

**Neutrino mixing model based on an  $A_4 \times Z_3 \times Z_4$  flavor symmetry**Nguyen Anh Ky,<sup>1,2,\*</sup> Phi Quang Văn,<sup>2,†</sup> and Nguyen Thi Hồng Vân<sup>2,3,‡</sup><sup>1</sup>*Duy Tan University, K7/25 Quang Trung Street, Hai Chau, Da Nang, Vietnam*<sup>2</sup>*Mathematical and High Energy Physics Group, Institute of physics,**Vietnam Academy of Science and Technology (VAST), 10 Dao Tan, Ba Dinh, Hanoi, Vietnam*<sup>3</sup>*Laboratory of High Energy Physics and Cosmology, Faculty of Physics,**VNU University of Science, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam*

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A model of a neutrino mixing with an  $A_4 \times Z_3 \times Z_4$  flavor symmetry is suggested. In addition to the standard model fields, the present model contains six new fields that transform under different representations of  $A_4 \times Z_3 \times Z_4$ . The model is constructed to slightly deviate from a tribimaximal model in agreement with the current experimental data; thus, all analysis can be done in the base of the perturbation method. Within this model, as an application, a relation between the mixing angles ( $\theta_{12}$ ,  $\theta_{23}$ ,  $\theta_{13}$ ) and the Dirac  $CP$ -violation phase ( $\delta_{CP}$ ) is established. This relation allows a prediction of  $\delta_{CP}$  and the Jarlskog parameter ( $J_{CP}$ ). The predicted value  $\delta_{CP}$  is in the  $1\sigma$  region of the global fit for both the normal and inverse neutrino mass ordering and gives  $J_{CP}$  to be within the bound  $|J_{CP}| \leq 0.04$ . For an illustration, the model is checked numerically and gives values of the neutrino masses (of the order of 0.1 eV) and the mixing angle  $\theta_{13}$  (about  $9^\circ$ ) very close to the current experimental data.

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After the discovery of the Higgs boson, called also the Brout-Englert-Higgs boson, [1,2] by the LHC collaborations ATLAS and CMS (for a review, see, for example, [3]), the particle content of the standard model (SM) seems to have been completely confirmed by the experiment. The SM is an excellent model of elementary particles and their interactions as it can explain and predict many phenomena, at least until the energy scale around the top quark mass. However, there are open problems that cannot be solved within the SM and thus call for modifying or extending the latter. The problem of neutrino masses and mixings [4–9] is among such problems beyond the SM. This problem is important for not only particle physics but also nuclear physics, astrophysics, and cosmology; therefore, it has attracted much interest [10–14]. The neutrino mixing means that the flavor neutrinos (flavor eigenstates of neutrinos) are superpositions of massive neutrinos (mass eigenstates of neutrinos) encoded in the so-called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix in terms of mixing angles  $\theta_{ij}$  and a given number of phases, while in the SM the neutrinos are massless and not mixing. One of the ways to try to explain this phenomenon is to add a flavor symmetry to the gauge symmetry  $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$  of the SM (see [15,16] for a review). A popular flavor symmetry intensively investigated in the literature is that described by the group  $A_4$  (see, for instance, [16–24]) allowing one to

obtain a tribimaximal (TBM) neutrino mixing corresponding to the mixing angles  $\theta_{12} \approx 35.26^\circ$  ( $\sin^2 \theta_{12} = 1/3$ ),  $\theta_{13} = 0^\circ$ , and  $\theta_{23} = 45^\circ$  (see [25]). The recent experimental data such as those from T2K [26,27], RENO [28], DOUBLE-CHOOZ [29], and DAYA-BAY [30,31] showing a nonzero mixing angle  $\theta_{13}$  and a possible nonzero Dirac  $CP$ -violation (CPV) phase  $\delta_{CP}$ , reject, however, the TBM scheme [32,33]. There have been many attempts to explain these experimental phenomena. In particular, for this purpose, various models with a discrete flavor symmetry [15,34–36], including an  $A_4$  flavor symmetry, have been suggested [15–24,37–47].

In general, the models, based on  $A_4$  flavor symmetry, have extended lepton and scalar sectors containing new fields in addition to the SM ones that now may have an  $A_4$  symmetry structure. Therefore, besides undergoing the SM symmetry, these fields may also transform under  $A_4$ . At the beginning, the  $A_4$ -based models were build to describe a TBM neutrino mixing (see, for example, [18]) but later many attempts, such as those in [15–17,19–24,38,39,43–46], to find a model fitting the non-TBM phenomenology, were made. On these models, however, are often imposed some assumptions, for example, the vacuum expectation values (VEVs) of some of the fields, especially those generating neutrino masses, have a particular alignment [40–44]. These assumptions may lead to a simpler diagonalization of a mass matrix but restrict the generality of the model. Since, according to the current experimental data, the discrepancy of a phenomenological model from a TBM model (i.e., a model in which the neutrino mixing matrix has a TBM form [25]) is quite small, we can think about a perturbation approach to building a new, realistic, model [45].

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The perturbative approach has been used by several authors (see, for example, [48,49]) but their methods mostly are model independent, that is, no model realizing the experimentally established neutrino mixing has been shown. On the other hand, most of the  $A_4$ -based models are analyzed in a nonperturbative way. There are a few cases such as [50] where the perturbative method is applied but the authors' approach is different from ours and their analysis, sometimes, is not precise (for example, the conditions imposed in Sec. IV of [50] are not always possible). Besides that, in many works done so far, the neutrino mixing has been investigated with a less general vacuum structure of scalar fields.

In this paper we introduce an  $A_4$  flavor symmetric standard model, which can generate a neutrino mixing, deviating from the TBM scheme slightly, as requested and explained above. Since the deviation is small we can use a perturbation method in elaborating such a non-TBM neutrino mixing model. The model field content is that of the SM extended with six new fields, all are  $SU(2)_L$  singlets (isosinglets), transforming under different representations of  $A_4$ : an  $A_4$  triplet fermion  $N$ , two  $A_4$  triplet scalars  $\varphi_E$  and  $\varphi_N$ , and three  $A_4$  singlet scalars  $\xi$ ,  $\xi'$ , and  $\xi''$ . In order to exclude unwanted interactions two additional symmetries, namely,  $Z_3$  and  $Z_4$ , are imposed, and, as a result, the model is based on an  $A_4 \times Z_3 \times Z_4$  flavor symmetry times the SM symmetry (see Table I for more detailed group transformations of the lepton and scalar fields in this extended model). For generality we consider a scalar sector containing all possible representations of  $A_4$ . The presence of the fields  $\xi'$  and  $\xi''$  guarantees a nontrivial mass matrix of the charged leptons; otherwise, the latter would become massless. The corresponding neutrino mass matrix can be developed perturbatively around a neutrino mass matrix diagonalizable by a TBM mixing matrix. As a consequence, a relation between the Dirac CPV phase  $\delta_{CP}$  and the mixing angles  $\theta_{ij}$ ,  $i, j = 1, 2, 3$  (for a three-neutrino mixing model) are established. Based on the experimental data of the mixing angles, this relation allows us to determine  $\delta_{CP}$  numerically in both normal ordering (NO) and inverse ordering (IO). It is very important as the existence of a Dirac CPV phase indicates a difference between the probabilities  $P(\nu_l \rightarrow \nu_{l'})$  and  $P(\bar{\nu}_l \rightarrow \bar{\nu}_{l'})$ ,  $l \neq l'$ , of the neutrino and antineutrino transitions (oscillations) in vacuum  $\nu_l \rightarrow \nu_{l'}$  and  $\bar{\nu}_l \rightarrow \bar{\nu}_{l'}$ , respectively, thus, a  $CP$  violation in the

TABLE I. Lepton and scalar sectors of the model and their group transformations, where  $\omega^k = e^{2k\pi/3}$ ,  $k = 0, 1, 2$ .

