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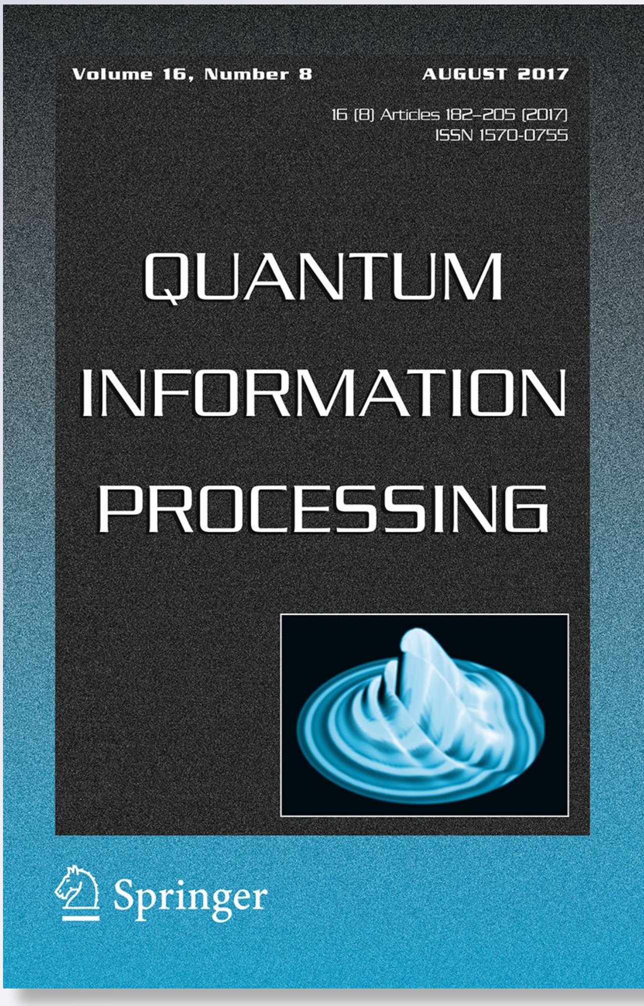

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Hierarchically controlling quantum teleportations

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Abstract
Controlling execution of certain quantum tasks in a quantum manner is necessary to ensure correctness of the final decision in case some doubts or/and wrong things might happen at the last-minute situation. As the importance of different tasks in practice may differ, controlling such tasks must be properly hierarchized: A lesser important task can be controlled in a looser way while a more important task must be put under a more severe control. Here, for concreteness, we deal with teleportations of three unknown quantum states containing information of different levels of importance. We design a scheme such that teleportation of the state of the lowest level of importance is allowed only by one agent, the general supervisor, but teleportation of the state of the middle level of importance is controlled by two agents, the general supervisor plus one more agent, and teleportation of the state of the highest level of importance requires permissions from three agents, the general supervisor plus two more agents. For that purpose, we construct quantum circuits to produce a genuine multipartite entangled state to be shared among all the authorized agents and show its relevance for the hierarchical control mentioned above.

Keywords Hierarchical control · Multiple quantum teleportations · Multiqubit entanglement

1 Introduction

Exploiting quantum entanglement as resource plus quantum logic gates and quantum measurements as tools, remote agents within a quantum network are able to perform various intriguing and useful global tasks without the need of gathering together in

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one place. What the agents have to do is just sitting at their own nodes and carrying out local operations/measurements combined with communicating with each other by classical means. This way of cooperation is very powerful and highly convenient in the mankind's future. Here the classical communication is used to broadcast the probabilistic local measurement outcomes which are meaningless to any unauthorized outsiders, so this does not cause any leakage of the content of intended information, despite classical communication can perfectly be eavesdropped.

Initially, only two-agent protocols such as quantum key distribution [1, 2], superdense coding [3], quantum teleportation [4], remote state preparation [5–8], quantum secure direct communication [9–11] and quantum dialog [12, 13] were of interest. Then, thanks to multipartite entangled states, more than two agents were involved into play. As an example, in the protocol proposed in Ref. [14] teleporting a quantum state from one agent to another is decided by cooperation of a third agent who is regarded as a controller. The controller, though does not need to know the detailed information contained in the quantum state under process, decides whether a given protocol is executed or not by cooperating with the protocol's implementers. The cooperation can be made at any time, but preferably at the last minute when the controller is sure that everything is alright. Otherwise, if there appears any kind of doubts, the controller changes his/her mind by ceasing the cooperation. This ensures correctness of the protocol execution. The protocol in Ref. [14] is thus referred to as controlled quantum teleportation in the literature. Another well-known example is the so-called quantum information splitting [15], in which one sender symmetrically distributes a secret quantum state to two receivers in such a way that neither of the receivers alone can retrieve the state, but any one of them can with assistance of the other. In this case, one of the receivers can be regarded as a controller for the other to get the quantum secrecy. The above controlled quantum teleportation and quantum information splitting protocols can be generalized to the multiparty setting (see, e.g., [16–19]) where there are many controllers and probably also many senders as in joint remote state preparation protocols [20–24]. The controllers in the above-cited references, though govern the protocols' completion, play an equal role, i.e., they have the same power or are of the same rank of significance.

