittel-Kasuya-Yosida type interaction between incompletely compensated atomic moments on different sites, the materials may undergo a magnetic phase transition at a finite temperature. In some materials, the heavy quasiparticles condense into the SC state at very low temperatures due to residual interactions among them. This seems to indicate that the attractive effective interaction between the strongly renormalized heavy quasiparticles in $UGe_2$ is not provided by the electron-phonon interaction as in ordinary superconductors, but rather is mediated by electronic spin fluctuations. In the vicinity of a ferromagnetic quantum critical point, critical magnetic fluctuations can mediate superconductivity by pairing the electrons in spin-triplet Cooper pairs, that is, equal-spin pairing states that have a nonzero total spin angular momentum ($S = 1$): $|\uparrow\uparrow\rangle$ ($L = 1, S_z = 1$), $|\downarrow\downarrow\rangle$ ($L = 1, S_z = -1$), and $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ ($L = 1, S_z = 0$). These spin-triplet Cooper pairs have quantum states with parallel electron spins; therefore, they can survive in the presence of magnetic moments.

### 1. The Hamiltonian-model and mean-field approximation

Our starting point is an interacting fermion model. In the terms of second quantization, the Hamiltonian of the system can be written as

$$H = H_0 + H_1, \tag{4}$$

where,

$$H_0 = \sum_{\sigma}\sum_{\mathbf{k}} \epsilon_\sigma(\mathbf{k})\psi^\dagger_\sigma(\mathbf{k})\psi_\sigma(\mathbf{k}). \tag{5}$$

Here, $H_0$ is the unperturbed Hamiltonian describing the system of free fermions; $\psi_\sigma(\mathbf{k})$ and $\psi^\dagger_\sigma(\mathbf{k})$ are the annihilation and the creation operators of the fermions with spin projections $\sigma = \uparrow, \downarrow$ respectively; $\epsilon_\sigma(\mathbf{k})$ is the dispersion of the free fermions;

$$H_1 = \sum_{\sigma_1,\sigma_2,\sigma_1',\sigma_2'}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q}) \times \psi^\dagger_{\sigma_1}(\mathbf{k}-\frac{\mathbf{q}}{2})\psi^\dagger_{\sigma_2}(\mathbf{k}'+\frac{\mathbf{q}}{2})\psi_{\sigma_2'}(\mathbf{k}'-\frac{\mathbf{q}}{2})\psi_{\sigma_1'}(\mathbf{k}+\frac{\mathbf{q}}{2}). \tag{6}$$

Here, $H_1$ is a generic effective two-body interaction term of interacting fermions written in the normal order. $V_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q})$ are general pair scattering matrix elements, which may cover the contributions of different interactions in the system, such as electron - phonon, spin - spin, ..., and even impurity - electron and impurity - impurity interactions. From both mathematical and physical aspects, the generic effective interaction term is very complicated and has no exact and explicit analytical expression.

In order to convert our Hamiltonian into an effectively non-interacting one, we decouple the quartic fermion term into quadratic ones. Different approximations can be used to tackle this task. In the low-temperature region where the elementary excitations are dominant, the fluctuations around the average value of the order parameter are so small that the second-order fluctuation term can be neglected and the simplest method is the mean-field approximation. At least, there are three inequivalent choices of pairing up the fermionic operators can be used to construct the fermionic bilinear term of the generic two-body interaction. Those are pairings in the direct channel $(\psi^\dagger\psi)_{\sigma_1\sigma_1'}(\mathbf{k},-\mathbf{q},\tau)$, in the exchange channel $(\psi^\dagger\psi)_{\sigma_2\sigma_1'}(\mathbf{k}',\mathbf{k},\mathbf{q},\tau)$, and in the cooper channel $(\psi\psi)_{\sigma_2'\sigma_1'}(\mathbf{k}',\mathbf{k},\mathbf{q})$ and $(\psi\psi)^\dagger_{\sigma_2\sigma_1}(\mathbf{k}',\mathbf{k},-\mathbf{q})$ [27]. Nevertheless, the "right" choice of the decoupling field should be only motivated by physical reasoning, *i.e.*, one has to proceed to derive an effective theory based on the coupling field later. In the simplest case, without the spin-dependences of the two-body matrix elements ($V_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q}) = V(\mathbf{k},\mathbf{k}',\mathbf{q})$ ), the decoupling of all three channels is possible, and all order parameters can coexist in the system considered. In this case, theoretical calculations, however, lead to different results. In other words, an apparent ambiguity exists. In order to avoid this fault, we have to include the spin-dependence of the two-body matrix element $V_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q})$ in the microscopic model. The simplest physical reason for including the spin-dependence of the two-body matrix element is the contribution of exchange bosons reflecting the interactions between conducting electrons and bosonic background fluctuations which cause the rise of selected competing channels. Depending on how the spin indices are split ($\sigma, \sigma'$ are $\downarrow\downarrow$, $\uparrow\uparrow$ or $\downarrow\uparrow \pm \uparrow\downarrow$), we will have a singlet superconducting order or a triplet superconducting order, a ferromagnetic order or an antiferromagnetic order. If the order is singlet, the system can exist in an antiferromagnetic phase and singlet superconducting phase as in $CeRhIn_5$ and $CeIrIn_5$. If the order is triplet, the system can fully exist in an ferromagnetic phase plus triplet superconducting phase as in $UGe_2$ without depending on whether the electron is localized or not. That is why, for this problem, the generic effective two-body interaction term $H_1\left[\psi^\dagger,\psi\right]$ can be broken down to a summation of two possible fermionic bilinear terms with arbitrary parameters $\{\gamma_i\}$, where $i \in \{\text{d,C}\}$, as

$$H_1\left[\psi^\dagger,\psi\right] = \gamma_d^2 H_1^{\text{d}}\left[\psi^\dagger,\psi\right] + \gamma_C^2 H_1^{\text{C}}\left[\psi^\dagger,\psi\right], \quad (7)$$

where the Hamiltonians $H_1^{\text{d}}$ and $H_1^{\text{C}}$ are the generic two-body interaction (they are responsible for magnetism and superconductivity, respectively) rewritten in different fermionic bilinear terms:

