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# Optimal quantum channel for perfect controlled super-dense coding protocol

**Van Hop Nguyen**[1] **and Ba An Nguyen**[2,3]

[1] Department of Physics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam
[2] Thang Long Institute of Mathematics and Applied Sciences (TIMAS), Thang Long University (TLU), Hoang Mai District Vietnam
[3] Center for Theoretical Physics, Institute of Physics, Vietnam Academy of Science and Technology (VAST), 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam

E-mail: nban@iop.vast.ac.vn

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**Abstract**
Perfect super-dense coding protocol allows transmission of two classical bits (cbits) by sending just one quantum bit (qubit) with unit success probability. Perfect controlled super-dense coding protocol should be such that perfect super-dense coding is achieved with the controller's cooperation. There are different possible types of quantum channels to be shared among the authorized parties to perform perfect controlled super-dense coding. Of our interest is the quantum channel which is optimal in the sense that no cbits can be transmitted with certainty without the controller's cooperation. We attack this problem by means of the so-called control power. By calculating and analyzing the control power for specific quantum channels we figure out the one that is optimal. We also address the issue of how to prepare such optimal quantum channel.

Keywords: controlled super-dense coding, control power, optimal quantum channel
Classification numbers: 3.00, 3.01

## 1. Introduction

One of the most fascinating applications of quantum entanglement is super-dense coding protocol, which was first proposed in 1992 by Bennett and Wiesner [1]. In the protocol, a sender and a receiver share beforehand a known two-qubit maximally entangled Bell state [2–4], each holds a qubit. The sender performs one of the four unitary operations on his/her qubit to encode two cbits and then sends the qubit to the receiver. Upon receiving the qubit from the sender, the receiver having at hand two qubits performs a Bell-basis measurement on them to decode the two cbits encoded by the sender. Thus, by sending only one qubit the sender is able to transfer two cbits of information to the receiver. Clearly, this amount of transmitted information is twice that of the Holevo's bound [5]. That is why such quantum protocol is referred to as 'super-dense coding' one, which has been extensively studied both theoretically [6–14] and experimentally [15–19] till now.

As controlling quantum protocols is really necessary in practice, a developed version of the super-dense coding protocol, the so-called controlled super-dense coding one, was put forward by Hao *et al* [20], which has since then received much attention. The basic feature that makes the controlled super-dense coding protocol different from the original one is the addition of a third party, called the controller, who can control the quantum channel shared among the participants and, therefore, has the ability to control the amount of classical information to be transmitted. The controlled super-dense coding was experimentally demonstrated by Jing *et al* [21] and Zhang *et al* [22]. Later, researchers have generalized the controlled super-dense coding protocols to multi-partite quantum channels to extend various coding tasks within a quantum network [23–41]. The quantum channel of the controlled super-dense coding protocol may be an entangled state of different kinds such as Greenberger-Horne-Zeilinger (GHZ) state [23], GHZ-class state [24], extended GHZ state [26], four-partite non-maximally entangled state [27], five-qubit cluster state [28], generalized GHZ-type state [29], six-qubit cluster state [30], a genuine five-atom entangled state in cavity QED [34] and so on. The quantum entanglement between the sender and the receiver is controlled by the controller by means of local operations (including unitary transformations, measurements, use of quantum ancillas…) and classical communication. It is well known that in ideal situations a given protocol should be executed perfectly with the controller's cooperation. But, more interestingly, the role of the controller must be assessed in how good is the performance of the protocol without the controller's cooperation. Such an aspect of control power has still been little investigated. For example, the role of the controller in controlled super-dense coding was studied recently in [41] with regard to the minimal control power.

Before going into the content of the present work, let us make clearer in what sense cooperation and non-cooperation are meant. As mentioned above, the concerned task is governed by means of the controller's local operation and classical communication. By cooperation we mean that the controller carries out the necessary local operations and publicly publishes useful classical communication (to be specified below). Non-cooperation consists of various actions: the controller might perform undesired local operations and/or broadcast fake announcements in which case the receiver obtains meaningless information. We shall exclude such kinds of disturbative actions because the function of the controller is to control but not to disturb. Instead, in this work, by non-cooperation we mean that the controller simply does nothing.

In this article, we consider quantum channels which are always capable of successfully transferring two cbits by sending one qubit if the controller cooperates. However, if the controller refuses to cooperate (i.e., he/she provides non-cooperation in the sense clarified above), then the amount of information that can be transferred depends on the structure of the quantum channel shared between the controller, the sender and the receiver. Here, following the Refs. [42–46], we make use of a quantity called control power which is a newly introduced indicator reflecting the power of the controller in a quantitative manner. By calculating and analyzing the control power in dependence on the parameters of various possible quantum channels we can find out which quantum channel is optimal from the point of view of the amount of transferrable information without the cooperation of the controller. In addition, we also propose ways to prepare such an optimal quantum channel for the perfect controlled super-dense coding problem.

We organize our article as follows. After this introduction section, in section 2 we setup the problem, specify the class of quantum channels of interest and define the corresponding control power. Then, section 3 calculates the control power for each possible quantum channel of the class of channels specified in section 2. By analyzing dependence of the control power on the quantum channel parameters the optimal channel is figured out. Next, in section 4, ways to prepare the optimal quantum channel are presented. Conclusion is finally given in section 5.

