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ELEMENTARY PARTICLES AND FIELDS
Theory

Fermion Mass and Mixing in a Simple Extension of the Standard Model Based on T_7 Flavor Symmetry*

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Abstract—We build a simple Standard Model extension based on T_7 flavor symmetry which accommodates lepton mass, mixing with non-zero θ_{13}, and CP violation phase. The lepton mixing matrix is obtained from three triplets and one singlet under T_7 symmetry, and the charged-lepton mass is derived through the spontaneous symmetry breaking by just one T_7 triplet (ϕ), while neutrinos get small masses from one $SU(2)_L$ doublet and two $SU(2)_L$ singlets in which one is in $\underline{1}$ and the two others are in $\underline{3}$ and $\underline{3}^*$ under T_7, respectively. There exist viable parameters of the model that predict the effective Majorana neutrino mass with values $m_\beta \simeq 10^{-2}$ eV and 4.95×10^{-2} eV as well as a lightest neutrino mass $m_{\text{light}} \simeq 4.97 \times 10^{-3}$ eV and 1.61×10^{-3} eV for the normal and inverted neutrino mass hierarchies, respectively. The model also gives a remarkable prediction of Dirac CP violation $\delta_{CP} \simeq 303.3°$ in the normal hierarchy and $\delta_{CP} \simeq 56.69°$ in the inverted hierarchy which is still missing in the neutrino mixing matrix. The quark mixing angles of the model are closed to the experimental data, whereas the obtained values for the quark masses are consistent with with the experimental data at the tree level.

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1. INTRODUCTION

The discovery of neutrino mass is a great breakthrough for particle physics, and up to now, this is one of the most important evidences of new physics. Neutrinos have tiny masses and this is probably related to the existence of a new mass scale in physics. Recently, it has been shown that neutrinos can also play a key role in providing an answer for the Baryon Asymmetry of Universe (BAU). Theoretically, there exist various models describing the smallness of neutrino mass and large θ_{13} mixing[3)]. Among the possible extensions of the Standard Model (SM), probably the simplest one is the neutrino minimal SM which has been studied in [2–6]. However, these extensions do not provide a natural explanation for large mass splitting between neutrinos and the lepton mixing was not explicitly explained [7].

There are five well-known patterns of lepton mixing [8], however, the Tri-bimaximal one proposed by Harrison–Perkins–Scott (HPS) [9–12] seems to be the most popular and can be considered as a leading-order approximation for the recent neutrino experimental data. Up to now, the absolute values of the entries of the lepton mixing matrix U_{PMNS} have not yet been determined exactly, however, their scales are given in [13]:

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.801 \to 0.845 & 0.514 \to 0.580 & 0.137 \to 0.158 \\ 0.225 \to 0.517 & 0.441 \to 0.699 & 0.614 \to 0.793 \\ 0.246 \to 0.529 & 0.464 \to 0.713 & 0.590 \to 0.776 \end{pmatrix}. \quad (1)$$

The best-fit values of neutrino mass squared differences and leptonic mixing angles are given in [14] as below:

$$\sin^2\theta_{12} = 0.304 \pm 0.014,$$
$$\sin^2\theta_{13} = (2.19 \pm 0.12) \times 10^{-2},$$
$$\sin^2\theta_{23} = 0.51 \pm 0.05(\text{NH}),$$
$$\sin^2\theta_{23} = 0.50 \pm 0.05(\text{IH}), \quad (2)$$
$$\Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5}\ \text{eV}^2,$$
$$\Delta m^2_{32} = (2.44 \pm 0.06) \times 10^{-3}\ \text{eV}^2(\text{NH}),$$
$$\Delta m^2_{32} = (2.51 \pm 0.06) \times 10^{-3}\ \text{eV}^2(\text{IH}). \quad (3)$$

Here NH and IH stand for normal and inverted hierarchies, respectively.

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3)The references for these models are mentioned in [1].

To explain the specific neutrino mixings, it is simple to use discrete symmetry such as A_4, S_3, S_4, etc. The use of non-Abelian discrete symmetries to construct the models describing the lepton masses and mixings is a new method firstly proposed by Ma and Rajasekaran in 2001 [15]. In this treatment, there are various models which have been proposed, see for example A_4 [15–35], S_3 [36–39], S_4 [40–43], D_4 [44–54], T' [55–64], T_7 [65–69]. However, in all above-mentioned works, the fermion masses and mixings are generated from *non-renormalizable* interactions[4)] or at loop levels but not at tree-level. The models involving only renormalizable interactions were implemented in our previous works [38, 39, 41–43, 70–77] in which the discrete symmetries have been added to the 3-3-1 models. As we know, the 3-3-1 model itself is an extension of the SM where the gauge group $SU(2)_L$ is extended to $SU(3)_L$.

In this paper, we construct a simple extension of the SM based on T_7 symmetry that leads to lepton mass, mixing with non-zero θ_{13}, and CP violation phase. Note that, T_7 symmetry has not been previously considered in this kind of the model with the mentioned scenario. Furthermore, this model is different from our previous works [71, 73] because the 3-3-1 model [based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$] itself is an extension of the SM. For this purpose, two $SU(2)_L$ doublets and two $SU(2)_L$ singlets are introduced. The result follows without perturbation and the number of scalars required to generate lepton masses are fewer than those in [1].

The future content of this paper reads as follows. In Section 2 we present the fundamental elements of the model and introduce necessary Higgs fields responsible for the fermion mass and mixing. We make conclusions in Section 3. Appendix A presents the scalar potential of the model. Appendices B and C provide detail solutions for neutrino masses in normal and inverted hierarchies, respectively.

2. RESULTS AND DISCUSSION

2.1. Lepton Mass and Mixing

The lepton content of the model, under $SU(2)_L \otimes U(1)_Y \otimes U(1)_X \otimes T_7$ symmetries, is given in Table 1.

The charged lepton masses arise from the couplings of $\bar{\psi}_L l_{1R}, \bar{\psi}_L l_{2R}$ and $\bar{\psi}_L l_{3R}$ to scalars, where $\bar{\psi}_L l_{iL} (i = 1, 2, 3)$ transforms as 2 under $SU(2)_L$ and $\underline{3}^*$ under T_7. In order to generate masses for charged leptons, we need only one $SU(2)_L$ Higgs doublets (ϕ) lying in $\underline{3}$ under T_7, as given in Table 1. The Yukawa interaction of the scalar field with charged leptons takes the form:

$$-\mathcal{L}_l = h_1(\bar{\psi}_L\phi)_{\underline{1}} l_{1R} + h_2(\bar{\psi}_L\phi)_{\underline{1}''} l_{2R} + h_3(\bar{\psi}_{iL}\phi)_{\underline{1}'} l_{3R} + \text{H.c.}$$
$$= h_1(\bar{\psi}_{1L}\phi_1 + \bar{\psi}_{2L}\phi_2 + \bar{\psi}_{3L}\phi_3) l_{1R} + h_2(\bar{\psi}_{1L}\phi_1 + \omega^2\bar{\psi}_{2L}\phi_2 + \omega\bar{\psi}_{3L}\phi_3) l_{2R}$$
$$+ h_3(\bar{\psi}_{1L}\phi_1 + \omega\bar{\psi}_{2L}\phi_2 + \omega^2\bar{\psi}_{3L}\phi_3) l_{3R} + \text{H.c.} \quad (4)$$

In this work we impose only the breaking $T_7 \to Z_3$ in charged lepton sector, and this can be achieved with the alignment of ϕ under T_7, $\langle\phi\rangle = (\langle\phi_1\rangle, \langle\phi_1\rangle, \langle\phi_1\rangle)$, where

$$\langle\phi_1\rangle = (0 v)^T. \quad (5)$$

With the vacuum expectation value (VEV) of ϕ_1 in Eq. (5), the mass Lagrangian for the charged leptons can be written in matrix form as

$$-\mathcal{L}_l^{\text{mass}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L}) M_l (l_{1R}, l_{2R}, l_{3R})^T + \text{H.c.}, \quad (6)$$

where

$$M_l = \begin{pmatrix} h_1 v & h_2 v & h_3 v \\ h_1 v & \omega^2 h_2 v & \omega h_3 v \\ h_1 v & \omega h_2 v & \omega^2 h_3 v \end{pmatrix}. \quad (7)$$

The mass matrix M_l in Eq. (7) is diagonalized:

$$U_L^\dagger M_l U_R = \text{diag}\left(\sqrt{3}h_1 v, \sqrt{3}h_2 v, \sqrt{3}h_3 v\right) \equiv \text{diag}\left(m_e, m_\mu, m_\tau\right), \quad (8)$$

where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad U_R = 1, \quad (9)$$

and ω is the cube root of unity, $\omega = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. The Yukawa couplings $h_{1,2,3}$ in charged-lepton sector are defined:

$$h_1 = \frac{m_e}{\sqrt{3}v}, \quad h_2 = \frac{m_\mu}{\sqrt{3}v}, \quad h_3 = \frac{m_\tau}{\sqrt{3}v}. \quad (10)$$

Table 1. Lepton content of the model

	ψ_L	$l_{(1,2,3)R}$	ν_R	ϕ	φ	χ	ζ
$SU(2)_L$	2	1	1	2	2	1	1
$U(1)_Y$	−1	−2	0	1	1	0	0
$U(1)_X$	1	1	0	0	−1	0	0
T_7	$\underline{3}$	$\underline{1}, \underline{1}', \underline{1}''$	$\underline{3}$	$\underline{3}$	$\underline{1}$	$\underline{3}$	$\underline{3}^*$

4) In [19], tribimaximal form obtained at the tree level but the realistic lepton mixing obtained with radiative corrections.

