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On the Role of the Controller in Controlled Quantum Teleportation

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Abstract We consider a set of quantum channels which are though partially entangled but can perform perfectly quantum teleportation of two-qubit states with the assistance of a controller. The quantum channels are designed so that only the controller is able to correctly control the task and without his/her cooperation the receiver cannot obtain *with certainty* a state with quality better than that obtained classically. The key point enhancing the role of the controller is that he/she is the only one who is allowed to know the quantum channel parameters.

Keywords Controlled teleportation · Two-qubit state · Partially entangled quantum channel · Role of the controller

1 Introduction

One of the purposes of quantum information sciences is to discover protocols by which different remote authorized parties can work together securely towards a target project only by dual usage of local operations and traditional classical communication. Such tasks remained mythic until recently when they have turned out to be realistic if the parties share in advance a 'spooky' correlated resource called quantum entanglement [1] . The first and perhaps most famous protocol as such is the quantum teleportation [2] which differs conceptually from the long-known fictitious teleportation in the star trek movies in that it is realizable [3–5]

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because it does not violate the nocloning theorem [6] and rests in peace with the Einstein's special relativity theory. In fact, quantum teleportation is the best way for a sender Alice to securely transfer unknown quantum information to a receiver Bob, no matter how far the two people are apart. It is perfect if both its fidelity and success probability are equal to 1. Perfect quantum teleportation of an N-qubit state requires at least a $2N$-qubit maximally entangled quantum channel which is beforehand shared equally between Alice and Bob [7]. As is known, after Alice announces her Bell-measurement outcomes, Bob is able to obtain a replica of the desired state by application of appropriate unitary operations to his qubits. Bob's recovery action cannot be affected or postponed or stopped by anyone.

In practice, last-minute situations, which generally depend on many factors including both technical and non-technical ones, may occur that lead to the need to stop Bob's receiving the target state even after the announcement of Alice's data. To meet such a need, the so-called controlled quantum teleportation protocols [8–22] have been proposed in which one or more people, the controller(s), will join the game with Alice and Bob. Now the quantum resource for perfectly teleporting an N-qubit state should contain more than $2N$ qubits. Namely, the working channel must be a $(2N + M)$-qubit maximally entangled state of which the M extra qubits are held and manipulated by the controllers. According to the well-established recipe, first Alice measures the qubits at her hand in the Bell-state basis, then the controllers measure their qubits in the basis $\left\{|\pm\rangle_m = 2^{-1/2}\left(|0\rangle \pm |1\rangle\right)_m ; m = 1, 2, ..., M\right\}$, and finally Bob reconstructs the collapsed state of his qubits to the right one using both the measurement outcomes publicly published by Alice and the controllers. The role the controllers play is clear: even knowing Alice's data Bob cannot get the correct state without the controllers' data. Therefore, the controllers are in the position to control the task at their will (i.e., to permit, to postpone or even to forbid Bob's getting the intended state). However, the controllers' role is limited in the following aspects.

(i) Although the teleportation process requires a prior establishment of the entangled quantum channel, there may happen a situation in which, after the quantum channel has been safely shared among the authorized parties, someone else could access to the M qubits of the $(2N+M)$-qubit entangled quantum channel (because of, say, the controller's negligent guarding). If so, they can correctly act on those M qubits since the measurement basis $\left\{|\pm\rangle_m\right\}$ is known to everybody.

(ii) It may also happen that due to certain unfavorable circumstances arising at last minutes the controllers purposively decide not to participate in the task (i.e., they do not do their measurements or do them but do not let the outcomes broadcasted). If so, Bob can still solely use Alice's data to obtain *with certainty* an approximate state whose fidelity is greater than the 'classical' fidelity $F_{class} = 2/(2^N + 1)$ which is achievable without sharing any entanglements [23, 24].

In order to overcome the two above-mentioned limitations, thereby enhancing the controllers' role, as well as to retain perfection of quantum teleportation, proper partially entangled states can be designed to serve as the quantum channels and, more importantly, the parameters characterizing such quantum channels are kept confidentially to everybody but the controllers.

In this paper, we shall demonstrate these ideas through the case of teleporting a general unknown two-qubit state of the form

$$|\psi\rangle_{uv} = \left(x|00\rangle + ye^{i\varphi_1}|01\rangle + ze^{i\varphi_2}|10\rangle + te^{i\varphi_3}|11\rangle\right)_{uv}, \tag{1}$$

where $\varphi_{1,2,3} \in [0, 2\pi]$ and $x^2+y^2+z^2+t^2 = 1$ to satisfy the normalization condition, under supervision of a single controller named Charlie. The next section, Section 2, is devoted to the case when Charlie is capable of controlling one bit of useful information. The case of controlling two informative bits is presented in Section 3 and Conclusion is the final section. There is a big bundle of cumbersome formulae which are not all shown explicitly to not delute the main content.