## 2. The control power

In this section, we shall consider a super-dense coding protocol executed by a sender Alice and a receiver Bob under control of a controller Charlie. Initially, Alice, Bob and Charlie are 'quantumly connected' via a quantum channel by sharing an entangled state which consists at least of three qubits: Alice and Bob each holds one qubit and the rest qubits are possessed by Charlie. This quantum channel should be engineered so that, only after the controller performs appropriate local transformations and measurements on his/her qubit(s) followed by revealing the measurement outcome, can the sender and the receiver execute the super-dense coding procedures with 100% success probability. In order to satisfy the above requirement, the concerned quantum channel state must be of the following general form

$$|Q\rangle_{ABC_1C_2\,\ldots\,C_M} = \sum_j \alpha_j\,|B_j\rangle_{AB}|\Phi_j\rangle_{C_1C_2\,\ldots\,C_M}, \tag{1}$$

where qubit $A$ ($B$) belongs to Alice (Bob), while qubits $C_1$, $C_2$, …, $C_M$, with $M \geqslant 1$, are fully manipulatable by Charlie. The complex numbers $\alpha_j$ are arbitrary up to the normalization condition $\sum_j |\alpha_j|^2 = 1$. The states $|B_j\rangle_{AB}$ with $j = 0, 1, 2, 3$ are the four well-known maximally entangled Bell state [2] shared between Alice and Bob: $|B_0\rangle_{AB} = |B_{00}\rangle_{AB}$, $|B_1\rangle_{AB} = |B_{01}\rangle_{AB}$, $|B_2\rangle_{AB} = |B_{10}\rangle_{AB}$ and $|B_3\rangle_{AB} = |B_{11}\rangle_{AB}$, with

$$|B_{mn}\rangle_{AB} = \frac{1}{\sqrt{2}}\sum_{k=0}^{1}(-1)^{mk}|k,\, n \oplus k\rangle_{AB}, \tag{2}$$

where $\oplus$ denotes an addition mod 2. As for $|\Phi_{\mathbf{j}}\rangle_{C_1C_2\,\ldots\,C_M}$, they are $M$-qubit entangled states which are orthonormal to each other so that they can be perfectly discriminated by a projective measurement. To ensure nonzero degree of entanglement of state $|Q\rangle_{ABC_1C_2\ldots C_M}$ in equation (1), with respect to the cut $AB|C_1C_2\ldots C_M$, the number $N$ of terms in the sum over $j$ in the right-hand-side of equation (1) should be greater than 1. Nevertheless, since there are only four Bell states at most, $N$ should not be greater than 4. This means that $N$ must be bound by the inequalities $1 < N \leqslant 4$.

In the ideal situation, Charlie measures her qubits $C_1$, $C_2$, …, $C_M$ and finds $|\Phi_j\rangle_{C_1C_2\ldots C_M}$ with a certain $j$. If Charlie publicly announces the outcome $j$, then Alice and Bob become aware of sharing the Bell state $|B_j\rangle_{AB}$ and confidently perform the standard super-dense coding protocol [1] which allows transmitting two cbits of information by sending only one qubit. Indeed, the probability of obtaining the measurement outcome $j$ is $P_j = |\alpha_j|^2$ and the number of transmitted cbits in this event is $I_j = 2$ (which is independent of $j$). Thus, the total number of transmitted cbits is

$$I = \sum_j P_j I_j = 2\sum_j |\alpha_j|^2 = 2, \tag{3}$$

which is trivial. Now we turn to a nontrivial situation when Charlie by some reason (for instance, she observes something wrong) decides to change her mind by not measuring qubits $C_1$, $C_2$, …, $C_M$ or by measuring them but keeping the outcome unrevealed. If so, what turns out to be shared between Alice and Bob is not a pure Bell state but a mixed state of the form

$$\rho_{AB} = \mathrm{Tr}_{C_1C_2\ldots C_M}(|Q\rangle_{ABC_1C_2\ldots C_M}\langle Q|) = \sum_j |\alpha_j|^2 |B_j\rangle_{AB}\langle B_j|. \tag{4}$$

Without Charlie's cooperation, the largest number $I_{\max}(\rho_{AB})$ of cbits, which can be transmitted by means of sharing the mixed state $\rho_{AB}$, can be quantified by the channel capacity $C(\rho_{AB})$ [9, 12]:

$$I_{\max}(\rho_{AB}) = C(\rho_{AB}) = 1 - S(\rho_{AB}) + S(\rho_B), \tag{5}$$

where $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy of a state described by density matrix $\rho$. From equation (4) it immediately follows that $|B_j\rangle_{AB}$ and $|\alpha_j|^2$ are eigenvectors and eigenvalues of the density matrix $\rho_{AB}$. Hence, by definition, the von Neumann entropy of the mixed state $\rho_{AB}$ is merely

$$S(\rho_{AB}) = -\sum_j |\alpha_j|^2 \log_2 |\alpha_j|^2 \tag{6}$$

and that of the reduced density matrix of Bob's qubit, $\rho_B = \mathrm{Tr}_A(\rho_{AB}) = I_B/2$ ($I_B$ is a $2\times 2$ unity matrix), is

$$S(\rho_B) = 1. \tag{7}$$

On account of equation (7), the maximum amount of transmitted cbits $I_{\max}(\rho_{AB})$ without Charlie's cooperation (see equation (5)) simplifies to

$$I_{\max}(\rho_{AB}) = 2 - S(\rho_{AB}). \tag{8}$$

To assess the role of the controller, we exploit the concept of the so-called controller's power (or just control power for short) $CP$ which was defined previously in [41] as the difference between the number of cbits that can be transmitted with and without the controller's cooperation, i.e.,

$$CP = 2 - I_{\max}(\rho_{AB}) = S(\rho_{AB}). \tag{9}$$

As $|Q\rangle_{ABC_1C_2\ldots C_M}$ is a pure state, we have

$$S(\rho_{C_1C_2\ldots C_M}) = S(\rho_{AB}), \tag{10}$$

with

$$\rho_{C_1C_2\ldots C_M} = \mathrm{Tr}_{AB}(|Q\rangle_{ABC_1C_2\ldots C_M}\langle Q|), \tag{11}$$