	$\ell_L$	$\tilde{e}_R$	$\tilde{\mu}_R$	$\tilde{\tau}_R$	$\phi_h$	$N$	$\varphi_E$	$\varphi_N$	$\xi$	$\xi'$	$\xi''$
$SU(2)_L$	2	1	1	1	2	1	1	1	1	1	1
$A_4$	3	1	1'	1''	1	3	3	3	1	1'	1''
$Z_3$	$\omega^2$	1	1	1	$\omega^2$	$\omega$	1	$\omega$	$\omega$	$\omega$	$\omega^2$
$Z_4$	i	1	1	1	1	i	i	-1	-1	i	i

neutrino subsector of the lepton sector. We should note that for a three-neutrino mixing model, as considered in this paper, the mixing matrix in general has one Dirac and two Majorana CPV phases [51] (for a more general,  $n$ -neutrino mixing, case, see [52,53]). Since the Majorana CPV phases do not effect these transition probabilities they are not a subject of detailed analysis here.

In the framework of the suggested model and the perturbation method our approach allows us to obtain  $\delta_{CP}$  within the  $1\sigma$  region of the best fit value [33]. This approach is different but our result is quite consistent with that obtained by other authors (see, for example, [54–60] and references therein). Further, knowing  $\delta_{CP}$  we can determine the Jarlskog parameter ( $J_{CP}$ ) measuring a  $CP$  violation. The determination of  $\delta_{CP}$  and  $J_{CP}$  represents an application of the present model and, in this way, verifies the latter (of course, it is not a complete verification). A numerical test of the model gives values of the neutrino masses, the mixing angle  $\theta_{13}$  and the Dirac  $CP$ -violation phase consistent with the current experimental results.

This paper has the following plan. A brief introduction to the representations of  $A_4$  and their application to building an extended standard model is made in the next section. Neutrino masses and mixing within this model are considered in Sec. III via a perturbation method. Section IV is devoted to the investigation of the Dirac CPV phase and Jarlskog parameter. The last section is designed for some discussions and conclusions.

## II. EXTENDED STANDARD MODEL WITH AN $A_4 \times Z_3 \times Z_4$ FLAVOR SYMMETRY

Here, we deal with an extended SM acquiring an additional  $A_4$  flavor symmetry. An extra  $Z_3 \times Z_4$  symmetry is also introduced to constrain the model not to deviate too much from the SM. As mentioned above, the flavor symmetry, in particular, that is based on the group  $A_4$ , has attracted much interest during the last ten years or so (see [15,16] for a review). Let us first summarize here representations of  $A_4$  [15,22,61] and then review briefly the model that is considered.

### A. Summary of representations of $A_4$

The group  $A_4$  is a group of even permutations on four objects and thus it has 12 elements ( $12 = 4!/2$ ). This group is also called the tetrahedral group as it can describe the orientation-preserving symmetry of a regular tetrahedron. It can be generated by two basic permutations  $S$  and  $T$  having properties

$$S^2 = T^3 = (ST)^3 = 1. \quad (1)$$

The group representations are relatively simple and include three one-dimensional unitary representations 1, 1', and 1'' with the generators  $S$  and  $T$  given, respectively, as follows,

$$1: S = 1, \quad T = 1, \quad (2a)$$

$$1': S = 1, \quad T = e^{i2\pi/3} \equiv \omega, \quad (2b)$$

$$1'': S = 1, \quad T = e^{i4\pi/3} \equiv \omega^2, \quad (2c)$$

and a three-dimensional unitary representation with the generators

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \quad (3)$$

Here we use the three-dimensional representation where the generator  $T$  has a diagonal form [18]. The reason for choosing this representation is that the latter ensures the diagonal mass matrix of the charged leptons (see the next section).

Representation theory and applications of a group often require knowing a multiplication and decomposition rule of a product of its (irreducible) representations. In the case of  $A_4$  these rules read

$$1 \times 1 = 1, \quad (4a)$$

$$1' \times 1'' = 1, \quad (4b)$$

$$1'' \times 1' = 1, \quad (4c)$$

$$1' \times 1' = 1'', \quad (4d)$$

$$1'' \times 1'' = 1', \quad (4e)$$

$$3 \times 3 = 1 + 1' + 1'' + 3_s + 3_a. \quad (4f)$$

While the first five rules are trivial, let us give more explicit expressions for the multiplication and decomposition rule for a product (4f) between two triplets, say  $3_a \sim (a_1, a_2, a_3)$  and  $3_b \sim (b_1, b_2, b_3)$ . This direct product can be decomposed into three singlets and two triplets as follows:

$$1 = a_1 b_1 + a_2 b_3 + a_3 b_2, \quad (5a)$$

$$1' = a_3 b_3 + a_1 b_2 + a_2 b_1, \quad (5b)$$

$$1'' = a_2 b_2 + a_1 b_3 + a_3 b_1, \quad (5c)$$

$$3_s \sim \frac{1}{3} (2a_1 b_1 - a_2 b_3 - a_3 b_2, 2a_3 b_3 - a_1 b_2 - a_2 b_1, 2a_2 b_2 - a_1 b_3 - a_3 b_1), \quad (5d)$$

$$3_a \sim \frac{1}{3} (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_1 b_3 - a_3 b_1). \quad (5e)$$

The above-given information is used for the construction of a Lagrangian, as the one in (24), of a model with an  $A_4$  symmetry.

## B. The model

Compared with the SM, the model studied here contains an extended lepton and scalar sector (the quark sector is not considered here yet). The lepton sector includes an  $A_4$  triplet  $N$  (its components are referred to as right-handed neutrinos), which is an isosinglet, in addition to the SM leptons among which the left-handed lepton isodoublets  $\ell_L$ ,  $\ell = \tilde{e}, \tilde{\mu}, \tilde{\tau}$ , all together form an  $A_4$  triplet, while the right-handed lepton isosinglets  $\tilde{e}_R, \tilde{\mu}_R$ , and  $\tilde{\tau}_R$  transform as  $A_4$  singlets 1, 1', and 1'', respectively. In general, the basis  $\ell = \tilde{e}, \tilde{\mu}, \tilde{\tau}$ , in which the charged lepton mass matrix may not be diagonal, is different from the standard basis of the mass states  $l = e, \mu, \tau$ . Besides the original SM Higgs field  $\phi_h$ , which is an  $A_4$  singlet, the scalar sector of the model has five additional isosinglet fields: two  $A_4$  triplets  $\varphi_E$  and  $\varphi_N$ , and three  $A_4$  singlets  $\xi, \xi', \xi''$ . Our choice of the model field content thus covers all irreducible representations of  $A_4$ . To keep maximally the SM interaction structure (as many of its consequences have been experimentally verified very well) an additional  $Z_3 \times Z_4$  symmetry is introduced. The transformation rules under  $SU(2)_L, A_4, Z_3$ , and  $Z_4$  of the leptons and the scalars in this model are summarized in Table I. Let us look at a closer distance the scalar and the lepton sector.