2 The One-bit Control Case

We start with the case when Charlie holds one qubit. This implies that he controls the task by one bit of information. The quantum channel must therefore be a five-qubit state. First, we consider the following state

$$|Q_1(\theta)\rangle_{12345} = |B_{00}\rangle_{12}|G(\theta)\rangle_{345}, \tag{2}$$

where $|B_{00}\rangle_{12}$ is one of the four Bell-states [2] defined compactly by $|B_{mn}\rangle_{12} = 2^{-1/2}\sum_{j=0}^{1}(-1)^{mj}\,|j, j\oplus n\rangle_{12}$, with $m, n \in \{0, 1\}$ and $\oplus$ an addition mod 2, while

$$|G(\theta)\rangle_{345}=\frac{1}{\sqrt{2}}\,(|000\rangle + \cos\theta\,|110\rangle - \sin\theta\,|111\rangle)_{345}\,, \tag{3}$$

which can be generated from the Greenberger-Horne-Zeilinger state $|GHZ\rangle_{345} = 2^{-1/2}\sum_{j=0}^{1}|jjj\rangle_{345}$ [25] as follows:

$$|G(\theta)\rangle_{345} = R_5(\theta)CNOT_{45}R_5(-\theta)|GHZ\rangle_{345}, \tag{4}$$

where $CNOT_{45}$ is a controlled-NOT gate acting on a two-qubit state as $CNOT_{45}\,|m, n\rangle_{45} = |m, m\oplus n\rangle_{45}$ and $R_5(\theta)$ a rotation gate acting on a single-qubit state as $R_5(\theta)|m\rangle_5 = \cos(\theta/2)|m\rangle_5 - (-1)^m\sin(\theta/2)|m\oplus 1\rangle_5$. We assume that the state (2) (as well as other quantum channel states to be used later in this work) is generated by some bank of states and then distributed to Alice, Bob and Charlie with a careful checking process or a proper distillation procedure so that the state is successfully shared among the three parties before they proceed to perform their task. Of the quantum channel state (2), Alice holds qubits 1 and 3, Bob qubits 2 and 4, while qubit 5 belongs to Charlie. Notably and of crucial importance in our protocol, the quantum channel $|Q_1(\theta)\rangle_{12345}$ is partially entangled and characterized by one parameter, the angle θ, whose value is known by Charlie but kept secret to anyone else.

To fulfill the task, Alice performs two Bell-state measurements, one on qubits $(u, 1)$ and the other on qubits $(v, 3)$, with the outcomes $\{a, b\}$ and $\{c, d\}$ corresponding to finding $|B_{ab}\rangle_{u1}$ and $|B_{cd}\rangle_{v3}$, each of which occurs with an equal probability of $P_{ab} = P_{cd} = 1/4$ for any $a, b, c, d \in \{0, 1\}$. As for the controller Charlie, he applies $R(-\theta)$ to the qubit at his hand (i.e., qubit 5), then measures it in its computational basis. Note at this point that noone except Charlie is able to manipulate qubit 5 correctly because only Charlie knows the value of θ, thereby the issue (i) mentioned in Introduction when using maximally entangled quantum channel is avoided. The outcome will be $m \in \{0, 1\}$ if Charlie finds $|m\rangle_5$, which occurs with probability

$$P_m = \delta_{m0}\cos^2(\theta/2) + \delta_{m1}\sin^2(\theta/2). \tag{5}$$

It is worth noting that Alice and Charlie can act independently. However, Alice should always broadcast her outcomes $\{a, b, c, d\}$ publicly, whereas Charlie would not do so. Actually, Charlie has to carry out an overall analysis. If he doubts in something, he keeps the outcome m with himself, otherwise he also publicly broadcasts it. In the latter case, conditioned on the broadcasted data $\{a, b, c, d, m\}$, Bob is able to obtain the desired state by applying to his qubits 2 and 4 the following recovery operator

$$R_{abcdm} = \left(X^b Z^a\right)_2 \otimes \left(X^d Z^{c\oplus m}\right)_4, \tag{6}$$

where X and Z are the usual Pauli bit-flip and phase-flip matrices. The total success probability is $P = \sum_{a,b,c,d,m=0}^{1} P_{ab} P_{cd} P_m = 1$.