and $S(\rho_{C_1C_2\ldots C_M})$ the von Neumann entropy of the reduced density matrix of Charlie's qubits $C_1, C_2, \ldots, C_M$. So, the control power can also be computed by

$$CP = S(\rho_{C_1C_2\ldots C_M}). \tag{12}$$

Obviously, in case the controller does not cooperate, his/her power is stronger if the maximum amount of transmitted cbits is smaller. Or, as seen from equations (9) and (12), the von Neumann entropies $S(\rho_{AB}) = S(\rho_{C_1C_2\ldots C_M})$ should be as large as possible to rank the power of the controller. Yet, certain conditions are to be imposed on the control power for the controller to be relevant/powerful. Since a classical protocol (i.e., protocol that does not require any shared quantum states) can always be used to transfer at most one cbit by sending one qubit, in order to ensure the relevance of the controller, the number of cbits that can be transmitted by a quantum protocol without Charlie's permission should be smaller than that by classical protocols which is equal to 1. This requires that the control power must satisfy the following conditions

$$1 < CP \leqslant 2. \tag{13}$$

From the above conditions, we categorize three types of the control power. The case of $CP \leqslant 1$ implies that the quantum protocol exhibits no advantages over the classical one. In this case, we can say that the controller is powerless (i.e., his/her role is not guaranteed). Meaningful are quantum protocols for which $1 < CP < 2$. In this case, the controller's role is guaranteed, because without the controller's participation the sender is unable to transfer even one cbit of information. Finally, if $CP = 2$, the corresponding quantum controlled protocol is perfect and the controller is most powerful because without the controller's cooperation, no useful classical information (i.e., zero cbits) can be transmitted.

## 3. Optimizing the control power

The working quantum channel $|Q\rangle_{ABC_1C_2\ldots C_M}$ in equation (1) can be characterized by two parameters $N$ (the number of terms in the sum over $j$) and $M$ (the number of qubits possessed by Charlie). To make the super-dense coding protocol controllable exclusively only by Charlie we propose to consider a more general form of the quantum channel as

$$|Q(\theta_1, \theta_2,\ldots,\theta_M)\rangle_{ABC_1C_2\ldots C_M} = \sum_j \alpha_j\, |B_j\rangle_{AB} \otimes |\Phi_j(\theta_1, \theta_2,\ldots,\theta_M)\rangle_{C_1C_2\ldots C_M}, \tag{14}$$

with introduction of the angles $\theta_1, \theta_2, \ldots, \theta_M$ whose values are unknown for anyone but Charlie. This reserves only for Charlie the ability to control the protocol by measuring his/her qubits in the right basis determined by the values of $\theta_1, \theta_2, \ldots, \theta_M$, a feature that was exploited in [44, 47, 48]. The angles $\theta_1, \theta_2,\ldots,\theta_M$ may simply be introduced into a given state $|\Phi_j\rangle_{C_1C_2\ldots C_M}$ by acting on it the operator $U(\theta_1, \theta_2,\ldots,\theta_M)_{C_1C_2\ldots C_M} = \prod_{m=1}^{M} R_{C_m}(\theta_m)$ with $R_{C_m}(\theta_m)|n\rangle_{C_m} = \cos\frac{\theta_m}{2}|n\rangle_{C_m} - (-1)^n \sin\frac{\theta_m}{2}|n\oplus 1\rangle_{C_m};\quad n \in \{0, 1\}$ being a rotation by an angle $\theta_m$. That is, $|\Phi_j(\theta_1, \theta_2,\ldots,\theta_M)\rangle_{C_1C_2\ldots C_M} = U(\theta_1, \theta_2, \ldots, \theta_M)_{C_1C_2\ldots C_M}|\Phi_j\rangle_{C_1C_2\ldots C_M}$. Therefore, measurement in the right basis $\{|\Phi_j(\theta_1, \theta_2,\ldots,\theta_M)\rangle_{C_1C_2\ldots C_M}\}$ is tantamount to measurement in the basis $\{|\Phi_j\rangle_{C_1C_2\ldots C_M}\}$ after application of $U^{-1}(\theta_1, \theta_2, \ldots, \theta_M)_{C_1C_2\ldots C_M} = U(-\theta_1, -\theta_2, \ldots, -\theta_M)_{C_1C_2\ldots C_M}$ on the states $|\Phi_j(\theta_1, \theta_2,\ldots,\theta_M)\rangle_{C_1C_2\ldots C_M}$. As mentioned above, the angles $\theta_1, \theta_2, \ldots, \theta_M$ are assumed to be known only by Charlie, no one else is able to control the protocol.

As clarified in the Introduction section, if the controller cooperates he/she should carry out the necessary local operations. Here the necessary local operations are measurements in the correct basis determined by the angles

$\theta_1, \theta_2, \ldots, \theta_M$. Measurements in any other bases, including $\{|\pm\rangle\}$, result in failure of super-dense coding protocol. Note that, in accordance with the definition of cooperation/non-cooperation, measurement in a wrong basis implies non-cooperation which we are not interested in.