As commonly demanded in practice, a specially important problem would be solved better by several committees of different ranks, from a lower rank one to a higher rank one. In this spirit, quantum information splitting, say, can be designed by distributing the secret state to two distant receivers in an asymmetrical manner so that the authorities of the receivers to get the state become biased [25], or even the quantum state is distributed among many receivers who are divided into a number of groups belonging to different grades with different degrees of difficulty to obtain the secret state [26, 27]. Such asymmetrical quantum information splitting is named hierarchical quantum-information splitting. Later, hierarchical quantum communication schemes are investigated with an observation that schemes of probabilistic hierarchical quantum information splitting and hierarchical quantum secret sharing can be obtained by modifying the scheme of hierarchical quantum information splitting [28]. An integrated hierarchical dynamic quantum secret sharing protocol is also considered and shown that it can be implemented using any existing protocol of quantum key distribution, quantum key agreement or secure direct quantum communication [29]. Recently, controlled remote state preparation was extended to include the aspect of hierarchy as well [30]. And, of late, researches on hierarchical joint remote state preparation in a deterministic manner [31] and in noisy environment [32] have been done as well. Generally speaking, the hierarchy's issue is of great necessity, say, in real situations when there are different problems that need most rationally be solved: The final solution should be made not by a single agent or by several agents with the same power but rather by multiple agents with hierarchical powers, depending on the importance level of the problem.

In this paper, we address multiple quantum teleportations under quantum control of different agents in a certain hierarchical fashion. The key point is construction of the quantum channel state to be shared among the legitimate agents and suitable for the required hierarchy. In Sect. 2, we set out the problem and specify the desired hierarchy. We then propose in Sect. 3 quantum circuits to produce the multiqubit entangled state that can be served as the right quantum channel. Section 4 explains how to exploit the multiqubit entangled state generated in Sect. 3 for hierarchically controlling the multiple teleportations. The final section, Sect. 5, is the conclusion.

2 The problem

Let a general supervisor (Sam) and six other people (Alice, Bob, Charlie, David, Eric and Felix) be different agents in remote nodes of a global quantum network. Alice, Bob and Charlie are, respectively, supplied three qubits x, y and z in the quantum states

$$|\psi\rangle_x = \big(\alpha_\psi|0\rangle + \beta_\psi|1\rangle\big)_x, \tag{1}$$

$$|\phi\rangle_y = \big(\alpha_\phi|0\rangle + \beta_\phi|1\rangle\big)_y \tag{2}$$

and

$$|\varphi\rangle_z = \big(\alpha_\varphi|0\rangle + \beta_\varphi|1\rangle\big)_z, \tag{3}$$

with unknown complex parameters α_ψ, β_ψ, α_ϕ, β_ϕ, α_φ, β_φ satisfying the standard normalization conditions $|\alpha_\psi|^2+|\beta_\psi|^2= |\alpha_\phi|^2+|\beta_\phi|^2= |\alpha_\varphi|^2+|\beta_\varphi|^2= 1$. The task the general supervisor Sam wants is that Alice teleports $|\psi\rangle_x$ to David, Bob teleports $|\phi\rangle_y$ to Eric and Charlie teleports $|\varphi\rangle_z$ to Felix. The point is that the information contained in state $|\psi\rangle_x$ is not very important, but the information contained in state $|\phi\rangle_y$ is important and the information contained in state $|\varphi\rangle_z$ is very important. Thus, Sam intentionally manages the process of the above multiple quantum teleportations in such a way that Alice–David's job is allowed only by himself, but Bob–Eric's job is under control of both Sam and Alice, while Charlie and Felix cannot proceed their job without permission of any of the three agents Sam, Alice and Bob (i.e., Charlie and Felix need the permission of all the three). The hierarchical control described above is sketched in Fig. 1. That is, there is a decentralization among the participating agents. Or, in other words, the authorities of the teleporters to perform their jobs are hierarchized. This kind of tasks management, in this context, can be named hierarchically controlling quantum teleportations. Here, Sam is the general controller, while Alice and Bob are at the same time teleporters and (sub-)controllers. If all the controllers cooperate, the three pairs of agents, Alice–David, Bob–Eric and Charlie–Felix, can easily follow the standard teleportation protocol [4], because each pair in this case is shared by a maximally entangled Bell (or Einstein–Podolsky–Rosen) state [33]. Hence, the hierarchical controlled problem set out above can be solved if it is possible to construct a suitable genuine multiqubit entangled state to be shared among all the agents. As can be imagined, to meet the hierarchical control specified above, Sam possesses 1 qubit, each of Alice and Bob needs holding 2 qubits, while only one qubit is required to be possessed by each of Charlie, David, Eric and Felix. Therefore, in total there are nine qubits in the entangled state to be served as the quantum channel. Furthermore, the quantum channel state should be such that, after Sam does his quantum control by measuring his qubit in a proper basis and reveals his measurement outcome, one of Alice's two qubits and the qubit of David must become factorized in a Bell state, but the other qubit of Alice, called Alice's control qubit, remains entangled with the qubits hold by Bob, Charlie, Eric and Felix. Likewise, after Alice measures her control qubit and reveals her measurement outcome, one of Bob's two qubits and the qubit of Eric should appear in a separate Bell state, but the other qubit of Bob, called Bob's control qubit, remains entangled with the qubits of Charlie and Felix. Only after hearing the outcome of Bob's measurement on his control qubit, Charlie and Felix will be aware of their sharing a Bell state. In the next section, we shall present two different schemes to produce such nine-qubit entangled state.

