Nonstandard protocols for joint remote preparation of a general quantum state and hybrid entanglement of any dimension

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Joint remote state preparation (JRSP) is a useful way to securely transfer quantum information encoded in quantum states between distant places without physically sending the states themselves. In this paper we study JRSP of the most general D-dimensional quantum state called quDit state, with arbitrary integer D. We first show that, by standard procedures, i.e., by exploiting projective measurements on the Hilbert space of the systems of concern, this task can be completed for the dimensions 2, 4, or 8 only. We then propose a nonstandard protocol by means of a positive operator-valued measurement (POVM), which we suitably design so that our protocol works deterministically for any dimension. We also propose a nonstandard protocol for deterministic JRSP of arbitrary hybrid quDit-quNit entanglement with another POVM. Moreover, we construct quantum circuits to realize the two POVMs in the two nonstandard JRSP protocols mentioned above. Our results may be of interest not only from a theoretical point of view but also from an experimental one.

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I. INTRODUCTION

A salient power in the quantum world is the possibility of performing, by means of local operation and traditional (i.e., classical) communication, a number of global tasks that cannot be performed in the classical world. Most of those tasks, such as superdense coding [1], quantum-algorithm-based computation [2,3], quantum teleportation [4], quantum secret sharing [5], remote state preparation (RSP) [6,7], quantum dialogue [8–10], joint remote state preparation (JRSP) [11–18], and so on, are thanks to the so-called quantum entanglement [19,20] which has no counterparts at all in our everyday life. As a prerequisite requirement for accomplishment of any quantum global task, an appropriate quantum entanglement bridging all the parties authorized to participate in a given task has to be established prior to the task's execution. Thus generation of entangled states is of paramount importance. As for quantum entanglements, they may exist not only among systems of similar natures or of the same dimension but also among systems of distinct natures or of different dimensions. Since quantum information can be encoded and manipulated by various means based on hardware platforms that may be largely dissimilar in nature or in dimension, preparation of such hybrid entanglements is necessary enabling a flexible interfacing between different physical systems for the purpose of achieving the internet of things in the era of the industrial revolution 4.0. Many useful hybrid protocols relying on discrete-continuous-variable techniques have been realized in the laboratory [21]. Applications of micro-macro hybrid entangled states to perform quantum teleportation [22,23] and test Bell-Clauser-Horne-Shimony-Holt inequality [24] have been theoretically considered. Their mechanisms and realization have also attracted great attention from experimentators [25–27]. In particular, schemes to generate quantum entanglement of photons in their dual wave-particle nature have been devised theoretically and confirmed experimentally [28,29].

In this work, we are interested in hybrid quDit-quNit entanglements [30,31] with arbitrary dimensions D and N. In very recent publications [32,33] such a type of quantum entanglement has been investigated via the technique of JRSP, but only equatorial states of the form

$$|\psi\rangle_{12} = \frac{1}{\sqrt{DN}} \sum_{d=0}^{D-1} \sum_{n=0}^{N-1} e^{i\varphi_{dn}} |d,n\rangle_{12}, \tag{1}$$

with φ_{dn} real angles carrying the phase information and $|d,n\rangle_{12} \equiv |d\rangle_1 |n\rangle_2$, have been dealt with. A natural extension is to construct a JRSP protocol for the most general hybrid quDit-quNit entangled state of the form

$$|\Psi\rangle_{12} = \sum_{d=0}^{D-1} \sum_{n=0}^{N-1} a_{dn} e^{i\varphi_{dn}} |d,n\rangle_{12}, \tag{2}$$

where a_{dn} are real numbers carrying the amplitude information and satisfying the normalization condition $\sum_{d=0}^{D-1} \sum_{n=0}^{N-1} a_{dn}^2 = 1$. In comparison with $|\psi\rangle_{12}$ in Eq. (1), the state $|\Psi\rangle_{12}$ in Eq. (2) is the most general one because it is characterized by both the phase $\{\varphi_{dn}\}$ and the amplitude $\{a_{dn}\}$ information. Before designing JRSP protocol of $|\Psi\rangle_{12}$, let us first consider the case of JRSP of a general single-quDit state

$$|\Phi\rangle = \sum_{d=0}^{D-1} a_d e^{i\varphi_d} |d\rangle, \tag{3}$$

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with φ_d and a_d real numbers satisfying the conditions $0 \leqslant \varphi_d \leqslant 2\pi$ and $\sum_{d=0}^{D-1} a_d^2 = 1$. This seemingly simple task turns out to be interestingly nontrivial. In fact, it was rigorously proved in [34] that the standard procedure for RSP (and also for JRSP as will be shown shortly) of the state (3) with $a_d = 1/\sqrt{D}\ \forall d$ [i.e., $|\Phi\rangle = (1/\sqrt{D})\sum_{d=0}^{D-1} e^{i\varphi_d}|d\rangle$] applies to any dimension D, but that with $\varphi_d = 0\ \forall d$ (i.e., $|\Phi\rangle = \sum_{d=0}^{D-1} a_d|d\rangle$) can only be implemented for $D = 2$, 4, or 8. In the next section, Sec. II, we will design a JRSP protocol for the state (3) with arbitrarily possible $\{a_d\}$ and $\{\varphi_d\}$ by means of a positive operator-valued measurement (POVM) combined with projective measurement (PM), which works for any dimension D. Since POVMs are not used in a standard RSP-JRSP protocol, ours can be referred to as nonstandard protocols. Section III describes the actual procedure of how to realize the POVM introduced in Sec. II. Subsequently, Sec. IV presents a JRSP protocol for the most general hybrid quDit-quNit state (2) whose POVM implementation is described in Sec. V. The final section, Sec. VI, is the conclusion.