$$H_1^{\text{d}} = \sum_{\sigma_1,\sigma_2,\sigma_1',\sigma_2'}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \psi^\dagger_{\sigma_1}(\mathbf{k}-\frac{\mathbf{q}}{2})\psi_{\sigma_1'}(\mathbf{k}+\frac{\mathbf{q}}{2}) \times (V_d)_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q})\psi^\dagger_{\sigma_2}(\mathbf{k}'+\frac{\mathbf{q}}{2})\psi_{\sigma_2'}(\mathbf{k}'-\frac{\mathbf{q}}{2}), \quad (8)$$

$$H_1^{\text{C}} = \sum_{\sigma_1,\sigma_2,\sigma_1',\sigma_2'}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \psi^\dagger_{\sigma_1}(\mathbf{k}-\frac{\mathbf{q}}{2})\psi^\dagger_{\sigma_2}(\mathbf{k}'+\frac{\mathbf{q}}{2}) \times (V_C)_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q})\psi_{\sigma_2'}(\mathbf{k}'-\frac{\mathbf{q}}{2})\psi_{\sigma_1'}(\mathbf{k}+\frac{\mathbf{q}}{2}). \quad (9)$$

Here, the values of the parameters $\gamma_i$ should satisfy the identity

$$\gamma_{\text{d}}^2 + \gamma_{\text{C}}^2 = 1. \quad (10)$$

This identity ensures that the potential $V_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q})$ can be written in many ways, but sum of there ways has to be equal to the initial potential. We can find parameters $\gamma_i$ by minimizing the Ginzburg-Landau free-energy functional with respect to $\gamma_i$.

For definiteness, we take the interaction matrix in the simple form

$$(V_d)_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q}) = (V_d)(\mathbf{k},\mathbf{k}',\mathbf{q})\boldsymbol{\sigma}_{\sigma_1\sigma_1'}.\boldsymbol{\sigma}_{\sigma_2\sigma_2'} \quad (11)$$

with a constant $(V_d)(\mathbf{k},\mathbf{k}',\mathbf{q}) = V_d$. Then, we define the spin-density-wave (SDW) order parameter as

$$\mathbf{M}_{\sigma_1\sigma_1'}(\mathbf{k},\mathbf{q}) = \sum_{\mathbf{k}',\sigma_2,\sigma_2'}(V_d)(\mathbf{k},\mathbf{k}',\mathbf{q})\boldsymbol{\sigma}_{\sigma_2\sigma_2'} \times \left\langle \psi^\dagger_{\sigma_2}(\mathbf{k}'+\frac{\mathbf{q}}{2})\psi_{\sigma_2'}(\mathbf{k}'-\frac{\mathbf{q}}{2})\right\rangle. \quad (12)$$

Here $\sigma^{(i)}_{ss'}$ denotes the $(ss')$ element of the $i$th Pauli matrix with $s,s'=\pm1$, and $\boldsymbol{\sigma} = \hat{x}\sigma_x+\hat{y}\sigma_y+\hat{z}\sigma_z$ denotes the vector that has the usual Pauli matrices are components. We consider also superconducting (SC) interaction only in the triplet channel, *i.e.*,

$$(V_C)_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q}) = (V_C)(\mathbf{k},\mathbf{k}',\mathbf{q})(i\boldsymbol{\sigma}\sigma_y)_{\sigma_1\sigma_2}(i\boldsymbol{\sigma}\sigma_y)^\dagger_{\sigma_1'\sigma_2'} \quad (13)$$

with a constant $(V_C)(\mathbf{k},\mathbf{k}',\mathbf{q}) = V_C$. The triplet order parameter is

$$\triangle_{\sigma_1\sigma_2}(\mathbf{k},\mathbf{q}) = -\sum_{\mathbf{k}',\sigma_1'\sigma_2'} V_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\mathbf{k},\mathbf{k}',\mathbf{q}) \times \left\langle \psi_{\sigma_2'}(\mathbf{k}'-\frac{\mathbf{q}}{2})\psi_{\sigma_1'}(\mathbf{k}+\frac{\mathbf{q}}{2})\right\rangle = \left[i\left(\mathbf{d}(\mathbf{k})\boldsymbol{\sigma}\right)\sigma_y\right]_{\sigma_1\sigma_2}. \quad (14)$$

Using the forms of SC and SDW order parameters, we write the interaction parts in quadratic forms as

$$H_1^{\text{d}} = \sum_{\sigma_1,\sigma_2}\sum_{\mathbf{k},\mathbf{q}}\Big[(\mathbf{M}.\boldsymbol{\sigma})_{\sigma_1\sigma_2}\psi^\dagger_{\sigma_1}(\mathbf{k}-\frac{\mathbf{q}}{2})\psi_{\sigma_2}(\mathbf{k}+\frac{\mathbf{q}}{2}) +(\mathbf{M}.\boldsymbol{\sigma})^\dagger_{\sigma_1\sigma_2}\psi^\dagger_{\sigma_1}(\mathbf{k}+\frac{\mathbf{q}}{2})\psi_{\sigma_2}(\mathbf{k}-\frac{\mathbf{q}}{2})\Big] + \frac{M^2}{V_d}, \quad (15)$$

$$H_1^{\text{C}} = \sum_{\sigma_1,\sigma_2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\Big[\triangle_{\sigma_1\sigma_2}(\mathbf{k},\mathbf{q})\psi^\dagger_{\sigma_1}(\mathbf{k}-\frac{\mathbf{q}}{2})\psi^\dagger_{\sigma_2}(\mathbf{k}'+\frac{\mathbf{q}}{2}) +\triangle^*_{\sigma_1\sigma_2}(\mathbf{k},\mathbf{q})\psi_{\sigma_1}(\mathbf{k}+\frac{\mathbf{q}}{2})\psi_{\sigma_2}(\mathbf{k}'-\frac{\mathbf{q}}{2})\Big] + \frac{|\triangle|^2}{V_C}. \quad (16)$$

