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Ginzburg-Landau functional for three order parameter problem

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Abstract. A model is presented utilizing a generic Hamiltonian with equal pairings in channels based on quantum field theory and functional integral formalism, to show the correlation among the order parameters which are described in multi-component Ginzburg-Landau functional. In the vicinity of the phase transition, the further perturbative expansions of the functional around the mean-field theory in the auxiliary fields are carried out with the aim of leading to a possible solution for the coexistence of many phases. The work is motivated by the recent theoretical researches and experimental evidences of the coexistence of superconductivity and ferromagnetism in U and Ce compounds.

1. Introduction

Conventional superconductivity was discovered in 1911 but till 1957 it was transparently understood thanks to appearance of BCS theory. Alternatively, this theory also caused a thought that superconductivity was incompatible with ferromagnetism for a long time. The physical reason for this incompatibility is the same as in the case of magnetic impurities in a superconductor[1]. Namely, in the framework of the standard BCS theory at low enough temperatures superconductivity is formed under condensation of singlet Cooper pairs. A singlet Cooper pair is a bound state of two electrons having opposite momenta and spins and which is formed indirectly through electron-phonon-electron interaction (an electron emits a phonon, an other electron absorbs this phonon). While, in ferromagnetic ground state all electron spins align parallel to produce a net spontaneous magnetization. Thus, when a standard ferromagnetic phase arises within a conventional superconductor, the spontaneous magnetization M will break down singlet Cooper pairs i.e., destroy conventional superconductivity.

In the background, the discovery of UGe_2 in 2000, and after that, UIr , $URhGe$, and $UCoGe$ with coexistence of ferromagnetism (FM) and superconductivity (SC) came as a big surprise and it appeared to require novel concepts to interpret. According to previous theories, ferromagnetism is induced by spin moments of localized 4f electrons, whereas superconductivity comes from Cooper pairs that formed by conduction electrons. Such discovery together with a number of reliable experimental data about coherence length and superconducting gap [2, 3, 4, 5], is in favor of the conclusion that 5f- electrons from U-atoms responsible for both FM and SC. Cooper



pairs in these metallic compounds be long to spin-triplet and magnetic-fluctuation induced pairing is a possible mechanism. In recent years, beside experimental investigations which examine the dependence of the phase transition on the applied pressure and magnetic field, there are also theoretical researches that concentrate on finding out phase transition mechanism, natures of phases and the dependence of temperature of phase transition and spontaneous magnetization moment on the parameters of materials. Different mechanisms have been proposed such as: coupled charge density waves and spin density waves [7], magnon exchange [8], electron interaction mediated by ferromagnetically aligned localized moments [9,10], screened phonon interactions[11], d-electron exchange [12], M-trigger [13, 14]....However, there have been no reliable answer for these important issues thus, we will confine our consideration to a phenomenological level.

The above consideration motivates transforming the fermionic field theory to an effective one based on the coupling fields which are expressed in terms of order parameter fields for different channels. The main purpose of this paper is formulation of a multi-component G-L functional which can describe the coexistence of many phases. In our research, through HS transformation, a microscopic Hamiltonian will be split into possible channels, then we can get a functional which only depends on order parameters. Thereby, we will arrive at a generic performance of Ginzburg-Landau (G-L) functional for three order parameter system through calculations based on Green function. Based on the specific problems of ferromagnetic superconductivity or antiferromagnetic superconductivity, we will draw the formal performance for G-L functional and create a graph showing the dependence of order parameters.

2. The model Hamiltonian of the system

2.1. The Hamiltonian

Our starting point is an interacting electron gas model [15]. In the terms of second quantization, the Hamiltonian of the system can be written as:

$$H = H_0 + H_1, \quad (1)$$

where,

$$H_0 = \sum_{\sigma} \sum_{\mathbf{k}} \epsilon_{\sigma}(\mathbf{k}) \psi_{\sigma}^{\dagger}(\mathbf{k}) \psi_{\sigma}(\mathbf{k}), \quad (2)$$