We now optimize the quantum channel for the controlled super-dense coding protocol by choosing suitable values of $N$ and $M$ so that the corresponding control power is maximum. Firstly, consider $N = 2$ for which case $M = 1$ suffices and the quantum channel (14) reads

$$|Q_{ij}(\theta)\rangle_{ABC} = \alpha_i\,|B_i\rangle_{AB}|\Phi_i(\theta)\rangle_C + \alpha_j\,|B_j\rangle_{AB}|\Phi_j(\theta)\rangle_C, \quad (15)$$

with $i \neq j$. There are six different pairs $\{|B_i\rangle_{AB}, |B_j\rangle_{AB};\ i, j \in \{0, 1, 2, 3\};\ i \neq j\}$ in total. Because any one of the four Bell states does equally well for the super-dense coding, we can, without loss of generality, choose $i = 0$ and $j = 1$. As for the Charlie's state we can set $|\Phi_0(\theta)\rangle_C = \left(\cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle\right)_C$ and $|\Phi_1(\theta)\rangle_C = \left(\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle\right)_C$. Thus, equation (15) becomes

$$|Q_{01}(\theta)\rangle_{ABC} = \alpha_0\,|B_0\rangle_{AB}\left(\cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle\right)_C + \alpha_1\,|B_1\rangle_{AB}\left(\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle\right)_C. \quad (16)$$

It is worth reminding that only Charlie knows the value of $\theta$ in equation (16). So, to distinguish the two states of Charlie's qubit, he/she first applies $R_C(-\theta)$ to the qubit $C$, transforming equation (16) to

$$|Q_{01}\rangle_{ABC} = \alpha_0\,|B_0\rangle_{AB}|0\rangle_C + \alpha_1\,|B_1\rangle_{AB}|1\rangle_C. \quad (17)$$

After that Charlie measures qubit $C$ in the computational basis $\{|0\rangle_C, |1\rangle_C\}$. If she finds $|0\rangle_C$, with a probability of $|\alpha_0|^2$, the state of qubit $A$ of Alice and qubit $B$ of Bob is projected onto the Bell state $|B_0\rangle_{AB}$. Otherwise, the state of Alice's and Bob's qubits is $|B_1\rangle_{AB}$, if Charlie finds $|1\rangle_C$, with a probability of $|\alpha_1|^2$. In either case, Alice is able to transfer to Bob two cbits, following the standard super-dense coding protocol [1]. According to the formulae in section 2, the control power in this case ($N = 2$, $M = 1$) is equal to

$$CP_1 = -|\alpha_0|^2\log_2|\alpha_0|^2 - |\alpha_1|^2\log_2|\alpha_1|^2, \quad (18)$$

where the sub-index 1 in $CP_1$ signals that the case with $N = 2$ and $M = 1$ is under consideration. In order to see more clearly the value of the control power in this case, we set

$$\alpha_0 = \cos\frac{\gamma}{2}, \quad \alpha_1 = \sin\frac{\gamma}{2}. \quad (19)$$

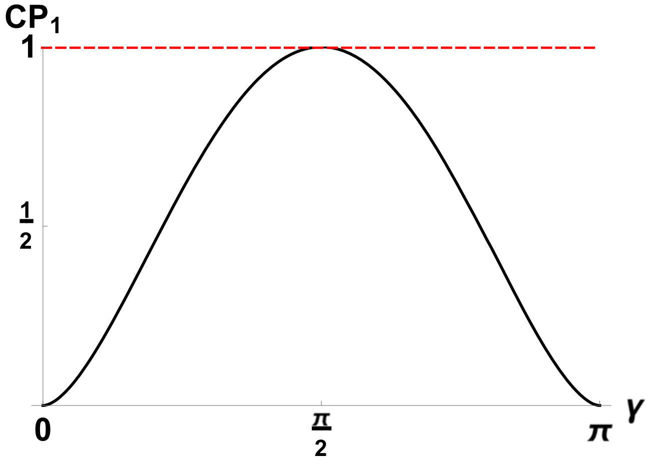


**Figure 1.** The dependence on $\gamma$ of the control power $CP_1$. The horizontal dashed line at 1 marks the maximal number of cbits that can be transmitted by classical means.

Then the control power explicitly depends on a single variable $\gamma$. We plot in figure 1 the dependence of $CP_1$ on $\gamma$ ($0 \leqslant \gamma \leqslant \pi$) and realize that the control power cannot exceed 1. In fact, it is confined between 0 and 1, i.e.,

$$0 \leqslant CP_1 \leqslant 1. \quad (20)$$

The maximum value of $CP_1$ is equal to 1 when $\gamma = \pi/2$ This result is understandable because $\gamma = \pi/2$ leads to $\alpha_0 = \alpha_1 = 1/\sqrt{2}$ making the quantum channel state $|Q_{01}\rangle_{ABC}$ in equation (17) maximally entangled with respect to the bipartite cut $AB|C$. Also transparent is the result that $CP_1 = 0$ when $\gamma = 0, \pi$. This is understandable because $\gamma = 0 \to \{\alpha_0 = 1, \alpha_1 = 0\}$ and $\gamma = \pi \to \{\alpha_0 = 0, \alpha_1 = 1\}$ that destroys all the entanglement between $AB$ and $C$, disabling the function of Charlie as a controller. Since the maximum value of $CP_1$ is just 1, the quantum protocol via the quantum channel with $N = 2$ and $M = 1$ does not outperform the classical one. The controller in this case can be regarded as powerless. We shall therefore consider the next possibility for $N$.