3 The quantum channel

At a quick thought, one might presume that the parties who are supposed to perform a (controlled) quantum teleportation protocol can generate (a GHZ trio [34]) a EPR pair [33] as their own quantum channel, but this awkwardly leads to independent (controlled) quantum teleportations. Here we are interested in a unified single protocol in which teleportation of states of different importance levels is put under controls by different numbers of agents in a hierarchical manner as described in Fig. 1. For that purpose, a proper nine-qubit entangled state should be generated to be used as the relevant quantum channel.

After careful consideration, we find out that the above-mentioned nine-qubit entangled state can be generated from the initial separable state $|\Phi\rangle_{123456789} = \otimes_{i=1}^{9}|0\rangle_i = |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9$ (i.e., all the nine qubits are in the "zero" states) in the following way. First, we flip the bit of qubit 5 by applying on it a NOT gate X and then apply two Hadamard gates H ($H|x\rangle = 2^{-1/2}\sum_{y=0}^{1}(-1)^{xy}|y\rangle; x = 0, 1$), one on qubit 1 and the other on qubit 5. Under such unitary operations, the state $|\Phi\rangle_{123456789}$ becomes $|\Omega\rangle_{123456789}$,

$$|\Omega\rangle_{123456789} = \frac{1}{2}(|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9$$
$$- |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|1\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9$$

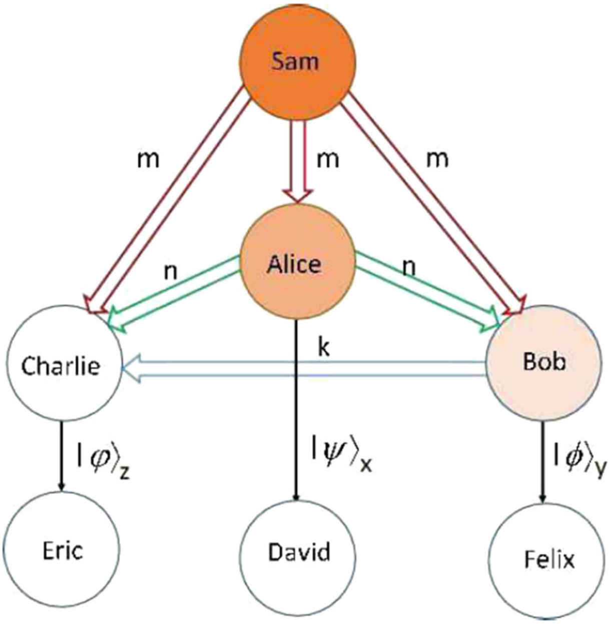

Fig. 1 Sketch of the control power. Sam has the highest power controlling the job of three pairs, Alice–David, Bob–Eric and Charlie–Felix; Alice has the middle power controlling the job of two pairs, Bob–Eric and Charlie–Felix; Bob has the lowest power controlling the job of just one pair, Charlie–Felix, while the remaining agents have no control power. The color brightness level indicates the agent's control power (no color indicates no control power). The double-line arrows represent classical communication signaling the measurement outcomes m, n and k of Sam, Alice and Bob, respectively. The single-line arrows show the direction of teleportation of states $|\psi\rangle_x$, $|\phi\rangle_y$ and $|\varphi\rangle_z$

$$+ \ |1\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9$$
$$- \ |1\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|1\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9). \quad (4)$$

Note that the above state (as well as other multiqubit states) can be formulated in a compact form like in Ref. [35] as

$$|\Omega\rangle_{123456789} = \frac{1}{2}(|\mathbf{0}\rangle - |\mathbf{16}\rangle + |\mathbf{256}\rangle - |\mathbf{272}\rangle)_{123456789}, \quad (5)$$

where the bold numbers are decimal numbers $\boldsymbol{N} = \sum_{n=0}^{8} x_n 2^n$, i.e., $|\boldsymbol{N}\rangle = |x_8\rangle_1|x_7\rangle_2|x_6\rangle_3|x_5\rangle_4|x_4\rangle_5|x_3\rangle_6|x_2\rangle_7|x_1\rangle_8|x_0\rangle_9$ with $x_n \in \{0, 1\}$. Next, applying $\otimes_{m=2}^{3}\mathrm{CNOT}_{1m} \otimes_{n=6}^{9} \mathrm{CNOT}_{5n}$ on $|\Omega\rangle_{123456789}$, where $\mathrm{CNOT}_{\mathrm{ab}}$ is a controlled-NOT (CNOT) gate acting on control qubit a and target qubit b in the following manner

$$\mathrm{CNOT}_{\mathrm{ab}}|x\rangle_{\mathrm{a}}|y\rangle_{\mathrm{b}} = |x\rangle_{\mathrm{a}}|x \oplus y\rangle_{\mathrm{b}}, \quad (6)$$

with $x,\ y \in \{0,\ 1\}$ and $\oplus$ addition mod 2, transforms $|\Omega\rangle_{123456789}$ to

$$|\Lambda\rangle_{123456789} = \frac{1}{2}(|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9 - |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|1\rangle_5|1\rangle_6|1\rangle_7|1\rangle_8|1\rangle_9 + |1\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9 - |1\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4|1\rangle_5|1\rangle_6|1\rangle_7|1\rangle_8|1\rangle_9) \quad (7)$$

or

$$|\Lambda\rangle_{123456789} = \frac{1}{2}(|\mathbf{0}\rangle - |\mathbf{31}\rangle + |\mathbf{448}\rangle - |\mathbf{479}\rangle)_{123456789}. \quad (8)$$