II. JRSP OF A GENERAL QUANTUM STATE OF ANY DIMENSION

The simplest JRSP protocol involves three distant parties: two preparers Alice and Bob plus one receiver Charlie. Charlie is able to securely and faithfully receive the state (3) only when Alice and Bob cooperate. A quantum state of dimension D is referred to as quDit, whose most general form is given by $|\Phi\rangle$ in Eq. (3). To jointly prepare $|\Phi\rangle$ the three parties must priorly share a quantum resource, say, in terms of a tripartite quantum state of the form

$$|Q\rangle_{ABC} = \frac{1}{\sqrt{D}}\sum_{d=0}^{D-1}|d,d,d\rangle_{ABC}, \tag{4}$$

which is a high-dimensional version of the well-known two-dimensional GHZ state [35]. Of the state $|Q\rangle_{ABC}$ quDit A belongs to Alice, B to Bob, and C to Charlie. In a standard JRSP protocol each party can only carry out a PM on his or her own quDit or/and apply unitary operators on it, without exploiting any extra Hilbert spaces besides those of A, B, and C. Therefore, a standard protocol begins with Alice who measures her quDit in an orthonormal basis $\{|\omega_p\rangle_A; p = 0, 1, \dots, D-1\}$ (i.e., the measurement is projective) so that when she obtains an outcome p the state of Bob's and Charlie's quDits is disentangled from $|Q\rangle_{ABC}$ and projected onto

$$|\Gamma_p\rangle_{BC} = U_{BC}^{(p)}\sum_{d=0}^{D-1} a_d|d,d\rangle_{BC}, \tag{5}$$

where operators $U_{BC}^{(p)} = U_B^{(p)} \otimes U_C^{(p)}$ should be unitary, dependent on the outcome p but independent of the amplitude information $\{a_d\}$ of the state to be prepared. Note at this point that we have assumed that Alice is able to find such a proper set of orthonormal states $|\omega_p\rangle_A$ with $p = 0, 1, \dots, D-1$. If so, when Alice publicly announces p the other two parties can make themselves share an entangled state of the form

$$|W\rangle_{BC} = \sum_{d=0}^{D-1} a_d|d,d\rangle_{BC}. \tag{6}$$

After that Bob performs an appropriate PM of his quDit to project Charlie's quDit onto a state that can be transformed into the desired state (3) by applying a correct recovering operator conditioned on Bob's measurement outcome. The first stage of the above JRSP strategy can be looked upon as RSP of the state $|W\rangle = \sum_{d=0}^{D-1} a_d|\widetilde{d}\rangle$ with $|\widetilde{d}\rangle \equiv |d,d\rangle$ from Alice to BoCha, where BoCha serves as one and the same company owned by both Bob and Charlie (i.e., of interest is the global two-quDit state $\sum_{d=0}^{D-1} a_d|d,d\rangle$ rather than a single-quDit one $\sum_{d=0}^{D-1} a_d|d\rangle$). As learnt from Ref. [34], if such an RSP is implementable (i.e., the operators $U_{BC}^{(p)}$ mentioned above exist) for a dimension D, then the sphere S^{D-1} is parallelizable. Since the sphere S^{D-1} is parallelizable only for $D = 2,\ 4$, or 8 (the case of $D = 1$ is trivial and will not be included in our consideration), the RSP of $|W\rangle$ from Alice to BoCha [which later leads to the JRSP of $|\Phi\rangle$ in Eq. (3) from Alice and Bob to Charlie] is implementable if and only if $D = 2,\ 4$, or 8 (see the main theorem in Ref. [34]).

Approaching the first stage of the standard JRSP strategy under another angle it can be verified that the operators $U_{BC}^{(p)}$ in Eq. (5) are guaranteed to exist if Alice's orthonormal basis states $\{|\omega_p\rangle_A; p = 0, 1, \dots, D-1\}$ are related to the computational basis ones $\{|p\rangle_A; p = 0, 1, \dots, D-1\}$ as

$$\begin{pmatrix} |\omega_0\rangle_A \\ |\omega_1\rangle_A \\ \vdots \\ |\omega_{D-1}\rangle_A \end{pmatrix} = \mathcal{O}^{(D)}\begin{pmatrix} |0\rangle_A \\ |1\rangle_A \\ \vdots \\ |D-1\rangle_A \end{pmatrix}, \tag{7}$$

where $\mathcal{O}^{(D)}$ is called a real orthogonal design of size D [36], which is a $D \times D$ unitary matrix with each of its entries being $\pm a_0, \pm a_1, \dots$ or $\pm a_{D-1}$. For $D = 2$, for example, one easily finds

$$\mathcal{O}^{(2)} = \begin{pmatrix} a_0 & a_1 \\ -a_1 & a_0 \end{pmatrix}. \tag{8}$$

Yet, already for the next higher dimension $D = 3$ no such $\mathcal{O}^{(3)}$ can be found. The success of RSP of $|W\rangle$ from Alice to BoCha (JRSP of $|\Phi\rangle$ from Alice and Bob to Charlie) is thus absolutely decided by the existence of $\mathcal{O}^{(D)}$. In fact, $\mathcal{O}^{(D)}$ exists if and only if $D = 2,\ 4$, or 8, that is the content of an important theorem which was rigorously proved by Hurwitz and Radon in Ref. [37].

Having been aware of the impossibility of the standard JRSP of the quDit state (3) with the dimension $D \neq 2$, 4, and 8, we now propose a nonstandard protocol by using POVM combined with PM that works for any dimension. The quantum state shared beforehand among the three parties is the same in Eq. (4). The measurement that Alice first performs on quDit A is now not a PM as in a standard protocol but a POVM of D^2 elements

$$E_A^{(k,l)} = M_A^{(k,l)\dagger} M_A^{(k,l)}, \tag{9}$$

with $k, l \in \{0, 1, \dots, D-1\}$ and

$$M_A^{(k,l)} = \frac{1}{D}\sum_{r,s=0}^{D-1} a_{r\oplus k}\exp\left[\frac{2\pi i}{D} l(r-s)\right]|s\rangle_A\langle r|, \tag{10}$$

with $\oplus$ denoting an addition modulo D, is the measurement (or detection) operator. It can be verified that this is indeed a D^2-element POVM, since $E_A^{(k,l)}$ are positive and satisfy the condition $\sum_{k,l=0}^{D-1} E_A^{(k,l)} = I_A$ with I_A the $D \times D$ unit matrix [38]. Different from the PM, the POVM's elements may be nonorthogonal as in our case $E_A^{(k,l)\dagger} E_A^{(k',l')} \neq \delta_{kk'}\delta_{ll'} E_A^{(k,l)\dagger} E_A^{(k,l)}$.