Now we are passing to study the case when Charlie observes something wrong. In that case Bob lacks the data m and what at his hand after Alice's measurements will be a two-qubit mixed state ρ_{abcd} resulted from tracing out over the states of Charlie's qubit. These mixed states are unitarily related so Bob can transform them around from one to another by appropriate unitary operators. The quality of ρ_{abcd} can be assessed by the fidelity $F_{abcd} = \langle\psi|\,\rho_{abcd}\,|\psi\rangle$ whose explicit expressions have been derived and are provided in the appendix. As seen from (35)–(50), the fidelities depend not only on Alice's data $\{a, b, c, d\}$ and on the quantum channel parameter θ but also on the state to be teleported through x, y, z, t, φ_1, φ_2 and φ_3. Then, of relevant interest is the fidelity averaged over all input states. To calculate such an averaged fidelity $\overline{F}_{abcd}$, we adopt the Hurwitz parameterization [26], changing the variables $\{x, y, z, t\}$ to $\{\gamma_1, \gamma_2, \gamma_3\}$ with $\gamma_{1,2,3} \in [0, \pi/2]$ as

$$x = \cos\gamma_3, \tag{7}$$

$$y = \sin\gamma_3 \cos\gamma_2, \tag{8}$$

$$z = \sin\gamma_3 \sin\gamma_2 \cos\gamma_1, \tag{9}$$

$$t = \sin\gamma_3 \sin\gamma_2 \sin\gamma_1. \tag{10}$$

The distribution of φ_k is assumed to be uniform with a probability density $\mathcal{P}(\varphi_k) = (2\pi)^{-1}$. As for the distribution of γ_k, it is taken to be nonuniform with a probability density $\mathcal{P}(\gamma_k) = k\sin(2\gamma_k)\sin^{2k-2}(\gamma_k)$, like the volume element on the sphere [26]. The averaged fidelity $\overline{F}_{abcd}$ is then determined by [27]

$$\overline{F}_{abcd} = \frac{3!}{\pi^3}\prod_{k=1}^{3}\int_0^{\pi/2}\cos\gamma_k(\sin\gamma_k)^{2k-1}d\gamma_k\prod_{q=1}^{3}\int_0^{2\pi}F_{abcd}d\varphi_q. \tag{11}$$

Substituting (35)–(50) into (11) yields

$$\overline{F}_{0000} = \frac{1}{5}(3 + 2\cos\theta), \tag{12}$$

$$\overline{F}_{0010} = \frac{1}{5}(3 - 2\cos\theta) \tag{13}$$

and

$$\overline{F}_{abcd\neq 0000,0010} = \frac{1}{5}. \tag{14}$$

Figure 1 shows the dependence on θ of the various $\overline{F}_{abcd}$. For any values of θ, $\overline{F}_{abcd\neq 0000,0010} \le \min\{\overline{F}_{0000}, \overline{F}_{0010}\}$. For $\pi/3 \le \theta \le 2\pi/3$ or $4\pi/3 \le \theta \le 5\pi/3$, both $\overline{F}_{0000}$ and $\overline{F}_{0010}$ are greater than or equal to $F_{class} = 2/(2^2+1) = 2/5$. If θ is chosen outside those domains, then among $\overline{F}_{0000}$ and $\overline{F}_{0010}$ one is greater and the other is smaller than F_{class}. Although, for a given θ, Bob has a chance to obtain a state with fidelity greater than $2/5$, he cannot obtain it *with certainty,* because he has no ideas about the value of θ. That is, he does not know which one of $\overline{F}_{0000}$ and $\overline{F}_{0010}$ is greater than F_{class}. In particular, if θ is chosen such that $0 < \theta < \pi/3$ or $5\pi/3 < \theta < 2\pi$ (for which only $\overline{F}_{0000}$ is greater than $2/5$ but all the other averaged fidelities are smaller than $2/5$) or $2\pi/3 < \theta < 4\pi/3$ (for

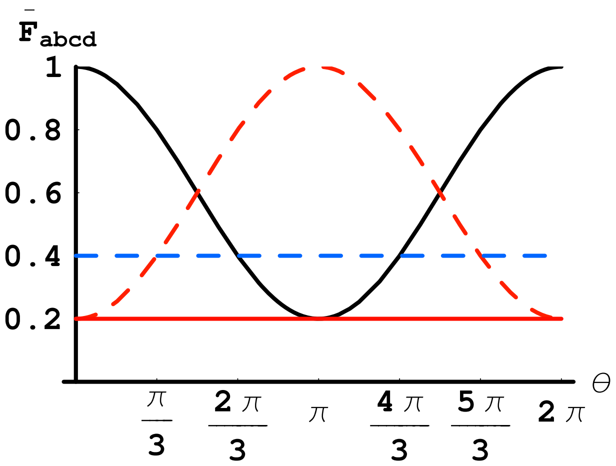

Fig. 1 The dependence on θ of the various averaged fidelities $\overline{F}_{abcd}$ when the state $|Q_1(\theta)\rangle_{12345}$ in (2) is used as the quantum channel. The *solid curve* is for $\overline{F}_{0000}$ in (12), the *dashed curve* is for $\overline{F}_{0010}$ in (13), and the *solid horizontal line* is for $\overline{F}_{abcd\neq 0000,0010}$ in (14). The *dashed horizontal line* is for $F_{class} = 2/5$