### 1. Scalar sector

The scalar potential has the form

$$V(\phi_h, \varphi_E, \varphi_N, \xi, \xi', \xi'') = V_1(\phi_h) + V_2(\varphi_E, \xi', \xi'') + V_3(\varphi_N, \phi_h, \xi, \xi', \xi'') + V_4(\xi, \phi_h), \quad (6)$$

with

$$V_1(\phi_h) = \mu_h^2 (\phi_h^\dagger \phi_h) + \lambda_h (\phi_h^\dagger \phi_h)^2, \quad (7)$$

$$V_2(\varphi_E, \xi', \xi'') = \alpha_1 (\varphi_E \varphi_E)_1 (\varphi_E \varphi_E)_1 + \alpha_2 (\varphi_E \varphi_E)_{1'} (\varphi_E \varphi_E)_{1''} + \alpha_3 (\varphi_E \varphi_E)_{3_s} (\varphi_E \varphi_E)_{3_s} + \alpha_4 (\varphi_E \varphi_E)_{3_a} (\varphi_E \varphi_E)_{3_a} + \alpha_5 (\varphi_E \varphi_E)_{3_s} (\varphi_E \varphi_E)_{3_a} + \left[ \frac{\alpha_6}{2} (\varphi_E \varphi_E)_1 (\xi' \xi'')_1 + \text{H.c.} \right], \quad (8)$$

$$V_3(\varphi_N, \phi_h, \xi, \xi', \xi'') = \mu^2 (\varphi_N^\dagger \varphi_N)_1 + \lambda_1 (\varphi_N^\dagger \varphi_N)_1^2 + 2\lambda_2 (\varphi_N^\dagger \varphi_N)_{1'} (\varphi_N^\dagger \varphi_N)_{1''} + \lambda_3 (\varphi_N^\dagger \varphi_N)_{3_s} (\varphi_N^\dagger \varphi_N)_{3_s} + \lambda_4 (\varphi_N^\dagger \varphi_N)_{3_a} (\varphi_N^\dagger \varphi_N)_{3_a} + 2\lambda_5 (\varphi_N^\dagger \varphi_N)_{3_s} (\varphi_N^\dagger \varphi_N)_{3_a} + \gamma_1 (\varphi_N^\dagger \varphi_N)_1 (\xi^\dagger \xi)_1 + \gamma_2 (\varphi_N^\dagger \varphi_N)_{1'} (\xi''^\dagger \xi'')_{1'} + \gamma_3 (\varphi_N^\dagger \varphi_N)_{1''} (\xi'^\dagger \xi')_{1''} + \gamma (\varphi_N^\dagger \varphi_N)_1 (\phi_h^\dagger \phi_h)_1, \quad (9)$$

and

$$V_4(\xi, \phi_h) = \eta_1^2(\xi^\dagger \xi)_1 + \chi_1(\xi^\dagger \xi)_1^2 + \chi_2(\xi^\dagger \xi)_1(\phi_h^\dagger \phi_h)_1. \quad (10)$$

Here, the coefficients  $\lambda_2$  and  $\lambda_5$  are multiplied by 2 just for further convenience. The additional  $Z_3 \times Z_4$  symmetry is introduced in order to avoid interactions between the scalar fields  $\varphi_E$  and  $\varphi_N$ , which would be

$$V_5(\varphi_E, \varphi_N) = \rho_1(\varphi_E \varphi_E)_{\mathbf{3}_s}(\varphi_N^\dagger \varphi_N)_{\mathbf{3}_s} + \rho_2(\varphi_E \varphi_E)_{\mathbf{3}_s}(\varphi_N^\dagger \varphi_N)_{\mathbf{3}_a} + \rho_3(\varphi_E \varphi_E)_{\mathbf{3}_a}(\varphi_N^\dagger \varphi_N)_{\mathbf{3}_s} + \rho_4(\varphi_E \varphi_E)_{\mathbf{3}_a}(\varphi_N^\dagger \varphi_N)_{\mathbf{3}_a} \\ + \rho_5(\varphi_E \varphi_E)_1(\varphi_N^\dagger \varphi_N)_1 + \rho_6(\varphi_E \varphi_E)_{1'}(\varphi_N^\dagger \varphi_N)_{1'} + \rho_7(\varphi_E \varphi_E)_{1''}(\varphi_N^\dagger \varphi_N)_{1''} + \text{H.c.}, \quad (11)$$

$$V_6(\varphi_E, \varphi_N) = \kappa_1(\varphi_E \varphi_E)_{\mathbf{3}_s} \varphi_N + \kappa_2(\varphi_E \varphi_E)_{\mathbf{3}_a} \varphi_N \\ + \kappa_3(\varphi_N^\dagger \varphi_N)_{\mathbf{3}_s} \varphi_E + \kappa_4(\varphi_N^\dagger \varphi_N)_{\mathbf{3}_a} \varphi_E + \text{H.c.}, \quad (12)$$

and Yukawa interactions involving  $\varphi_E$ ,  $\varphi_N$ , and charged leptons,

$$-\mathcal{L}_Y^f = \lambda_e^f(\bar{l}_L \phi_h) \tilde{e}_R \frac{\varphi_N}{\Lambda} + \lambda_\mu^f(\bar{l}_L \phi_h) \tilde{\mu}_R \frac{\varphi_N}{\Lambda} \\ + \lambda_\tau^f(\bar{l}_L \phi_h) \tilde{\tau}_R \frac{\varphi_N}{\Lambda} + g_N^f(\bar{N}^c N) \varphi_E + \text{H.c.}, \quad (13)$$

because such interactions would destroy too much the charged lepton mass structure, which is already described relatively well by the SM (see more below).

Let us denote the VEVs of these scalar fields  $\xi$ ,  $\xi'$ ,  $\xi''$ ,  $\varphi_E := (\phi_1, \phi_2, \phi_3)$  and  $\varphi_N := (\varphi_1, \varphi_2, \varphi_3)$  as follows:

$$\langle \xi \rangle = \sigma_a, \quad \langle \xi' \rangle = \sigma_b, \quad \langle \xi'' \rangle = \sigma_c, \\ \langle \phi_h \rangle = v_h, \quad \langle \varphi_E \rangle = (v_1, v_2, v_3), \quad \langle \varphi_N \rangle = (u_1, u_2, u_3). \quad (14)$$

Getting a VEV of  $\varphi_E = (\phi_1, \phi_2, \phi_3)$  imposes an extremum condition on the potential  $V$ ,

$$\left. \frac{\partial V}{\partial \phi_i} \right|_{\langle \phi_i \rangle = v_i} = 0, \quad (i = 1, 2, 3), \quad (15)$$

leading to the equation system of  $v_i$ ,

$$\begin{cases} 2(\alpha_1 + \alpha'_3)v_1^3 + (\alpha_2 - \alpha'_3)(v_2^3 + v_3^3) + 4(\alpha_1 + \alpha_2)v_1v_2v_3 + \alpha_6v_1\sigma_b\sigma_c = 0, \\ 2(\alpha_1 + \alpha_2)v_1^2v_3 + 3(\alpha_2 - \alpha'_3)v_1v_2^2 + (4\alpha_1 + \alpha_2 + 3\alpha'_3)v_2v_3^2 + \alpha_6v_3\sigma_b\sigma_c = 0, \\ 2(\alpha_1 + \alpha_2)v_1^2v_2 + 3(\alpha_2 - \alpha'_3)v_1v_3^2 + (4\alpha_1 + \alpha_2 + 3\alpha'_3)v_2^2v_3 + \alpha_6v_2\sigma_b\sigma_c = 0, \end{cases} \quad (16)$$

where

$$\alpha'_3 = \frac{4\alpha_3}{9}, \quad \alpha_{6,} = \frac{1}{2}(\alpha_6 + \alpha_6^*). \quad (17)$$

In principle, this equation system has several solutions but we choose the one satisfying the equality

$$v_1^2 = v^2 = \frac{-\alpha_{6,} \sigma_b \sigma_c}{2(\alpha_1 + \alpha'_3)}, \quad v_2 = v_3 = 0, \quad (18)$$

in order to get, as shown below, a diagonalized mass matrix of the charged leptons. We note that if the fields  $\xi'$  and  $\xi''$  are excluded from the model, the VEV in (18) becomes a trivial one,  $v_1 = v_2 = v_3 = 0$ , leading, as seen in (25), to massless charged leptons.