The experimental values for masses of the charged leptons are given in [14]:

$$m_e \simeq 0.510998928 \text{ MeV},$$
$$m_\mu = 105.6583715 \text{ MeV},$$
$$m_\tau = 1776.86 \text{ MeV}. \quad (11)$$

It follows that $h_1 \ll h_2 \ll h_3$. Furthermore, if we choose[5)] the VEV $v \sim 100$ GeV, then

$$h_1 \sim 10^{-6}, \quad h_2 \sim 10^{-4}, \quad h_3 \sim 10^{-2}, \quad (12)$$

i.e., in the model under consideration, the hierarchy between the masses for charged-leptons can be achieved if there exists a hierarchy between Yukawa couplings $h_i (i = 1, 2, 3)$ in charged-lepton sector as given in Eq. (12). We note that the masses of charged leptons are self-separated by only one T_7 triplet ϕ, and this is a good feature of the T_7 group. We remind that in the other models with discrete symmetries (for example, see [35, 78, 79]), the charged masses are generated from non-renormalizable interactions or at loop levels.

The neutrino masses arise from the couplings of $\bar{\psi}_L \nu_R$ and $\bar{\nu}^c_R \nu_R$ to scalars, where $\bar{\psi}_L \nu_R$ transforms as 2 under $SU(2)_L$ and $\underline{1} \oplus \underline{1}' \oplus \underline{1}'' \oplus \underline{3} \oplus \underline{3}^*$ under T_7; $\bar{\nu}^c_R \nu_R$ transform as 1 under $SU(2)_L$ and $\underline{3} \oplus \underline{3}^* \oplus \underline{3}^*$ under T_7. Note that $\underline{3} \otimes \underline{3} \otimes \underline{3}$ has two invariants and $\underline{3} \otimes \underline{3} \otimes \underline{3}^*$ has one invariant under T_7. In order to generate mass for neutrinos, we additionally introduce one $SU(2)_L$ doublet (φ) and two $SU(2)_L$ singlets (χ, ζ), respectively, put in $\underline{1}$, $\underline{3}$, and $\underline{3}^*$ under T_7 as given in Table 1. We note that the $U(1)_X$ symmetry forbids the Yukawa terms of the form $(\bar{\psi}_L \tilde{\phi})_{\underline{3}_s} \nu_R$ and yields the expected results in the neutrino sector, and this is an interesting feature of X-symmetry. It is also interesting to note that φ contributes to the Dirac mass matrix, χ and ζ contribute to the Majorana mass matrix of the right-handed neutrinos. In fact, there exist no one-dimensional representation in $\underline{3} \otimes \underline{3}$ under T_7. Hence, ζ put in $\underline{3}^*$ will be responsible for a realistic neutrino spectrum without any perturbation and soft breaking in both lepton and neutrino sectors. This feature is different from that in [35]. It needs to note that φ contributes to the Dirac mass matrix in the neutrino sector, χ and ζ contribute to the Majorana mass matrix of the right-handed neutrinos. The interesting feature of X symmetry is that it prevents the unwanted interaction of the form $(\bar{\psi}_L \tilde{\phi})_{\underline{3}_s} \nu_R$ and provides the expected results in the neutrino sector[6)].

In this work we impose that the breaking $T_7 \to$ {identity} must take place, i.e., T_7 is completely broken in the neutrino sector. This can be achieved within each case below.

(1) A new $SU(2)_L$ singlet χ lies in $\underline{3}$ under T_7 and the VEV is given by $\langle \chi \rangle = (0, \langle \chi_2 \rangle, 0)^T$ under T_7, where

$$\langle \chi_2 \rangle = v_\chi. \quad (13)$$

(2) Another singlet ζ lies in $\underline{3}^*$ under T_7 and the VEV is given by $\langle \zeta \rangle = (\langle \zeta_1, \langle \zeta_2 \rangle, \langle \zeta_3 \rangle)^T$ under T_7, i.e. $\langle \zeta_1 \rangle \neq \langle \zeta_2 \rangle \neq \langle \zeta_3 \rangle \neq 0$, where

$$\langle \zeta_i \rangle = u_i (i = 1, 2, 3). \quad (14)$$

The neutrino Yukawa interactions are given by

$$\begin{aligned} -\mathcal{L}_\nu &= x(\bar{\psi}_L \tilde{\varphi})_{3^*} \nu_R + \frac{y}{2} (\bar{\nu}^c_R \chi)_{3^*} \nu_R \\ &+ \frac{z}{2} (\bar{\nu}^c_R \zeta)_{3^*} \nu_R + \text{H.c.} = x(\bar{\psi}_{1L} \tilde{\varphi} \nu_{1R} \\ &+ \bar{\psi}_{2L} \tilde{\varphi} \nu_{2R} + \bar{\psi}_{3L} \tilde{\varphi} \nu_{3R}) + \frac{y}{2} \Big[(\bar{\nu}^c_{2R} \chi_3 \\ &+ \bar{\nu}^c_{3R} \chi_2) \nu_{1R} + (\bar{\nu}^c_{3R} \chi_1 + \bar{\nu}^c_{1R} \chi_3) \nu_{2R} \\ &+ (\bar{\nu}^c_{1R} \chi_2 + \bar{\nu}^c_{2R} \chi_1) \nu_{3R} \Big] + \frac{z}{2} (\bar{\nu}^c_{1R} \zeta_2 \nu_{1R} \\ &+ \bar{\nu}^c_{2R} \zeta_3 \nu_{2R} + \bar{\nu}^c_{3R} \zeta_1 \nu_{3R}) + \text{H.c.} \end{aligned} \quad (15)$$

From (15), we find the following neutrino mass terms

$$\begin{aligned} -\mathcal{L}^{\text{mass}}_\nu &= xv(\bar{\nu}_{1L} \nu_{1R} + \bar{\nu}_{2L} \nu_{2R} + \bar{\nu}_{3L} \nu_{3R}) \\ &+ \frac{y}{2} \Big(v_\chi \bar{\nu}^c_{3R} \nu_{1R} + v_\chi \bar{\nu}^c_{1R} \nu_{3R} + v_\chi \bar{\nu}^c_{2R} \nu_{1R} \\ &+ v_\chi \bar{\nu}^c_{1R} \nu_{2R} \Big) + \frac{z}{2} (u_2 \bar{\nu}^c_{1R} \nu_{1R} + u_3 \bar{\nu}^c_{2R} \nu_{2R} \\ &+ u_1 \bar{\nu}^c_{3R} \nu_{3R}) + \text{H.c.}, \end{aligned} \quad (16)$$

which can be rewritten in the matrix form as

$$-\mathcal{L}^{\text{mass}}_\nu = \frac{1}{2} \bar{\chi}^c_L M_\nu \chi_L + \text{H.c.},$$

$$\chi_L \equiv \begin{pmatrix} \nu^c_L \\ \nu_R \end{pmatrix}, \quad M_\nu \equiv \begin{pmatrix} 0 & M_D \\ M^T_D & M_R \end{pmatrix},$$

$$\nu^c_L = (\nu^c_{1L} \nu^c_{2L} \nu^c_{3L})^T, \quad \nu_R = (\nu_{1R} \nu_{2R} \nu_{3R})^T, \quad (17)$$

5) In the SM, the Higgs VEV v is 246 GeV, fixed by the W-boson mass and the gauge coupling $m^2_W = \frac{g^2}{4} v^2_{\text{weak}}$. In the model under consideration $M^2_W \simeq \frac{3}{2} g^2 v^2$. Therefore, we can identify $v^2_{\text{weak}} = 6v^2 = (246 \text{ GeV})^2$. It follows $v \simeq 100$ GeV.