H_0 is unperturbed Hamiltonian describing the system of free electrons; $\psi_{\sigma}(\mathbf{k})$ and $\psi_{\sigma}^{\dagger}(\mathbf{k})$ are the annihilation and creation operators of the electrons with the spin projection $\sigma = \uparrow, \downarrow$ respectively; $\epsilon_{\sigma}(\mathbf{k})$ is the dispersion of the free electrons; and

$$H_1 = \sum_{\sigma_1, \sigma_2, \sigma'_1, \sigma'_2} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} V_{\sigma_1 \sigma_2 \sigma'_1 \sigma'_2}(\mathbf{k}, \mathbf{k}', \mathbf{q}) \psi_{\sigma_1}^{\dagger}(\mathbf{k} - \frac{\mathbf{q}}{2}) \psi_{\sigma_2}^{\dagger}(\mathbf{k}' + \frac{\mathbf{q}}{2}) \psi_{\sigma'_2}(\mathbf{k}' - \frac{\mathbf{q}}{2}) \psi_{\sigma'_1}(\mathbf{k} + \frac{\mathbf{q}}{2}); \quad (3)$$

H_1 is a generic effective two-body interaction term of interacting electrons written in the normal order, which may cover the contributions of other interactions in the system.

2.2. Grand partition function

In a previous short paper[15], the authors Nguyen Tri Lan and Nguyen Toan Thang have shown that there are, at least, three inequivalent choices of pairing up the fermionic operators to

construct the fermionic bilinear term of the generic two-body interaction. Those are pairings in the direct channel $(\psi^\dagger\psi)_{\sigma\sigma'}(\mathbf{k}, -\mathbf{q}, \tau)$, in the exchange channel $(\psi^\dagger\psi)_{\sigma_2\sigma'_1}(\mathbf{k}', \mathbf{k}, \mathbf{q}, \tau)$, and in the cooper channel $(\psi\psi)_{\sigma'_2\sigma'_1}(\mathbf{k}', \mathbf{k}, \mathbf{q})$ and $(\psi\psi)_{\sigma_2\sigma_1}^\dagger(\mathbf{k}', \mathbf{k}, -\mathbf{q})$. Nevertheless, the "right" choice of the decoupling field should be only motivated by physical reasoning, i.e one has to proceed to derive an effective theory based on the coupling field later. For this problem, the generic effective two-body interaction term $H_1[\psi^\dagger, \psi]$ can be broken down to a summation of all possible fermionic bilinear terms including all of three channels with arbitrary parameters $\{\gamma_i\}$, where $i \in \{d, ex, C\}$ as

$$H_1[\psi^\dagger, \psi] = \gamma_d^2 H_1^d[\psi^\dagger, \psi] - \gamma_{ex}^2 H_1^{ex}[\psi^\dagger, \psi] + \gamma_C^2 H_1^C[\psi^\dagger, \psi]. \quad (4)$$

By introducing HS decoupling on the two-body interaction term, the authors are able to express the grand partition function as a functional integral of the auxiliary fields $\{\varphi_i\}$ as follow

$$\mathcal{Z} = \frac{1}{\mathcal{W}} \int [\mathcal{D}\varphi] [\mathcal{D}\varphi^*] \exp\{-S_0[\{\varphi_i\}]\} \exp\left\{\ln \det\left(\frac{1}{2}[\mathcal{G}_0]^{-1}\right)\right\}. \quad (5)$$

The logarithmic contributions in Eq.(5) can be expanded as if it is function (a consequence trace operator), i.e.,

$$\text{Tr} \ln([\mathcal{G}_0]^{-1}) = \ln[G_0^e]^{-1} + \ln[G_0^h]^{-1} + \sum_{N \geq 1} \text{Tr}[\mathcal{G}]^N, \quad (6)$$

where the trace operator is understood as spin, momentum and frequency diagonal operator whose matrix elements give the free Green's function of the free electrons.