Next, for the VEV of  $\varphi_N = (\varphi_1, \varphi_2, \varphi_3)$  we have the equations

$$\begin{cases} \lambda_0 u_1 + 2(\lambda_1 + \lambda'_3)u_1^3 + (2\lambda_2 - \lambda'_3 + \lambda'_5)(u_2^3 + u_3^3) + 2(2\lambda_1 + 4\lambda_2 - \lambda'_5)u_1u_2u_3 + \beta_2u_3 + \beta_3u_2 = 0, \\ \lambda_0 u_3 + 2(\lambda_1 + 2\lambda_2 + \lambda'_5)u_1^2u_3 + (6\lambda_2 - 3\lambda'_3 - \lambda'_5)u_1u_2^2 + (4\lambda_1 + 2\lambda_2 + 3\lambda'_3 - \lambda'_5)u_2u_3^2 + \beta_2u_2 + \beta_3u_1 = 0, \\ \lambda_0 u_2 + 2(\lambda_1 + 2\lambda_2)u_1^2u_2 + (6\lambda_2 - 3\lambda'_3 - \lambda'_5)u_1u_3^2 + (4\lambda_1 + 2\lambda_2 + 3\lambda'_3 + \lambda'_5)u_2^2u_3 + \beta_2u_1 + \beta_3u_3 = 0, \end{cases} \quad (19)$$

where

$$\lambda_0 = \mu^2 + \gamma_1 \sigma_a^2 + \gamma v_h^2, \quad \lambda'_3 = \frac{4\lambda_3}{9}, \quad \lambda'_5 = \frac{4\lambda_5}{9}, \quad (20)$$

$$\beta_2 = \gamma_2 \sigma_c^2, \quad \beta_3 = \gamma_3 \sigma_b^2. \quad (21)$$

This equation system has a special solution with

$$u_1^2 = u_2^2 = u_3^2 = -\frac{\lambda_0 + \beta_2 + \beta_3}{6(\lambda_1 + 2\lambda_2)} \equiv u^2 \quad (22)$$

and another solution with

$$u_1 \neq u_2 \neq u_3 \neq u_1, \quad (23)$$

which, however, has too long of an expression to be written down here (in fact, we do not need its explicit analytical expression but below numerical calculations are done). As we see later, the solution (22) leads to a TBM model, while the solution (23) leads to a non-TBM model.

## 2. Lepton sector

Based on the  $A_4 \times Z_3 \times Z_4$  flavor symmetry we can construct the following Yukawa terms of the effective Lagrangian for the lepton sector of the present model:

$$-\mathcal{L}_Y = \lambda_e (\bar{l}_L \phi_h) \tilde{e}_R \frac{\varphi_E}{\Lambda} + \lambda_\mu (\bar{l}_L \phi_h)' \tilde{\mu}_R \frac{\varphi_E}{\Lambda} + \lambda_\tau (\bar{l}_L \phi_h)' \tilde{\tau}_R \frac{\varphi_E}{\Lambda} + \lambda_D \bar{\ell}_L \tilde{\phi}_h N + g_N (\bar{N}^c N) \varphi_N + g_\xi (\bar{N}^c N)_1 \xi + \text{H.c.} \quad (24)$$

From this Lagrangian we get the following mass matrix of the charged leptons:

$$M_l = v_h \begin{pmatrix} \frac{\lambda_e v_1}{\Lambda} & \frac{\lambda_\mu v_2}{\Lambda} & \frac{\lambda_\tau v_3}{\Lambda} \\ \frac{\lambda_e v_3}{\Lambda} & \frac{\lambda_\mu v_1}{\Lambda} & \frac{\lambda_\tau v_2}{\Lambda} \\ \frac{\lambda_e v_2}{\Lambda} & \frac{\lambda_\mu v_3}{\Lambda} & \frac{\lambda_\tau v_1}{\Lambda} \end{pmatrix}. \quad (25)$$

As explained above, we choose the VEV alignment (18),

$$\langle \varphi_E \rangle = (v, 0, 0). \quad (26)$$

This choice of the VEV of  $\varphi_E$  breaks the symmetry  $A_4$  down to its subgroup  $G_S$  [19]. The corresponding charged lepton mass matrix automatically has a diagonal form,

$$M_l = \begin{pmatrix} y_e v_h & 0 & 0 \\ 0 & y_\mu v_h & 0 \\ 0 & 0 & y_\tau v_h \end{pmatrix}, \quad (27)$$

where

$$y_e = \frac{\lambda_e v}{\Lambda}, \quad y_\mu = \frac{\lambda_\mu v}{\Lambda}, \quad y_\tau = \frac{\lambda_\tau v}{\Lambda}. \quad (28)$$

It is obvious that  $v$  must be nonzero ( $v \neq 0$ ); otherwise, the charged leptons are massless (this case happens when  $\xi'$  and  $\xi''$  are absent or they develop no VEV).

For the neutrino mass matrix, the Majorana part  $M_N$  and the Dirac part  $M_D$  are respectively

$$M_N = \begin{pmatrix} 2b_1 + d & -b_3 & -b_2 \\ -b_3 & 2b_2 & -b_1 + d \\ -b_2 & -b_1 + d & 2b_3 \end{pmatrix}, \quad (29)$$

and

$$M_D = \lambda_D v_h \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (30)$$

where

$$d = 2g_\xi \sigma_a, \quad b_1 = \frac{2}{3} g_N u_1, \quad b_2 = \frac{2}{3} g_N u_2, \quad b_3 = \frac{2}{3} g_N u_3. \quad (31)$$

From the seesaw mechanism [10–12,52,62–65], we get a neutrino mass matrix of the form

$$M_\nu = -M_D^T M_N^{-1} M_D. \quad (32)$$

As the scale of  $M_M$  is very large but not fixed yet [however, the relative scale (32) is important] we can work, for a further convenience, in a scale where  $M_D$  is normalized to 1, that is,  $(\lambda_D v_h)^2 \sim 1$ . It is not difficult to see that for the VEV alignment  $u_1 = u_2 = u_3 = u$  in (22), that is,  $b_1 = b_2 = b_3 \equiv b$ , the matrix (32) has the form

$$M_{\nu 0} = \frac{1}{\mathcal{D}_0} \begin{pmatrix} 3b^2 + 2bd - d^2 & -3b^2 + bd & -3b^2 + bd \\ -3b^2 + bd & 3b^2 + 2bd & 3b^2 - bd - d^2 \\ -3b^2 + bd & 3b^2 - bd - d^2 & 3b^2 + 2bd \end{pmatrix} \equiv \frac{1}{\mathcal{D}_0} M'_0, \quad (33)$$

where  $\mathcal{D}_0 \equiv \det(M_N^0)$ , taking the value

$$\mathcal{D}_0 = 9b^2 d - d^3, \quad (34)$$

is the determinant  $\mathcal{D} \equiv \det(M_N)$  of the matrix  $M_N$  for  $u_1 = u_2 = u_3$ . It can be checked that the mass matrix  $M_{\nu 0}$ , as noted above, can be diagonalized by the TBM matrix (up to a phase factor)

$$U_{\text{tbm}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \quad (35)$$

For the VEV alignment  $u_1 \neq u_2 \neq u_3 \neq u_1$  of  $\varphi_N$  in (23) the neutrino mass has a general form

$$M_\nu = -M_D^T M_N^{-1} M_D = \begin{pmatrix} A & B & C \\ B & E & D \\ C & D & F \end{pmatrix}, \quad (36)$$

where  $A, B, C, D, E,$  and  $F$  in general are complex numbers but here we do not need their explicit expressions. One of the key problems of a neutrino mass and mixing model is diagonalizing the corresponding neutrino mass matrix. Customarily, instead of (36), the matrix

$$\mathbb{M}_\nu \equiv M_\nu M_\nu^\dagger \quad (37)$$

must be diagonalized. Let  $U_{\text{pmns}}$  be the matrix diagonalizing the matrix (37),

$$\text{diag}(\mathbb{M}_\nu) = U_{\text{pmns}}^\dagger \mathbb{M}_\nu U_{\text{pmns}}. \quad (38)$$

$$U_{\text{pmns}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (39)$$

where  $s_{ij} = \sin \theta_{ij}$ ,  $c_{ij} = \cos \theta_{ij}$  with  $\theta_{ij} \in [0, \pi/2]$  being mixing angles, and  $\delta \equiv \delta_{CP} \in [0, 2\pi]$  being the Dirac CPV phase. In a TBM model (for which  $s_{13} = 0$ ,  $s_{23}^2 = \frac{1}{2}$ ,  $s_{12}^2 = \frac{1}{3}$ ) this matrix  $U_{\text{pmns}}$  becomes the matrix  $U_{\text{tbm}}$  in (35). Here we work with the choice  $s_{23} = -\sqrt{\frac{1}{2}}$ ,  $s_{12} = \sqrt{\frac{1}{3}}$  but another choice, for example,  $s_{23} = \sqrt{\frac{1}{2}}$ ,  $s_{12} = \sqrt{\frac{1}{3}}$ , can be made.