which only $\overline{F}_{0010}$ is greater than $2/5$ but all the other averaged fidelities are smaller than $2/5$), and its exact value is kept secret to Bob, then Bob is unable to obtain *with certainty* a state with quality better than that achievable by classical means, resolving the issue (ii) in Introduction.

It is worth emphasizing again that in our above protocol we only allow Charlie know the value of θ. This is the key point to resolve both the issues (i) and (ii) mentioned in Introduction, thereby enhancing the role of the controller compared with the case of using maximally entangled state or the case of using partially entangled state whose characteristics are known by both Clarlie and Bob.

The quantum channel $|Q_1(\theta)\rangle$ in (2) turns out not good in the case when $xt = yze^{i(\varphi_1+\varphi_2-\varphi_3)}$ for which $|\psi\rangle_{uv} = (yt^{-1}e^{i(\varphi_1-\varphi_3)}|0\rangle + |1\rangle)_u(zt^{-1}e^{i(\varphi_2-\varphi_3)}|0\rangle + |1\rangle)_v$ is a product state, because qubit u becomes beyond any control. To be valid to an arbitrary two-qubit state, including the product one, the following genuine five-qubit entangled quantum channel,

$$|Q_2(\theta)\rangle_{12345} = \frac{1}{2}[|00000\rangle + \cos\theta(|00110\rangle - |11110\rangle) + |11000\rangle - \sin\theta(|00111\rangle - |11111\rangle)]_{12345}, \tag{15}$$

can be used, which is produced by application of a sequence of operators to $|GHZ\rangle_{123}|00\rangle_{45}$ like this:

$$|Q_2(\theta)\rangle_{12345} = R_5(\theta)CNOT_{45}R_5(-\theta)CNOT_{45}CNOT_{34}H_3|GHZ\rangle_{123}|00\rangle_{45}, \tag{16}$$

where H is the Hadamard gate acting on a single-qubit state as $H|j\rangle = 2^{-1/2}((-1)^j|j\rangle + |j\oplus 1\rangle)$. The same procedures as for $|Q_1(\theta)\rangle_{12345}$ apply here, but the obtained results are different. Namely, with the controller's permission Bob's recovery operators read

$$R_{abcdm} = \left[\left(X^bZ^a\right)_2 \otimes \left(X^dZ^{c\oplus m}\right)_4\right]CZ_{24}, \tag{17}$$

where CZ_{24} is a controlled-Z gate acting on a two-qubit state as $CZ_{24}|mn\rangle_{24} = (-1)^{mn}|mn\rangle_{24}$, while without the controller's permission the fidelities of Bob's mixed states depend on the parameters of both the input state and the quantum channel (we have explicitly derived those formulae but will not show them because they are cumbersome). The corresponding averaged fidelities have been calculated to be

$$\overline{F}_{0000} = \overline{F}_{0010} = \overline{F}_{1010} = \frac{2}{5} \tag{18}$$

and

$$\overline{F}_{abcd \neq 0000,0010,1010} = \frac{1}{5}. \tag{19}$$

Since all the averaged fidelities are independent of θ, Bob can, for any Alice's possible measurement outcomes, obtain a state with the highest fidelity equal to $2/5 = F_{class}$, but a state with fidelity higher than $2/5$ he cannot. In this sense, the role of the controller in this case is marginal.

3 The Two-bit Control Case

In the previous section we showed that by holding only one qubit the controller is useful in the sense that without his permission the receiver can never obtain *with certainty* a state with fidelity better than the 'classical' one. To see whether any added benefits may arise when the controller holds two qubits, we shall consider in this section the following six-qubit quantum channels:

$$|Q_3(\theta_1, \theta_2)\rangle_{123456} = |G(\theta_1)\rangle_{125} |G(\theta_2)\rangle_{346} \tag{20}$$

and

$$\begin{aligned} |Q_4(\theta_1, \theta_2)\rangle_{123456} = & \frac{1}{2}[|000000\rangle + \cos\theta_1 |110000\rangle - \sin\theta_1 |110011\rangle \\ & + \cos\theta_2 |001100\rangle - \sin\theta_2 |001101\rangle \\ & - \cos\theta_1 \cos\theta_2 |111100\rangle + \cos\theta_1 \sin\theta_2 |111101\rangle \\ & + \sin\theta_1 \cos\theta_2 |111110\rangle - \sin\theta_1 \sin\theta_2 |111111\rangle]_{123456}. \end{aligned} \tag{21}$$