Performing matrix multiplications, then totalizing elements on the main diagonal (see Appendix A), in the energy-momentum space, we can write:

$$\begin{aligned} \text{Tr}[\mathcal{G}]^N &= (-1)^{N-1} \frac{1}{N} \sum_{\sigma_i=\uparrow,\downarrow} \sum_{\alpha_i=e,h} \sum_{p_i,\omega_{n_i}} G_{\sigma_1\sigma_1}^{\alpha_1}(\mathbf{p}_1, \omega_{n_1}) \Delta_{\sigma_1\sigma_2}^{\alpha_1\alpha_2}(\mathbf{p}_1, \omega_{n_1}; \mathbf{p}_2, \omega_{n_2}) \times \\ &G_{\sigma_2\sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \Delta_{\sigma_2\sigma_3}^{\alpha_2\alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \dots G_{\sigma_N\sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N\sigma_1}^{\alpha_N\alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \end{aligned} \quad (7)$$

where

$$\Delta_{\sigma_i\sigma_j}^{\alpha_i\alpha_j} = \delta_{\alpha_i\alpha_j} \Phi_{\sigma_i\sigma_j}^{\alpha_i\alpha_j} + (1 - \delta_{\alpha_i\alpha_j}) \mathcal{F}_{\sigma_i\sigma_j}^{\alpha_i\alpha_j}. \quad (8)$$

As it was discussed in [15], the independence of physical quantities on HS coupling's parameters $\{\gamma_i\}$ can be expressed in mathematical language by all order zero-derivatives of the grand thermodynamic potential, i.e., for any set of HS coupling's parameters $\{\gamma_i\}$ following conditions must be satisfied

$$\begin{aligned} \frac{d}{d\gamma_i} \Omega &= -\beta \frac{d}{d\gamma_i} \ln \mathcal{Z} = -\beta \left\langle \frac{d}{d\gamma_i} \ln \det\left(\frac{1}{2}[\mathcal{G}_0]^{-1}\right) \right\rangle \\ &\equiv 0. \end{aligned} \quad (9)$$

To avoid the complexity and the complication in further calculations, these parameters can be presented, without the loss of generality, by two new independent parameters ϕ and θ as follows

$$\begin{cases} \gamma_d &= \cosh \theta \sin \phi, \\ \gamma_{\text{ex}} &= \sinh \theta, \\ \gamma_C &= \cosh \theta \cos \phi. \end{cases} \quad (10)$$

In the terms of parameters ϕ and θ , the equations (9) are rewritten as

$$\left\langle \frac{d}{d\phi} \ln \det \left([\mathcal{G}_0]^{-1} \right) \right\rangle = 0, \quad (11)$$

and

$$\left\langle \frac{d}{d\theta} \ln \det \left([\mathcal{G}_0]^{-1} \right) \right\rangle = 0. \quad (12)$$

In general, it is impossible to evaluate the self-consistent system of equations(11) and (12) without some further approximation. A good and convenient idea of such approximation is some mean-field theory obtained by seeking for solution to the set of saddle point equations of the effective action in Eq. (5). We have

$$\frac{\delta}{\delta \varphi_i} \left(S_0 [\{\varphi_i\}] - \ln \det \left(\frac{1}{2} [\mathcal{G}]^{-1} \right) \right) = 0. \quad (13)$$

Eliminating parameters θ, ϕ from Eq. (11), Eq. (12) and Eq. (13) (see Appendix B), we obtain the system of three equations

$$\begin{aligned} & \sum_{\substack{\sigma_2 \\ \sigma'_2}} \sum_{\substack{p_2 \cdot p'_2 \\ \omega_{n_2}, \omega'_{n_2}}} (\varphi_1)_{\sigma_2 \sigma'_2}^* (p_2, \omega_{n_2}; p'_2, \omega'_{n_2}) (V_1)_{\sigma_2 \sigma'_2 \sigma_l \sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) = \\ &= \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l}) \underbrace{\delta_{\alpha_l \alpha_k} (-1)^{\alpha_l} i \cosh \theta \sin \phi}_{\sigma_l = \sigma_1, \sigma_k = \sigma_2} \right. \\ & G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\ & \quad + \cdots \\ & \left. + \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \cdots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) \right. \\ & \left. G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \underbrace{\delta_{\alpha_l \alpha_k} (-1)^{\alpha_l} i \cosh \theta \sin \phi}_{\sigma_l = \sigma_N, \sigma_k = \sigma_1} \right\}, \quad (14) \end{aligned}$$

$$\sum_{\substack{\sigma_2 \\ \sigma'_1}} \sum_{\substack{p_2 \cdot p'_1 \\ \omega_{n_2}, \omega'_{n_1}}} (\varphi_2)_{\sigma_2 \sigma'_1}^* (p_2, \omega_{n_2}; p'_1, \omega'_{n_1}) (V_2)_{\sigma_2 \sigma'_1 \sigma_l \sigma_k}^{-1} (p'_1, \omega'_{n_1}; p_l, \omega_{n_l}) =$$