To prevent any of the preparers from knowing the full detail of $|\Phi\rangle$, its amplitude information $\{a_d\}$ is given to Alice, while its phase information $\{\varphi_d\}$ is given to Bob. Having $\{a_d\}$ with herself Alice is able to perform on her quDit A the above specified POVM (9). With a probability of $1/D^2$ she obtains an outcome k, l that partially disentangles $|Q\rangle_{ABC}$ as

$$|Q\rangle_{ABC} \to \frac{M_A^{(k,l)}|Q\rangle_{ABC}}{\sqrt{{}_{ABC}\langle Q|M_A^{(k,l)\dagger} M_A^{(k,l)}|Q\rangle_{ABC}}} = |\Omega_l\rangle_A |\Pi_{kl}\rangle_{BC}, \tag{11}$$

where

$$|\Omega_l\rangle_A = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} \exp\left(-\frac{2\pi i}{D} lj\right) |j\rangle_A \tag{12}$$

and

$$|\Pi_{kl}\rangle_{BC} = \sum_{d=0}^{D-1} a_{d\oplus k} \exp\left(\frac{2\pi i}{D} ld\right) |d, d\rangle_{BC}. \tag{13}$$

To make the present JRSP successful Alice must broadcast via any public media her outcome k, l.

Next, making use of the broadcasted outcome k, l Bob applies on his quDit B an operator which is defined in dependence on k, l as

$$V_B^{(k,l)} = \sum_{j=0}^{D-1} \exp\left(-\frac{2\pi i}{D} lj\right) |j \oplus k\rangle_B \langle j| \tag{14}$$

and then projectively measures B in the $\{\varphi_d\}$-dependent orthonormal basis $\{|\Lambda_m\rangle_B; m = 0, 1, \ldots, D-1\}$ with

$$|\Lambda_m\rangle_B = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} \exp\left(-\frac{2\pi i}{D} mj - i\varphi_j\right) |j\rangle_B. \tag{15}$$

Bob is capable of managing the basis $\{|\Lambda_m\rangle_B\}$ because the phase information $\{\varphi_d\}$ is in his hand. As a consequence of Bob's action, if the outcome m occurs, with a probability of $1/D$, the entangled state $|\Pi_{kl}\rangle_{BC}$ in Eq. (13) becomes separable as

$$|\Pi_{kl}\rangle_{BC} \to |\Lambda_m\rangle_B |\Phi_{km}\rangle_C, \tag{16}$$

where

$$|\Phi_{km}\rangle_C = \sum_{d=0}^{D-1} a_d \exp\left(\frac{2\pi i}{D} md + i\varphi_d\right) |d \ominus k\rangle_C, \tag{17}$$

with $\ominus$ a subtraction modulo D. To allow Charlie to get the desired state $|\Phi\rangle$ Bob must also publicly broadcast his measurement outcome m. Note that the operator $V_B^{(k,l)}$ in Eq. (14) is nothing else but $X_B^k Z_B^{l\dagger}$ because $X_B = \sum_{j=0}^{D-1} |j \oplus 1\rangle_B \langle j|$ and $Z_B = \sum_{j=0}^{D-1} \exp\left(\frac{2\pi i}{D} j\right) |j\rangle_B \langle j|$. So $V_B^{(k,l)}$ is readily available to Bob with the knowledge of k and l.

Obviously, to complete the JRSP protocol, Charlie, after hearing about k from a previous announcement by Alice and

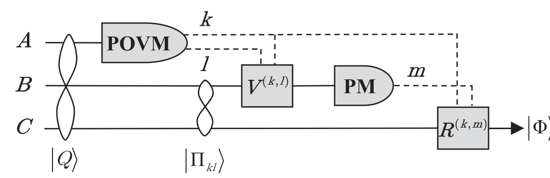

FIG. 1. Quantum circuit for JRSP of the general state $|\Phi\rangle$ of dimension D in Eq. (3). A solid line represents a quDit, while a dashed line represents a classical dit. A, B, and C are quDits of the quantum channel $|Q\rangle \equiv |Q\rangle_{ABC}$ of Eq. (4). $|\Pi_{kl}\rangle \equiv |\Pi_{kl}\rangle_{BC}$ is given by Eq. (13), $V^{(k,l)}$ by Eq. (14), and $R^{(k,m)}$ by Eq. (18). POVM is the positive operator-valued measurement defined by Eq. (9), while PM is the projective measurement in the basis $\{|\Lambda_m\rangle_B\}$ defined by Eq. (15). The wavy line embraces quDits that are entangled with each other.

m from a recent broadcasting by Bob, applies on the state $|\Phi_{km}\rangle_C$ of Eq. (17) the operator

$$R_C^{(k,m)} = \sum_{j=0}^{D-1} \exp\left(-\frac{2\pi i}{D} m(j+k)\right) |j \oplus k\rangle_C \langle j| \tag{18}$$

to transform the state $|\Phi_{km}\rangle_C$ of Eq. (17) to the target state $|\Phi\rangle_C$ of Eq. (3). The operator $R_C^{(k,m)}$ defined in Eq. (18) is also manageable by Charlie, since it is nothing else but $Z_C^{m\dagger} X_C^k$. The quantum circuit for our JRSP of the general single-quDit state $|\Phi\rangle$ is sketched in Fig. 1.

III. REALIZATION OF THE POVM (9)

The nonstandard JRSP protocol described in the preceding section is valid for any dimension D, as opposed to the standard one which is applicable for $D = 2$, 4, or 8 only. The key component in our nonstandard JRSP protocol is the POVM (9). In theory it seems very transparent and simple. But, what matters in practice is how to actually execute the POVM. In a recent publication [39] another formally different POVM is employed for a controlled JRSP protocol. Its realization has, however, remained unelucidated. Our purpose in this section is thus to provide a quantum circuit to implement our POVM defined in Eqs. (9) and (10).