The exact values of θ_1 and θ_2 are known only by Charlie. The state $|Q_4(\theta_1, \theta_2)\rangle_{123456}$, which is a genuine six-qubit entangled state, can be generated starting from $|GHZ\rangle_{123}|000\rangle_{456}$ as

$$\begin{aligned} |Q_4(\theta_1, \theta_2)\rangle_{123456} = & R_6(\theta_2) R_5(\theta_1) CNOT_{36} CNOT_{25} R_6(-\theta_2) R_5(-\theta_1) \\ & CNOT_{36} CNOT_{25} CNOT_{34} H_3 |GHZ\rangle_{123} |000\rangle_{456}. \end{aligned} \tag{22}$$

Of $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ and $|Q_4(\theta_1, \theta_2)\rangle_{123456}$ qubits 1 (2) and 3 (4) are possessed by Alice (Bob), while qubits 5 and 6 by Charlie. For these quantum channels Alice will do the same way as in the case of using $|Q_1(\theta)\rangle_{12345}$ and $|Q_2(\theta)\rangle_{12345}$, but Charlie will do a little more: he will apply $R(-\theta_1)$ to qubit 5, $R(-\theta_2)$ to qubit 6, followed by measuring the two qubits in the basis $\{|00\rangle_{56}, |01\rangle_{56}, |10\rangle_{56}, |11\rangle_{56}\}$. The outcome will be $m, n \in \{0, 1\}$ if he finds $|mn\rangle_{56}$, which occurs with probability

$$\begin{aligned} P_{mn} = & \delta_{m0}\delta_{n0} \cos^2(\theta_1/2) \cos^2(\theta_2/2) \\ & + \delta_{m0}\delta_{n1} \cos^2(\theta_1/2) \sin^2(\theta_2/2) \\ & + \delta_{m1}\delta_{n0} \sin^2(\theta_1/2) \cos^2(\theta_2/2) \\ & + \delta_{m1}\delta_{n1} \sin^2(\theta_1/2) \sin^2(\theta_2/2). \end{aligned} \tag{23}$$

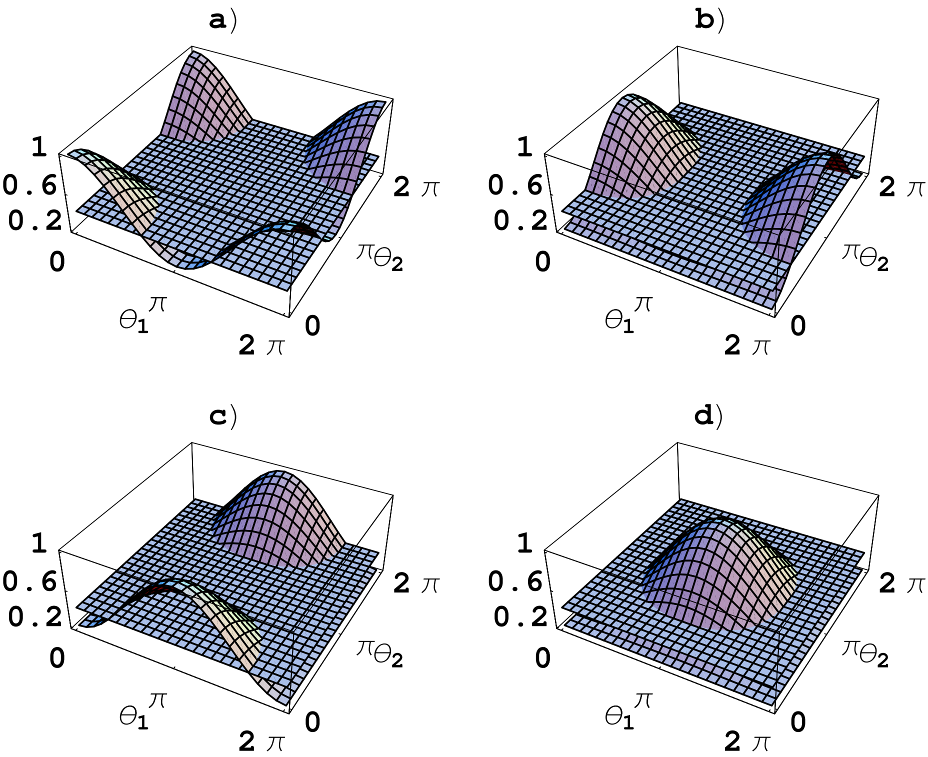

Fig. 2 The dependence on θ_1 and θ_2 of the various averaged fidelities: a) $\overline{F}_{0000}$ in (26), b) $\overline{F}_{0010}$ in (27), c) $\overline{F}_{1000}$ in (28), and d) $\overline{F}_{1010}$ in (29), when $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ in (20) is used as the quantum channel. The plane at $2/5$ is the 'classical' fidelity