In order to define the Ginzburg-Landau free energy, we have to integrate out the fermionic fields of the mean-field grand canonical partition function. For convenience, we rewrite Eqs. (15) and (16) in terms of compact quadratic forms of the four-component spinor $\Psi$, which is introduced as follows:

$$\Psi\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right) = \Big(\psi_\uparrow(\mathbf{k}+\tfrac{\mathbf{q}}{2})\ \ \psi_\downarrow(\mathbf{k}+\tfrac{\mathbf{q}}{2})\ \ \psi^\dagger_\uparrow(-\mathbf{k}+\tfrac{\mathbf{q}}{2})\ \ \psi^\dagger_\downarrow(-\mathbf{k}+\tfrac{\mathbf{q}}{2})\Big)^T, \quad (17)$$

$$\Psi^\dagger\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right) = \Big(\psi^\dagger_\uparrow(\mathbf{k}+\tfrac{\mathbf{q}}{2})\ \ \psi^\dagger_\downarrow(\mathbf{k}+\tfrac{\mathbf{q}}{2})\ \ \psi_\uparrow(-\mathbf{k}+\tfrac{\mathbf{q}}{2})\ \ \psi_\downarrow(-\mathbf{k}+\tfrac{\mathbf{q}}{2})\Big). \quad (18)$$

Now, the mean-field Hamiltonian can be written in the bilinear form as

$$H_{\text{MF}} = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\Psi^\dagger\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)D(\mathbf{k},\mathbf{k}',\mathbf{q})\Psi\left(\mathbf{k}'-\frac{\mathbf{q}}{2}\right) + \left[\frac{|\triangle|^2}{V_C}+\frac{\mathbf{M}^2}{V_d}\right], \quad (19)$$

where $D(\mathbf{k},\mathbf{k}',\mathbf{q})$ is a $4\times 4$ matrix given by

$$D(\mathbf{k},\mathbf{k}',\mathbf{q}) = \begin{pmatrix} -\epsilon_{\uparrow\uparrow}-\Phi_{\uparrow\uparrow} & -\Phi_{\uparrow\downarrow} & \mathcal{F}_{\uparrow\uparrow} & \mathcal{F}_{\uparrow\downarrow} \\ -\Phi_{\downarrow\uparrow} & -\epsilon_{\downarrow\downarrow}-\Phi_{\downarrow\downarrow} & \mathcal{F}_{\downarrow\uparrow} & \mathcal{F}_{\downarrow\downarrow} \\ \mathcal{F}^*_{\uparrow\uparrow} & \mathcal{F}^*_{\uparrow\downarrow} & \epsilon_{\uparrow\uparrow}+\Phi_{\uparrow\uparrow} & \Phi_{\uparrow\downarrow} \\ \mathcal{F}^*_{\downarrow\uparrow} & \mathcal{F}^*_{\downarrow\downarrow} & \Phi_{\downarrow\uparrow} & \epsilon_{\downarrow\downarrow}+\Phi_{\downarrow\downarrow} \end{pmatrix}. \tag{20}$$

In Eq. (20), the two off-diagonal blocks of the matrix $D(\mathbf{k},\mathbf{k}',\mathbf{q})$ correspond to a purely triplet SC system, and the two diagonal blocks contain the SDW field $\mathbf{M}$ that couples fermions between the two bands of electrons and holes. Here, for simplicity, these short notations are used:

$$\begin{aligned} \mathcal{F}_{\sigma\sigma'} &= 2\gamma_{\rm C}^2(\Delta)_{\sigma_1\sigma_2}(-\mathbf{k}',\mathbf{k},\mathbf{q}), \\ \mathcal{F}^*_{\sigma\sigma'} &= 2\gamma_{\rm C}^2(\Delta)^*_{\sigma_2\sigma_1}(-\mathbf{k},\mathbf{k}',-\mathbf{q}), \\ (\Phi)_{\sigma\sigma'} &= \gamma_{\rm d}^2\left((\mathbf{M}.\boldsymbol{\sigma})^*_{\sigma'\sigma}(-\mathbf{k},-\mathbf{q}) + (\mathbf{M}.\boldsymbol{\sigma})_{\sigma\sigma'}(\mathbf{k},\mathbf{q})\right). \end{aligned} \tag{21}$$

### 2. Derivation of the free energy

To derive the free energy, we start with the grand canonical partition function of the system, which can be represented via a functional integral

$$\mathcal{Z} = \int [\mathcal{D}\psi]\left[\mathcal{D}\psi^\dagger\right] e^{-S[\psi^\dagger,\psi]}, \tag{22}$$

where the action $S$ has the form

$$S = \int_0^\beta \mathrm{d}\tau \left[\sum_\sigma\sum_{\mathbf{k}} \psi^\dagger_\sigma(\mathbf{k},\tau)\partial_\tau\psi_\sigma(\mathbf{k},\tau) + H\left[\psi^\dagger,\psi\right](\tau)\right\}. \tag{23}$$

With the mean-field Hamiltonian in Eq. (19), the functional integral in Eq. (22) has a Gaussian form, and the fermionic fields can be integrated out. Then, the Ginzburg-Landau free energy can be obtained in a straightforward way, yielding

$$F = \frac{|\Delta|^2}{V_C} + \frac{\mathbf{M}^2}{V_d} - \int_\Omega \ln\det\frac{1}{2}\left([\mathcal{G}_0]^{-1}\right). \tag{24}$$

In Eq. (24) the Greens function is given by $[\mathcal{G}_0]^{-1}(\mathbf{k},\mathbf{k}',\mathbf{q}) = i\omega_n I_{4\times4} + D(\mathbf{k},\mathbf{k}',\mathbf{q})$, where $I_{4\times4}$ is the $4\times4$ identity matrix. $\int_\Omega = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{V}\sum_k\sum_{\sigma_i=\uparrow,\downarrow}$ $\sum_{\alpha_i=e,h}\sum_{q_i,\omega_{\nu_i}}$ is a notation of sum over momentum, Matsubara frequencies, spin and the band indices of electrons and holes. The inverse temperature and the volume of the system are denoted by $\beta = 1/k_{\rm B}T$ and $V$, respectively. $V_d$ is the exchange interaction constant, and $V_C$ is the triplet SC coupling constant (the momentum dependence of which we dropped for simplicity). Here, we denoted the bosonic Matsubara frequencies by $\omega_\nu = 2\nu\pi T$ , $\nu = 0,\pm1,\pm2,\ldots$, and fermionic ones by $\omega_n = (2n+1)\pi T$ , $n = 0,\pm1,\pm2,\ldots$