6) There is an unwanted Goldstone boson generated from the spontaneous breaking of the continuous group $U(1)_X$, however, it is harmless because this boson can be removed by interactions violating the T_7 symmetry or by additional introduction of a new standard model singlet scalar charged only under another discrete subgroup Z_n that was in detail discussed in [19] and [20], respectively.

where the Dirac neutrino mass matrix (M_D) and the right-handed Majorana neutrino mass matrix (M_R) are given by

$$M_D = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad M_R = \begin{pmatrix} N_2 & 0 & b \\ 0 & N_3 & 0 \\ b & 0 & N_1 \end{pmatrix}, \quad (18)$$

with

$$a = v_\varphi x, \quad b = v_\chi y,$$
$$N_i = u_i z \quad (i = 1, 2, 3). \quad (19)$$

The seesaw mechanism generates small masses for neutrinos is given by

$$M_{\rm eff} = -M_D M_R^{-1} M_D^T = \begin{pmatrix} A_1 & 0 & B \\ 0 & A_3 & 0 \\ B & 0 & A_2 \end{pmatrix}, \quad (20)$$

where

$$A_1 = \frac{a^2 N_1}{b^2 - N_1 N_2}, \quad A_2 = \frac{a^2 N_2}{b^2 - N_1 N_2},$$
$$A_3 = -\frac{a^2}{N_3}, \quad B = \frac{a^2 b}{N_1 N_2 - b^2}. \quad (21)$$

The matrix $M_{\rm eff}$ in Eq. (20) has three exact eigenvalues given by

$$m_{1,3} = \frac{1}{2}\left(A_1 + A_2 \mp \sqrt{(A_1 - A_2)^2 + 4B^2}\right),$$
$$m_2 = A_3, \quad (22)$$

and the corresponding eigenstates are

$$R_\nu = \begin{pmatrix} \frac{K}{\sqrt{K^2+1}} & 0 & \frac{1}{\sqrt{K^2+1}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{K^2+1}} & 0 & -\frac{K}{\sqrt{K^2+1}} \end{pmatrix}, \quad (23)$$

with K being real and

$$K = \left[A_1 - A_2 - \sqrt{(A_1 - A_2)^2 + 4B^2}\right]/(2B), \quad (24)$$

and $A_{1,2}, B$ are given in Eq. (21).

The lepton mixing matrix is then expressed as

$$U_\nu^\dagger M_{\rm eff} U_\nu = \begin{cases} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad U_\nu \equiv U_\nu^N = \begin{pmatrix} \frac{K}{\sqrt{K^2+1}} & 0 & \frac{1}{\sqrt{K^2+1}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{K^2+1}} & 0 & -\frac{K}{\sqrt{K^2+1}} \end{pmatrix}, \quad \text{for NH}, \\ \begin{pmatrix} m_3 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{pmatrix}, \quad U_\nu \equiv U_\nu^I = \begin{pmatrix} \frac{1}{\sqrt{K^2+1}} & 0 & \frac{K}{\sqrt{K^2+1}} \\ 0 & 1 & 0 \\ -\frac{K}{\sqrt{K^2+1}} & 0 & \frac{1}{\sqrt{K^2+1}} \end{pmatrix}, \quad \text{for IH}, \end{cases} \quad (25)$$

where $m_i (i = 1, 2, 3)$ and K are given in Eqs. (22) and (24), respectively.

Using the rotation matrices in the charged lepton and in the neutrino sectors given by Eqs. (9) and (26) for the normal and inverted neutrino mass hierarchies, respectively, we find that **the leptonic** mixing matrix takes the form

$$U_{\rm Lep} = \begin{cases} R_{lL}^\dagger U_\nu^N = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1+K}{\sqrt{K^2+1}} & 1 & \frac{1-K}{\sqrt{K^2+1}} \\ \frac{K+\omega^2}{\sqrt{K^2+1}} & \omega & \frac{1-K\omega^2}{\sqrt{K^2+1}} \\ \frac{K+\omega}{\sqrt{K^2+1}} & \omega^2 & \frac{1-K\omega}{\sqrt{K^2+1}} \end{pmatrix}, \quad \text{for NH}, \\ R_{lL}^\dagger U_\nu^I = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1-K}{\sqrt{K^2+1}} & 1 & \frac{1+K}{\sqrt{K^2+1}} \\ \frac{1-K\omega^2}{\sqrt{K^2+1}} & \omega & \frac{K+\omega^2}{\sqrt{K^2+1}} \\ \frac{1-K\omega}{\sqrt{K^2+1}} & \omega^2 & \frac{K+\omega}{\sqrt{K^2+1}} \end{pmatrix}, \quad \text{for IH}. \end{cases} \quad (26)$$

We see that all the elements of the matrix U_{PMNS} defined in Eq. (26) depend only on one parameter K and $|U_{i2}| = \frac{1}{\sqrt{3}}(i = 1, 2, 3)$ for both normal and inverted hierarchies. From the experimental constraints on the other elements of the lepton mixing matrix given in [13] as given in Eq. (1), we can find out the regions of K that satisfy the experimental data on the lepton mixing matrix.

The neutrino mass spectrum can be the normal or inverted hierarchy or nearly degenerate. The mass ordering of neutrino depends on the sign of Δm^2_{23} which is currently unknown. However, some tight upper limits on the total neutrino mass $\sum m_\nu$ have been given by the recent studies. For example, the total mass of three degenerate neutrinos was given by the Planck satellite mission [80], $\sum m_\nu < 0.72$ eV (95% CL) by using Planck TT + lowP data, and $\sum m_\nu < 0.49$ eV (95% CL) by using Planck TT, TE, EE + lowP data. While the improved constraints are given by adding the baryon acoustic oscillation (BAO) measurements [81], i.e., $\sum m_\nu < 0.21$ eV (95% CL) and $\sum m_\nu < 0.17$ eV (95% CL), respectively. Another upper limit was given in [82], $\sum m_\nu < 0.113$ eV (95% CL).

As will see, in the model under consideration, the two possible signs of Δm^2_{23} corresponding to *two types of the neutrino mass spectrum can be provided*. Combining (31) and the two experimental constraints on squared mass differences of neutrinos as shown in (3), we obtain the solutions as shown bellow.

2.1.1. Normal spectrum ($\Delta m^2_{23} > 0$). In the normal Hierarchy, the range of the elements $|U_{i1}|$ and $|U_{i3}|$ $(i = 1, 2, 3)$ in Eq. (26) are depicted in Fig. 1 with $K \in (0.675, 0.710)$. In the case $K = 0.7$, the lepton mixing matrix in (26) takes the form

$$U^N = \begin{pmatrix} 0.804072 & 0.57735 & 0.141895 \\ 0.0945968 - 0.409616i & -0.288675 + 0.5i & 0.638528 + 0.286731i \\ 0.0945968 + 0.409616i & -0.288675 - 0.5i & 0.638528 - 0.286731i \end{pmatrix}, \tag{27}$$

which is unitary and consistent with the constraint given in Eq. (1). This result implies that in the model under consideration, the value of the Jarlskog invariant J_{CP} which determines the magnitude of CP violation in neutrino oscillations is determined as [14, 83]:

$$J^N_{CP} = \text{Im}\left[U^N_{23}(U^N_{13})^* U^N_{12}(U^N_{22})^*\right] = -0.0329361. \tag{28}$$

On the other hand, in the standard parametrization of the three neutrino mixing matrix, J_{CP} is determined [14]:

$$J_{CP} = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23} \times \sin 2\theta_{13}\sin\delta_{CP}. \tag{29}$$

Substituting the best-fit values of leptonic mixing angles given in (2) for the nomal hierarchy to J_{CP} in Eq. (29) and comparing to J_{CP} in Eq. (28), we get $\sin\delta^N_{CP} = -0.98956$ or $\delta^N_{CP} = 303.302°$.

From standard parametrization of the leptonic mixing matrix, it follows that the lepton mixing parameters of our model take the form:

$$\sin^2\theta^N_{12} = \left|U^N_{12}\right|^2 / \left(1 - \left|U^N_{13}\right|^2\right) = 0.340183,$$
$$\sin^2\theta^N_{13} = \left|U^N_{13}\right|^2 = 0.0201342,$$
$$\sin^2\theta^N_{23} = \left|U^N_{23}\right|^2 / \left(1 - \left|U^N_{13}\right|^2\right) = 0.5, \tag{30}$$

which is consistent with the experimental data given in Eq. (2).

Now, by combining Eq. (22) and the two experimental constraints on the squared mass differences of neutrinos in the normal mass hierarchy as shown in Eq. (3), we get the other parameters of the model. Firstly, substituting $K = 0.7$ into Eq. (24) we get the following relations[7)]:

$$A_1 = A_2 - 0.728571B, \quad m^N_1 = A_2 - 1.42857B,$$
$$m^N_2 = A_3, \quad m^N_3 = A_2 + 0.7B, \tag{31}$$

$$\left(\Delta m^2_{21}\right)_N = \left|A^2_3 - 1.1327(0.939597A_2 - 1.34228B)^2\right|,$$
$$\left(\Delta m^2_{32}\right)_N = \left|0.5A^2_2 - 0.5A^2_3 + 2.82857A_2B - 0.530408B^2\right|. \tag{32}$$

[7)] Here, we we assumed that B is real and positive.