$$\begin{aligned}
 &= \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \underbrace{\delta_{\alpha_l \alpha_k} (-1)^{\alpha_l} \sinh \theta}_{\sigma_l = \sigma_1, \sigma_k = \sigma_2} \right. \\
 &G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 &\quad + \cdots \\
 &+ \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \cdots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) \\
 &G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \underbrace{\delta_{\alpha_l \alpha_k} (-1)^{\alpha_l} \sinh \theta}_{\sigma_l = \sigma_N, \sigma_k = \sigma_1} \left. \right\}, \quad (15)
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{\substack{\sigma'_1 \\ \sigma'_2}} \sum_{\substack{p'_1, p'_2 \\ \omega'_{n'_1}, \omega'_{n'_2}}} (\varphi_3)_{\sigma_2 \sigma'_2}^* (p'_1, \omega'_{n'_1}; p'_2, \omega'_{n'_2}) (V_3)^{-1}_{\sigma'_1 \sigma'_2 \sigma_l \sigma_k} (p'_2, \omega'_{n'_2}; p_l, \omega_{n_l}) = \\
 &= \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \underbrace{(1 - \delta_{\alpha_l \alpha_k}) (-2i \cosh \theta \cos \phi)}_{\sigma_l = \sigma_1, \sigma_k = \sigma_2} \right. \\
 &G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 &\quad + \cdots \\
 &+ \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \cdots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) \\
 &G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \underbrace{(1 - \delta_{\alpha_l \alpha_k}) (-2i \cosh \theta \cos \phi)}_{\sigma_l = \sigma_N, \sigma_k = \sigma_1} \left. \right\}. \quad (16)
 \end{aligned}$$

which describe many-body relations between the physical quantities of the system where many kinds of fluctuations of corresponding order parameters are considered. Further analysis of this self-consistent system of equations shows the concrete constraints of diagram cancellation between diagram families generated by different HS couplings, which will be presented in a forthcoming paper.

3. Three - component Ginzburg-Landau functional

Expanding effective action in the Hubbard-Stratonovich auxiliary fields $\{\varphi_i\}$ to quartic one which only includes terms allowed by symmetry of system and retains minimum numbers of the simplest terms to get the results with multiple meanings, we will obtain G-L free energy functional with the participation of several order parameters describing relationship of density wave, spin wave and superconductivity phases.

$$\begin{aligned}
 f_{GL} [\{\varphi_i\}] &= a_1 |\varphi_1|^2 + a_2 |\varphi_2|^2 + a_3 |\varphi_3|^2 + a_4 (\varphi_1^* \varphi_2) + a_5 (\varphi_2^* \varphi_1) \\
 &+ b_1 (\varphi_1^*) |\varphi_3|^2 + b_2 (\varphi_1) |\varphi_3|^2 + b_3 (\varphi_2^*) |\varphi_3|^2 + b_4 (\varphi_2) |\varphi_3|^2 \\
 &+ c_1 |\varphi_1|^4 + c_2 |\varphi_2|^4 + c_3 |\varphi_3|^4 + c_4 |\varphi_1|^2 |\varphi_2|^2 + c_5 |\varphi_2|^2 |\varphi_3|^2 + c_6 |\varphi_3|^2 |\varphi_1|^2
 \end{aligned}$$

$$\begin{aligned}
& + \left[c_7 |\varphi_1|^2 + c_8 |\varphi_2|^2 + c_9 |\varphi_3|^2 \right] (\varphi_1^* \varphi_2) \\
& + \left[c_{10} |\varphi_1|^2 + c_{11} |\varphi_2|^2 + c_{12} |\varphi_3|^2 \right] (\varphi_2^* \varphi_1) \\
& + c_{13} (\varphi_1^* \varphi_2) (\varphi_1^* \varphi_2) + c_{14} (\varphi_2^* \varphi_1) (\varphi_2^* \varphi_1)
\end{aligned} \tag{17}$$

Here, all three quadratic terms $|\varphi_1|^2, |\varphi_2|^2, |\varphi_3|^2$ must be equivalent, and all three quartic terms $|\varphi_1|^4, |\varphi_2|^4, |\varphi_3|^4$ too, they describe each special channel respectively. The other terms describe interaction between channels that result in the coexistence of equilibrium phases.