Near the critical point, both the FM and the SC order parameters are small, so a GL functional approach is justified. In this case, by the expanding effective action (see Appendix A) in the order parameters $\{\mathbf{M},\Delta\}$ to a quartic one that only includes terms allowed by the symmetry of the system and retains minimum numbers of the simplest terms to get the results with multiple meanings, we will obtain the G-L free-energy functional with the participation of two order parameters describing the relationship between the spin-density-wave and the superconductivity phases:

$$\begin{aligned} f(\mathbf{M},\Delta) = \alpha_f.\left|\mathbf{M}.\boldsymbol{\sigma}\right|^2 + \beta_f\left|\mathbf{M}.\boldsymbol{\sigma}\right|^4 + \alpha_s\left|\Delta\right|^2 + \beta_s\left|\Delta\right|^4 \\ + u_{fs}.(\mathbf{M}.\boldsymbol{\sigma})\left|\Delta\right|^2 + v_{fs}.\left|\mathbf{M}.\boldsymbol{\sigma}\right|^2\left|\Delta\right|^2. \end{aligned} \tag{25}$$

Here, both quadratic terms $|\mathbf{M}.\boldsymbol{\sigma}|^2$ and $|\Delta|^2$ must be equivalent, and both quartic terms $|\mathbf{M}.\boldsymbol{\sigma}|^4$ and $|\Delta|^4$ too; they describe each special channel. The other terms describe interactions between channels that result in the coexistence of equilibrium phases. The microscopic expressions for the GL coefficients, which are functions of temperature (and pressure *etc.*), are listed in Appendix B.

In the remaining part of this paper, we will convert the functional in Eq. (25) into the phenomenological Ginzburg-Landau functional in Eq. (1) proposed by Shopova and Uzunov [15] to investigate the coexistence of magnetism and superconductivity in heavy fermions. Concretely, we will use the approach presented in Ref. 15 to study the conditions causing the Meissner phase in the presence of ferromagnetic order when no external magnetic field exists in $UGe_2$.

### 3. Ginzburg-Landau energy functional for superconductivity and ferromagnetism

Here, we are only interested in the uniform phases, i.e, order parameters $\mathbf{d}$ and $\mathbf{M}$ that do not depend on the spatial vector $\mathbf{x}$. Therefore, we will neglect all anisotropic terms that are additional Landau invariants and gradient terms in the Ginzburg-Landau free-energy functional of unconventional superconductors. In order to present a detailed investigation of the coexistence of Meissner superconductivity and ferromagnetic order, we should focus on a particular problem that has enough information from experiments to make a detailed comparison of the theoretical parameters with the existing experimental data.

Now, we construct the specific Ginzburg-Landau free-energy functional for $UGe_2$ starting from Eq. (25). From the definitions of the order parameters in Sec. II, we find that

$$\Delta_{\mathbf{k}}.\Delta_{\mathbf{k}}^{\dagger} = |\mathbf{d}|^2\,\sigma_0 + i\,(\mathbf{d}\times\mathbf{d})\,\boldsymbol{\sigma}, \quad (\mathbf{M}.\boldsymbol{\sigma})\,(\mathbf{M}.\boldsymbol{\sigma})^{\dagger} = M^2.\sigma_0. \tag{26}$$

$UGe_2$ is a ferromagnet that has an orthorhombic structure with magnetic moments oriented along one of the crystallographic axes. If we choose a coordinate system $x//b, y//c, z//a$, where the magnetic easy axis is the $a$-axis, then $\mathbf{M} = (0,0,M)$. Because of the pair-breaking effect of the strong exchange field $\mathbf{M}$, only the Cooper pairs with parallel spins will survive. In this case of equal-spin pairing, we can write vector $\mathbf{d}$ in the form $\mathbf{d} = (d_1, d_2, 0)$, implying that the Cooper pair spin-orientation points to the $\mathbf{M}$ direction. Then, we have

$$|\mathbf{M}.\boldsymbol{\sigma}|^2 = M^2.\sigma_0, \quad |\mathbf{M}.\boldsymbol{\sigma}|^4 = M^4.\sigma_0, \quad (\mathbf{M}.\boldsymbol{\sigma})\,|\Delta|^2 = |\mathbf{d}|^2\,|\mathbf{M}|\,\sigma_z + i\mathbf{M}.\,(\mathbf{d}\times\mathbf{d}^*)\,\sigma_0, \tag{27}$$

by using the conditions of equilibrium and stable phases for the coexistence of ferromagnetic and superconducting order (FS phase) given by $\sin\theta = -1$ (for $\sin\theta = 1$, we have another phase domain that is thermodynamically equivalent, so we will not consider it here) and by $\phi_1 = \phi_2 = \phi/\sqrt{2}$ (where $\phi = |\mathbf{d}|, \phi_j = |d_j|$, and $\theta = \theta_1 - \theta_2$ is the phase angle between the complex $d_1 = \phi_1 e^{i\theta_1}$ and $d_2 = \phi_2 e^{i\theta_2}$); we also have

$$|\Delta|^2 = |\mathbf{d}|^2\,\sigma_0 + |\mathbf{d}|^2\,\sigma_z, \quad |\Delta|^4 = 2\,|\mathbf{d}|^4\,\sigma_0 + 2\,|\mathbf{d}|^4\,\sigma_z. \tag{28}$$