Next, consider $N = 3$. In this case Charlie should hold a qutrit or at least two qubits. Let us deal with the latter situation, i.e., with $M = 2$, for which the quantum channel (14) reads

$$|Q_{ijk}(\theta_1, \theta_2)\rangle_{ABC_1C_2} = \alpha_i\,|B_i\rangle_{AB}|\Phi_i(\theta_1, \theta_2)\rangle_{C_1C_2} + \alpha_j\,|B_j\rangle_{AB}|\Phi_j(\theta_1, \theta_2)\rangle_{C_1C_2} + \alpha_k\,|B_k\rangle_{AB}|\Phi_k(\theta_1, \theta_2)\rangle_{C_1C_2}, \quad (21)$$

with $i \neq j \neq k \neq i$. Clearly, there are four different trios $\{|B_i\rangle_{AB}, |B_j\rangle_{AB}, |B_k\rangle_{AB}\}$ in total. By the same reason as in the case of $N = 2$ and $M = 1$, we can, without loss of generality, choose $i = 0$, $j = 1$ and $k = 2$, then work with the following

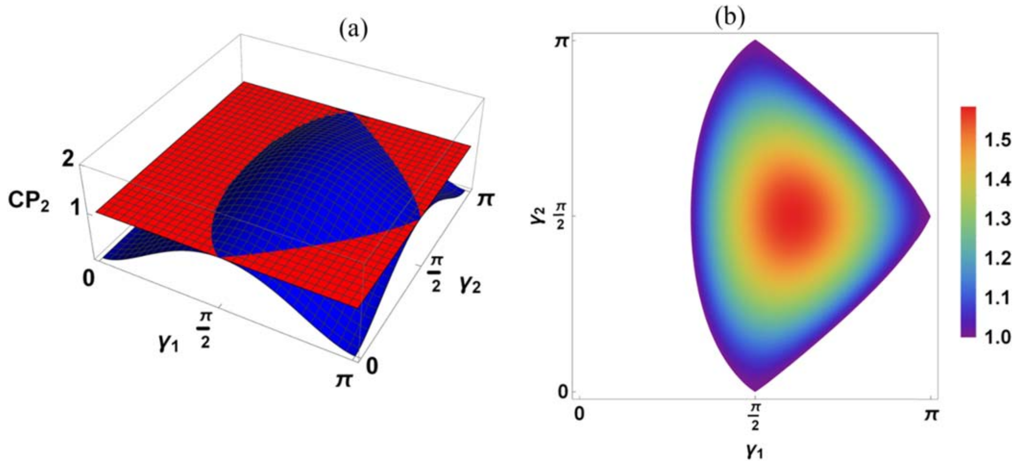


**Figure 2.** (a) The dependence on $\gamma_1$ and $\gamma_2$ of the control power $CP_2$. The horizontal plane at 1 marks the classical limit of the control power. (b) The corresponding phase diagram in the $\gamma_1$–$\gamma_2$ space. The quantum domain is described in colors, while the classical one in the white background.

quantum channel

$$
\begin{aligned}
|Q_{012}(\theta_1, \theta_2)\rangle_{ABC_1C_2} = \alpha_0\, |B_0\rangle_{AB}\Big(&\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle \\
&- \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\
&- \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|10\rangle \\
&+ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle\Big)_{C_1C_2} \\
&- \alpha_1\, |B_1\rangle_{AB}\Big(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|00\rangle \\
&+ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|01\rangle \\
&+ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|10\rangle \\
&+ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|11\rangle\Big)_{C_1C_2} \\
&+ \alpha_2\, |B_2\rangle_{AB}\Big(\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle \\
&- \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\
&+ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|10\rangle \\
&- \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle\Big)_{C_1C_2}.
\end{aligned}
\tag{22}
$$

To control the super-dense coding between Alice and Bob, Charlie first applies $R_{C_m}(-\theta_m)$ to the qubit $C_m$ where m = 1, 2. As a result, equation (22) is transformed to

$$
\begin{aligned}
|Q_{012}\rangle_{ABC_1C_2} = \alpha_0\, |B_0\rangle_{AB}|00\rangle_{C_1C_2} + \alpha_1\, |B_1\rangle_{AB}|10\rangle_{C_1C_2} \\
- \alpha_2\, |B_2\rangle_{AB}|11\rangle_{C_1C_2}.
\end{aligned}
\tag{23}
$$

Then Charlie measures qubit $C_m$ in the computational basis $\{|0\rangle_{C_m}, |1\rangle_{C_m}\}$ to find either $|00\rangle_{C_1C_2}$, $|10\rangle_{C_1C_2}$ or $|11\rangle_{C_1C_2}$. Each outcome happens with a nonzero probability, but in any case Alice and Bob are left in a known Bell state, which is ready for them to carry out the bipartite super-dense coding perfectly. According to the formulae in section 2, the control power is calculated as

$$
\begin{aligned}
CP_2 = -|\alpha_0|^2 \log_2 |\alpha_0|^2 - |\alpha_1|^2 \log_2 |\alpha_1|^2 \\
- |\alpha_2|^2 \log_2 |\alpha_2|^2,
\end{aligned}
\tag{24}
$$

where the sub-index 2 in $CP_2$ signals that the case with $N = 3$ and $M = 2$ is under consideration. To explicitly analyze the dependence of the control power on the coefficients $\alpha_{0,1,2}$ we set

$$
\begin{aligned}
\alpha_0 &= \cos\frac{\gamma_1}{2}, \\
\alpha_1 &= \sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}, \\
\alpha_2 &= \sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2},
\end{aligned}
\tag{25}
$$

with $0 \leqslant \gamma_{1,2} \leqslant \pi$. In figure 2(a) we plot $CP_2$ as a function of $\gamma_1$ and $\gamma_2$. The phase diagram in the figure 2(b) shows that, although there is domain of parameters within which the quantum control is meaningful (i.e., $CP_2 > 1$), no parameters can be found that make $CP_2 > \log_2 3 \simeq 1.585$. Actually, $CP_2$ is strictly confined between 0 and $\log_2 3$, i.e.,

$$
0 \leqslant CP_2 \leqslant \log_2 3. \tag{26}
$$

To have a more visual view of the parameter domain in which the quantum control is relevant, let us consider a particular situation with $\gamma_1 = \gamma_2 = \gamma$. Then, as seen from figure 3, $1 < CP_2 \leqslant \log_2 3$, if $\gamma$ is chosen such that $0.329\pi < \gamma < 0.774\pi$, otherwise the control is not guaranteed at all.