The current experimental data ( $\theta_{13} \approx 9^\circ$ ,  $\theta_{23} \approx 42^\circ$ ,  $\theta_{12} \approx 33^\circ$ ) [32] show that the matrix  $U_{\text{pmns}}$  can be obtained from  $U_{\text{tbm}}$  by a small correction as seen from their difference,

$$\begin{pmatrix} 0.006 & -0.029 & 0.153e^{-i\delta} \\ -0.008 + 0.084e^{i\delta} & 0.047 + 0.056e^{i\delta} & 0.054 \\ 0.041 - 0.095e^{i\delta} & -0.027 - 0.064e^{i\delta} & 0.034 \end{pmatrix}. \quad (40)$$

Therefore, we can consider  $U_{\text{pmns}}$  as a perturbative development around  $U_{\text{tbm}}$ . This requirement imposes a restriction on the construction of a model, in particular, on its parameters. Working in the basis of the diagonalized charged lepton mass matrix (i.e., in the basis  $l = e, \mu, \tau$ ) and with a neutrino mixing matrix treated as a small deviation from the TBM form, one can write a perturbative expansion of  $\mathbb{M}_\nu$  around a nonperturbative TBM mass matrix  $\mathbb{M}_0$ , which can be diagonalized (cf., [45,48]),

$$U_{\text{TBM}}^\dagger \mathbb{M}_0 U_{\text{TBM}} = \text{diag}(|m_{01}|^2, |m_{02}|^2, |m_{03}|^2), \quad (41)$$

Here,  $U_{\text{pmns}}$  is a mixing matrix, which may differ from the PMNS matrix, denoted as  $U_{\text{PMNS}}$ , by a phase factor. It is a difficult task to find a realistic (phenomenological) model to realize  $U_{\text{pmns}}$ , i.e.,  $U_{\text{PMNS}}$ . To solve this problem, different methods and tricks have been used. Since, as discussed earlier,  $U_{\text{pmns}}$  slightly differs from the TBM form (35) we follow a perturbation approach. This approach allows us to find a theoretical mixing matrix, say  $U$ , which must be compared with the empirical PMNS matrix.

### III. NEUTRINO MASSES AND MIXING

The standard (three) neutrino mixing matrix, the PMNS matrix, has the canonical form (up to a diagonal phase matrix to be specified below)

by the matrix

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} \times P_0 \sim (|1^0\rangle, |2^0\rangle, |3^0\rangle), \quad (42)$$

where  $m_{0i}$ ,  $i = 1, 2, 3$ , are nonperturbative masses, and

$$P_0 = \text{diag}\left(e^{i\frac{\alpha_{01}}{2}}, e^{i\frac{\alpha_{02}}{2}}, 1\right), \quad (43)$$

with  $\alpha_{01}$  and  $\alpha_{02}$  being Majorana phases. We note that  $U_{\text{tbm}}$  given in (35) differs from  $U_{\text{TBM}}$  (42), used frequently in the literature, by the factor  $P_0$ . Thus,  $M_\nu$  in (36) can be written as

$$M_\nu = M_0 + \mathcal{V}, \quad (44)$$

with

$$M_0 = \frac{M'_0}{\mathcal{D}}, \quad \mathcal{D} = \det(M_N), \quad (45)$$

where  $M'_0$  is defined in (33) and  $\mathcal{V}$  is a small matrix to be specified below. At the first order of perturbation the matrix  $\mathbb{M}$  is developed around  $\mathbb{M}_0$  as follows:

$$\mathbb{M}_\nu = \mathbb{M}_0 + (M_0^\dagger \mathcal{V} + \mathcal{V}^\dagger M_0). \quad (46)$$

Thus, the squared masses  $|m_i|^2$  obtained by a diagonalization of  $\mathbb{M}$  represent a perturbative shift

$$|m_i|^2 = |m_{0i}|^2 + \delta|m_i|^2 \quad (47)$$

from the nonperturbative squared masses  $|m_{0i}|^2$ , where  $m_{0i}$  now have the form

$$\begin{aligned} m_{01} &= \frac{(3b-d)d}{\mathcal{D}}, & m_{02} &= \frac{9b^2-d^2}{\mathcal{D}}, \\ m_{03} &= \frac{(3b+d)d}{\mathcal{D}}. \end{aligned} \quad (48)$$

Since a homogeneous VEV alignment  $\langle \varphi_N \rangle = (u, u, u)$  such as that in (22) leads to a TBM mixing but the experiment tells us a mixing slightly deviating from the TBM one, we must consider an inhomogeneous VEV alignment (23) to deviate from a homogeneous alignment with an appropriate amount, that is,

$$(u_1, u_2, u_3) = (u_1, u_1 + \epsilon_2, u_1 + \epsilon_3), \quad (49)$$

where  $(0, \epsilon_2, \epsilon_3)$  is an appropriate shift of  $\langle \varphi_N \rangle$  from the level  $(u_1, u_1, u_1)$ . It can be shown that it is enough this shift

to obey the condition  $\epsilon_2, \epsilon_3 \ll \mathcal{D}/g_N$ , if not stronger,  $\epsilon_2, \epsilon_3 \ll 1$ . The latter can be satisfied if  $\lambda_1, \lambda_2, \lambda'_3$ , and  $\lambda'_5$  are chosen to have the same order of magnitude but much bigger than that of  $\lambda_0$ , i.e.,

$$\lambda_0 \ll \lambda_1 \approx \lambda_2 \approx \lambda'_3 \approx \lambda'_5 \equiv \lambda \quad (50)$$

as well as  $\beta_2$  and  $\beta_3$  being chosen to be at the same order of magnitude but much smaller than that of  $\lambda$ , i.e.,

$$\beta_2 \approx \beta_3 \ll \lambda. \quad (51)$$

It is observed from (31) that an alignment  $(u_1, u_2, u_3)$  is proportional to an alignment  $(b_1, b_2, b_3)$ ; therefore, a homogeneous alignment  $(b, b, b)$  corresponds to a TBM mixing. That means that a realistic alignment  $(b_1, b_2, b_3)$  must deviate from a homogeneous alignment by only a small amount,

$$(b_1, b_2, b_3) = (b_1, b_1 + e_2, b_1 + e_3), \quad (52)$$

where  $e_2, e_3 \ll \mathcal{D}$  [see (69) below for a numerical illustration]. Taking into account (50)–(52) we get

$$\mathcal{V} = \frac{1}{\mathcal{D}} \begin{pmatrix} 4b(e_2 + e_3) & -de_3 + b(4e_2 + e_3) & -de_2 + b(e_2 + 4e_3) \\ -de_3 + b(4e_2 + e_3) & 4be_2 + 2de_2 - 2be_3 & b(e_2 + e_3) \\ -de_2 + b(e_2 + 4e_3) & b(e_2 + e_3) & 4be_3 + 2de_3 - 2be_2 \end{pmatrix}. \quad (53)$$