Substituting Eqs. (27) and (28) back in Eq. (25), we obtain the G-L energy functional of the triplet ferromagnetic superconductor as follows:

$$f_{\rm GL}(\mathbf{d},\mathbf{M}) = a_f\,|\mathbf{M}|^2 + \frac{b_f}{2}\,|\mathbf{M}|^4 + a_s\,|\mathbf{d}|^2 + \frac{b_s}{2}\,|\mathbf{d}|^4 + i\gamma_0\mathbf{M}.\,(\mathbf{d}\times\mathbf{d}^*) + \delta_0\,|\mathbf{d}|^2\,|\mathbf{M}|^2\,, \tag{29}$$

where

$$a_f = \alpha_f;\quad b_f = 2\beta_f;\quad a_s = 2\alpha_s,\quad b_s = 8\beta_s;\quad \gamma_0 = 2u_{fs};\quad \delta_0 = v_{fs}. \tag{30}$$

The functional in Eq. (29) is the same as the phenomenological Ginzburg-Landau functional in Eq. (1) proposed by Shopova and Uzunov [15] when we ignore effects of anisotropy from both Cooper pairs and crystals, simultaneously restricting consideration of the uniform order parameters. However, the functional in Eq. (29) has a microscopic derivation that is established on the basic of the mean-field theory and the functional integral formalism. The constants that couple with the FM and SC order parameters in the microscopic theory are products of the free Green functions of electrons and holes. They can be summed over $\omega_n = (2n+1)\frac{\pi}{\beta}$ and the wave vectors $\mathbf{k}$ on the basis of the Taylor series expansion technique and the application of the residue theorem.

## IV. CONCLUSIONS AND DISCUSSION

We have microscopically derived the two-component Ginzburg-Landau functional that describes the relationship between the feromagnetic order parameter and the superconducting order parameter. The model in Eq. (29) is the same as Shopova's phenomenological model and it describes the coexistence of Meissner superconductivity and ferromagnetic order in $UGe_2$ well. In ferromagnetic metal $UGe_2$ systems, the interactions that lead to the formations of superconducting (SC) and magnetic spin-density-wave (SDW) orders, the pull and push of particles, are same, and as a result, influence each other. In particular, the two orders may support each other and lead to homogeneous local coexistence of SC and SDW states, or one order may completely suppress the other order, resulting in a state with spatially separated regions of pure SDW or SC orders. The transitions between various states may also be either continuous second order or abrupt first order.

The Ginzburg-Landau functional reveals not only that the triplet gap amplitude couples quadratically with the magnetization magnitude ($|\mathbf{d}|^2\,|\mathbf{M}|^2$) but also that the triplet $\mathbf{d}$-vector couples linearly with the magnetization direction ($i\mathbf{M}.\,(\mathbf{d}\times\mathbf{d}^*)$). The mean-field level in which coupling forces the $\mathbf{d}$-vector to align parallel or antiparallel to the magnetization is suitable. Although we focus on a microscopic model that has been widely employed in studies of ferromagnetic metal $UGe_2$ systems, most of our results follow from a Ginzburg-Landau analysis and, as such, should be applicable to other systems of interest, such as Ce-based and U-based compounds.

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## APPENDIX A: MICROSCOPIC DERIVATION OF THE TERMS IN THE G-L FREE-ENERGY FUNCTIONAL FOR FERROMAGNETISM AND SUPERCONDUCTIVITY

In this appendix, we show how to calculate the terms that appear in the GL functional for ferromagnetism and superconductivity. Let's start with the Ginzburg-Landau free-energy functional

$$F = \frac{|\Delta|^2}{V_C} + \frac{\mathbf{M}^2}{V_d} - \int_{\Omega} \ln\det\frac{1}{2}\left([\mathcal{G}_0]^{-1}\right). \tag{A1}$$

By using the result of the calculation in Appendix A of a previous paper [24], we have

$$\begin{aligned}
&\ln\det\frac{1}{2}\left([\mathcal{G}_0]^{-1}\right) \\
&= \mathrm{Tr}\ln\frac{1}{2}\left([\mathcal{G}_0]^{-1}\right) \\
&= \ln\left(\frac{1}{2}\right) + \ln\left[G_0^e\right]^{-1} + \ln\left[G_0^h\right]^{-1} + \sum_{N\geq 1}\mathrm{Tr}\,[\mathcal{G}]^N,
\end{aligned} \tag{A2}$$

where

$$\begin{aligned}
&\mathrm{Tr}\,[\mathcal{G}]^N \\
&= (-1)^{N-1}\frac{1}{N}\left[\begin{pmatrix} G^e & 0 \\ 0 & G^h\end{pmatrix}\begin{pmatrix}\Phi^{ee} & \mathcal{F}^{eh} \\ \mathcal{F}^{he} & \Phi^{hh}\end{pmatrix}\right]^N \\
&= (-1)^{N-1}\frac{1}{N}G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)\Delta^{\alpha_1\alpha_2}_{\sigma_1\sigma_2}(\mathbf{q_1},\omega_{\nu_1}) \\
&\times G^{\alpha_2}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1})\Delta^{\alpha_2\alpha_3}_{\sigma_2\sigma_3}(\mathbf{q_2},\omega_{\nu_2})\dots \\
&\times G^{\alpha_N}_{\sigma_N\sigma_N}(\mathbf{k}-\mathbf{q_1}-\cdots-\mathbf{q_{N-1}},\omega_n-\omega_{\nu_1}-\cdots-\omega_{\nu_{N-1}}) \\
&\times \Delta^{\alpha_N\alpha_1}_{\sigma_N\sigma_1}(-\mathbf{q_1}-\cdots-\mathbf{q_{N-1}},-\omega_{\nu_1}-\cdots-\omega_{\nu_{N-1}}),
\end{aligned}$$

with

$$\begin{aligned}
\Phi^{ee} &= -\Phi, \quad \mathcal{F}^{eh} = \mathcal{F}, \quad G^e = G_0^e, \\
\Phi^{hh} &= \Phi, \quad \mathcal{F}^{he} = \mathcal{F}^*, \quad G^h = G_0^h,
\end{aligned} \tag{A3}$$

which have an explicit form as in Eq. (21), and

$$\mathbf{q_N} = -\mathbf{q_1} - \cdots - \mathbf{q_{N-1}}. \tag{A4}$$

Tracing the matrix, retaining only pair order parameter terms that have closed momentum, boson Matsubara frequency, and spin, we obtain the following:

first-order expansion

$$\mathrm{Tr}\,[\mathcal{G}]^1 = G^{\alpha_1}_{\sigma_1\sigma_1}\Phi^{\alpha_1\alpha_1}_{\sigma_1\sigma_1} = 2\gamma_\mathrm{d}^2 G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{M}.\boldsymbol{\sigma})_{\sigma\sigma'}(\mathbf{k},\mathbf{q}), \tag{A5}$$

quadratic expansion

$$\begin{aligned}
&\mathrm{Tr}\,[\mathcal{G}]^2 \\
&= -\frac{1}{2}\Big\{ G^{\alpha_1}_{\sigma_1\sigma_1}\Phi^{\alpha_1\alpha_1}_{\sigma_1\sigma_2}G^{\alpha_1}_{\sigma_2\sigma_2}\Phi^{\alpha_1\alpha_1}_{\sigma_2\sigma_1} \\
&\qquad + G^{\alpha_1}_{\sigma_1\sigma_1}\mathcal{F}^{\alpha_1\alpha_2}_{\sigma_1\sigma_2}G^{\alpha_2}_{\sigma_2\sigma_2}\mathcal{F}^{\alpha_2\alpha_1}_{\sigma_2\sigma_1}\Big\} \\
&= -\left(\gamma_\mathrm{d}^2\right)^2\left(G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_1}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1})\right) \\
&\qquad\times\left[(\mathbf{M}.\boldsymbol{\sigma})^*_{\sigma_1\sigma_2}(\mathbf{q_1},\omega_{\nu_1})(\mathbf{M}.\boldsymbol{\sigma})_{\sigma_2\sigma_1}(-\mathbf{q_1},-\omega_{\nu_1})\right] \\
&\quad -\frac{1}{2}\left(2\gamma_\mathrm{C}^2\right)^2\left(G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_2}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1})\right) \\
&\qquad\times\left[(\Delta)^*_{\sigma_1\sigma_2}(\mathbf{q}_1,\omega_{\nu_1})(\Delta)_{\sigma_2\sigma_1}(-\mathbf{q_1},-\omega_{\nu_1})\right],
\end{aligned} \tag{A6}$$

cubic expansion

$$\begin{aligned}
&\mathrm{Tr}\,[\mathcal{G}]^3 \\
&= \frac{1}{3}\Big\{3G^e_{\sigma_1\sigma_1}\Phi^{ee}_{\sigma_1\sigma_2}G^e_{\sigma_2\sigma_2}\mathcal{F}^{eh}_{\sigma_2\sigma_3}G^h_{\sigma_3\sigma_3}\mathcal{F}^{he}_{\sigma_3\sigma_1} \\
&\qquad + 3G^h_{\sigma_1\sigma_1}\Phi^{hh}_{\sigma_1\sigma_2}G^h_{\sigma_2\sigma_2}\mathcal{F}^{he}_{\sigma_2\sigma_3}G^e_{\sigma_3\sigma_3}\mathcal{F}^{eh}_{\sigma_3\sigma_1}\Big\} \\
&= -2\left(\gamma_\mathrm{d}^2\right)\left(2\gamma_\mathrm{C}^2\right)^2\left\{\begin{pmatrix} G^e_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^e_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) \\ \times G^h_{\sigma_3\sigma_3}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3})\end{pmatrix}\right. \\
&\quad\left. -2\left(\gamma_\mathrm{d}^2\right)\left(2\gamma_\mathrm{C}^2\right)^2\begin{pmatrix} G^h_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^h_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) \\ \times G^e_{\sigma_3\sigma_3}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3})\end{pmatrix}\right\} \\
&\quad\times(\mathbf{M}.\boldsymbol{\sigma})_{\sigma_1\sigma_1}(0,0)\left[(\Delta)^*_{\sigma_1\sigma_3}(\mathbf{q_3},\omega_{\nu_3})\right. \\
&\qquad\qquad\left.\times(\Delta)_{\sigma_3\sigma_1}(-\mathbf{q_3},-\omega_{\nu_2})\right],
\end{aligned} \tag{A7}$$