Finally, acting on $|\Lambda\rangle_{123456789}$ the operations $\otimes_{m=7}^{9}\mathrm{CNOT}_{4m} \otimes_{n=2}^{3} \mathrm{CNOT}_{4n} H_4$ yields

$$|\Lambda\rangle_{123456789} \rightarrow |\Psi\rangle_{123456789} = \frac{1}{2\sqrt{2}}(|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9 - |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|1\rangle_5|1\rangle_6|1\rangle_7|1\rangle_8|1\rangle_9 + |0\rangle_1|1\rangle_2|1\rangle_3|1\rangle_4|0\rangle_5|0\rangle_6|1\rangle_7|1\rangle_8|1\rangle_9 - |0\rangle_1|1\rangle_2|1\rangle_3|1\rangle_4|1\rangle_5|1\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9 + |1\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9 - |1\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4|1\rangle_5|1\rangle_6|1\rangle_7|1\rangle_8|1\rangle_9 + |1\rangle_1|0\rangle_2|0\rangle_3|1\rangle_4|0\rangle_5|1\rangle_6|1\rangle_7|1\rangle_8|1\rangle_9 - |1\rangle_1|0\rangle_2|0\rangle_3|1\rangle_4|1\rangle_5|1\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9) \quad (9)$$

or

$$|\Psi\rangle_{123456789} = \frac{1}{2\sqrt{2}}(|\mathbf{0}\rangle - |\mathbf{31}\rangle + |\mathbf{231}\rangle - |\mathbf{248}\rangle + |\mathbf{448}\rangle - |\mathbf{479}\rangle + |\mathbf{303}\rangle - |\mathbf{312}\rangle)_{123456789}, \quad (10)$$

which is the desired state to perform the hierarchical controlled quantum teleportations. The quantum circuit describing the evolution process from $|\Phi\rangle_{123456789}$ to $|\Psi\rangle_{123456789}$ is depicted in Fig. 2.

We shall elucidate the relevance of the generated state $|\Psi\rangle_{123456789}$ shortly. But, before doing that, let us evaluate the scheme for generating the quantum channel state $|\Psi\rangle_{123456789}$ in Eq. (9). By counting the number of CNOT gates (see Fig. 2), we observe that the scheme uses eleven CNOT gates in total, so it is expensive because even one CNOT gate already costs a lot. We therefore propose another more economical quantum circuit, which goes as schematically represented in Fig. 3. Namely, starting again from $|\Phi\rangle_{123456789}$, but applying $\mathrm{CNOT}_{23} H_2$ on qubits 2 and 3, $\mathrm{CNOT}_{56} H_5 X_5$ on qubits 5 and 6, and $\mathrm{CNOT}_{89} H_8$ on qubits 8 and 9, we obtain $|\Theta\rangle_{123456789}$ from $|\Phi\rangle_{123456789}$, i.e.,

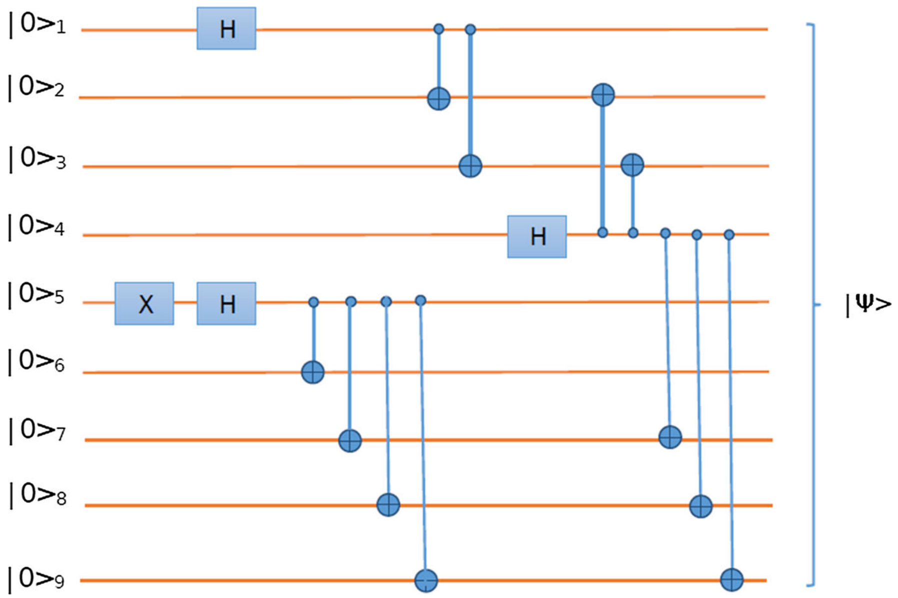

Fig. 2 A quantum circuit for generation of the nine-qubit state $|\Psi\rangle = |\Psi\rangle_{123456789}$ of Eq. (9). Qubits are represented by straight lines, and time goes from left to right. There are one NOT gate X, three Hadamard gates H and eleven CNOT gates

$$\begin{aligned}
&|\Phi\rangle_{123456789} \rightarrow |\Theta\rangle_{123456789}\\
&= \frac{1}{2\sqrt{2}}(|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9\\
&+ |0\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9\\
&+ |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|1\rangle_8|1\rangle_9\\
&- |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4|1\rangle_5|1\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9\\
&- |0\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4|1\rangle_5|1\rangle_6|0\rangle_7|0\rangle_8|0\rangle_9\\
&- |0\rangle_1|0\rangle_2|0\rangle_3|1\rangle_4|0\rangle_5|0\rangle_6|0\rangle_7|1\rangle_8|1\rangle_9\\
&- |0\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4|1\rangle_5|1\rangle_6|0\rangle_7|1\rangle_8|1\rangle_9)
\end{aligned} \tag{11}$$

or

$$\begin{aligned}
|\Theta\rangle_{123456789} = \frac{1}{2\sqrt{2}}&(|\mathbf{0}\rangle + |\mathbf{192}\rangle + |\mathbf{3}\rangle + |\mathbf{195}\rangle\\
&- |\mathbf{24}\rangle - |\mathbf{216}\rangle - |\mathbf{35}\rangle - |\mathbf{219}\rangle)_{123456789}.
\end{aligned} \tag{12}$$