Thanks to decoupling of two-body interaction terms using H-S transformations, we have above functional in general manner. In fact, auxiliary fields $\{\varphi_i\}$ are not any specific phases because above spin indexes, σ, σ' are common for all possible phases. In addition, above inversed potential V^{-1} which is included in decoupling method in almost physical systems does not always exist. Depending on how spin indexes are split, we will have singlet superconducting order or triplet superconducting order, ferromagnetic order or antiferromagnetic order. If the order is singlet, the system can fully exist an antiferromagnetic phase plus singlet superconducting phase as in $CeIn_5$ and $CeIrIn_5$. If the order is triplet, the system can fully exist an ferromagnetic phase plus triplet superconducting phase as in UGe_2 without depending on whether electronic is localized or not.

The remaining part of this paper will apply the model (17) to study the conditions causing Meissner phase in the presence of ferromagnetic order. In addition, we will establish a phase diagram corresponding to the ferromagnetic superconductor model when no external magnetic field exists in UGe_2 .

4. Unconventional Superconductivity in UGe_2

4.1. Experimental studies

The experimental researches about UGe_2 [3, 16, 17, 18] show that at ambient pressure UGe_2 is an itinerant ferromagnet having Curie temperature $T_{curie} = 52K$, and the spontaneous moment $\mu_s = 1.4\mu_B/U - \text{atom}$. An easy axis is the a-axis in orthorhombic crystal [7]. When the pressure increases the system passes through two quantum phase transition, one is from ferromagnetic phase to FS phase at $P = 1GPa$, and the other from ferromagnetic phase to paramagnetic phase at higher pressure $P_c = 1.6GPa$. The superconducting phase exists entirely within the ferromagnetic domain at low temperature and pressure interval between $1.0 \div 1.6$ GPa with a maximum $T_{sc} = 0.8K$ near 1.2 GPa. In ferromagnetic domain, there are two distinct ferromagnetic phases usually denoted by FM2 (highly polarized phase) and FM1 (weakly polarized phase). The line T_x separate two phases ends at $T_x = T_{sc}$ and the pressure $P_x = 1.2$ GPa. The order of transition from FM1 to paramagnetic changes from second to first at the tricritical point T_{cr} on the $T(P)$ diagram, and at critical pressure P_c . The more P increases, the more both T_{FM} and T_{FS} drop down to disappear almost simultaneously around $P \sim 1.7$ GPa.

4.2. Theoretical studies

With the experimental evidences about UGe_2 , as presented above, people believe that the ferromagnetism and superconductivity are caused by itinerant 5f-electrons in the same band. The Cooper pairs formed from the electrons to be of spin-triplet type, and magnetic-fluctuations are responsible for inducing pairing. Driving from the general multi-components G-L functional (17), we formulate specific functional which is capable of describing the coexistence of ferromagnetism and superconductivity in UGe_2 . We know that, UGe_2 is ferromagnet having orthorhombic structure with the magnetic easy axis is the a-axis. The order parameters in orthorhombic ferromagnet have general form as the same one allowed by symmetry given by Machida and Ohmi [19], and others [20, 13, 14]. According to these authors the Ginzburg-Landau (GL) free energy density part which describes the interaction between the ferromagnetic order parameter

\mathbf{M} and the superconducting order parameter ψ has to be coupled linearly and quadratically, namely, $i\gamma\mathbf{M} \cdot (\psi \times \psi^*)$ and $\mathbf{M}^2 \cdot \psi \cdot \psi^*$ (corresponding to $(\varphi_2^*)|\varphi_3|^2 + (\varphi_2)|\varphi_3|^2$ and $|\varphi_2|^2|\varphi_3|^2$ in the model (17)) respectively. If we choose a coordinate system with $x//b, y//c, z//a$ where a-axis direction is easy to be magnetized, then $\mathbf{M} = (0, 0, M)$ and $\psi = (\psi_1, \psi_2, 0)$, implying that the Cooper pair spin orientation points to the \mathbf{M} direction. Therefor, we considerably decreased complexity of analysis of the free energy in the below part.