quartic expansion

$$\begin{aligned}
&\mathrm{Tr}\,[\mathcal{G}]^4 \\
&= -\frac{1}{4}\Big\{G^{\alpha_1}_{\sigma_1\sigma_1}\Phi^{\alpha_1\alpha_1}_{\sigma_1\sigma_2}G^{\alpha_1}_{\sigma_2\sigma_2}\Phi^{\alpha_1\alpha_1}_{\sigma_2\sigma_3}G^{\alpha_1}_{\sigma_3\sigma_3}\Phi^{\alpha_1\alpha_1}_{\sigma_3\sigma_4}G^{\alpha_1}_{\sigma_4\sigma_4}\Phi^{\alpha_1\alpha_1}_{\sigma_4\sigma_1} \\
&\quad + 4G^{\alpha_1}_{\sigma_1\sigma_1}\Phi^{\alpha_1\alpha_1}_{\sigma_1\sigma_2}G^{\alpha_1}_{\sigma_2\sigma_2}\Phi^{\alpha_1\alpha_1}_{\sigma_2\sigma_3}G^{\alpha_1}_{\sigma_3\sigma_3}\mathcal{F}^{\alpha_1\alpha_2}_{\sigma_3\sigma_4}G^{\alpha_2}_{\sigma_4\sigma_4}\mathcal{F}^{\alpha_2\alpha_1}_{\sigma_4\sigma_1} \\
&\quad + G^{\alpha_1}_{\sigma_1\sigma_1}\mathcal{F}^{\alpha_1\alpha_2}_{\sigma_1\sigma_2}G^{\alpha_2}_{\sigma_2\sigma_2}\mathcal{F}^{\alpha_2\alpha_1}_{\sigma_2\sigma_3}G^{\alpha_1}_{\sigma_3\sigma_3}\mathcal{F}^{\alpha_1\alpha_2}_{\sigma_3\sigma_4}G^{\alpha_2}_{\sigma_4\sigma_4}\mathcal{F}^{\alpha_2\alpha_1}_{\sigma_4\sigma_1}\Big\} \\
&= -2\left(\gamma_\mathrm{d}^2\right)^4 G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_1}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1}) \\
&\qquad\times G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_1}_{\sigma_4\sigma_4}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3}) \\
&\qquad\times\left[(\mathbf{M}.\boldsymbol{\sigma})^*_{\sigma_1\sigma_2}(\mathbf{q_1},\omega_{\nu_1})(\mathbf{M}.\boldsymbol{\sigma})_{\sigma_2\sigma_1}(-\mathbf{q_1},-\omega_{\nu_1})\right] \\
&\qquad\times\left[(\mathbf{M}.\boldsymbol{\sigma})^*_{\sigma_1\sigma_4}(\mathbf{q_3},\omega_{\nu_3})(\mathbf{M}.\boldsymbol{\sigma})_{\sigma_4\sigma_1}(-\mathbf{q_3},-\omega_{\nu_3})\right] \\
&\quad -4\left(\gamma_\mathrm{d}^2\right)^2\left(2\gamma_\mathrm{C}^2\right)^2 G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_1}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1}) \\
&\qquad\times G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_2}_{\sigma_4\sigma_4}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3}) \\
&\qquad\times\left[(\mathbf{M}.\boldsymbol{\sigma})^*_{\sigma_1\sigma_2}(\mathbf{q_1},\omega_{\nu_1})(\mathbf{M}.\boldsymbol{\sigma})_{\sigma_2\sigma_1}(-\mathbf{q_1},-\omega_{\nu_1})\right] \\
&\qquad\times\left[(\Delta)^*_{\sigma_1\sigma_4}(\mathbf{q_3},\omega_{\nu_3})(\Delta)_{\sigma_4\sigma_1}(-\mathbf{q_3},-\omega_{\nu_3})\right] \\
&\quad -\frac{1}{2}\left(2\gamma_\mathrm{C}^2\right)^4 G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_2}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1}) \\
&\qquad\times G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_2}_{\sigma_4\sigma_4}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3}) \\
&\qquad\times\left[(\Delta)^*_{\sigma_1\sigma_2}(\mathbf{q_1},\omega_{\nu_1})(\Delta)_{\sigma_2\sigma_1}(-\mathbf{q_1},-\omega_{\nu_1})\right] \\
&\qquad\times\left[(\Delta)^*_{\sigma_1\sigma_4}(\mathbf{q_3},\omega_{\nu_3})(\Delta)_{\sigma_4\sigma_1}(-\mathbf{q_3},-\omega_{\nu_3})\right].
\end{aligned} \tag{A8}$$

## APPENDIX B: COEFFICIENTS IN THE G-L FREE-ENERGY FUNCTIONAL FOR FERROMAGNETISM AND SUPERCONDUCTIVITY

This appendix demonstrates how to calculate the coefficients in the GL free-energy functional for ferromagnetism and superconductivity. The coefficients to be considered are: $\alpha_f, \beta_f, \alpha_s, \beta_s, u_{fs}$, and $v_{fs}$. From now on, both the zero-temperature and the infinite-volume limits are understood:

$$\begin{aligned}\tfrac{1}{V}\textstyle\sum_{\boldsymbol{k}} &\to \frac{1}{8\pi^3}\int_{\boldsymbol{k}} \mathrm{d}\boldsymbol{k};\\ \lim_{\beta\to\infty}\tfrac{\partial f_e(\xi_{\mathbf{k}})}{\partial \xi_{\mathbf{k}}} &= -\delta\left(\xi_{\mathbf{k}}\right) = -\delta\left(\frac{\mathbf{k}^2}{2m}-\frac{k_{\mathrm{F}}^2}{2m}\right)\\ &= -\frac{1}{\upsilon_{\mathrm{F}}}\delta\left(|\mathbf{k}|-k_{\mathrm{F}}\right).\end{aligned} \tag{B1}$$

Furthermore, the long-wavelength limit $|\mathbf{q_i}| \ll k_{\mathrm{F}}$ will also be assumed for some given transfer momentum $\mathbf{q_i}$. Choosing a spherical coordinate system in $\mathbf{k}$-space with the angle between $\mathbf{q}$ and $\mathbf{k}$ being the polar angle $\theta$, we have

$$\begin{aligned}\mathbf{k}.\mathbf{q} &= |\mathbf{k}|\,.\,|\mathbf{q}|\,.\cos\theta\\ &= |\mathbf{k}|\,.\,|\mathbf{q}|\,.\lambda;\\ \tfrac{1}{8\pi^3}\int_{\boldsymbol{k}}\mathrm{d}\boldsymbol{k} &= \frac{1}{8\pi^3}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} k^2\mathrm{d}k\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\\ &= \frac{1}{4\pi^2}\int_{k=0}^{\infty}k^2\mathrm{d}k\int_{-1}^{+1}\mathrm{d}\lambda;\end{aligned} \tag{B2}$$

and

$$\frac{|\mathbf{q}|^2}{2m} \ll |\mathbf{q}|\,\upsilon_{\mathrm{F}}. \tag{B3}$$

Denoting by $\mathcal{N}(0)$ the density of state at the Fermi energy, we find that

$$\mathcal{N}(0) \equiv Z(E_{\mathrm{F}}) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}E_{\mathrm{F}}^{1/2} = \frac{mk_{\mathrm{F}}}{\pi^2\hbar^2} = \frac{3}{2}\frac{n}{\mu}, \tag{B4}$$

and using the short notations

$$\begin{aligned}\sum_{\substack{\alpha_1,\alpha_2=e,h\\ \alpha_1\neq\alpha_2}}\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{\beta}\sum_{\omega_n} &\equiv \sum_a,\\ \frac{1}{V}\sum_{\mathbf{k}}\frac{1}{\beta}\sum_{\omega_n} &\equiv \sum_b.\end{aligned} \tag{B5}$$