The general idea comes up from Neumark's theorem [40] whose spirit is the following. To implement a POVM on a system of interest, the Hilbert space of it must first be enlarged to embrace extra dimensions of an auxiliary system called ancilla. Then the system and the ancilla are forced to unitarily evolve so that their degrees of freedom, which are separable initially, become entangled with each other. Afterwards, depending on the concrete type of the POVM, PM(s) will be carried out in the combined system-ancilla Hilbert space or in the Hilbert space of the ancilla or/and the system. These procedures will result in realization of the POVM in the system of concern (see, e.g., Ref. [41] for more details).

In our case the extra dimension is supplied by an ancilla a which is of the same dimension D as that of quDit A.

Moreover, the ancilla state should have the form

$$|\chi\rangle_a = \sum_{j=0}^{D-1} a_j |j\rangle_a, \tag{19}$$

with $\{a_j\}$ coincident with those in the state $|\Phi\rangle$ of Eq. (3). Since the POVM (9) is to be carried out on Alice's quDit A, she has to create $|\chi\rangle_a$ locally by herself. Of course, she is in the position to do that because she knows the values of $\{a_j\}$. There are several methods to create the state $|\chi\rangle_a$ (see, e.g., Refs. [42,43]). Nevertheless, most of the existing methods suffer from certain limitations, requiring even further additional dimensions plus entangling transformations and/or recursive postselection measurements; hence the whole process is costly and not deterministic (may be asymptotically deterministic but consumes a lot of quantum resource [42]). Here we propose a deterministic and economical method that only needs an initial quDit a in the simplest state $|0\rangle_a$ on which a proper unitary operator $\mathcal{U}_a$ will act. To imprint the wanted amplitude information into $|0\rangle_a$, the operator $\mathcal{U}_a$ should depend on $\{a_j\}$. The first column of $\mathcal{U}_a$ is the normalized vector $(a_0, a_1, a_2, \ldots, a_{D-1})^T$ and the remaining column vectors could be constructed to be mutually orthonormal vectors using the well-known method such as Gram-Schmidt process so that $\mathcal{U}_a$ becomes a unitary operator. Here we have designed an explicit form of such an $\{a_j\}$-dependent unitary operator as

$$\mathcal{U}_a = \begin{pmatrix} a_0 & \frac{a_1}{S_2} & \frac{a_0 a_2}{S_2 S_3} & \ldots & \frac{a_0 a_{D-3}}{S_{D-3} S_{D-2}} & \frac{a_0 a_{D-2}}{S_{D-2} S_{D-1}} & \frac{a_0 a_{D-1}}{S_{D-1}} \\ a_1 & -\frac{a_0}{S_2} & \frac{a_1 a_2}{S_2 S_3} & \ldots & \frac{a_1 a_{D-3}}{S_{D-3} S_{D-2}} & \frac{a_1 a_{D-2}}{S_{D-2} S_{D-1}} & \frac{a_1 a_{D-1}}{S_{D-1}} \\ a_2 & 0 & -\frac{S_2}{S_3} & \ldots & \frac{a_2 a_{D-3}}{S_{D-3} S_{D-2}} & \frac{a_2 a_{D-2}}{S_{D-2} S_{D-1}} & \frac{a_2 a_{D-1}}{S_{D-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{D-3} & 0 & 0 & \ldots & -\frac{S_{D-3}}{S_{D-2}} & \frac{a_{D-3} a_{D-2}}{S_{D-2} S_{D-1}} & \frac{a_{D-3} a_{D-1}}{S_{D-1}} \\ a_{D-2} & 0 & 0 & \ldots & 0 & -\frac{S_{D-2}}{S_{D-1}} & \frac{a_{D-2} a_{D-1}}{S_{D-1}} \\ a_{D-1} & 0 & 0 & \ldots & 0 & 0 & -S_{D-1} \end{pmatrix}, \tag{20}$$

with

$$S_q = \sqrt{\sum_{j=0}^{q-1} a_j^2} \tag{21}$$

for $q = 2, 3, \ldots, D-1$. Transparently, all the columns of $\mathcal{U}_a$ depend explicitly on $\{a_j\}$ and a direct check gives

$$\mathcal{U}_a |0\rangle_a = |\chi\rangle_a, \tag{22}$$

as we need. In principle the unitary operator $\mathcal{U}_a$ that we explicitly derived in Eq. (20) can be implemented in the laboratory. Most transparent is the case in the optical domain where the quDit is employed in terms of multirail coding and the desired operator $\mathcal{U}_a$ can be realized by means of linear-optics devices such as beam splitters, phase shifters, and mirrors [44]. Yet, what Alice still needs is the so-called D-dimensional controlled-backward-shift (CBS) gate that acts on control quDit c and target quDit t in the following manner:

$$G_{ct}^{CBS} |r, s\rangle_{ct} = |r, s \ominus r\rangle_{ct}. \tag{23}$$

The inverse of that CBS gate is obviously the D-dimensional controlled-forward-shift (CFS) one whose action on control quDit c and target quDit t reads

$$G_{ct}^{CFS} |r, s\rangle_{ct} = |r, s \oplus r\rangle_{ct}. \tag{24}$$

Now, having the state $|\chi\rangle_a$ at hand, Alice applies a CBS gate on quDit A (served as the control one) of the quantum state $|Q\rangle_{ABC}$ and quDit a (served as the target one), making all the quDits A, B, C, and a entangled in a single four-quDit state:

$$\begin{aligned} |\Theta\rangle_{ABCa} &= G_{Aa}^{CBS} |Q\rangle_{ABC} |\chi\rangle_a \\ &= \frac{1}{\sqrt{D}} \sum_{d,j=0}^{D-1} a_j G_{Aa}^{CBS} |d, j\rangle_{Aa} |d, d\rangle_{BC} \\ &= \frac{1}{\sqrt{D}} \sum_{d,j=0}^{D-1} a_j |d, j \ominus d\rangle_{Aa} |d, d\rangle_{BC} \\ &= \frac{1}{\sqrt{D}} \sum_{d,k=0}^{D-1} a_{d \oplus k} |k, d, d, d\rangle_{aABC}. \end{aligned} \tag{25}$$