At this point, a natural question arises: is there a quantum channel that allows the control power to be greater than

$\log_2 3$? To answer this question, we finally aim at considering the case with $N = 4$. The minimum value of $M$ in this situation is 2. The quantum channel (14) is then of the explicit form

$$
\begin{aligned}
|Q_{0123}(\theta_1, \theta_2)\rangle_{ABC_1C_2} = \alpha_0\, |B_0\rangle_{AB}\Big(&\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle \\
&- \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\
&- \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|10\rangle \\
&+ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle\Big)_{C_1C_2} \\
&+ \alpha_1\, |B_1\rangle_{AB}\Big(\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|00\rangle \\
&+ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|01\rangle \\
&- \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|10\rangle \\
&- \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|11\rangle\Big)_{C_1C_2} \\
&+ \alpha_2\, |B_2\rangle_{AB}\Big(\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle \\
&- \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\
&+ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|10\rangle \\
&- \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle\Big)_{C_1C_2} \\
&+ \alpha_3\, |B_3\rangle_{AB}\Big(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|00\rangle \\
&+ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|01\rangle \\
&+ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|10\rangle \\
&+ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|11\rangle\Big)_{C_1C_2}.
\end{aligned} \quad (27)
$$

By acting $R_{C_1}(-\theta_1)R_{C_2}(-\theta_2)$ on the qubits $C_1$ and $C_2$ Charlie transforms equation (27) to

$$
\begin{aligned}
|Q_{0123}\rangle_{ABC_1C_2} = \alpha_0\, |B_0\rangle_{AB}|00\rangle_{C_1C_2} + \alpha_1\, |B_1\rangle_{AB}|01\rangle_{C_1C_2} \\
+ \alpha_2\, |B_2\rangle_{AB}|10\rangle_{C_1C_2} + \alpha_3\, |B_3\rangle_{AB}|11\rangle_{C_1C_2}.
\end{aligned} \quad (28)
$$

Note again that the transformation from equation (27) to equation (28) can be done only by the controller Charlie because she is the only one who knows the precise values of $\theta_1$ and $\theta_2$. As in the previous case, Charlie proceeds by measuring qubits $C_1$ and $C_2$ in the computational bases $\{|0\rangle_{C_1}, |1\rangle_{C_1}\}$ and $\{|0\rangle_{C_2}, |1\rangle_{C_2}\}$. As an outcome, Charlie may find one of the four two-qubit states $|00\rangle_{C_1C_2}$, $|01\rangle_{C_1C_2}$, $|10\rangle_{C_1C_2}$ or $|11\rangle_{C_1C_2}$, with a probability of $|\alpha_0|^2$, $|\alpha_1|^2$, $|\alpha_2|^2$ or $|\alpha_3|^2$, respectively. Yet, whatever outcome happens, Alice and Bob are disentangled from Charlie but become maximally entangled with each other via a known Bell state. The control power is then determined by the following formula

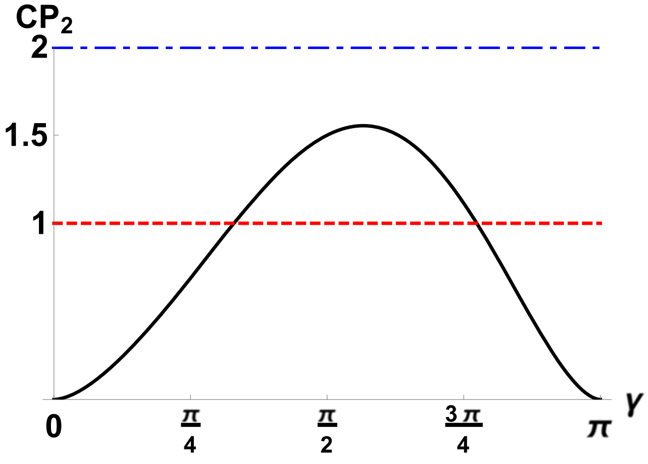


**Figure 3.** The dependence on $\gamma = \gamma_1 = \gamma_2$ of the control power $CP_2$. The horizontal dashed line at 1 marks the classical limit, while the dash-dotted one at 2 indicates the maximal value of $CP_2$.

$$CP_3 = -\sum_{i=0}^{3} |\alpha_i|^2 \log_2 |\alpha_i|^2, \quad (29)$$

where the sub-index 3 in $CP_3$ signals that the case with $N = 4$ and $M = 2$ is under consideration.