Then, applying on $|\Theta\rangle_{123456789}$ six more CNOT gates, $CNOT_{64}CNOT_{94}CNOT_{97}$ $CNOT_{91}CNOT_{61}CNOT_{31}$, we arrive at the same state $|\Phi\rangle_{123456789}$ as in Eq. (9). Obviously, the latter quantum circuit in Fig. 3 is much cheaper than that in Fig. 2 since only nine (not eleven) CNOT gates are to be implemented. In the next section, we shall show how it works with such nine-qubit entangled state.

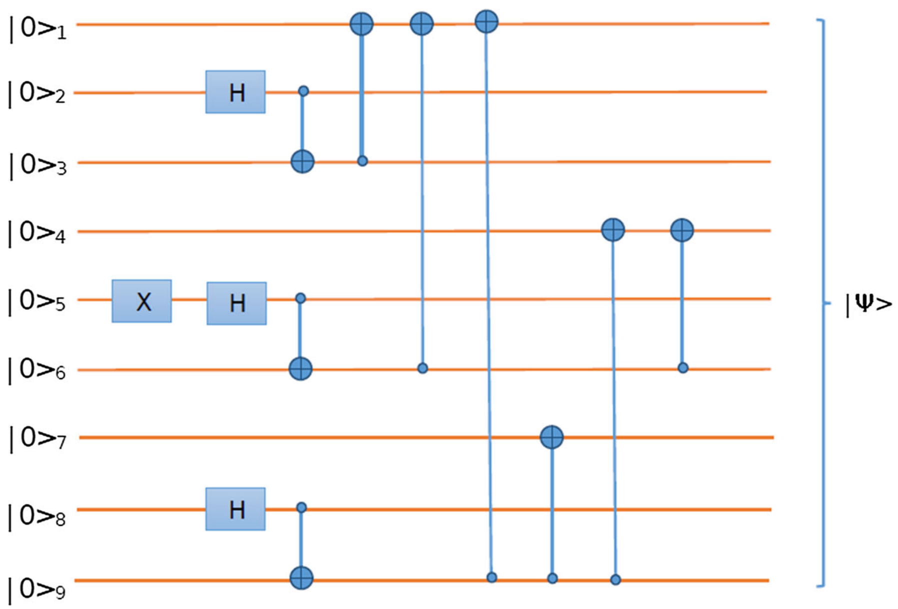

Fig. 3 Another quantum circuit for generation of the nine-qubit state $|\Psi\rangle_{123456789}$ of Eq. (9). Qubits are represented by straight lines and time goes from left to right. Compared to the quantum circuit in Fig. 2, there are the same numbers of X and H gates but only nine (not eleven) CNOT gates here

4 The hierarchical control

Before presenting the necessary steps of our protocol, it is worth noting that the nine-qubit entangled state $|\Psi\rangle_{123456789}$ in Eqs. (9) or (10) can be prepared by any of the participants or even by an outsider, following the quantum circuits in Figs. 2 or 3. Of more importance is how to distribute the qubits? Clearly, no matter who prepares the nine-qubit entangled state, after the preparation the qubits should be distributed correctly among the participants (see later). But, due to unavoidable influences of surrounding environments during the qubits distribution process, the resulting state shared among the participants becomes mixed one that does not guarantee perfect hierarchical control. So, many such nine-qubit entangled states should be prepared and distributed. Then, from a large number of the resulting mixed states, some proper procedure [36] needs to be carried out to distill a desired state (i.e., the pure state $|\Psi\rangle_{123456789}$). In fact, the whole task consists of two stages. In the first stage, after local preparation of a large enough number of the nine-qubit entangled states, these states are distributed through noisy environments followed by an appropriate distillation protocol to guarantee the correct sharing of a necessary pure nine-qubit entangled state. Only after that, the second stage starts. In our paper, we implicitly assume that the first stage has been successfully completed and deal only with the second stage.

To easily follow how the hierarchical control works, let us change the qubits' labeling as $1 \to S$ (Sam), $2 \to A$ (Alice), $3 \to D$ (David), $4 \to A'$ (Alice$'$), $5 \to B$ (Bob), $6 \to E$ (Eric), $7 \to B'$ (Bob$'$), $8 \to C$ (Charlie) and $9 \to F$ (Felix). This re-labeling visually signals that the qubits in the state $|\Psi\rangle_{123456789}$ of Eq. (9) must be distributed so that qubit S is held by Sam, qubits A, A$'$ go to Alice, qubits B, B$'$ to Bob, qubit C to Charlie, while qubits D, E and F belong to David, Eric and Felix, respectively.