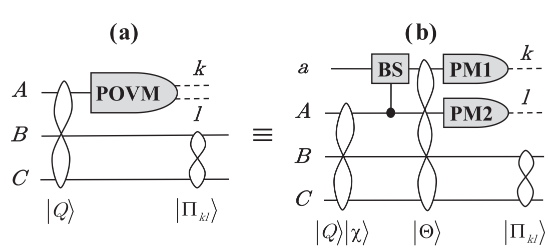

FIG. 2. Quantum circuit for the POVM defined by Eq. (9): (a) the formal performance and (b) the actual realization. A solid line represents a quDit, while a dashed line represents a classical dit. A, B, and C are quDits of the quantum channel $|Q\rangle \equiv |Q\rangle_{ABC}$ of Eq. (4), while a is an ancillary quDit in state $|\chi\rangle \equiv |\chi\rangle_a$ of Eq. (19). $|\Theta\rangle \equiv |\Theta\rangle_{ABCa}$ and $|\Pi_{kl}\rangle \equiv |\Pi_{kl}\rangle_{BC}$ are given by Eqs. (25) and (13), respectively. BS is short for backward-shift gate, while PM1 and PM2 are projective measurements in the bases $\{|k\rangle_a\}$ and $\{|\Omega_l\rangle_A\}$ of Eq. (12). The wavy line embraces quDits that are entangled with each other.

Next, Alice measures quDit a in its computational basis $\{|k\rangle_a; k = 0, 1, \ldots, D-1\}$ and quDit A in the basis $\{|\Omega_l\rangle_A; l = 0, 1, \ldots, D-1\}$, which was defined in Eq. (12). The outcome may be one of the D^2 possibilities. If an outcome k, l happens, with a probability of $1/D^2$, the two unmeasured quDits B and C turn out to be disentangled from $|\Theta\rangle_{ABCa}$ but remain entangled with each other, notably, in a state identical to $|\Pi_{kl}\rangle_{BC}$ defined by Eq. (13) in Sec. II, which is precisely the product brought about by the POVM (9) on quDit A. The amount of additional dimension in our circuit of realization of the POVM (9) is calculated as subtraction between dimension of $\mathcal{H}_A \otimes \mathcal{H}_a$ and that of $\mathcal{H}_A$ ($\mathcal{H}_X$ denotes the Hilbert space of system X), which is equal to $D(D-1)$. This is the lower bound for our scheme which makes use of the tensor product extension, i.e., the working Hilbert space is the tensor product of the Hilbert space of quDit A and the Hilbert space of quDit a [41]. The quantum circuit for realization of our POVM defined by Eq. (9) is sketched in Fig. 2.

IV. JRSP OF AN ARBITRARY HYBRID QUDIT-QUNIT ENTANGLEMENT

As secure exchange of quantum information between physical systems of different dimensions may be of benefit in future applications, entangling such systems turns out necessary. In this section, we are concerned with the problem of how to jointly prepare at a remote location an entangled state between a system of dimension D and another system of dimension N, with arbitrary D and N. This kind of entanglement can be named hybrid quDit-quNit that lives in a Hilbert space, which is the tensor product of the Hilbert spaces of the two different-in-dimension systems. A special case of it was dealt with in Refs. [32,33] and here we address its most general form as given in Eq. (2), $|\Psi\rangle_{12} = \sum_{d=0}^{D-1}\sum_{n=0}^{N-1} a_{dn} e^{i\varphi_{dn}}|d,n\rangle_{12}$, of which, to remind the reader, system labeled 1 has dimension D while system labeled 2 has dimension N. It is worthy to note that, if $a_{dn} = b_d c_n$ and $\varphi_{dn} = \mu_d + \nu_n$, then state $|\Psi\rangle_{12}$ factors out as $\sum_{d=0}^{D-1} b_d e^{i\mu_d}|d\rangle_1 \otimes \sum_{n=0}^{N-1} c_n e^{i\nu_n}|n\rangle_2$ (i.e., $|quDit\rangle_1 \otimes |quNit\rangle_2$) and the problem is just a simple consequence of the protocol proposed in Sec. II [i.e., two independent protocols: one deals with quDit state $|quDit\rangle_1 = \sum_{d=0}^{D-1} b_d e^{i\mu_d}|d\rangle_1$ via the shared quantum state $|Q_D\rangle_{A_1B_1C_1}$ of Eq. (26) and the other deals with quNit state $|quNit\rangle_2 = \sum_{n=0}^{N-1} c_n e^{i\nu_n}|n\rangle_2$ via the state $|Q_N\rangle_{A_2B_2C_2}$ of Eq. (26)]. Here, we are interested in the case with $a_{dn} \neq b_d c_n$ and $\varphi_{dn} \neq \mu_d + \nu_n$ in which $|\Psi\rangle_{12}$ is the genuine hybrid quDit-quNit entangled state and the protocol for JRSP of $|\Psi\rangle_{12}$ can by no means be split into two independent ones. As mentioned before, to keep the secrecy of $|\Psi\rangle_{12}$, Alice is allowed to know only the amplitudes $\{a_{dn}\}$, while Bob only the phases $\{\varphi_{dn}\}$. Furthermore, in order that Alice and Bob can jointly prepare $|\Psi\rangle_{12}$ for a remote Charlie, the three parties have to be quantumly connected via, say, a pair of high-dimensional GHZ states $|Q_D\rangle_{A_1B_1C_1}|Q_N\rangle_{A_2B_2C_2}$, one of dimension D and the other of dimension N. The shared quantum state can be represented in the following way:

$$\begin{aligned}|\overline{Q}\rangle_{A_1B_1C_1A_2B_2C_2} &\equiv |Q_D\rangle_{A_1B_1C_1}|Q_N\rangle_{A_2B_2C_2}\\ &= \frac{1}{\sqrt{DN}}\sum_{d=0}^{D-1}\sum_{n=0}^{N-1}|d,n\rangle_{A_1A_2}|d,n\rangle_{B_1B_2}|d,n\rangle_{C_1C_2}, \quad (26)\end{aligned}$$

with quDit A_1 and quNit A_2 (quDit B_1 and quNit B_2, quDit C_1 and quNit C_2) possessed by Alice (Bob, Charlie).