If both Alice and Charlie announce their measurement outcomes $\{a, b, c, d, m, n\}$, then the recovery operators of Bob are

$$R_{abcdmn} = \left(X^b Z^{a\oplus m}\right)_2 \otimes \left(X^d Z^{c\oplus n}\right)_4 \tag{24}$$

when using $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ and

$$R_{abcdmn} = \left[\left(X^b Z^{a\oplus m}\right)_2 \otimes \left(X^d Z^{c\oplus n}\right)_4\right] CZ_{24} \tag{25}$$

when using $|Q_4(\theta_1, \theta_2)\rangle_{123456}$. The total success probability is $P = \sum_{a,b,c,d,m,n=0}^{1} P_{ab} P_{cd} P_{mn} = 1$.

In case Charlie does not announce his outcomes the mixed states Bob would obtain have the fidelities depending on the parameters of both the input state and the quantum channel (whose explicit formulae have been derived but not shown). The corresponding averaged fidelities can then be calculated to be

$$\overline{F}_{0000} = \frac{1}{5}(2 + \cos\theta_1 + \cos\theta_2 + \cos\theta_1 \cos\theta_2), \tag{26}$$

$$\overline{F}_{0010} = \frac{1}{5}(2 + \cos\theta_1 - \cos\theta_2 - \cos\theta_1 \cos\theta_2), \tag{27}$$

$$\overline{F}_{1000} = \frac{1}{5}(2 - \cos\theta_1 + \cos\theta_2 - \cos\theta_1 \cos\theta_2), \tag{28}$$

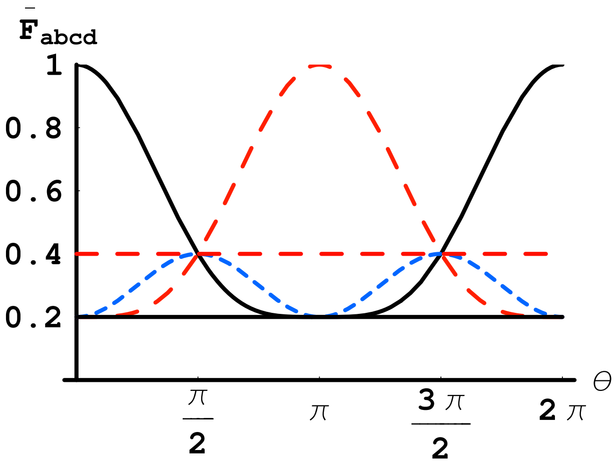

Fig. 3 The θ-dependence of the various averaged fidelities when $|Q_3(\theta_1 = \theta_2 = \theta)\rangle_{123456}$ in (20) is used as the quantum channel. The *solid curve* is for $\overline{F}_{0000}$ in (32), the *short-dashed curve* is for $\overline{F}_{0010}$ and $\overline{F}_{1000}$ in (33), the *long-dashed curve* is for $\overline{F}_{1010}$ in (34) and the *solid horizontal line* is for $\overline{F}_{abcd\neq 0000,0010,1000,1010}$ in (31). The *dashed horizontal line* is for $F_{classical} = 2/5$

$$\overline{F}_{1010} = \frac{1}{5}(2 - \cos\theta_1 - \cos\theta_2 + \cos\theta_1\cos\theta_2) \tag{29}$$

for $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ and

$$\overline{F}_{0000} = \overline{F}_{0010} = \overline{F}_{1000} = \overline{F}_{1010} = \frac{2}{5} \tag{30}$$

for $|Q_4(\theta_1, \theta_2)\rangle_{123456}$, while

$$\overline{F}_{abcd\neq 0000,0010,1000,1010} = \frac{1}{5} \tag{31}$$

for both $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ and $|Q_4(\theta_1, \theta_2)\rangle_{123456}$.