Starting from the initial genuine entangled state (Fig. 4a), in order to supervise the tasks in the hierarchical manner specified in Sect. 2, Sam first turns on the green light for Alice–David's job by measuring his qubit S in the rotated basis $\{|\widetilde{m}\rangle_S = [|0\rangle_S + (-1)^m|1\rangle_S]/\sqrt{2}; m = 0, 1\}$. If he finds his qubit S in state $|\widetilde{m}\rangle_S$, then qubits A, D become entangled with each other in state $|B_m\rangle_{AD}$, but the remaining qubits are left in a new entanglement $|\Gamma_m\rangle_{A'BEB'CF}$ (see Fig. 4b), i.e.,

$$|\Psi\rangle_{SADA'BEB'CF} \to |\widetilde{m}\rangle_S|B_m\rangle_{AD}|\Gamma_m\rangle_{A'BEB'CF}, \tag{13}$$

where

$$|B_m\rangle_{AD} = \frac{1}{\sqrt{2}}\big(|0\rangle_A|0\rangle_D + (-1)^m|1\rangle_A|1\rangle_A\big) \tag{14}$$

being maximally entangled Bell states and

$$\begin{aligned}|\Gamma_m\rangle_{A'BEB'CF} = &\frac{1}{2}(|0\rangle_{A'}|0\rangle_B|0\rangle_E|0\rangle_{B'}|0\rangle_C|0\rangle_F \\ &- (-1)^m|1\rangle_{A'}|1\rangle_B|1\rangle_E|0\rangle_{B'}|0\rangle_C|0\rangle_F \\ &+ (-1)^m|1\rangle_{A'}|0\rangle_B|0\rangle_E|1\rangle_{B'}|1\rangle_C|1\rangle_F \\ &- |0\rangle_{A'}|1\rangle_B|1\rangle_E|1\rangle_{B'}|1\rangle_C|1\rangle_F) \\ = &\frac{1}{2}\big(|\mathbf{0}\rangle - (-1)^m|\mathbf{56}\rangle + (-1)^m|\mathbf{39}\rangle - |\mathbf{31}\rangle\big)_{A'BEB'CF}.\end{aligned} \tag{15}$$

This means that, though Sam only finds $|\widetilde{0}\rangle_S$ or $|\widetilde{1}\rangle_S$ with an equal probability (i.e., the exact value of m cannot be expected), for whatever measurement outcome $m \in \{0, 1\}$, if he uses one bit to publicly announce the value of m, then Alice and David know with certainty that they share the Bell state $|B_m\rangle_{AD}$. Hence, the teleportation of $|\psi\rangle_x$ from Alice to David is just up to their convenient time. However, the teleportations of $|\phi\rangle_y$ and $|\varphi\rangle_z$ remain impossible.

The teleportation of $|\phi\rangle_y$ is now further decided by Alice, who will measure her qubit A$'$ in the rotated basis $\{|\widetilde{n}\rangle_{A'}; n = 0, 1\}$, with the outcome n if she finds $|\widetilde{n}\rangle_{A'}$. Note again that, due to the form of $|\Gamma_m\rangle_{A'BEB'CF}$ in Eq. (15), both the probabilities of obtaining $n = 0$ and $n = 1$ are equal to 1/2 for any value of m. Also, it can be verified that (i) if $\{m = n = 0\}$ or $\{m = n = 1\}$, then Bob and Eric are sure to be in the Bell state $|B_1\rangle_{BE}$, while qubits B$'$, C and F remain entangled in the Greenberger–Horne–Zeilinger (GHZ) state $|G_0\rangle_{B'CF}$ [34] and (ii) if $\{m = 0$ and $n = 1\}$ or $\{m = 1$ and $n = 0\}$, then Bob and Eric are sure to be in the Bell state $|B_0\rangle_{BE}$, while qubits B$'$, C and F remain entangled in the GHZ state $|G_1\rangle_{B'CF}$, where

$$\big|G_j\big\rangle_{B'CF} = \frac{1}{\sqrt{2}}\Big[|0\rangle_{B'}|0\rangle_C|0\rangle_F + (-1)^j|0\rangle_{B'}|0\rangle_C|0\rangle_F\Big], \tag{16}$$