Now a perturbation expansion is made around the TBM state (42). Here, we follow the perturbative approach described in [66]. Using the perturbation decomposition

$$|n\rangle = |n^0\rangle + \sum_{k \neq n} a_{kn} |k^0\rangle + \dots, \quad (54)$$

with  $|n^0\rangle$  defined in (42) and

$$\begin{aligned} a_{kn} &= (|m_{0n}|^2 - |m_{0k}|^2)^{-1} V_{kn}, \\ V_{kn} &= \langle k^0 | M_0^\dagger \mathcal{V} + \mathcal{V}^\dagger M_0 | n^0 \rangle, \end{aligned} \quad (55)$$

one can diagonalize the matrix  $\mathbb{M}_\nu$ ,

$$U^\dagger \mathbb{M}_\nu U = \text{diag}(|m_1|^2, |m_2|^2, |m_3|^2), \quad (56)$$

by the matrix

$$\begin{aligned} U &= U_{\text{TBM}} + \Delta U \\ &= \begin{pmatrix} \sqrt{\frac{2}{3}} + \Delta U_{11} & \sqrt{\frac{1}{3}} + \Delta U_{12} & \Delta U_{13} \\ -\sqrt{\frac{1}{6}} + \Delta U_{21} & \sqrt{\frac{1}{3}} + \Delta U_{22} & -\sqrt{\frac{1}{2}} + \Delta U_{23} \\ -\sqrt{\frac{1}{6}} + \Delta U_{31} & \sqrt{\frac{1}{3}} + \Delta U_{32} & \sqrt{\frac{1}{2}} + \Delta U_{33} \end{pmatrix} \times P_0, \end{aligned} \quad (57)$$

representing a perturbative expansion from  $U_{\text{TBM}}$  in (35), where (up to the first perturbation order)

$$\begin{aligned} \Delta U_{11} &= \sqrt{\frac{1}{3}} X^*, & \Delta U_{12} &= -\sqrt{\frac{2}{3}} X, \\ \Delta U_{13} &= -\sqrt{\frac{2}{3}} Y - \sqrt{\frac{1}{3}} Z, & \Delta U_{21} &= \sqrt{\frac{1}{3}} X^* - \sqrt{\frac{1}{2}} Y^*, \\ \Delta U_{22} &= \sqrt{\frac{1}{6}} X - \sqrt{\frac{1}{2}} Z^*, & \Delta U_{23} &= \sqrt{\frac{1}{6}} Y - \sqrt{\frac{1}{3}} Z, \\ \Delta U_{31} &= \sqrt{\frac{1}{3}} X^* + \sqrt{\frac{1}{2}} Y^*, & \Delta U_{32} &= \sqrt{\frac{1}{6}} X + \sqrt{\frac{1}{2}} Z^*, \\ \Delta U_{33} &= \sqrt{\frac{1}{6}} Y - \sqrt{\frac{1}{3}} Z, \end{aligned} \quad (58)$$

and

$$X = -a_{12}, \quad Y = -a_{13}, \quad Z = -a_{23}. \quad (59)$$

We note that the parameters  $a_{ij}$  defined in (55) and appearing in  $\Delta U$  are determined from the elements of the matrix  $\mathcal{V}$  in (53) derived under the condition (52) leading to imposing constraints (50) and (51) on the model parameters.

TABLE II. Experimental data for a NO and an IO [32,33].

Parameter	Best fit	$1\sigma$ range	$2\sigma$ range	$3\sigma$ range
$\Delta m_{21}^2/10^{-5} \text{ eV}^2$ (NO or IO)	7.54	7.32–7.80	7.15–8.00	6.99–8.18
$\sin^2\theta_{12}/10^{-1}$ (NO or IO)	3.08	2.91–3.25	2.75–3.42	2.59–3.59
$\Delta m_{31}^2/10^{-3} \text{ eV}^2$ (NO)	2.47	2.41–2.53	2.34–2.59	2.27–2.65
$ \Delta m_{32}^2 /10^{-3} \text{ eV}^2$ (IO)	2.42	2.36–2.48	2.29–2.55	2.23–2.61
$\sin^2\theta_{13}/10^{-2}$ (NO)	2.34	2.15–2.54	1.95–2.74	1.76–2.95
$\sin^2\theta_{13}/10^{-2}$ (IO)	2.40	2.18–2.59	1.98–2.79	1.78–2.98
$\sin^2\theta_{23}/10^{-1}$ (NO)	4.37	4.14–4.70	3.93–5.52	3.74–6.26
$\sin^2\theta_{23}/10^{-1}$ (IO)	4.55	4.24–5.94	4.00–6.20	3.80–6.41

To check how the model works, let us make a numerical analysis. It is enough (and for simplicity) to assume the parameters  $g_N, d, \lambda_0, \lambda_1$  to be real. Under this assumption the equation system (19) has 27 solutions  $(u_1, u_2, u_3)$  belonging to the following four types:

$$\text{Type 1: } (0, 0, 0), \quad \text{i.e., } u_1 = u_2 = u_3 = 0, \quad (60)$$

$$\text{Type 2: } (u, 0, 0), \quad u \neq 0, \quad (61)$$

$$\text{Type 3: } (u, u, u), \quad u \neq 0, \quad (62)$$

$$\text{Type 4: } (u_1, u_2, u_3); \quad u_1 \neq u_2 \neq u_3 \neq u_1, \quad u_i \neq 0. \quad (63)$$

It is observed that the solutions of type 1, type 2, and type 3 do not lead to the PMNS mixing as desired (where only the type-3 solutions give the TBM mixing); therefore, they are excluded from our consideration and only the solutions of type 4 remain as a choice.

One of the type-4 solutions having the form

$$(u_1, u_2, u_3) = \left( -(0.14 + 0.28i)\sqrt{\frac{\lambda_0}{\lambda}}, -(0.019 - 0.32i)\sqrt{\frac{\lambda_0}{\lambda}}, \right. \\ \left. -(0.17 - 0.26i)\sqrt{\frac{\lambda_0}{\lambda}} \right) \quad (64)$$

gives a result consistent with the current experimental data (see below). It follows that

$$(b_1, b_2, b_3) = (-(0.14 + 0.28i)K, \\ -(0.019 - 0.32i)K, -(0.17 - 0.26i)K), \\ K = \frac{2}{3}g_N\sqrt{\frac{\lambda_0}{\lambda}}. \quad (65)$$

The neutrino masses (47) now get the form [66]

$$m_1^2 = m_{01}^2 + V_{11}, \quad m_2^2 = m_{02}^2 + V_{22}, \\ m_3^2 = m_{03}^2 + V_{33}, \quad (66)$$

where  $V_{ii}$  are given in (55), namely,

$$V_{ii} = \langle i^0 | M_0^\dagger \mathcal{V} + \mathcal{V}^\dagger M_0 | i^0 \rangle, \quad i = 1, 2, 3.$$

Using the experimental data for the squared mass differences  $\Delta m_{21}^2$  and  $\Delta m_{32}^2$  (see Table II below),

$$\Delta m_{21}^2 = m_2^2 - m_1^2 = 7.54 \times 10^{-5}, \\ \Delta m_{31}^2 = m_3^2 - m_1^2 = 2.47 \times 10^{-3}, \quad (67)$$

we can find  $K$  in (65) and  $d$  in (29). Here, for a demonstration, we work with a normal neutrino mass ordering, but the case with an inverse neutrino mass ordering is similar. Since the equations (67) are nonlinear in  $K$  and in  $d$ , they may have more than one solution in  $K$  and in  $d$ . Below, as an illustration, we expose one of the numerical solutions,

$$K = 1.74 + 0.05i, \quad d = -9.01, \quad (68)$$

giving

$$\frac{e_2}{\mathcal{D}} = 0.0003 + 0.0015i, \quad \frac{e_3}{\mathcal{D}} = -0.0001 + 0.0014i \quad (69)$$

and

$$X = 0.326 + 0.034i, \quad Y = -0.007 + 0.003i, \\ Z = -0.082 + 0.251i. \quad (70)$$

The latter values of  $X, Y,$  and  $Z$  provide

$$U_{13} = 0.053 - 0.148i. \quad (71)$$

It is not difficult to find all other elements of  $U$  and  $\mathcal{V}$  that we do not expose here to save the paper's length. Further, using (68) in (67) we obtain absolute neutrino masses,

$$m_1 = 0.1109 \text{ eV}, \quad m_2 = 0.1114 \text{ eV}, \quad m_3 = 0.1217 \text{ eV}. \quad (72)$$

This result is consistent with the current experimental data [33] and it means that our model and method work quite well.