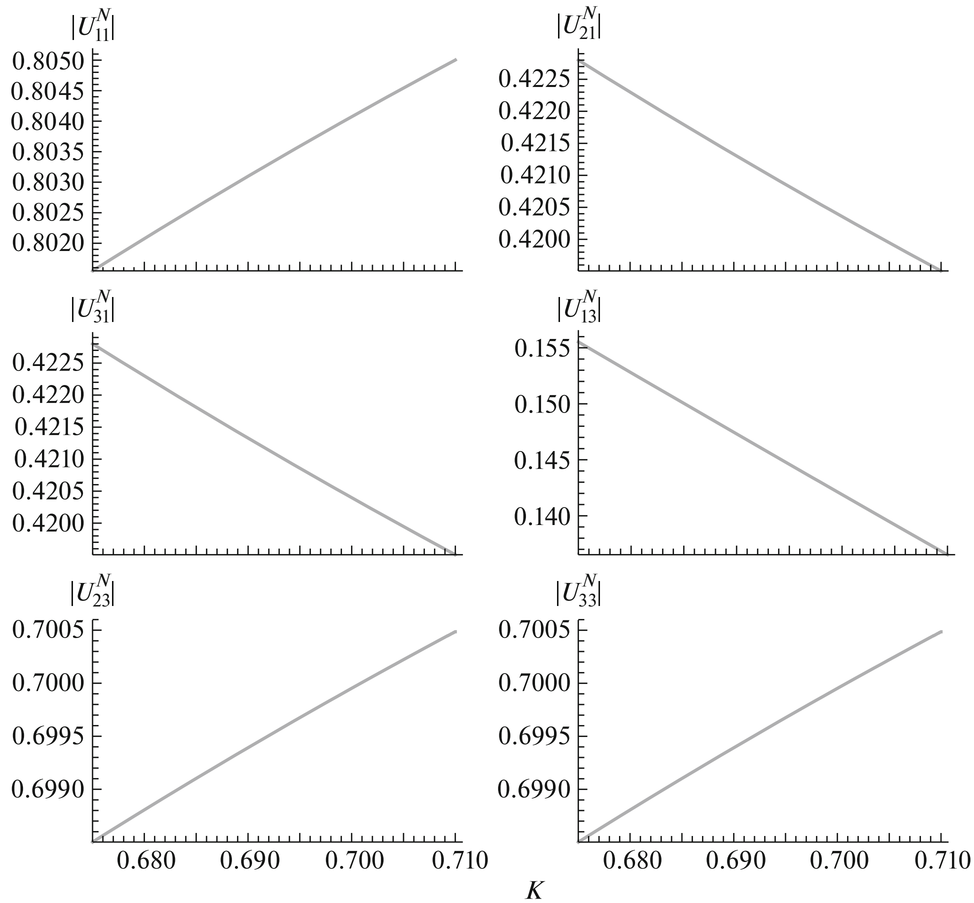

Fig. 1. U_{i1}^N and U_{i3}^N $(i = 1, 2, 3)$ as functions of K with $K \in (0.675, 0.710)$.

Next, comparing Eq. (32) and the two experimental constraints on the squared mass differences of neutrinos in the normal mass hierarchy as shown in Eq. (3), we get four solutions (in [eV]) given in Appendix B. These solutions have the same absolute values of $m_{1,2,3}$, the unique difference is the sign of them. Therefore we only consider in detail the case in Eq. (B.2)[8)]. Indeed, using the upper bound on the absolute value of the neutrino mass [84] we can restrict the values of $A_3 : A_3 \leq 0.6$ eV. However, in the normal hierarchy in Eq. (B.2), $A_3 \in (0.0087, 0.05)$ eV or $A_3 \in (-0.05, -0.0087)$ eV are good regions of A_3 that can reach the realistic neutrino mass hierarchies. $m_{1,2,3}$ as functions of A_3 are plotted in Fig. 2 with $A_3 \in (0.0087, 0.05)$ eV. This figure shows that there exist allowed regions of A_3 where either normal or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy[9)] is obtained when $A_3 \in (0.05\,\text{eV}, +\infty)$ or $A_3 \in (-\infty, -0.05\,\text{eV})$ ($|A_3|$ increases but must be small enough because of the scale of $m_{1,2,3}$). The normal mass hierarchy can be achieved if $A_3 \in (0.0087, 0.05)$ eV or $A_3 \in (-0.05, -0.0087)$ eV. The

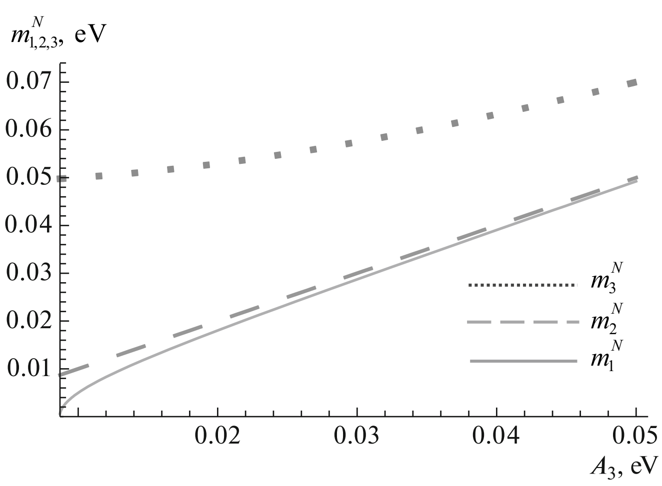

Fig. 2. $m_{1,2,3}$ as functions of A_3 in the normal spectrum with $A_3 \in (0.0087, 0.05)$ eV.

[8)]The expressions from Eq. (B.1) to Eq. (B.4) show that m_i $(i = 1, 2, 3)$ depend only on one parameter (A_3) so we consider $m_{1,2,3}$ as functions of A_3.

[9)]There is no clear limits between neutrino mass hierarchies by the recent experimental results on neutrino oscillations.

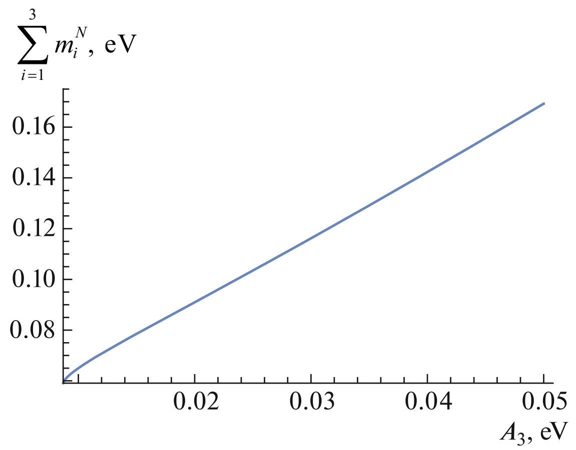

Fig. 3. The sum $\sum_{i=1}^{3} m_i$ as a function of A_3 with $A_3 \in (0.0087, 0.05)$ eV in the normal spectrum.

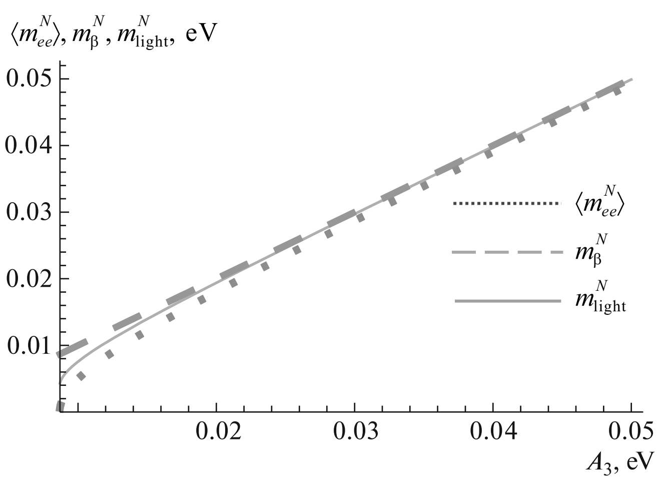

Fig. 4. $\langle m_{ee}\rangle$, m_β, and m_{light} as functions of A_3 with $A_3 \in (0.0087, 0.05)$ eV in the normal spectrum.

total neutrino masses in the model under consideration $\sum_{i=1}^{3} m_i$ with $A_3 \in (0.0087, 0.05)$ eV is depicted in Fig. 3.

It is easy to obtain the effective neutrino mass $\langle m_{ee}\rangle$ governing neutrinoless double beta decay [85–90] $\langle m_{ee}\rangle = \left|\sum_{i=1}^{3} U_{ei}^2 m_i\right|$ and $m_\beta = \left\{\sum_{i=1}^{3} |U_{ei}|^2 m_i^2\right\}^{1/2}$ from Eqs. (9) and (31). $\langle m_{ee}\rangle$ and m_β together with $m_{\text{light}} = m_1$ as functions of A_3 are plotted in Fig. 4 with $A_3 \in (0.0087, 0.05)$ eV.

To get explicit values of the model parameters, we assume $A_3 \equiv m_2 = 10^{-2}$ eV, which is safely small. Then the other neutrino masses and the other parameters are explicitly given in Table 2.