To satisfy the normalization condition $\sum_{i=0}^{3}|\alpha_i|^2 = 1$ we set

$$
\begin{aligned}
\alpha_0 &= \cos\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}, \quad & \alpha_1 &= \cos\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}, \\
\alpha_2 &= \sin\frac{\gamma_1}{2}\cos\frac{\gamma_3}{2}, \quad & \alpha_3 &= \sin\frac{\gamma_1}{2}\sin\frac{\gamma_3}{2}.
\end{aligned} \quad (30)
$$

By setting so the control power $CP_3$ becomes a function of three variables $\gamma_1$, $\gamma_2$ and $\gamma_3$. To make the dependence of $CP_3$ on its variables visual, in figure 4 we plot $CP_3$ versus $\gamma_2$ and $\gamma_3$ for several fixed values of $\gamma_1$. The figure shows that $CP_3$ reaches its maximal value equal to 2 when $\gamma_1 = \gamma_2 = \gamma_3 = \pi/2$, for which $|\alpha_0| = |\alpha_1| = |\alpha_2| = |\alpha_3| = 1/2$. In this case the entanglement with respect to the cut $AB|C_1C_2$ becomes maximal, rendering the quantum control to be perfect. The quantum channel corresponding to that case can be called optimal one. But, in general, within the whole parameters' range $CP_3$ obeys the following bounds

$$0 \leqslant CP_3 \leqslant 2. \quad (31)$$

From figure 4 it is also evident that there exist wide ranges of the parameters within which the quantum control is guaranteed. In particular, for $\gamma_1 = \pi/2$ the control power is always greater than 1 (but does not exceed 2, of course) for any possible values of $\gamma_2$ and $\gamma_3$, as seen from figure 4(b).

Of interest is a special case when $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ i.e., the single-parameter case. Figure 5 plots such a special case for $CP_3$ as a function of the single parameter $\gamma \in [0, \pi]$. For $0 \leqslant \gamma \leqslant \gamma_{\min} = 0.215\pi$ or $\gamma_{\max} = 0.785\pi \leqslant \gamma \leqslant \pi$ the quantum control is not guaranteed (i.e., the controller is powerless) since $CP_3 \leqslant 1$. However, the quantum control is

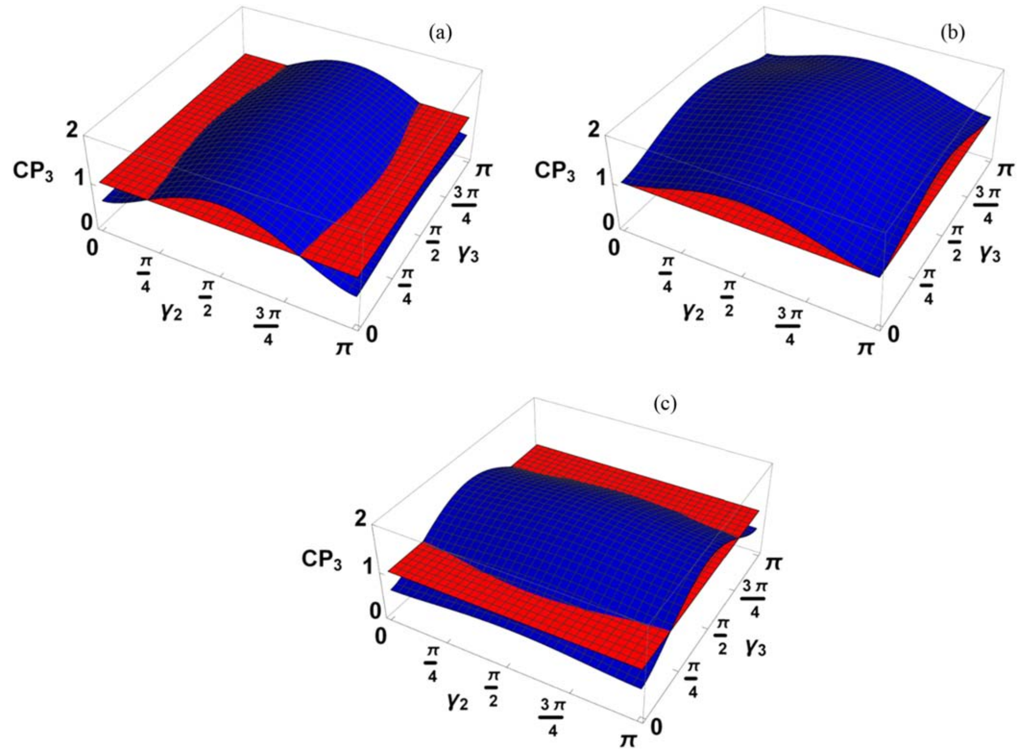


**Figure 4.** The dependence on $\gamma_2$ and $\gamma_3$ of the control power $CP_3$ for (a) $\gamma_1 = \pi/4$, (b) $\gamma_1 = \pi/2$ and (c) $\gamma_1 = 3\pi/4$. The horizontal plane at 1 marks the classical limit.

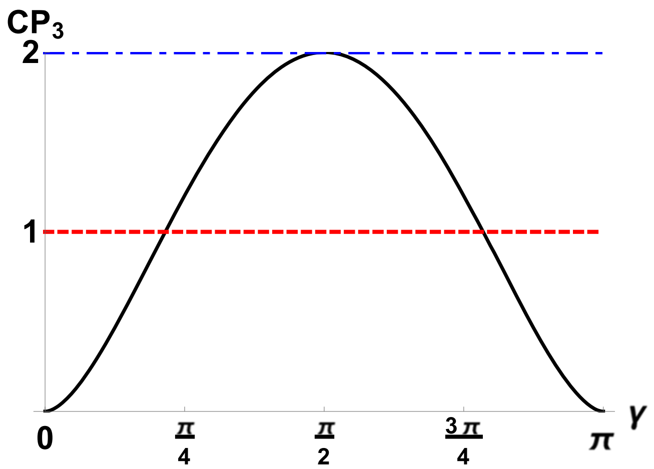


**Figure 5.** The dependence on $\gamma = \gamma_1 = \gamma_2 = \gamma_3$ of the control power $CP_3$. The horizontal dashed line at 1 marks the classical limit, while the dash-dotted one at 2 indicates the maximal value of $CP_3$.

guaranteed in general for $\gamma$ such that $\gamma_{\min} < \gamma < \gamma_{\max}$, and perfect for $\gamma = \pi/2$ in accordance with what said above.