Ignoring effects of anisotropy from both Cooper pairs and crystals, simultaneously restricting in consideration of the uniform order parameters, i.e., neglecting the x -dependence of ψ and M , in the traditional approach, G-L energy functional of the triplet ferromagnetic superconductor is split into a sum of three terms:

$$f_{GL}(\psi, \mathbf{M}) = f_S(\psi) + f_F(\mathbf{M}) + f_I(\psi, \mathbf{M}), \quad (18)$$

where

$$f_S(\psi) = a_s |\psi|^2 + \frac{b_s}{2} |\psi|^4, \quad (19)$$

$$f_F(\mathbf{M}) = a_f |\mathbf{M}|^2 + \frac{b_f}{2} |\mathbf{M}|^4, \quad (20)$$

and

$$f_I(\psi, \mathbf{M}) = i\gamma_0 \mathbf{M} \cdot (\psi \times \psi^*) + \delta_0 |\psi|^2 |\mathbf{M}|^2. \quad (21)$$

In Eq. (18): The first term $f_S(\psi)$ given by Eq. (19), describes the superconductivity for $M = H = 0$ (H is external magnetic field). $\psi = (\psi_1, \psi_2, 0)$ is the complex vector order parameter with three-components which are complex numbers. The complex numbers ψ_1, ψ_2 will be often expressed in terms of moduli and phase angles ($\psi_1 = \varphi_1 e^{i\theta_1}, \psi_2 = \varphi_2 e^{i\theta_2}$), where $\varphi = (\varphi_1^2 + \varphi_2^2)^{1/2}$ is magnitude of the complex vector order parameter $\psi = (\psi_1, \psi_2, 0)$; The second term $f_F(M)$ given by Eq. (20) is a part describing the free energy of a standard isotropic ferromagnet, where \mathbf{M} is the ferromagnetic order parameter; and finally the term $f_I(\psi, \mathbf{M})$ given by Eq. (21) describes the interaction between the ferromagnetic order parameter \mathbf{M} and the superconducting order parameter ψ . Where $\delta_0 > 0$ and the parameter γ_0 for ferromagnetic superconductors may take both positive and negative values.

In Eq. (19), choose $b_s > 0$ and $a_s = \alpha_s(T - T_s)$, where α_s is a positive material parameter and T_s is the critical temperature of a standard second order phase transition in case there is no magnet field. In Eq. (20) $b_f > 0$, T_f is critical temperature of the ferromagnet and $a_f = \alpha_f(T - T_f)$ depend on the material parameter $\alpha_f > 0$ and the temperature. In general, the values of the material parameters ($T_s, T_f, a_s, b_s, a_f, b_f, \gamma_0, \delta_0$) depends on selection of the concrete substance and thermodynamic parameters as temperature T and pressure P .

We assume that only T_f significantly depends on the pressure P . Expanding $T_f(P)$ in a Taylor series about the critical pressure point P_c at which it reach to zero. We keep only the

first term, enough for get non-trivial result, yields $T_f(P) = T_f'(P_c - P) = T_f(0) (1 - P/P_c)$ [20]. From (20) we find

$$M = (\alpha_f/b_f)^{1/2} [T_f(P) - T]^{1/2}. \quad (22)$$

Setting $M_0 = [\alpha_f T_{f0}/b_f]^{1/2}$ which is the value of the magnetization M corresponding to a pure magnetic subsystem $\psi = 0$ at $T = P = 0$ and $T_{f0} = T_f(0)$. Dividing both sides of Eq.(18) by $(b_f M_0^4)$, then calculating the right hand side, we obtain

$$f = r\phi^2 + \frac{1}{2}\phi^4 + tm^2 + \frac{1}{2}m^4 + 2\gamma m\phi_1\phi_2\sin\theta + \delta\phi^2m^2, \quad (23)$$

where $f = f_{GL}/(b_f M_0^4)$, $m = M/M_0$ and $\phi_j = |\varphi_j| / [(b_f/b_s)^{1/4} M_0]$, $\phi = (\phi_1^2 + \phi_2^2)^{1/2}$ and $\theta = \theta_2 - \theta_1$ is the phase angle between the complex ψ_1 and ψ_2 .