Now, comparing Eq. (21) and derived values in Table 2 we get the relations:

$$N_2 = 1.77938 N_1, \quad N_3 = 1.25649 N_1,$$
$$a = -0.112093 i\sqrt{N_1}, \quad b = -1.06974 N_1, \quad (33)$$

i.e., N_1, N_2, N_3, and b have the same order of magnitude, and approximately two orders of magnitude of a^2. N_2, N_3, $|a|$, and $|b|$ as functions of N_1 with $N_1 \in (10^8, 10^9)$ eV in the normal spectrum is depicted in Fig. 5. In the case, $N_1 = 10^9$ eV we get:

$$N_2 = 1.77938 \times 10^9, \quad N_3 = 1.25649 \times 10^9,$$
$$a = -3.54469 i \times 10^3, \quad b = -1.06974 \times 10^9. \quad (34)$$

Combining Eq. (34) and (19) we get the following relation:

$$x = -\frac{3.54469 i \times 10^3}{v_\varphi}, \quad y = -\frac{1.06974 \times 10^9}{v_\chi},$$
$$z = \frac{10^9}{u_1} = \frac{1.77938 \times 10^9}{u_2} = \frac{1.25649 \times 10^9}{u_3},$$
$$u_1 = 0.561994 u_2 = 0.795868 u_3. \quad (35)$$

2.1.2. Inverted spectrum ($\Delta m_{32}^2 < 0$). In inverted Hierarchy, the elements $|U_{i1}^I|$ and $|U_{i3}^I|$ ($i = 1, 2, 3$) in Eq. (26) are depicted in Fig. 6 with $K \in (-0.725, -0.675)$. In the case $K = -0.7$, the lepton mixing matrix in (26) takes the form

$$U^I = \begin{pmatrix} 0.804072 & 0.57735 & 0.141895 \\ 0.307439 - 0.286731i & -0.288675 + 0.5i & -0.567581 - 0.409616i \\ 0.307439 + 0.286731i & -0.288675 - 0.5i & -0.567581 + 0.409616i \end{pmatrix}. \quad (36)$$

The matrix U^I in Eq. (36) satisfies the unitary condition and has the same order in magnitude with U_{PMNS} in Eq. (1). The value of the Jarlskog invariant J_{CP} in this case is determined as [14, 83], $J_{CP}^I =$

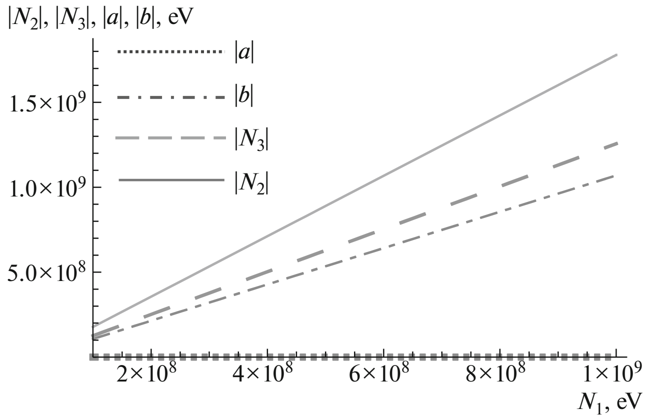

Fig. 5. $N_2, N_3, |a|$, and $|b|$ as functions of N_1 with $N_1 \in (10^8, 10^9)$ eV in the normal spectrum.

0.0329361. This value implies $\sin\delta^I_{CP} = 0.989362$ or $\delta^I_{CP} = 56.6863°$.

Substituting $K = -0.7$ into Eq. (24) we get:

$$A_1 = A_2 + 0.728571B, \quad m_1^I = A_2 + 1.42857B,$$
$$m_2^I = A_3, \quad m_3^I = A_2 - 0.7B, \qquad (37)$$
$$\left(\Delta m^2_{21}\right)_I = \left|A_3^2 - 1.1327(0.939597A_2 + 1.34228B)^2\right|,$$
$$\left(\Delta m^2_{32}\right)_I = \Big|0.5A_2^2 - 0.5A_3^2 - 2.82857A_2B - 0.530408B^2\Big|. \qquad (38)$$

Table 2. The model parameters in the case $A_3 \equiv m_2^N = 10^{-2}$ eV in the normal hierarchy

Parameter, eV	The derived value
A_1	1.97862×10^{-2}
A_2	3.52072×10^{-2}
B	2.11661×10^{-2}
$m^N_{\text{light}} \equiv m_1^N$	4.96991×10^{-3}
m_3	5.00235×10^{-2}
$\sum m_i^N$	6.49934×10^{-2}
$\langle m^N_{ee}\rangle$	7.55373×10^{-3}
m^N_β	9.98427×10^{-3}

Table 3. The model parameters in the case $A_3 \equiv m_2^I = 0.0505$ eV in the inverted hierarchy

Parameter, eV	The derived value
A_1	3.39188×10^{-2}
A_2	1.74426×10^{-2}
B	2.26144×10^{-2}
$m^I_{\text{light}} \equiv m_3^I$	1.61245×10^{-3}
m_1	4.97489×10^{-2}
$\sum m_i^I$	0.101861
$\langle m^I_{ee}\rangle$	4.90301×10^{-2}
m^I_β	4.95002×10^{-2}

Similar to the normal case, by combining Eq. (22) and the two experimental constraints on squared mass differences of neutrinos in the inverted mass hierarchy as shown in Eq. (3), we get four solutions (in [eV]) given in Appendix C. Here, we consider in detail the case in Eq. (C.4) with $A_3 \in (0.0505, 0.1)$ eV that can reach the inverted neutrino mass hierarchy which is plotted in Fig. 7. The total neutrino mass $\sum_{i=1}^3 m_i^I$ and the effective neutrino masses $\langle m^I_{ee}\rangle$, m^I_β together with $m^I_{\text{light}} = m^I_3$ in the inverted hierarchy model under consideration with $A_3 \in (0.0505, 0.1)$ eV are depicted in Figs. 8 and 9, respectively. To get explicit values of the model parameters, we assume $A_3 \equiv m_2^I = 0.0505$ eV, the other neutrino masses and the other parameters are explicitly given in Table 3.

Now, comparing Eq. (21) and derived values in Table 3 we get the relations:

$$N_2 = 0.514245N_1, \quad N_3 = 0.0468315N_1,$$
$$a = -0.0486312i\sqrt{N_1}, \quad b = -0.666723N_1. \quad (39)$$

Combining Eq. (39) and (19) we get the relation:

$$x = -1.53785i \times 10^3/v_\varphi,$$
$$y = -6.66723 \times 10^8/v_\chi,$$
$$z = 10^9/u_1 = 5.14245 \times 10^8/u_2$$
$$= 4.68315 \times 10^9/u_3,$$
$$u_1 = 1.9446u_2 = 21.3531u_3. \qquad (40)$$

2.2. Quark Mass

The quarks content of the model under $[SU(2)_L, U(1)_Y, U(1)_X, \underline{T}_7]$ symmetries, respectively, given in Table 4. The Yukawa interactions are[10]:

$$-\mathcal{L}_q = h_1^u\bar{Q}_{1L}(\tilde{\phi}u_R)_{\underline{1}} + h_2^u\bar{Q}_{2L}(\tilde{\phi}u_R)_{\underline{1}'} \qquad (41)$$
$$+ h_3^u\bar{Q}_{3L}(\tilde{\phi}u_R)_{\underline{1}''} + h_1^d\bar{Q}_{1L}(\phi d_R)_{\underline{1}}$$
$$+ h_2^d\bar{Q}_{2L}(\phi d_R)_{\underline{1}'} + h_3^d\bar{Q}_{3L}(\phi d_R)_{\underline{1}''} + \text{H.c.}$$
$$= h_1^u\bar{Q}_{1L}(\tilde{\phi}_1u_{1R} + \tilde{\phi}_2u_{2R} + \tilde{\phi}_3u_{3R})$$
$$+ h_2^u\bar{Q}_{2L}(\tilde{\phi}_1u_{1R} + \omega\tilde{\phi}_2u_{2R} + \omega^2\tilde{\phi}_3u_{3R})$$

[10] Here, $\tilde{\phi} = i\sigma_2\phi^* = \begin{pmatrix}\phi_2^0 \\ -\phi_1^-\end{pmatrix} \sim [2, -1, 0, \underline{3}]$, and $\tilde{\phi}' \sim [2, -1, 0, \underline{3}']$.