## 4. Preparation of the optimal quantum channel

As learnt from the analysis in section 3, the optimal quantum channel for the perfect controlled super-dense coding protocol is the four-qubit maximally entangled state of the form

$$|Q_{opt}\rangle_{ABC_1C_2} = \frac{1}{2}(|B_0\rangle_{AB}|\Phi_0\rangle_{C_1C_2} + |B_1\rangle_{AB}|\Phi_1\rangle_{C_1C_2} + |B_2\rangle_{AB}|\Phi_2\rangle_{C_1C_2} + |B_3\rangle_{AB}|\Phi_3\rangle_{C_1C_2}), \quad (32)$$

with $\{|\Phi_j\rangle_{C_1C_2}; j = 0, 1, 2, 3\}$ any complete set of four two-qubit orthonormal states. In this section we propose ways to prepare such optimal quantum channel state. The preparation is assumed to be done in a central laboratory and, after the state has been prepared, its qubits are distributed to the authorized remote parties Alice, Bob and Charlie.

The preparation of the optimal quantum state (32) starts from the four-qubit cluster state [49] which was experimentally produced in [50, 51]. First, two separate pairs of two entangled photons are generated from type-II spontaneous parametric down conversion by pumping a nonlinear crystal in a double-pass configuration. Then, the two pairs of photon are entangled into a single four-qubit state by application of a controlled-PHASE gate on two photons, each taken from a pair of entangled photons. These result in the following cluster state

$$|Q_{cluster}\rangle_{1234} = \frac{1}{2}(|0000\rangle_{1234} + |0011\rangle_{1234} + |1100\rangle_{1234} - |1111\rangle_{1234}). \quad (33)$$

A good feature of the state (33) is that any two photons of it is maximally entangled with the remaining two photons. However, in order to utilize the cluster state (33) as the optimal quantum channel state for perfect controlled super-dense coding, a correct qubits' distribution is essential. A wrong distribution of qubits among Alice, Bob and Charlie leads to incapacity of the shared quantum channel for the super-dense coding task. We shall examine all the possible ways of qubits' distribution to see which one is valid and which one is not. It is easy to verify that there are only six ways to distribute the qubits: (i) $1, 2 \to AB$ & $3, 4 \to C$, (ii) $1, 3 \to AB$ & $2, 4 \to C$, (iii) $1, 4 \to AB$ & $2, 3 \to C$, (iv) $2, 3 \to AB$ & $1, 4 \to C$, (v) $2, 4 \to AB$ & $1, 3 \to C$ and (vi) $3, 4 \to AB$ & $1, 2 \to C$, where $AB$ implies Alice and Bob while $C$ stands for Charlie. In the case i)

$$|Q_{\text{cluster}}\rangle_{1234} \to \frac{1}{\sqrt{2}}(|00\rangle_{AB}|B_0\rangle_{C_1C_2} + |11\rangle_{AB}|B_2\rangle_{C_1C_2}), \quad (34)$$

which obviously cannot be used for super-dense coding (i.e., this way of distributing qubits is wrong). In the case (ii), thanks to the equality

$$|mn\rangle_{XY} = \frac{1}{\sqrt{2}}(|B_{m\oplus n}\rangle + (-1)^m|B_{m\oplus n+2}\rangle)_{XY}, \quad (35)$$

one has

$$|Q_{\text{cluster}}\rangle_{1234} \to \frac{1}{2}(|B_0\rangle_{AB}|B_2\rangle_{C_1C_2} + |B_1\rangle_{AB}|B_1\rangle_{C_1C_2} + |B_2\rangle_{AB}|B_0\rangle_{C_1C_2} + |B_3\rangle_{AB}|B_3\rangle_{C_1C_2}), \quad (36)$$

which can be used as an optimal quantum channel for super-dense coding, with $|\Phi_j\rangle_{C_1C_2} = |B_j\rangle_{C_1C_2}$. Similarly, the cluster state in the cases (iii)–(v) can also be used as an optimal quantum channel. Finally, in the case (vi)

$$|Q_{\text{cluster}}\rangle_{1234} \to \frac{1}{\sqrt{2}}(|B_0\rangle_{AB}|00\rangle_{C_1C_2} + |B_2\rangle_{AB}|11\rangle_{C_1C_2}), \quad (37)$$

which can be used for super-dense coding but the controller is powerless in the sense that without his/her cooperation the number of transmitted cbits in this case is equal to 1 (i.e., the quantum protocol is still as good as the classical one). In other words, this way of distributing qubits is wrong too.

## 5. Conclusion

In conclusion, we have studied the problem of quantum controlled super-dense coding from the controller's point of view. Namely, we are concerned with the situation in which the controller declines cooperation with the sender and the receiver, and we are interested in the role of the controller quantified by the so-called control power. The controller is powerless, powerful, and most powerful, if the control power is less than or equal to 1, greater than 1 but still less than 2, and equal to 2, respectively. The quantum channels we deal with are those that collapse into a Bell state after the controller's measurement. These are a sum of $N$ $(2 \leqslant N \leqslant 4)$ terms, each is a product of a Bell state and a controller's state. Our calculation and analysis of the control power show that for $N = 2$ the controller is always powerless, while for $N = 3$ the controller may be powerful but never most powerful. Only for $N = 4$ the controller may be most powerful in which case the corresponding quantum channel is regarded as optimal one. We further recognize that the well-known four-qubit cluster state, which is available in the laboratory, can serve as the optimal quantum channel for a perfect controlled super-dense coding protocol, provided that its qubits are correctly distributed among the controller, the sender and the receiver.

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