Our protocol is intended to work for arbitrary D and N, so we need to devise a suitable POVM for Alice to measure her quDit A_1 and quNit A_2. Let us devise our POVM like this:

$$E_{A_1A_2}^{(k_1,k_2,l_1,l_2)} = M_{A_1A_2}^{(k_1,k_2,l_1,l_2)\dagger} M_{A_1A_2}^{(k_1,k_2,l_1,l_2)}, \quad (27)$$

with $k_1, l_1 \in \{0, 1, \ldots, D-1\}$, $k_2, l_2 \in \{0, 1, \ldots, N-1\}$, and

$$\begin{aligned}M_{A_1A_2}^{(k_1,k_2,l_1,l_2)} &= \frac{1}{DN}\sum_{r_1,s_1=0}^{D-1}\sum_{r_2,s_2=0}^{N-1} a_{r_1\oplus k_1 r_2+k_2}\exp\\ &\quad\times\left[\frac{2\pi i}{D}l_1(r_1-s_1)+\frac{2\pi i}{N}l_2(r_2-s_2)\right]|s_1\rangle_{A_1}\langle r_1|\\ &\quad\otimes|s_2\rangle_{A_2}\langle r_2|, \quad (28)\end{aligned}$$

where $+$ in the subindex of the coefficients $a_{j_1\oplus k_1 j_2+k_2}$ is understood as an addition modulo N. We can easily verify that the operators $E_{A_1A_2}^{(k_1,k_2,l_1,l_2)}$ in Eq. (27) constitute a relevant POVM with $(DN)^2$ elements because they are positive and sum up to a unit matrix [38]. Like the POVM in Eq. (9), the elements of the POVM in Eq. (27) are not mutually orthogonal either, i.e., $E_{A_1A_2}^{(k_1,k_2,l_1,l_2)\dagger}E_{A_1A_2}^{(k_1',k_2',l_1',l_2')} \neq \delta_{k_1k_1'}\delta_{k_2k_2'}\delta_{l_1l_1'}\delta_{l_2l_2'}E_{A_1A_2}^{(k_1,k_2,l_1,l_2)\dagger}E_{A_1A_2}^{(k_1,k_2,l_1,l_2)}$. Three following steps are to be proceeded to complete our JRSP of the hybrid entangled state (2).

Step 1. Alice, by utilizing the POVM (27), measures A_1, A_2 of the shared quantum state $|\overline{Q}\rangle_{A_1B_1C_1A_2B_2C_2}$. With a probability of $1/(DN)^2$ she obtains an outcome k_1, k_2, l_1, l_2, projecting B_1, B_2, C_1, C_2 onto a hybrid entangled state of the form

$$\begin{aligned}&|\Pi_{k_1k_2l_1l_2}\rangle_{B_1B_2C_1C_2}\\ &\quad= \sum_{d=0}^{D-1}\sum_{n=0}^{N-1} a_{d\oplus k_1 n+k_2}\exp\left[\frac{2\pi i}{D}l_1 d+\frac{2\pi i}{N}l_2 n\right]\\ &\quad\times|d,n\rangle_{B_1B_2}|d,n\rangle_{C_1C_2}. \quad (29)\end{aligned}$$

After the POVM measurement Alice publicly publishes her outcome so that both Bob and Charlie are able to know and use the values of k_1, k_2, l_1, l_2 in case of need.

Step 2. Conditioned on Alice's measurement outcome, Bob first applies $V_{B_1}^{(k_1,l_1)} = X_{B_1}^{k_1}Z_{B_1}^{l_1\dagger}$ on quDit B_1 and $V_{B_2}^{(k_2,l_2)} = X_{B_2}^{k_2}Z_{B_2}^{l_2\dagger}$ on quNit B_2 with the X, Z gates being D dimensional for B_1 and N dimensional for B_2. Then, Bob performs PM on B_1B_2 in the orthonormal basis

$$\begin{aligned}&|\Lambda_{m_1m_2}\rangle_{B_1B_2}\\ &\quad= \frac{1}{\sqrt{DN}}\sum_{j_1=0}^{D-1}\sum_{j_2=0}^{N-1}\exp\left(-\frac{2\pi i}{D}m_1j_1-\frac{2\pi i}{N}m_2j_2-i\varphi_{j_1j_2}\right)\\ &\quad\times|j_1,j_2\rangle_{B_1B_2}, \quad (30)\end{aligned}$$

with $m_1 \in \{0, 1, \ldots, D-1\}$ and $m_2 \in \{0, 1, \ldots, N-1\}$. If an outcome m_1, m_2 happens after the measurement of B_1 and B_2, with a probability of $1/(DN)$, the two remaining quDit C_1 and quNit C_2 are disentangled from $|\Pi_{k_1k_2l_1l_2}\rangle_{B_1B_2C_1C_2}$ with