The values of the averaged fidelities in (30) and (31) indicate that the role of the controller when employing $|Q_4(\theta_1, \theta_2)\rangle_{123456}$ as the quantum channel is marginal, similar to the case of employing $|Q_2(\theta)\rangle_{12345}$, explained in the previous section. Concerning the case of employing $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ as the quantum channel, it is obvious from (26)–(29) and (31) that $\overline{F}_{abcd\neq 0000,0010,1000,1010} \leq \min\{\overline{F}_{0000}, \overline{F}_{0010}, \overline{F}_{1000}, \overline{F}_{1010}\}$ for any θ_1 and θ_2. So we are left only with $\overline{F}_{0000}$, $\overline{F}_{0010}$, $\overline{F}_{1000}$ and $\overline{F}_{1010}$ in (26)–(29). A quick look at them reveals that they sum up to $8/5$ and hence not all of them can be larger than $F_{class} = 2/5$. For a closer look, we plot their dependences on θ_1 and θ_2 in Fig. 2. These figures show that, as a function of θ_1 and θ_2, each of $\overline{F}_{0000}$, $\overline{F}_{0010}$, $\overline{F}_{1000}$ and $\overline{F}_{1010}$ can be smaller or larger than $2/5$, but which one is larger than $2/5$ is unknown without specifying the values of θ_1 and θ_2. This means that Bob cannot sort out *with certainty* a state with fidelity larger than the 'classical' one. In particular, it is sufficient to use $\theta_1 = \theta_2 = \theta$ in which case

$$\overline{F}_{0000} = \frac{1}{5}(2 + 2\cos\theta + \cos^2\theta), \tag{32}$$

$$\overline{F}_{0010} = \overline{F}_{1000} = \frac{1}{5}(2 - \cos^2\theta) \tag{33}$$

and

$$\overline{F}_{1010} = \frac{1}{5}(2 - 2\cos\theta + \cos^2\theta), \tag{34}$$

whose θ-dependence is displayed in Fig. 3. As is visual from Fig. 3, for all θ neither $\overline{F}_{0010}$ nor $\overline{F}_{1000}$ can exceed F_{class}, while either $\overline{F}_{0000}$ or $\overline{F}_{1010}$ can be above F_{class}, but both of $\overline{F}_{0000}$ and $\overline{F}_{1010}$ cannot simultaneously. As elucidated in the previous section, this implies an enhanced role of the controller (i.e., a state with fidelity greater than $2/5$ cannot be obtained *with certainty* by Bob).

4 Conclusion

We have studied quantum teleportation of a two-qubit state under control of a controller via a set of partially entangled quantum channels. The partial entanglement, whose identity is allowed to be known only by the controller, is intentionally employed to avoid anyone else except the controller to be able to control the task. All the protocols considered are perfect as both their fidelity and total success probability are equal to 1, despite the partial entanglement. We have further analyzed the role the controller plays in our protocols. The controller is regarded as unuseful if without his/her cooperation the receiver is surely able to obtain an approximate state with fidelity higher than the 'classical' fidelity. Otherwise, if the receiver cannot obtain *with certainty* a state with fidelity equal to or higher than the 'classical' fidelity, then the controller is fully useful. In case, the receiver can certainly obtain only a state with fidelity equal to the 'classical' fidelity, the controller's role is marginal. According to such criteria, we have shown that the controller is fully useful when the quantum channels $|Q_1(\theta)\rangle_{12345}$ and $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ in (2) and (20) are used, but its role is marginal when using the quantum channels $|Q_2(\theta)\rangle_{12345}$ and $|Q_4(\theta_1, \theta_2)\rangle_{123456}$ in (15) and (21). With respect to the quantum resource cost, it is economical to use $|Q_1(\theta)\rangle_{12345}$ when the state to be teleported is entangled. Yet, to teleport a pair of single-qubit states, $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ is the right choice, though one extra qubit must be held by the controller, since in this case $|Q_1(\theta)\rangle_{12345}$ cannot control both qubits at the same time. In summary, in our proposed protocols, using $|Q_1(\theta)\rangle_{12345}$ or $|Q_3(\theta_1, \theta_2)\rangle_{123456}$ essentially enhances the controller's role. All the quantum channels under consideration in this paper are produceable within current technologies. We hope our study would shed some light also to other kinds of controlled quantum information processing protocols.