From (71), as  $U_{13} = s_{13}e^{-i\delta}$ , we obtain  $s_{13} \approx 0.157$  (or  $\theta_{13} \approx 9.03^\circ$ ) and  $\delta \approx 1.39\pi$ . The latter value of  $s_{13}$  is very close to the experimental data shown in (40). Interestingly, the Dirac CPV phase,  $\delta_{CP} \equiv \delta$ , obtained here, surprisingly (but hopefully not just accidentally) coincides with its global fit given in [32]. A more detailed analysis on  $\delta_{CP}$  is made in the next section.

#### IV. DIRAC CP VIOLATION PHASE AND JARLSKOG PARAMETER

In order to determine all variables in the matrix (57), or, at least, their relations, we must compare this matrix with the experimental one. Denoting the elements of the matrix (57) by  $U_{ij}$ ,  $i, j = 1, 2, 3$ , we get the equation (up to the first perturbation order)

$$2(|U_{21}|^2 - |U_{31}|^2) - (|U_{22}|^2 - |U_{32}|^2) = -2\sqrt{2}\text{Re}(U_{13}). \quad (73)$$

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times P \equiv U_{\text{pmns}} \times P, \quad (74)$$

where  $P$  (which in general is different from  $P_0$ ) is a diagonal matrix of the form

$$P = \text{diag}\left(e^{i\frac{\alpha_1}{2}}, e^{i\frac{\alpha_2}{2}}, 1\right)$$

with  $\alpha_1$  and  $\alpha_2$  being Majorana phases, we obtain the following relation between the Dirac CPV phase  $\delta_{CP} \equiv \delta$  and the neutrino mixing angles  $\theta_{ij}$ ,

$$(c_{23}^2 - s_{23}^2)(2s_{12}^2 - c_{12}^2) + 12s_{13}s_{23}c_{23}s_{12}c_{12} \cos \delta = -2\sqrt{2}s_{13} \cos \delta, \quad (75)$$

neglecting  $\mathcal{O}(\lambda^2)$  terms and higher order perturbation terms. Solving this equation for  $\cos \delta$  we get

$$\cos \delta = \frac{(s_{23}^2 - c_{23}^2)(2s_{12}^2 - c_{12}^2)}{2\sqrt{2}(3\sqrt{2}s_{23}c_{23}s_{12}c_{12} + 1)s_{13}}. \quad (76)$$

Below, this equation is used for  $s_{23} < 0$  because, as seen in (35) and (42), its TBM limit ( $-\sqrt{1/2}$ ) is negative, while the small perturbative fluctuation cannot change its sign. Since we work with  $\delta \in [0, 2\pi]$ , if  $\delta_0$  is a solution of the equation (76) so is  $2\pi - \delta_0$ . Having a value of  $\delta_{CP}$  we can obtain a value of the Jarlskog parameter  $J_{CP}$ .

Based on the relation (76) and experimental inputs (see Table II),  $\delta_{CP}$  can be calculated numerically. Using the

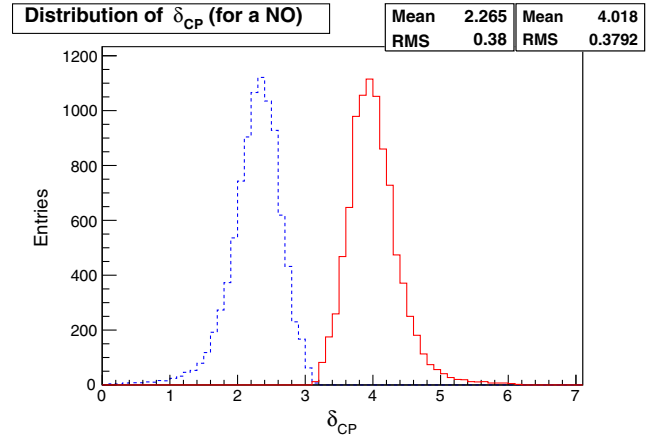


FIG. 1. Distribution of  $\delta_{CP}$  in a NO.

Further, comparing  $U_{ij}$  in (73) with the corresponding elements of the matrix  $U_{\text{PMNS}}$  given in the ‘‘trigonometric’’ form

experimental data of the mixing angles within  $1\sigma$  around the best fit value (BFV) [32,33], the distributions of  $\delta_{CP}$  are plotted in Figs. 1 and 2 for a NO and in Figs. 3 and 4 for an IO. Here, for each of these distributions, 10000 events are created and  $\delta_{CP}$  is calculated event by event with  $s_{ij}$  taken as random values generated on the base of a Gaussian distribution having the mean (best fit) value and sigmas given in Table II. Each of these distributions has two (sub) populations corresponding to two solutions of (76). In Figs. 1 and 2, the distributions corresponding to two solutions are distinguished by being plotted in blue and

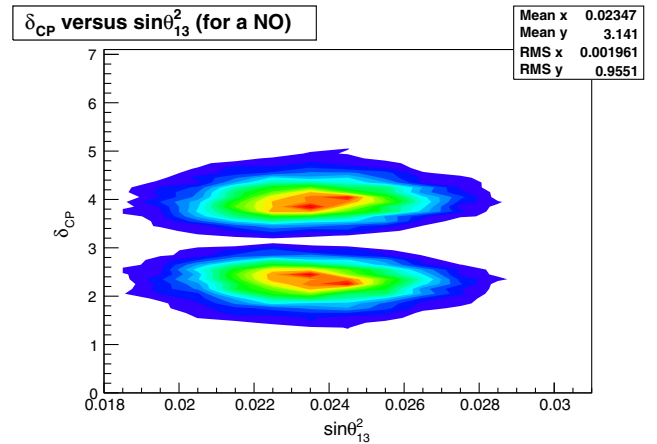
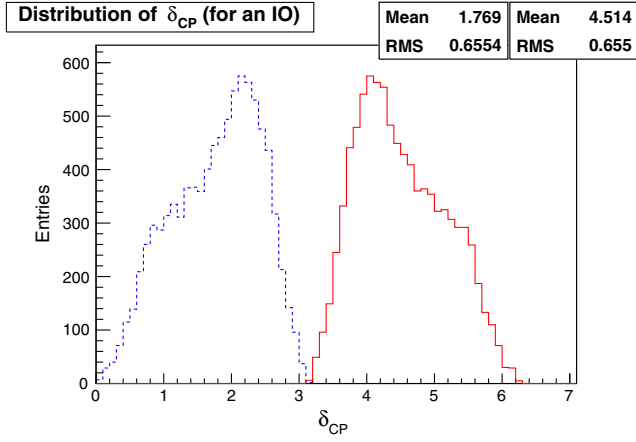


FIG. 2.  $\delta_{CP}$  versus  $\sin^2 \theta_{13}$  in a NO.


 FIG. 3. Distribution of  $\delta_{CP}$  in an IO.

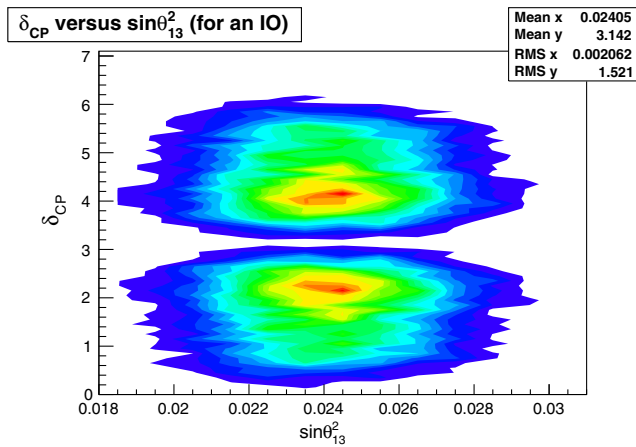
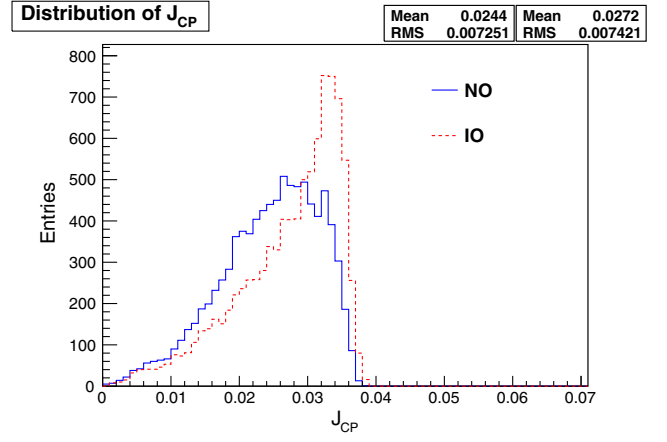
red. We see that the solution located in the range  $[\pi, 2\pi]$  is nearer the BFV (within  $1\sigma$  region). In Figs. 2 and 4, the three  $1\sigma$ ,  $2\sigma$ , and  $3\sigma$  regions are colored with different colors (red, green, and blue, respectively).