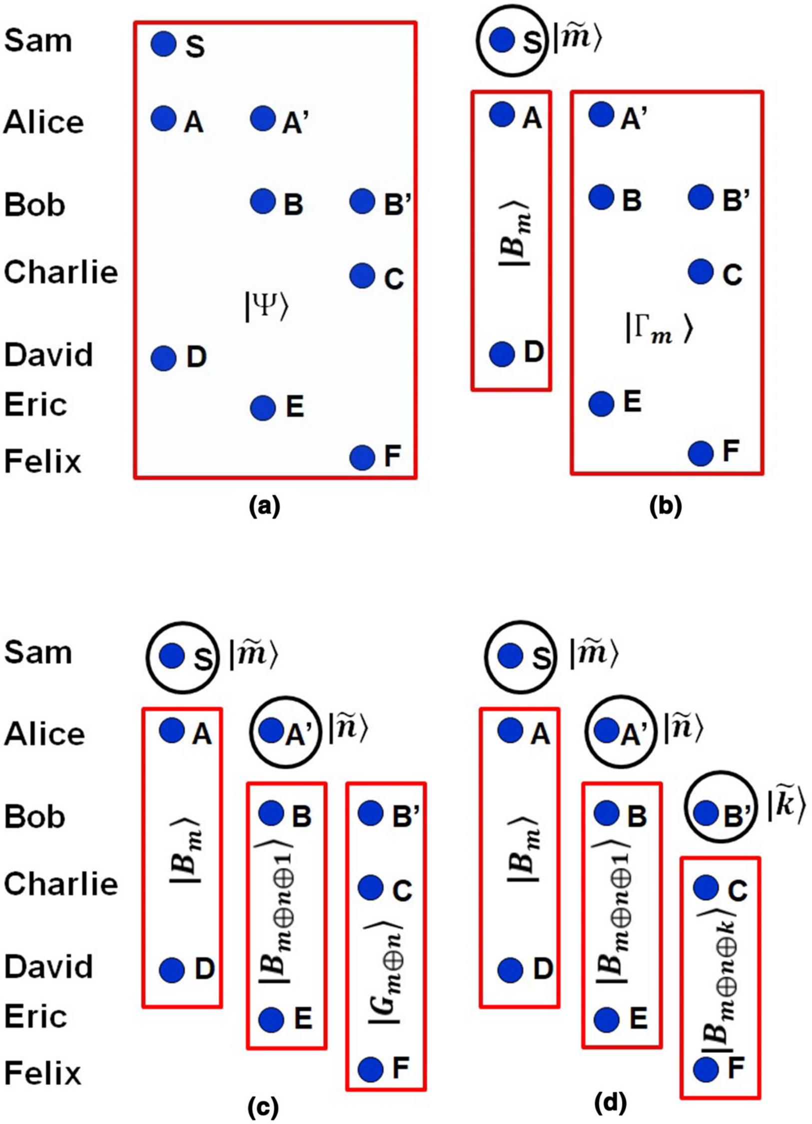

Fig. 4 The way of exploiting genuine multipartite entanglement to hierarchically control Alice–David, Bob–Eric and Charlie–Felix sharing a Bell state. A qubit is represented by a dot. Measured qubits are circled, and entangled qubits are surrounded by a rectangular. **a** The initial nine-qubit entangled state $|\Psi\rangle = |\Psi\rangle_{123456789}$ of Eq. (9) shared among all the seven agents. **b** Sam controls by measuring qubit S to find it in state $|\widetilde{m}\rangle_{\mathrm{S}}$ projecting Alice–David onto Bell state $|B_m\rangle = |B_m\rangle_{\mathrm{AD}}$ of Eq. (14), while the others remain entangled in state $|\Gamma_m\rangle = |\Gamma_m\rangle_{\mathrm{A'BEB'CF}}$ of Eq. (15). **c** Alice in turn controls by measuring qubit A′ to find it in state $|\widetilde{n}\rangle_{\mathrm{A'}}$ projecting Bob–Eric onto Bell state $|B_{m\oplus n\oplus 1}\rangle = |B_{m\oplus n\oplus 1}\rangle_{\mathrm{BE}}$, while Bob–Charlie–Felix are still entangled in state $|G_{m\oplus n}\rangle = |G_{m\oplus n}\rangle_{\mathrm{B'CF}}$ of Eq. (16). **d** The final control is done by Bob who measures qubit B′ to find it in state $|\widetilde{k}\rangle_{\mathrm{B'}}$ projecting Charlie–Felix onto Bell state $|B_{m\oplus n\oplus k}\rangle = |B_{m\oplus n\oplus k}\rangle_{\mathrm{CF}}$. Note that the knowledge of k makes no sense to the jobs of Alice and Bob, while without knowing k and n Alice can still do her job, but without the knowledge of m no jobs can be completed. It is what the hierarchical control means

with $j = 0, 1$ (see Fig. 4c). This implies a more stringent control for the teleportation of $|\phi\rangle_y$ because it depends on both Sam's and Alice's measurement outcomes m and n. In other words, Bob–Eric's job is under simultaneous control of two agents (Sam and Alice) compared with Alice–David's job which is governed only by Sam.

Finally, as the state $|\varphi\rangle_z$ is supposed to contain the most important information, its teleportation is designed to be controlled by Bob as well (see Fig. 4d). Actually, Bob will measure his qubit B′ in the rotated basis $\{|\widetilde{k}\rangle_{B'}; k = 0, 1\}$. With an equal probability, his measurement outcome may be $k = 0$ if he finds $|\widetilde{0}\rangle_{B'}$ or $k = 1$ if he finds $|1\rangle_{B'}$. By carefully considering all the situations that may happen, we observe the following: (i) if $\{m, n, k\} = \{0, 0, 0\}$ or $\{0, 1, 1\}$ or $\{1, 0, 1\}$ or $\{1, 1, 0\}$, then Charlie and Felix are certainly in the Bell state $|B_0\rangle_{CF}$ and (ii) if $\{m, n, k\} = \{0, 0, 1\}$ or $\{0, 1, 0\}$ or $\{1, 0, 0\}$ or $\{1, 1, 1\}$, then Charlie and Felix are certainly in the Bell state $|B_1\rangle_{CF}$. Transparently, the teleportation of $|\varphi\rangle_z$ is the most stringently controlled, requiring permission(s) not only from the general supervisor Sam but also from the two sub-supervisors Alice and Bob. Should either of the three refuses to cooperate, $|\varphi\rangle_z$ remained unteleportable.