The parameters t and r in Eq. (23) are given by

$$r = \kappa (\tilde{T} - \tilde{T}_s); \quad t = \tilde{T} - \tilde{T}_f(P) = \tilde{T} - 1 + \tilde{P}, \quad (24)$$

where $\kappa = \alpha_s b_f^{1/2}/\alpha_f b_s^{1/2}$, $\gamma = \gamma_0/[\alpha_f T_{f0}/b_s]^{1/2}$, $\delta = \delta_0/(b_s b_f)^{1/2}$ whereas $\tilde{T} = T/T_{f0}$, $\tilde{T}_f(P) = T_f(P)/T_{f0}$, $\tilde{T}_s = T_s/T_{f0}$ and $\tilde{P} = P/P_c$ are the reduced temperatures and pressure.

Imposing the conditions of equilibrium and stable phases (the free energy has to have minima, i.e, find out conditions in which multi-function has minima) on the system which has the free energy given by Eq.(23), for $r > 0$, i.e., $T > T_s$ there are three stable phases:

- the normal phase (N): it exists for $\phi = m = 0$ and the stability conditions: $t \geq 0, r \geq 0$;
- the pure ferromagnetic phase (FM): it exists for $\phi = 0, m = (-t)^{1/2} > 0$, where $t < 0$ and is stable provided $r \geq 0$ and $r \geq \delta t + \gamma(-t)^{1/2}$;
- the phase of coexistence of ferromagnetic order and superconductivity (FS): given by $\sin\theta = -1$ (for $\sin\theta = 1$, we has other phase domain thermodynamically equivalent, so we are not consideration here), $\phi_1 = \phi_2 = \phi/\sqrt{2}$, where the conductivity parameter satisfy the equation

$$\phi^2 = \kappa(\tilde{T}_s - \tilde{T}) + \gamma m - \delta m^2 \geq 0, \quad (25)$$

and the magnetization m is root of the equation

$$2(1 - \delta^2)m^3 + 3\gamma\delta m^2 + 2(t - \delta r - \frac{\gamma^2}{2})m + \gamma r = 0. \quad (26)$$

The stability conditions for the FS phase domain given by Eqs. (25) and (26) are

$$\kappa(\tilde{T}_s - \tilde{T}) + 2\gamma m - \delta m^2 \geq 0,$$

$$\gamma m \geq 0,$$

and

$$(3 - \delta^2) m^2 + \gamma \delta m + \tilde{T} + t - \delta r \geq 0.$$

From Eq.(25) and existence condition of FM phase, it is derived that the second order phase transition line $T_{FS}(P)$ separating the FM and FS phases is given by the solution of the equation

$$\tilde{T}_{FS}(\tilde{P}) = \tilde{T}_s + \frac{\delta}{\kappa} t(\tilde{T}_{FS}, \tilde{P}) + \frac{\gamma}{\kappa} [-t(\tilde{T}_{FS}, \tilde{P})]^{1/2} \quad (27)$$

The coefficients of this equation are find out with the help of experimental data. $T_{FS}(P)$ curve has a maximum at point (T_m, P_m) which we obtain by both derivative sides of Eq.(27) with respect to \tilde{P} . We have

$$\frac{\partial \tilde{T}_{FS}}{\partial \tilde{P}} = \frac{\frac{\partial \tilde{T}_s}{\partial \tilde{P}} + \frac{\delta}{\kappa} - \frac{\gamma}{2\kappa} [-t(\tilde{T}_{FS}, \tilde{P})]^{-1/2}}{1 - \frac{\delta}{\kappa} + \frac{\gamma}{\kappa} [-t(\tilde{T}_{FS}, \tilde{P})]^{-1/2}} \quad (28)$$

Setting $\frac{\partial \tilde{T}_{FS}}{\partial \tilde{P}} = 0$, and note that $\frac{\partial \tilde{T}_s}{\partial \tilde{P}} = 0$ since \tilde{T}_s does not depend on \tilde{P} we derive $t(\tilde{T}_m, \tilde{P}_m) = -\frac{\gamma^2}{4\delta^2}$. Substituting this expression back in Eq.(27), we obtain coordinate of maximum point of $T_{FS}(P)$ curve as

$$\begin{cases} \tilde{T}_m = \tilde{T}_s + \frac{\gamma^2}{4\kappa\delta} \\ \tilde{P}_m = 1 - \tilde{T}_m - \frac{\gamma^2}{4\delta^2} \end{cases} \quad (29)$$