In the case of a NO,  $\delta_{CP}$  has a mean value of  $2.265 \approx 0.72\pi$  for one of the solutions, and a mean value of  $4.018 \approx 1.28\pi$  for the other solution and its distribution gets maximums at  $2.35 \approx 0.75\pi$  and  $3.95 \approx 1.26\pi$ , respectively. We see that the second solution (for both its mean value and the value at its maximal distribution) lies in the  $1\sigma$  region from the BFV  $1.39\pi$  given in [32,33].

In the case of an IO,  $\delta_{CP}$  gets a mean value around  $1.769 \approx 0.56\pi$  (for the first solution), and around  $4.514 \approx 1.44\pi$  (for the second solution). Its distribution reaches maximums at about  $2.15 \approx 0.68\pi$  and  $4.17 \approx 1.33\pi$ . Again, the second solution lies within the  $1\sigma$  region of the BFV  $1.31\pi$  given in [32,33].

Having all the mixing angles and Dirac CPV phase, it is not difficult to determine the Jarlskog parameter  $J_{CP} \equiv J$ . Indeed, using the expression [10]

$$|J_{CP}| = |c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13} \sin \delta|, \quad (77)$$


 FIG. 4.  $\delta_{CP}$  versus  $\sin^2 \theta_{13}$  in an IO.

 FIG. 5. Distribution of  $J_{CP}$  in a NO and an IO.

we obtain  $|J_{CP}| \leq 0.038$  and  $|J_{CP}| \leq 0.039$  (rough bounds) for a NO and an IO, respectively (see the distribution of  $J_{CP}$  in Fig. 5). It, up to a sign, has a mean value and a maximum at

$$J_{\text{mean}}^{\text{NO}} = 0.024 \quad \text{and} \quad J_{\text{max}}^{\text{NO}} = 0.027, \quad (78)$$

respectively, for a NO, and

$$J_{\text{mean}}^{\text{IO}} = 0.027 \quad \text{and} \quad J_{\text{max}}^{\text{IO}} = 0.033, \quad (79)$$

respectively, for an IO. The result obtained here is similar to that obtained in [54–57,67] by other methods by other authors.

To have a better view in comparing the two cases, the NO and the IO, the BFVs of  $\delta_{CP}$  and  $J_{CP}$  for both cases are summarized in Table III. These mean values of  $\delta_{CP}$  and  $J_{CP}$  are closer to the global fits than their corresponding values obtained at the BFVs of the mixing angles [by inserting the latter in the analytical expressions (76) and (77) for  $\delta_{CP}$  and  $|J_{CP}|$ , respectively]. To avoid any confusion, let us stress that the mean values of  $\delta_{CP}$  and  $|J_{CP}|$  do not coincide, in fact and in principle, with their values obtained at the BFVs of the mixing angles. It means that a value of  $\delta_{CP}$  or  $|J_{CP}|$  obtained at a BFV of the mixing angles should not in any way be identified with the mean value of the quantity concerned, although in some cases they may be close to each other.

It is also important to note that Eq. (76) is ill defined in the  $3\sigma$  region of the mixing angles. It means that this equation of determination of  $\delta_{CP}$  restricts the dissipation of

 TABLE III. The mean values of  $\delta_{CP}$  and  $|J_{CP}|$  in a NO and an IO.

	Normal ordering	Inverse ordering
$\delta_{CP}/\pi$	1.28	1.44
$ J_{CP} $	0.024	0.027

the mixing angles (that is, the values scattered too far, in the  $3\sigma$  region of distribution, are automatically excluded).

## V. CONCLUSIONS

Based on the fact that the observed neutrino mixing differs from a TBM one just slightly, we have suggested a non-TBM neutrino mixing model corresponding to this observation. This model represents an extended standard model acquiring an additional  $A_4 \times Z_3 \times Z_4$  flavor symmetry. Besides the SM fields assumed now to have also an  $A_4 \times Z_3 \times Z_4$  symmetry structure (see Table I), this model contains six additional fields; all are  $SU(2)_L$  singlets, which are one  $A_4$ -triplet fermion  $N$  (right-handed neutrinos), two  $A_4$ -triplet scalars  $\varphi_E$  and  $\varphi_N$ , and three  $A_4$ -singlet scalars  $\xi$ ,  $\xi'$ , and  $\xi''$ . The presence of the fields  $\xi'$  and  $\xi''$  (along with the SM Higgs field  $\phi_h$ ) is very important as it guarantees nonzero masses of the charged leptons. To avoid unwanted Lagrangian terms two discrete symmetries  $Z_3$  and  $Z_4$  are also introduced. Then, neutrino masses can be generated via Yukawa couplings of neutrinos to all scalars but  $\varphi_E$ . The corresponding neutrino mass matrix is obtained for a general VEV structure of the scalar field  $\varphi_N$ . It is observed that the model in general is a non-TBM model, but it becomes a TBM model [25] under a given circumstance with a specific VEV alignment of  $\varphi_N$  as in (22). Because the current experimentally established neutrino mixing represents just a small deviation from a TBM mixing we must build a theoretical model to satisfy this requirement. The latter puts a restriction on the model, in particular, it imposes constraints on its parameters. Therefore, the model constructed can be perturbatively developed around a TBM model, and, thus, the perturbative method can be applied to our further analysis.

As usual, diagonalizing a mass matrix is a difficult task. Here, within the above-suggested model and via a perturbation approach, the obtained neutrino mass matrix can be diagonalized by a matrix  $U_{\text{PMNS}}$  perturbatively expanded around the tribimaximal matrix  $U_{\text{TBM}}$ . In this way, a relation, see (76), between the Dirac CPV phase and the mixing angles is established. Based on the experimental values of the mixing angles this relation allows us to determine the Dirac CPV phase and the Jarlskog invariant

in a quite good agreement (within the  $1\sigma$  region of the best fit) with the recent experimental data at both the normal and the inverse neutrino mass ordering. These hierarchies are not compatible with each other; hence, only one of them, at most, can be realized in nature; however, none of them, so far, has been confirmed or excluded experimentally. Therefore, we here consider both NO and IO, and have obtained results in both cases close to the global fit [32,33]. For an illustration checking the model, numerical calculations have been also done and give results that are in good agreement with the current experimental data.

The determination of  $\delta_{CP}$  and  $J_{CP}$  is often both a theoretically and experimentally difficult problem and it can be used to verify the corresponding theoretical neutrino mixing model. This paper's method allows us to obtain an explicit  $\delta_{CP}$  as a function of the mixing angles; thus,  $\delta_{CP}$  could be determined experimentally via the mixing angles. This function in turn isolates the mixing angles from dissipated values in their distribution, i.e., the latter should be excluded. Our approach is useful and its application to higher order perturbations that may give a better fit is our next consideration. Finally, the mass spectrum that can be obtained by diagonalizing the mass matrix (36) is a subject of analysis to be done in a separate work.

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*Note added.*—Recently, we learned about new results [68] from T2K that are in quite good agreement with our results; in particular, the value of  $\theta_{13} \approx 9.03^\circ$  obtained by us above is very close to that of  $\theta_{13} \approx 8.47^\circ$  given by T2K (and to  $\theta_{13} \approx 8.8^\circ$  in [33]).

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