Carrying out the integration over momentum and frequency, we obtain

$$\begin{aligned}&\alpha_f\\ &= \frac{1}{V_d} - \left(\gamma_{\mathrm{d}}^2\right)^2\sum_a G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_1}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1})\\ &\approx \frac{1}{V_d} - \left(\gamma_{\mathrm{d}}^2\right)^2\frac{\mathcal{N}(0)}{2}\times\pi\frac{\omega_{\nu_1}}{\upsilon_{\mathrm{F}}|\mathbf{q_1}|} + O\left[\left(\frac{|\mathbf{q}|}{c}\right)^2\right];\end{aligned} \tag{B6}$$

$$\begin{aligned}&\beta_f\\ &= -2\left(\gamma_{\mathrm{d}}^2\right)^4\\ &\times\sum_a\begin{pmatrix} G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_1}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1})\\ \times G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_1}_{\sigma_4\sigma_4}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3})\end{pmatrix}\\ &\approx 4\mathcal{N}(0)\times\frac{1}{\upsilon_{\mathrm{F}}|\mathbf{q_1}|}\times\frac{1}{\upsilon_{\mathrm{F}}|\mathbf{q_3}|}\\ &\times\left[\left(\frac{\pi}{2}\right)\left(\frac{\omega_{\nu_1}}{\upsilon_{\mathrm{F}}|\mathbf{q_1}|}\right)+\left(\frac{\pi}{2}\right)\left(\frac{\omega_{\nu_3}}{\upsilon_{\mathrm{F}}|\mathbf{q_3}|}\right)\right],\end{aligned} \tag{B7}$$

with the help of Eqs. (1.622.3) and (1.644.2) from Ref. 28 to reach the last equalities of (B6) and (B7). Also,

$$\begin{aligned}&\alpha_s\\ &= \frac{1}{V_C}\\ &\quad-\frac{1}{2}\left(2\gamma_{\mathrm{C}}^2\right)^2\sum_a G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_2}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1})\\ &= \left(2\gamma_{\mathrm{C}}^2\right)^2\frac{\mathcal{N}(0)}{2}\ln\left(\frac{T}{T_C}\right)\\ &\quad+\frac{1}{2}\left(2\gamma_{\mathrm{C}}^2\right)^2\mathcal{N}(0)\frac{1}{6\pi^2T^2}\frac{7}{8}\zeta(3)\upsilon_{\mathrm{F}}^2\mathbf{q}^2,\end{aligned} \tag{B8}$$

where

$$T_C = \left(\frac{2\gamma\omega_D}{\pi}\right)e^{-\frac{2}{V_C\cdot\mathcal{N}(0)}}, \tag{B9}$$

with the help of Eq. (3.527.3) from Ref. 28 to reach second line of equality (B8). Furthermore,

$$\begin{aligned}&\beta_s\\ &= -\frac{1}{2}\left(2\gamma_{\mathrm{C}}^2\right)^4\\ &\times\sum_a\begin{pmatrix} G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_2}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1})\\ \times G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n)G^{\alpha_2}_{\sigma_4\sigma_4}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3})\end{pmatrix}\\ &= -\frac{1}{2}\left(2\gamma_{\mathrm{C}}^2\right)^4\\ &\times 2\mathcal{N}(0)\left[\frac{\beta^2}{\pi^2}\frac{7}{8}\zeta(3) - \frac{31}{192}\left(\frac{|\mathbf{q_3}|\,|\mathbf{k}_{\mathrm{F}}|}{m}\right)^2\frac{\beta^4}{\pi^4}\zeta(5)\right.\\ &\qquad\left.-\frac{31}{192}\left(\frac{|\mathbf{q_1}|\,|\mathbf{k}_{\mathrm{F}}|}{m}\right)^2\frac{\beta^4}{\pi^4}\zeta(5)\right];\end{aligned} \tag{B10}$$

$$\begin{aligned} & u_{fs} \\ &= 2\gamma_{\mathrm{d}}^2 \left(2\gamma_{\mathrm{C}}^2\right)^2 \sum_b \left\{ \begin{pmatrix} G^h_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) G^h_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) \\ \times G^e_{\sigma_3\sigma_3}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3}) \end{pmatrix} \right. \\ & \qquad \left. - \begin{pmatrix} G^e_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) G^e_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) \\ \times G^h_{\sigma_3\sigma_3}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3}) \end{pmatrix} \right\} \\ &= 2\gamma_{\mathrm{d}}^2 \left(2\gamma_{\mathrm{C}}^2\right)^2 \times 2\mathcal{N}(0) \frac{\beta^2}{\pi^2} \frac{7}{8} \zeta(3) \frac{\mathbf{q}^2}{2m}; \end{aligned} \tag{B11}$$

$$\begin{aligned} & v_{fs} \\ &= 4\left(\gamma_{\mathrm{d}}^2\right)^2 \left(2\gamma_{\mathrm{C}}^2\right)^2 \\ & \quad \times \sum_a \begin{pmatrix} G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) G^{\alpha_1}_{\sigma_2\sigma_2}(\mathbf{k}-\mathbf{q_1},\omega_n-\omega_{\nu_1}) \\ G^{\alpha_1}_{\sigma_1\sigma_1}(\mathbf{k},\omega_n) G^{\alpha_2}_{\sigma_4\sigma_4}(\mathbf{k}-\mathbf{q_3},\omega_n-\omega_{\nu_3}) \end{pmatrix} \\ &= 4\left(\gamma_{\mathrm{d}}^2\right)^2 \left(2\gamma_{\mathrm{C}}^2\right)^2 \\ & \quad \times \mathcal{N}(0) \left[ \frac{7}{8}\frac{\beta^2}{\pi^2}\zeta(3) - \frac{7}{24} \times \frac{31}{32}\frac{\beta^4}{\pi^4}\zeta(5)\left(|\mathbf{q_1}|\,v_{\mathrm{F}}\right)^2 \right. \\ & \qquad \left. - \frac{7}{24} \times \frac{31}{32}\frac{\beta^4}{\pi^4}\zeta(5)\left(|\mathbf{q_3}|\,v_{\mathrm{F}}\right)^2 \right]. \end{aligned} \tag{B12}$$

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