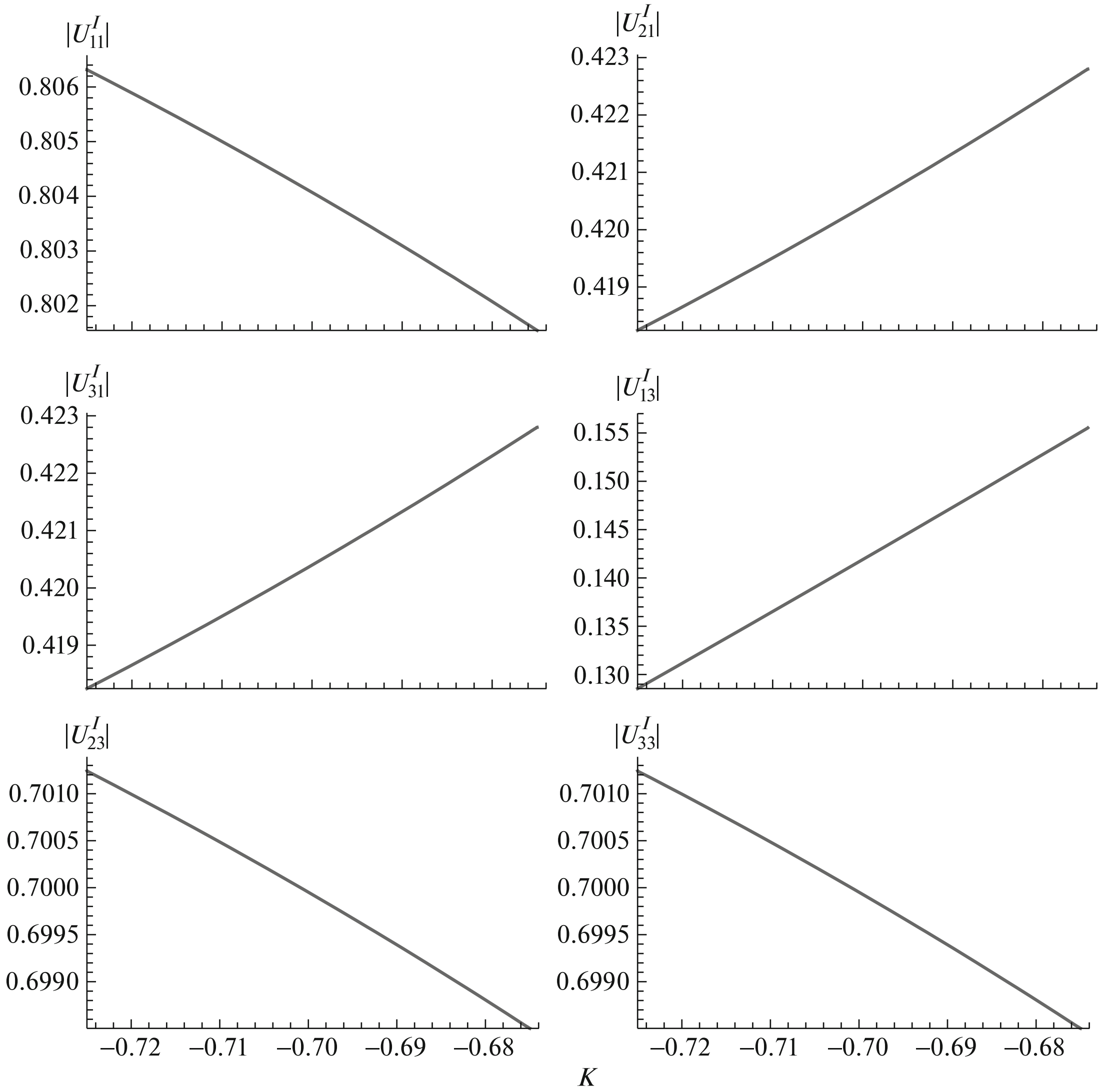

Fig. 6. U^I_{i1} and U^I_{i3} $(i = 1, 2, 3)$ as functions of K with $K \in (-0.725, -0.675)$.

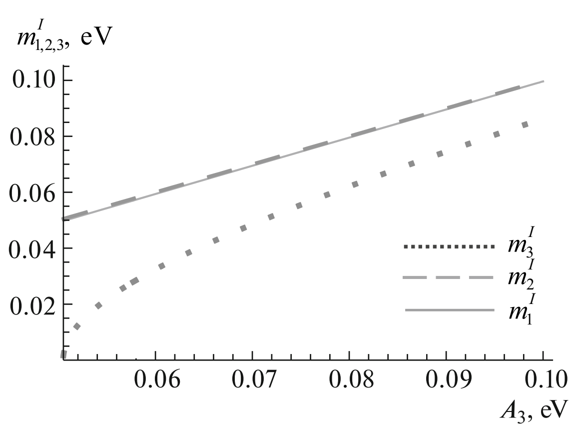

Fig. 7. $m^I_{1,2,3}$ as functions of A_3 in the inverted hierarchy with $A_3 \in (0.0505, 0.1)$ eV.

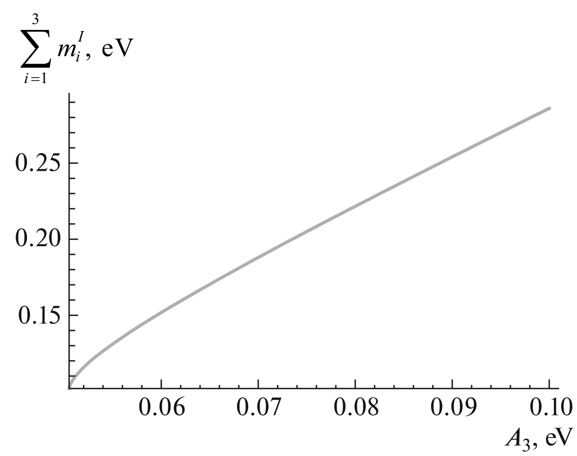

Fig. 8. The sum $\sum_{i=1}^{3} m^I_i$ as a function of A_3 with $A_3 \in (0.0505, 0.1)$ eV in the inverted spectrum.

Table 4. The quark content of the model

Field	$Q_{1,2,3L}$	$u_{1,2,3R}$	$d_{1,2,3R}$
$SU(2)_L$	2	1	1
$U(1)_Y$	1/3	4/3	−2/3
$U(1)_X$	0	0	0
$\underline{T}_7$	$\underline{1}, \underline{1}', \underline{1}''$	$\underline{3}$	$\underline{3}^*$

$$+ h^u_3\bar{Q}_{3L}(\tilde{\phi}_1 u_{1R} + \omega^2\tilde{\phi}_2 u_{2R} + \omega\tilde{\phi}_3 u_{3R})$$
$$+ h^d_1\bar{Q}_{1L}(\phi_1 d_{1R} + \phi_2 d_{2R} + \phi_3 d_{3R})$$
$$+ h^d_2\bar{Q}_{2L}(\phi_1 d_{1R} + \omega\phi_2 d_{2R} + \omega^2\phi_3 d_{3R})$$
$$+ h^d_3\bar{Q}_{3L}(\phi_1 d_{1R} + \omega^2\phi_2 d_{2R} + \omega\phi_3 d_{3R})$$
$$+ \text{H.c.}$$

With the VEV alignment of ϕ as given in Eq. (5), the mass Lagrangian of quarks reads

$$\begin{aligned}
-\mathcal{L}^{\text{mass}}_q &= h^u_1 v(\bar{u}_{1L}u_{1R} + \bar{u}_{1L}u_{2R} + \bar{u}_{1L}u_{3R}) \\
&+ h^u_2 v(\bar{u}_{2L}u_{1R} + \omega\bar{u}_{2L}u_{2R} + \omega^2\bar{u}_{2L}u_{3R}) \\
&+ h^u_3 v(\bar{u}_{3L}u_{1R} + \omega^2\bar{u}_{3L}u_{2R} + \omega\bar{u}_{3L}u_{3R}) \\
&+ h^d_1 v(\bar{d}_{1L}d_{1R} + \bar{d}_{1L}d_{2R} + \bar{d}_{1L}d_{3R}) \\
&+ h^d_2 v(\bar{d}_{2L}d_{1R} + \omega\bar{d}_{2L}d_{2R} + \omega^2\bar{d}_{2L}d_{3R}) \\
&+ h^d_3 v(\bar{d}_{3L}d_{1R} + \omega^2\bar{d}_{3L}d_{2R} + \omega\bar{d}_{3L}d_{3R}) \\
&+ \text{H.c.} \equiv (\bar{u}_{1L}, \bar{u}_{2L}, \bar{u}_{3L})M_u(u_{1R}, u_{2R}, u_{3R})^T \\
&+ (\bar{d}_{1L}, \bar{d}_{2L}, \bar{d}_{3L})M_d(d_{1R}, d_{2R}, d_{3R})^T + \text{H.c.}, \quad (42)
\end{aligned}$$

where the mass matrices for up- and down-quarks are, respectively, obtained as follows

$$M_u = \begin{pmatrix} h^u_1 v & h^u_1 v & h^u_1 v \\ h^u_2 v & \omega h^u_2 v & \omega^2 h^u_2 v \\ h^u_3 v & \omega^2 h^u_3 v & \omega h^u_3 v \end{pmatrix},$$
$$M_d = \begin{pmatrix} h^d_1 v & h^d_1 v & h^d_1 v \\ h^d_2 v & \omega h^d_2 v & \omega^2 h^d_2 v \\ h^d_3 v & \omega^2 h^d_3 v & \omega h^d_3 v \end{pmatrix}. \quad (43)$$