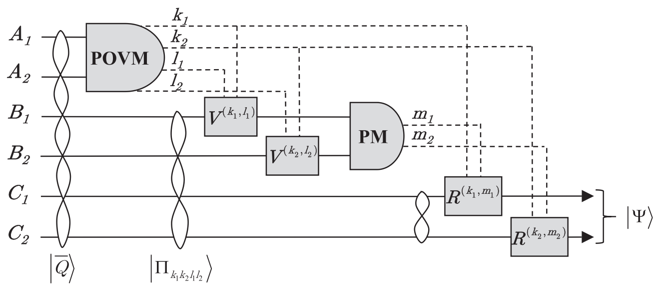

FIG. 3. Quantum circuit for JRSP of an arbitrary hybrid quDit-quNit entanglement $|\Psi\rangle$ of any dimensions D and N in Eq. (2). A solid line represents a quDit or a quNit, while a dashed line represents a classical dit or nit. QuDits A_1, B_1, C_1 and quNits A_2, B_2, C_2 belong to the quantum channel $|\overline{Q}\rangle \equiv |\overline{Q}\rangle_{A_1B_1C_1A_2B_2C_2}$ of Eq. (26). $|\Pi_{k_1k_2l_1l_2}\rangle \equiv |\Pi_{k_1k_2l_1l_2}\rangle_{B_1B_2C_1C_2}$ is given by Eq. (29), $V^{(k_1,l_1)} = X^{k_1}Z^{l_1\dagger}$, $V^{(k_2,l_2)} = X^{k_2}Z^{l_2\dagger}$, $R^{(k_1,m_1)} = Z^{m_1\dagger}X^{k_1}$, and $R^{(k_2,m_2)} = Z^{m_2\dagger}X^{k_2}$. POVM is the positive operator-valued measurement defined by Eq. (27), while PM is the projective measurement in the basis $\{|\Lambda_{m_1m_2}\rangle_{B_1B_2}\}$ defined by Eq. (30). The wavy line embraces quDits and quNits that are entangled with each other.

their state appearing like this:

$$\begin{aligned}
&|\Psi_{k_1k_2m_1m_2}\rangle_{C_1C_2}\\
&= \sum_{d=0}^{D-1}\sum_{n=0}^{N-1} a_{dn}\exp\left(\frac{2\pi i}{D}m_1 d + \frac{2\pi i}{N}m_2 n + i\varphi_{dn}\right)\\
&\quad\times |d\ominus k_1, n-k_2\rangle_{C_1C_2},
\end{aligned}\tag{31}$$

where $-$ within the ket for C_2 is understood as a subtraction modulo N. The obtained outcome m_1, m_2 should be publicly published too for Charlie's use in the next step.

Step 3. To complete the JRSP protocol, Charlie, after getting the values of k_1, k_2 from Alice's announcement and m_1, m_2 from Bob's one, applies $R_{C_1}^{(k_1,m_1)} = Z_{C_1}^{m_1\dagger}X_{C_1}^{k_1}$ on quDit C_1 and $R_{C_2}^{(k_2,m_2)} = Z_{C_2}^{m_2\dagger}X_{C_2}^{k_2}$ on quNit C_2 of the state $|\Psi_{k_1k_2m_1m_2}\rangle_{C_1C_2}$ to transform it into the target hybrid entangled state $|\Psi\rangle_{C_1C_2}$ of Eq. (2). Here, as before, the X, Z gates are D dimensional for C_1 and N dimensional for C_2.

The quantum circuit for our JRSP protocol of an arbitrary hybrid quDit-quNit entanglement $|\Psi\rangle$ is sketched in Fig. 3.

V. REALIZATION OF THE POVM (27)

In Sec. III we proposed a quantum circuit to implement the POVM (9). In this section, we shall propose another quantum circuit to implement the POVM in Eq. (27). As discussed in Sec. III, in general a POVM is realized by first extending the Hilbert space of the system of interest to include also the Hilbert space of some auxiliary system, then entangling the two systems into a proper entangled state followed by necessary PMs.

In the case of our POVM (27) Alice produces an auxiliary entangled state $|K\rangle_{a_1a_2}$ of two ancillas, quDit a_1 and quNit a_2,

$$|K\rangle_{a_1a_2} = \sum_{j_1=0}^{D-1}\sum_{j_2=0}^{N-1} a_{j_1j_2}|j_1, j_2\rangle_{a_1a_2},\tag{32}$$

with $\{a_{j_1j_2}\}$ nothing else but the amplitudes of $|\Psi\rangle_{12}$ in Eq. (2) which are supposedly exposed to Alice. She then applies a D-dimensional CBS gate on A_1, a_1 and an N-dimensional CBS gate on A_2, a_2, with A_1 (A_2) being the control quDit (quNit) and a_1 (a_2) the target one. This brings all the involved quDits and quNits into an entangled state of the form

$$\begin{aligned}
&|\Xi\rangle_{a_1a_2A_1A_2B_1B_2C_1C_2}\\
&= \frac{1}{\sqrt{DN}}\sum_{d,k_1=0}^{D-1}\sum_{n,k_2=0}^{N-1} a_{d\oplus k_1 n+k_2}|k_1, k_2\rangle_{a_1a_2}|d, n\rangle_{A_1A_2}\\
&\quad\times |d, n\rangle_{B_1B_2}|d, n\rangle_{C_1C_2}.
\end{aligned}\tag{33}$$

Next, Alice measures a_1, a_2 in their computational bases $\{|k_1\rangle; k_1 = 0, 1, \dots, D-1\}$, $\{|k_2\rangle_{a_2}; k_2 = 0, 1, \dots, N-1\}$ and A_1, A_2 in the bases $\{|\Omega_{l_1}^{(D)}\rangle_{A_1}; l_1 = 0, 1, \dots, D-1\}$, $\{|\Omega_{l_2}^{(N)}\rangle_{A_2}; l_2 = 0, 1, \dots, N-1\}$, with

$$|\Omega_{l_1}^{(D)}\rangle_{A_1} = \frac{1}{\sqrt{D}}\sum_{j_1=0}^{D-1}\exp\left(-\frac{2\pi i}{D}l_1 j_1\right)|j_1\rangle_{A_1},\tag{34}$$

$$|\Omega_{l_2}^{(N)}\rangle_{A_2} = \frac{1}{\sqrt{N}}\sum_{j_2=0}^{N-1}\exp\left(-\frac{2\pi i}{N}l_2 j_2\right)|j_2\rangle_{A_2}.\tag{35}$$

If an outcome k_1, k_2, l_1, l_2 occurs, the unmeasured part is separated to be in the state $|\Pi_{k_1k_2l_1l_2}\rangle_{B_1B_2C_1C_2}$, which is exactly that defined by Eq. (29). This is tantamount to realizing the POVM (27) on A_1, A_2. According to a theorem in Ref. [41], the additional dimension needed to implement the POVM (27) is $DN(DN-1)$.

The quantum circuit for realization of our POVM defined by Eq. (27) is sketched in Fig. 4.