Stringing all the above analyzed results together, we are in the position to work out a generic rule of how the teleporter becomes able to know in which Bell state he/she is with his/her partner, conditioned on the broadcasted measurement outcomes of the controller(s). The rule is illustrated in Fig. 4 and reads as follows. If Sam publicly discloses his measurement outcome m, then Alice and David are sure that they share the Bell state $|B_m\rangle_{AD}$:

$$m \to |B_m\rangle_{AD}. \tag{17}$$

If both Sam and Alice publicly reveal their measurement outcome m and n, then Bob and Eric in turn know with certainty of their sharing the Bell state $|B_{m\oplus n\oplus 1}\rangle_{BE}$ with $\oplus$ an addition mod 2, i.e.,

$$m, n \to |B_{m\oplus n\oplus 1}\rangle_{BE}. \tag{18}$$

Finally, if all the measurement outcomes m, n and k of Sam, Alice and Bob are broadcasted, then Charlie and Felix are certainly entangled in the Bell state $|B_{m\oplus n\oplus k}\rangle_{CF}$:

$$m, n, k \to |B_{m\oplus n\oplus k}\rangle_{CF}. \tag{19}$$

Of course, whenever the shared Bell state is known, teleportation is readily performable by the corresponding partners at their convenient time. Clearly from (17)–(19), to carry out her job, Alice needs to know neither n nor k. Bob's job is independent of k either. But Charlie has to know the values of all m, n and k. It is also clear that without knowing m no jobs can be completed. That is the merit of the hierarchical control under consideration in this paper. It is noteworthy that here the hierarchical control is non-trivially processed using a genuine entangled state $|\Psi\rangle_{123456789}$ in Eq. (9). This is totally different from the simple single-agent control in cyclic teleportations [37] where the solution turns out quite trivial because the state of the working quantum channel (see Eq. (4) in Ref. [37]) is nothing else but a product state of two Bell states and one GHZ state! Even three-agent controls can be realized via a product state which is a tensor product of three GHZ states, but this is just three independent controls, not hierarchical control.

5 Conclusion

To meet practical needs of controlling quantum tasks with unequal levels of importance, the control process must be designed in a demanded hierarchical fashion. The point of essence is existence of an appropriate multipartite entangled quantum channel connecting all the authorized agents, including both the controllers and the implementers (in many circumstances, an agent can act as a controller and an implementer at the same time). Depending on the task and the desired authority decentralization, the quantum state serving as the quantum channel must be constructed correctly. The task of concern in this paper is three teleportations between three pairs of agents. It is notoriously known that any pair of agents is able to execute a teleportation protocol [4] provided that the pair shares a known Bell state. From the relations (17)–(19), after the controller discloses his/her measurement outcome, the teleporter and his/her corresponding receiver know which Bell state they are entangled in. However, for certain reasons (say, in case of possible doubts), the controller might postpone the announcement of his/her outcome until the last minute to check whether everything is alright. In that case, though the corresponding partners do not know beforehand in which exact Bell state they are, teleportation steps can still be processed as usual except the last step of state's recovering. In other words, the teleporter should not care about the controller's measurement outcome. He/she keeps carrying out a Bell measurement jointly on the to-be-teleported qubit and his/her qubit and then publicly reveals the outcome of the Bell measurement. The corresponding receiver, however, needs the controller's measurement outcome to recover the desired state. For example, if Sam intentionally keeps his measurement outcome m unpublished for a while, Alice still makes a Bell measurement on qubits x and A with the outcome $p, q \in \{0, 1\}$ if she finds the Bell state $\left|\mathrm{B}_{pq}\right\rangle_{xA} = 2^{-1/2}\sum_{j=0}^{1}(-1)^{pj}|j\rangle_x|j \oplus q\rangle_{\mathrm{A}}$ (with such notations $|B_0\rangle = |\mathrm{B}_{00}\rangle$ and $|B_1\rangle = |\mathrm{B}_{10}\rangle$). Alice also publishes her outcome p, q. Yet, the knowledge of p, q is not sufficient for the state's recovering. In fact, David still needs Sam's measurement outcome m. Only if Sam finally announces m, David will use that m together with p, q to apply on his qubit D the recovering operators $Z^p X^q Z^m$ (Z is the phase-flip operator) to obtain the desired state $|\psi\rangle_{\mathrm{D}}$. Likewise, even without knowing the controlling measurement outcome n and k, Bob (Charlie) makes a Bell measurement on qubits y and B (z and C) and announces the outcome p', q' (p'', q''). At the final step, Eric (Felix) has to wait for Alice's (Bob's) measurement outcome n (k). Only if that outcome is broadcasted, can Eric (Felix) recover the desired state $|\phi\rangle_{\mathrm{E}}$ ($|\varphi\rangle_{\mathrm{F}}$) by acting $Z^{p'} X^{q'} Z^{m\oplus n\oplus 1}$ ($Z^{p''} X^{q''} Z^{m\oplus n\oplus k}$) on qubit E (F). In the main text, we do not explicitly address the issue of how the pairs Alice–David, Bob–Eric and Charlie–Felix perform their teleportation protocols, but instead we concentrate on the issue of how the authority of sharing a Bell state by a pair of agent can be hierarchically controlled in a desired manner. To suit the authority hierarchy in our task, we propose two schemes to produce a nine-qubit state entangling seven agents: one general controller (the supervisor Sam), three teleporters (Alice, Bob and Charlie) and three corresponding receivers (David, Eric and Felix). Here, Alice and Bob also play a role of (sub-)controllers. We show that, using our constructed nine-qubit entangled state as the quantum channel, Alice and David will share a Bell state if Sam permits it. However, sharing a Bell state between Bob and Eric is under control not only by Sam but also by Alice. Finally, only when all the three agents Sam, Alice and Bob are in consensus to cooperate, Charlie and Felix can share a Bell state. Note that the task of multiple teleportations and the way of hierarchical control set out here are just an example for concrete illustration. There are many other more complex tasks with more flexible ways of hierarchical control that could be studied with standard tool boxes and implemented with current technologies.

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