The matrices M_u and M_d in Eq. (43) are, respectively, diagonalized as

$$\begin{aligned}
&U^{u+}_L M_u U^u_R \\
&= \text{diag}\left(\sqrt{3}h^u_1 v, \sqrt{3}h^u_2 v, \sqrt{3}h^u_3 v\right) \\
&\equiv \text{diag}(m_u, m_c, m_t), \\
&U^{d+}_L M_d U^d_R \\
&= \text{diag}\left(\sqrt{3}h^d_1 v, \sqrt{3}h^d_2 v, \sqrt{3}h^d_3 v\right) \\
&\equiv \text{diag}(m_d, m_s, m_b), \quad (44)
\end{aligned}$$

where $U^u_L = U^d_L = 1$, i.e, the matrix that couples the left-handed up-and down-quarks to those in the mass bases are unit matrices, and $U^u_R = U^d_R = U_L$ with U_L given in (9). Therefore, the quark mixing matrix is an unit matrix, $U_{\text{CKM}} = U^{d\dagger}_L U^u_L = 1$. This is the common property for some models based on discrete symmetry groups [38, 39, 41–43, 71, 73, 75] and can be seen as an important result of the paper because the experimental quark mixing matrix is close to the unit matrix [14]. The current mass values for the quarks are given by [14]:

$$\begin{aligned}
&m_u = 2.2^{+0.6}_{-0.4}\ \text{MeV}, \quad m_c = 1.27 \pm 0.03\ \text{GeV}, \\
&m_t = 173.21 \pm 0.51 \pm 0.71\ \text{GeV}, \\
&m_d = 4.7^{+0.5}_{-0.4}\ \text{MeV}, \quad m_s = 96^{+8}_{-4}\ \text{MeV}, \\
&m_b = 4.18^{+0.04}_{-0.03}\ \text{GeV}, \quad (45)
\end{aligned}$$

With the help of Eqs. (44) and (45) we obtain the followings relations:

$$\begin{aligned}
&h^u_1 = 1.27017 \times 10^6/v, \quad h^u_2 = 7.33235 \times 10^8/v, \\
&h^u_3 = 1.00003 \times 10^{11}/v, \\
&h^d_1 = 2.71355 \times 10^6/v, \quad h^d_2 = 5.54256 \times 10^7/v, \\
&h^d_3 = 2.41332 \times 10^9/v, \quad (46)
\end{aligned}$$

or

$$\begin{aligned}
&h^u_1/h^u_2 = 1.73228 \times 10^3, \quad h^u_1/h^u_3 = 1.27013 \times 10^5, \\
&h^d_1/h^d_2 = 4.89583 \times 10^2, \\
&h^d_1/h^d_3 = 1.1244 \times 10^3, \quad (47) \\
&h^u_1/h^d_1 = 0.468085, \quad h^u_2/h^d_2 = 13.2292, \\
&h^u_3/h^d_3 = 41.4378, \quad (48)
\end{aligned}$$

i.e, h^u_1 is one order of magnitude less than h^d_1, but h^u_2 and h^u_3 are one order of magnitude larger than h^d_2 and h^d_3, respectively. Furthermore, h^u_1 is three order of magnitude less than h^u_2 and five order of magnitude less than h^u_3, while h^d_1 is two order of magnitude less than h^d_2 and three order of magnitude less than h^d_3. To get explicit values of the Yukawa couplings in the quark sector, we assume $v \sim 100$ GeV, then

$$\begin{aligned}
&h^u_1 = 1.27017 \times 10^{-5}, \quad h^u_2 = 7.33235 \times 10^{-3}, \\
&h^u_3 = 1.00003, \\
&h^d_1 = 2.71355 \times 10^{-5}, \quad h^d_2 = 5.54256 \times 10^{-4}, \\
&h^d_3 = 2.41332 \times 10^{-2}. \quad (49)
\end{aligned}$$

We note that, in the model under consideration, the quark mixing matrix $U_{\text{CKM}} = 1$ has no predictive power for quarks mixing but their masses are consistent with the experimental data given in [14].

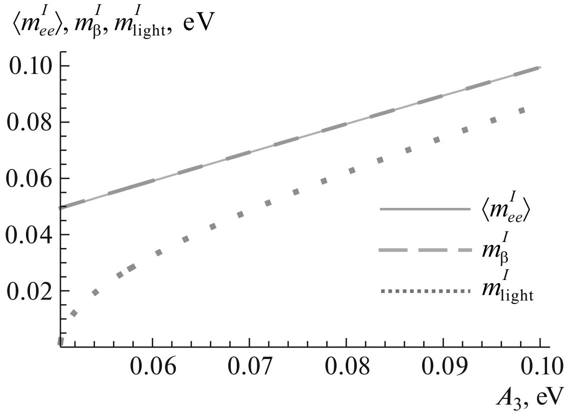

Fig. 9. $\langle m^I_{ee}\rangle$, m^I_β, and $m^I_{\rm light}$ as functions of A_3 with $A_3 \in (0.0505, 0.1)$ eV in the inverted spectrum.

3. CONCLUSION

We have constructed a simple Standard Model Extension based on T_7 flavor symmetry accommodating lepton mass and mixing with non-zero θ_{13} and CP violation phase. In difference from the other discrete groups, with T_7, the spontaneous symmetry breaking in the model is imposed to obtain the realistic lepton mass and mixing pattern at the tree-level with renormalizable interactions. The charged-lepton masses generated from one triplet (ϕ) under T_7, and the neutrinos get small masses from one $SU(2)_L$ doublet and two $SU(2)_L$ singlets in which one being in $\underline{1}$ and the two others in $\underline{3}$ and $\underline{3}^*$ under T_7, respectively. Furthermore, there exist the parameters of the model that predict an effective Majorana neutrino mass parameter with values $m_\beta \simeq 10^{-2}$ eV and 4.95×10^{-2} eV as well as a lightest neutrino mass $m_{\rm light} \simeq 4.97\times10^{-3}$ eV and 1.61×10^{-3} eV for the normal and inverted neutrino mass hierarchies, respectively. The model also gives a remarkable prediction of the Dirac CP violation $\delta_{CP} \simeq 303.3^\circ$ in the normal hierarchy and $\delta_{CP} \simeq 56.69^\circ$ in the inverted hierarchy. The quark mixing angles of the model are close to the experimental data, whereas the obtained values for the quark mases are consistent with with the experimental data at the tree level.

ACKNOWLEDGEMENTS

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Appendix A

HIGGS POTENTIAL

The renormalizable Higgs potential invariant under all symmetries, $SU(2)_L\otimes U(1)_Y\otimes U(1)_X\otimes \underline{T}_7$, is given by:

$$V = V(\phi)+V(\varphi)+V(\chi)+V(\zeta)+V(\phi,\varphi) + V(\phi,\chi)+V(\phi,\zeta)+V(\varphi,\chi) + V(\varphi,\zeta)+V(\chi,\zeta)+V(\phi,\varphi,\chi,\zeta), \tag{A.1}$$

where

$$V(\phi) = \mu^2_\phi\phi^\dagger\phi + \lambda^\phi_1(\phi^\dagger\phi)_1(\phi^\dagger\phi)_1 + \lambda^\phi_2(\phi^\dagger\phi)_{1'}(\phi^\dagger\phi)_{1''} + \lambda^\phi_3(\phi^\dagger\phi)_3(\phi^\dagger\phi)_{3^*}, \tag{A.2}$$

$$V(\varphi) = \mu^2_\varphi\varphi^\dagger\varphi + \lambda^\varphi(\varphi^\dagger\varphi)^2, \tag{A.3}$$

$$V(\chi) = \mu^2_\chi\chi^\dagger\chi + \lambda^\chi_1(\chi^\dagger\chi)_1(\chi^\dagger\chi)_1 + \lambda^\chi_2(\chi^\dagger\chi)_{1'}(\chi^\dagger\chi)_{1''} + \lambda^\chi_3(\chi^\dagger\chi)_3(\chi^\dagger\chi)_{3^*}, \tag{A.4}$$

$$V(\zeta) = \mu^2_\zeta\zeta^\dagger\zeta + \lambda^\zeta_1(\zeta^\dagger\zeta)_1(\zeta^\dagger\zeta)_1 + \lambda^\zeta_2(\zeta^\dagger\zeta)_{1'}(\zeta^\dagger\zeta)_{1''} + \lambda^\zeta_3(\zeta^\dagger\zeta)_3(\zeta^\dagger\zeta)_{3^*}, \tag{A.5}$$

$$V(\phi,\varphi) = \lambda^{\phi\varphi}_1(\phi^\dagger\phi)_1(\varphi^\dagger\varphi) + \lambda^{\phi\varphi}_2(\phi^\dagger\varphi)_{3^*}(\varphi^\dagger\phi)_3, \tag{A.6}$$

$$\begin{aligned} V(\phi,\chi) &= \delta^{\phi\chi}(\phi^\dagger\phi)_{3^*}\chi + \lambda^{\phi\chi}_1(\phi^\dagger\phi)_1(\chi^\dagger\chi)_1 + \lambda^{\phi\chi}_2(\phi^\dagger\phi)_{1'}(\chi^\dagger\chi)_{1''} \\ &+ \lambda^{\phi\chi*}_2(\phi^\dagger\phi)_{1''}(\chi^\dagger\chi)_{1'} + \lambda^{\phi\chi}_3(\phi^\dagger\phi)_3(\chi^\dagger\chi)_{3^*} + \lambda^{\phi\chi*}_3(\phi^\dagger\phi)_{3^*}(\chi^\dagger\chi)_3 + \lambda^{\phi\chi}_4(\phi^\dagger\chi)_3(\chi^\dagger\phi)_{3^*} \\ &+ \lambda^{\phi\chi*}_4(\phi^\dagger\chi)_{3^*}(\chi^\dagger\phi)_3, \end{aligned} \tag{A.7}$$