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Appendix

The fidelity F_{abcd} as a function of x, y, z, t and θ when $|Q_1(\theta)\rangle_{12345}$, (2), is used as the working quantum channel:

$$F_{0000} = \left(x^2 + z^2\right)^2 + (y^2 + t^2)^2 + 2\left(x^2 + z^2\right)(y^2 + t^2)\cos\theta, \tag{35}$$

$$
\begin{aligned}
F_{0001} =\ & 2(x^2y^2 + z^2t^2) + 4xyzt\cos(\varphi_1 + \varphi_2 - \varphi_3) \\
& +2\{x^2y^2\cos(2\varphi_1) + z^2t^2\cos[2(\varphi_2 - \varphi_3)] \\
& +2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta,
\end{aligned} \tag{36}
$$

$$
F_{0010} = \left(x^2 + z^2\right)^2 + (y^2 + t^2)^2 - 2\left(x^2 + z^2\right)(y^2 + t^2)\cos\theta, \tag{37}
$$

$$
\begin{aligned}
F_{0011} =\ & 2(x^2y^2 + z^2t^2) + 4xyzt\cos(\varphi_1 + \varphi_2 - \varphi_3) \\
& -2\{x^2y^2\cos(2\varphi_1) + z^2t^2\cos[2(\varphi_2 - \varphi_3)] \\
& +2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta,
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
F_{0100} =\ & 2(x^2z^2 + y^2t^2) + 2x^2z^2\cos(2\varphi_2) \\
& +2y^2t^2\cos[2(\varphi_1 - \varphi_3)] \\
& +4xyzt\cos(\varphi_2)\cos(\varphi_1 - \varphi_3)\cos\theta,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
F_{0101} =\ & 2(x^2t^2 + y^2z^2) + 4xyzt\cos(\varphi_1 - \varphi_2 - \varphi_3) \\
& +2\{x^2t^2\cos(2\varphi_3) + y^2z^2\cos[2(\varphi_1 - \varphi_2)] \\
& +2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
F_{0110} =\ & 2(x^2z^2 + y^2t^2) + 2x^2z^2\cos(2\varphi_2) \\
& +2y^2t^2\cos[2(\varphi_1 - \varphi_3)] \\
& -4xyzt\cos(\varphi_2)\cos(\varphi_1 - \varphi_3)\cos\theta,
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
F_{0111} =\ & 2(x^2t^2 + y^2z^2) + 4xyzt\cos(\varphi_1 - \varphi_2 - \varphi_3) \\
& -2\{x^2t^2\cos(2\varphi_3) + y^2z^2\cos[2(\varphi_1 - \varphi_2)] \\
& +2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta,
\end{aligned} \tag{42}
$$

$$
F_{1000} = \left(x^2 - z^2\right)^2 + (y^2 - t^2)^2 + 2\left(x^2 - z^2\right)(y^2 - t^2)\cos\theta, \tag{43}
$$

$$
\begin{aligned}
F_{1001} =\ & 2(x^2y^2 + z^2t^2) - 4xyzt\cos(\varphi_1 + \varphi_2 - \varphi_3) \\
& +2\{x^2y^2\cos(2\varphi_1) + z^2t^2\cos[2(\varphi_2 - \varphi_3)] \\
& -2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta,
\end{aligned} \tag{44}
$$

$$
F_{1010} = \left(x^2 - z^2\right)^2 + (y^2 - t^2)^2 - 2\left(x^2 - z^2\right)(y^2 - t^2)\cos\theta, \tag{45}
$$

$$
\begin{aligned}
F_{1011} =\ & 2(x^2y^2 + z^2t^2) - 4xyzt\cos(\varphi_1 + \varphi_2 - \varphi_3) \\
& -2\{x^2y^2\cos(2\varphi_1) + z^2t^2\cos[2(\varphi_2 - \varphi_3)] \\
& -2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta,
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
F_{1100} =\ & 2(x^2z^2 + y^2t^2) - 2x^2z^2\cos(2\varphi_2) \\
& -2y^2t^2\cos[2(\varphi_1 - \varphi_3)] \\
& -8xyzt\sin(\varphi_2)\sin(\varphi_1 - \varphi_3)\cos\theta,
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
F_{1101} =\ & 2(x^2t^2 + y^2z^2) - 4xyzt\cos(\varphi_1 - \varphi_2 - \varphi_3) \\
& -2\{x^2t^2\cos(2\varphi_3) + y^2z^2\cos[2(\varphi_1 - \varphi_2)] \\
& +2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta,
\end{aligned} \tag{48}
$$

$$F_{1110} = 2\left(x^2z^2 + y^2t^2\right) - 2x^2z^2\cos(2\varphi_2) \\ -2y^2t^2\cos[2(\varphi_1 - \varphi_3)] \\ +8xyzt\sin(\varphi_2)\sin(\varphi_1 - \varphi_3)\cos\theta, \tag{49}$$

$$F_{1111} = 2(x^2t^2 + y^2z^2) - 4xyzt\cos(\varphi_1 - \varphi_2 - \varphi_3) \\ +2\{x^2t^2\cos(2\varphi_3) + y^2z^2\cos[2(\varphi_1 - \varphi_2)] \\ -2xyzt\cos(\varphi_1 - \varphi_2 + \varphi_3)\}\cos\theta. \tag{50}$$

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