VI. CONCLUSION

We have devised nonstandard JRSP protocols for one-quDit and quDit-quNit states with all possible complex coefficients (i.e., $\alpha_d = \{a_d e^{i\varphi_d}\}$ and $\alpha_{dn} = \{a_{dn}e^{i\varphi_{dn}}\}$). Our protocols are nonstandard in the sense that they apply to arbitrary dimensions, while a standard one is valid only for the dimension 2, 4, or 8. The essential components that make our protocols nonstandard are POVM measurements to be carried out by one of the preparers at the beginning of the tasks. Since each concrete task demands a suitable POVM, we design the POVM (9) for the case of JRSP of a general quDit state and the POVM (27) for the case of JRSP of hybrid quDit-quNit entanglement. As the definition of the POVMs and the formal measurements in terms of them have a theoretical meaning only, we construct the quantum circuits in Figs. 2 and 4 to realize our POVMs, that are of importance making our protocols attractable from both theoretical and experimental points of view. Usually, a POVM-based protocol, like that for unambiguous quantum state discrimination (see, e.g., Ref. [45]), just succeeds probabilistically, but ours do deterministically. This is owing to special designs of our POVMs and the way our protocols proceed. Namely, the actions of the preparers must be taken in a rigid sequence: Alice should act first and then Bob because Bob's action

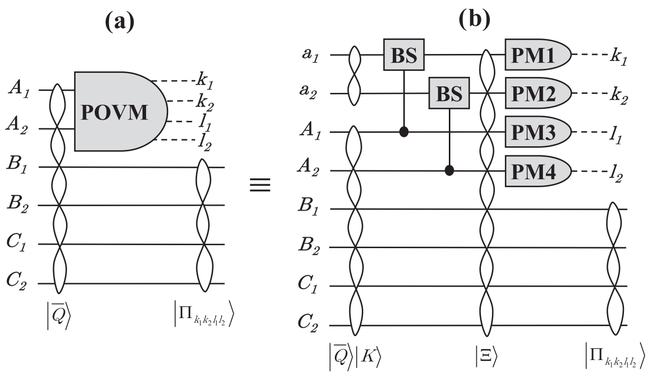

FIG. 4. Quantum circuit for the POVM defined by Eq. (27): (a) the formal performance and (b) the actual realization. A solid line represents a quDit or quNit, while a dashed line represents a classical dit or nit. QuDits A_1, B_1, C_1 and quNits A_2, B_2, C_2 belong to the quantum channel $|\overline{Q}\rangle \equiv |\overline{Q}\rangle_{A_1B_1C_1A_2B_2C_2}$ of Eq. (26), while ancillary quDit a_1 and quNit a_2 belong to the state $|K\rangle \equiv |K\rangle_{a_1a_2}$ of Eq. (32). $|\Xi\rangle \equiv |\Xi\rangle_{a_1a_2A_1A_2B_1B_2C_1C_2}$ and $|\Pi_{k_1k_2l_1l_2}\rangle \equiv |\Pi_{k_1k_2l_1l_2}\rangle_{B_1B_2C_1C_2}$ are given by Eqs. (33) and (29), respectively. BS is short for backward-shift gate, while PM1, PM2, PM3, and PM4 are projective measurements of a_1 in the basis $\{|k_1\rangle_{a_1}\}$, a_2 in the basis $\{|k_2\rangle_{a_2}\}$, A_1 in the basis $\{|\Omega_{l_1}^{(D)}\rangle_{A_1}\}$ of Eq. (34), and A_2 in the basis $\{|\Omega_{l_2}^{(N)}\rangle_{A_2}\}$ of Eq. (35). The wavy line embraces quDits and quNits that are entangled with each other.

is dictated by the result of Alice's action. Furthermore, for the case of JRSP of hybrid quDit-quNit entanglement no combined actions are to be taken simultaneously on both quDit C_1 and quNit C_2 (see Fig. 3). This means that C_1 and C_2 can be separated arbitrarily far away from each other. That is, our protocol can be served as an entangler of distant parties, which is practically advantageous, since there is no need to bring them together to the same place to implement entangling transformations. Theoretical extensions of our proposals to more complicated contexts to involve more preparers, more receivers, and also additional controllers as well as to use different types of quantum resources, say, in terms of high-dimensional EPR pairs, are straightforward.

Last but not least, we would like to address a quite delicate issue as follows. Formally, the state (2) can be relabeled as $|\Psi\rangle = \sum_{d=0}^{D-1}\sum_{n=0}^{N-1} a_{dn}e^{i\varphi_{dn}}|d,n\rangle \to |\Psi'\rangle = \sum_{m=0}^{DN-1} c_m|m\rangle$, where $a_{dn}e^{i\varphi_{dn}} \to c_m$ and $|d,n\rangle \to |m\rangle$. Then $|\Psi'\rangle = \sum_{m=0}^{DN-1} c_m|m\rangle$ deems to be a qu$M$it state (with $M = D \times N$), which could be jointly and remotely prepared using the scheme developed in Secs. II and III, provided that the shared quantum state (4) is replaced by that of three qu$M$its of the form $|Q\rangle_{ABC} \to \frac{1}{\sqrt{M}}\sum_{m=0}^{M-1}|m,m,m\rangle_{ABC}$. However, the intrinisic natures of the above two formal ways of labelings are totally distinct. Namely, $|\Psi'\rangle$ can be dealt with by the scheme in Secs. II and III iff it describes a single object with $M = D \times N$ orthonormal states (e.g., an M-level atom or an M-rail single photon) and is present only at one place as a whole. On the other hand, our state $|\Psi\rangle$ in Eq. (2) describes a system of two objects, one with D orthonormal states and the other with N orthonormal states (e.g., two atoms, one with D levels and the other with N levels, or two photons, one in D-rail coding and the other in N-rail coding), and can be present in two spatially separated places, thus generally establishing a hybrid entanglement of different dimensions. As such, our state $|\Psi\rangle$ must be jointly and remotely prepared by the scheme in Secs. IV and V using a couple of shared quantum states as in Eq. (26).

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