$$\begin{aligned} V(\phi,\zeta) &= \delta^{\phi\zeta}(\phi^\dagger\phi)_3\zeta + \lambda^{\phi\zeta}_1(\phi^\dagger\phi)_1(\zeta^\dagger\zeta)_1 + \lambda^{\phi\zeta}_2(\phi^\dagger\phi)_{1'}(\zeta^\dagger\zeta)_{1''} \\ &+ \lambda^{\phi\zeta*}_2(\phi^\dagger\phi)_{1''}(\zeta^\dagger\zeta)_{1'} + \lambda^{\phi\zeta}_3(\phi^\dagger\phi)_3(\zeta^\dagger\zeta)_{3^*} + \lambda^{\phi\zeta*}_3(\phi^\dagger\phi)_{3^*}(\zeta^\dagger\zeta)_3 + \lambda^{\phi\zeta}_4(\phi^\dagger\zeta^\dagger)_3(\zeta\phi)_{3^*} \\ &+ \lambda^{\phi\zeta*}_4(\phi^\dagger\zeta^\dagger)_{3^*}(\zeta\phi)_3, \end{aligned} \tag{A.8}$$

$$V(\varphi,\chi) = \lambda^{\varphi\chi}_1(\varphi^\dagger\varphi)(\chi^\dagger\chi)_1 + \lambda^{\varphi\chi}_2(\varphi^\dagger\chi)_3(\chi^\dagger\varphi)_{3^*}, \tag{A.9}$$

$$V(\varphi,\zeta) = \lambda^{\varphi\zeta}_1(\varphi^\dagger\varphi)(\zeta^\dagger\zeta)_1 + \lambda^{\varphi\zeta}_2(\varphi^\dagger\zeta)_{3^*}(\zeta^\dagger\varphi)_3, \tag{A.10}$$

$$\begin{aligned} V(\chi,\zeta) &= \delta^{\chi\zeta}(\chi^\dagger\chi)_3\zeta + \lambda^{\chi\zeta}_1(\chi^\dagger\chi)_1(\zeta^\dagger\zeta)_1 + \lambda^{\chi\zeta}_2(\chi^\dagger\chi)_{1'}(\zeta^\dagger\zeta)_{1''} \\ &+ \lambda^{\chi\zeta*}_2(\chi^\dagger\chi)_{1''}(\zeta^\dagger\zeta)_{1'} + \lambda^{\chi\zeta}_3(\chi^\dagger\chi)_3(\zeta^\dagger\zeta)_{3^*} + \lambda^{\chi\zeta*}_3(\chi^\dagger\chi)_{3^*}(\zeta^\dagger\zeta)_3 + \lambda^{\chi\zeta}_4(\chi^\dagger\zeta^\dagger)_3(\zeta\chi)_{3^*} \\ &+ \lambda^{\chi\zeta*}_4(\chi^\dagger\zeta^\dagger)_{3^*}(\zeta\chi)_3, \end{aligned} \tag{A.11}$$

$$V(\chi, \varphi, \chi, \zeta) = 0. \tag{A.12}$$

In the model under consideration, there are total 44 coupling constants in the renormalizable Higgs potential invariant under all symmetries, $SU(2)_L \otimes U(1)_Y \otimes U(1)_X \otimes \underline{T}_7$, including four of $V(\phi)$, two of $V(\varphi)$, four of $V(\chi)$, four of $V(\zeta)$, two of $V(\phi, \varphi)$, eight of $V(\phi, \chi)$, eight of $V(\phi, \zeta)$, two of $V(\varphi, \chi)$, two of $V(\varphi, \zeta)$, and eight of $V(\chi, \zeta)$, whereas there are only 37 equations for the potential minimization, including:

$$\frac{\partial V}{\partial v_i} = 0, \quad \frac{\partial V}{\partial v_i^*} = 0, \quad \frac{\partial V}{\partial v_{\chi i}} = 0,$$
$$\frac{\partial V}{\partial v_{\chi i}^*} = 0, \quad \frac{\partial V}{\partial u_i} = 0, \quad \frac{\partial V}{\partial u_i^*} = 0,$$

$$\frac{\partial V}{\partial v_\varphi} = 0 \quad (i = 1, 2, 3). \tag{A.13}$$

Because the number of equations are less than the number of Higgs potential parameters (the coupling constants and the VEVs), so the system of Eqs. (A.13) always have a nontrivial solution as expected. It is also noted that the above alignment is only one of the solutions to be imposed to have the desired results.

In general, the coupling constants and mass parameters are independent, however, the experimental data on lepton masses and mixings, and the discrete symmetry T_7 force them being related. This is the common property of the discrete flavor symmetries.

Appendix B

FOUR SOLUTIONS IN THE NORMAL SPECTRUM

The first case:

$$A_2 = -1.44358 \times 10^{-3}\sqrt{\alpha - 34.5969\sqrt{\beta}},$$
$$B = -\frac{\left(53.4131 + 3.3645 \times 10^4 A_3^2 + 5.49523\sqrt{\beta}\right)\sqrt{\alpha - 34.5969\sqrt{\beta}}}{5.40863 \times 10^4 + 1.69805 \times 10^7 A_3^2}, \tag{B.1}$$

The second case:

$$A_2 = 1.44358 \times 10^{-3}\sqrt{\alpha - 34.5969\sqrt{\beta}},$$
$$B = \frac{\left(53.4131 + 3.3645 \times 10^4 A_3^2 + 5.49523\sqrt{\beta}\right)\sqrt{\alpha - 34.5969\sqrt{\beta}}}{5.40863 \times 10^4 + 1.69805 \times 10^7 A_3^2}, \tag{B.2}$$

The third case:

$$A_2 = -1.44358 \times 10^{-3}\sqrt{\alpha + 34.5969\sqrt{\beta}},$$
$$B = -\frac{\left(53.4131 + 3.3645 \times 10^4 A_3^2 - 5.49523\sqrt{\beta}\right)\sqrt{\alpha + 34.5969\sqrt{\beta}}}{5.40863 \times 10^4 + 1.69805 \times 10^7 A_3^2}, \tag{B.3}$$

The fourth case:

$$A_2 = 1.44358 \times 10^{-3}\sqrt{\alpha + 34.5969\sqrt{\beta}},$$
$$B = \frac{\left(53.4131 + 3.3645 \times 10^4 A_3^2 - 5.49523\sqrt{\beta}\right)\sqrt{\alpha + 34.5969\sqrt{\beta}}}{5.40863 \times 10^4 + 1.69805 \times 10^7 A_3^2}, \tag{B.4}$$

where

$$\alpha = 5.15348 \times 10^2 + 2.68042 \times 10^5 A_3^2,$$
$$\beta = -6.78108 + 8.72315 \times 10^4 A_3^2 + 3.74859 \times 10^7 A_3^4. \tag{B.5}$$

Appendix C

FOUR SOLUTIONS IN THE INVERTED SPECTRUM

The first case:

$$A_2 = -1.44358 \times 10^{-3}\sqrt{\alpha' - 34.5969\sqrt{\beta'}},$$
$$B = \frac{\left(11.6721 - 6.72899 \times 10^3 A_3^2 - 1.09904\sqrt{\beta'}\right)\sqrt{\alpha' - 34.5969\sqrt{\beta'}}}{1.1305 \times 10^4 - 3.3961 \times 10^6 A_3^2}, \tag{C.1}$$

The second case:

$$A_2 = 1.44358 \times 10^{-3}\sqrt{\alpha' - 34.5969\sqrt{\beta'}},$$
$$B = -\frac{\left(11.6721 - 6.72899 \times 10^3 A_3^2 - 1.09904\sqrt{\beta'}\right)\sqrt{\alpha' - 34.5969\sqrt{\beta'}}}{1.1305 \times 10^4 - 3.3961 \times 10^6 A_3^2}, \tag{C.2}$$

The third case:

$$A_2 = -1.44358 \times 10^{-3}\sqrt{\alpha' + 34.5969\sqrt{\beta'}},$$
$$B = \frac{\left(11.6721 - 6.72899 \times 10^3 A_3^2 + 1.09904\sqrt{\beta'}\right)\sqrt{\alpha' + 34.5969\sqrt{\beta'}}}{1.1305 \times 10^4 - 3.3961 \times 10^6 A_3^2}, \tag{C.3}$$

The fourth case:

$$A_2 = 1.44358 \times 10^{-3}\sqrt{\alpha' + 34.5969\sqrt{\beta'}},$$
$$B = -\frac{\left(11.6721 - 6.72899 \times 10^3 A_3^2 + 1.09904\sqrt{\beta'}\right)\sqrt{\alpha' + 34.5969\sqrt{\beta'}}}{1.1305 \times 10^4 - 3.3961 \times 10^6 A_3^2}, \tag{C.4}$$

where

$$\alpha' = -5.5457 \times 10^2 + 2.68042 \times 10^5 A_3^2,$$
$$\beta' = 7.19122 - 9.83236 \times 10^4 A_3^2 + 3.74859 \times 10^7 A_3^4. \tag{C.5}$$

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