On the other hand, the $T_{FS}(P)$ curve cut \tilde{P} -axis at two points having coordinate:

$$\begin{cases} (\tilde{P} = 1, \tilde{T}_{FS} = 0) \\ (\tilde{P} = 1 - \frac{\gamma^2}{\delta^2}, \tilde{T}_{FS} = 0) \end{cases} \quad (30)$$

In order to outline a $T - P$ diagram of UGe_2 we need information about the values of $P_c, T_{f0}, T_s, \gamma/\kappa, \delta/\kappa$. From experimental phase diagrams [17,18,19,20], we have: $P_c = 1.6$ GPa, $T_{f0} = 52$ K, $T_m \approx 0.8$ K and $P_m \approx 1.2$ GPa. The experimental evidences show that it has not pure superconducting phase coexisting with ferromagnetism in UGe_2 . This indicates that critical temperature T_s of the pure superconductivity state is too small, and so considered as $T_s = 0$. Using these given parameters, we plot the $T - P$ diagram of UGe_2 shown in Fig.C1.

5. Conclusions and Discussions

Through calculations based on quantum field theory and functional integral formalism, the multi-component G-L functional is established. Based on the specific problem of ferromagnetic superconductivity, we transform the correlation among the order parameters described in multi-component Ginzburg-Landau functional into the correlation among order parameters T and P (physical quantities can be obtained directly from the experimental phase diagrams). $T - P$ diagram of UGe_2 outlined on the basis of theoretical calculations and shown in Fig.C1 has a agreement with the main experimental findings, although P_m corresponding to the maximum (found at ~ 1.45 GPa) is about 0.25 GPa higher than experimental data [3]. If the experimental plots are accurate, this difference may result from not including contribution of anisotropy of the spin-triplet Cooper pair and crystal or from any effect which is outside the scope of our current

model. The relative importance of this effect on the phase diagram needs to be investigated. However, in this context, the introduction of specific problem of ferromagnetic superconductivity only aims to illustrate that the multi-component G-L functional (17) can describe the coexistence of many phases in many-body system. Also, we obtain the system of three equations (14), (15) and (16) which describes many-body relations between the physical quantities of the system where many kinds of fluctuations of corresponding order parameters are considered. This self-consistent system of equations provides a general framework, which may serve as starting points for any further investigations considering the coexistence of many phases in many-body system and will be presented in a forthcoming paper.

Appendix A. Trace of matrix whose matrix elements are free Green's function of the free electron

$$\begin{aligned} \text{Tr} \ln \left([\mathcal{G}_0]^{-1} \right) &= \text{Tr} \ln \begin{pmatrix} [G_0^e]^{-1} - \Phi & \mathcal{F} \\ \mathcal{F}^* & [G_0^h]^{-1} + \Phi \end{pmatrix} \\ &= \ln [G_0^e]^{-1} + \ln [G_0^h]^{-1} + \text{Tr} \sum_{n \geq 1} (-1)^{n-1} \frac{1}{n} \left[\begin{pmatrix} G_0^e & 0 \\ 0 & G_0^h \end{pmatrix} \begin{pmatrix} -\Phi & \mathcal{F} \\ \mathcal{F}^* & \Phi \end{pmatrix} \right]^n. \end{aligned} \quad (\text{A.1})$$

Setting

$$[\mathcal{G}]^n = (-1)^{n-1} \frac{1}{n} \left[\begin{pmatrix} G^e & 0 \\ 0 & G^h \end{pmatrix} \begin{pmatrix} \Phi^{ee} & \mathcal{F}^{eh} \\ \mathcal{F}^{he} & \Phi^{hh} \end{pmatrix} \right]^n,$$

where

$$\begin{aligned} \Phi^{ee} &= -\Phi, & \mathcal{F}^{eh} &= \mathcal{F}, \\ \Phi^{hh} &= \Phi, & \mathcal{F}^{he} &= \mathcal{F}^*, \\ G^e &= G_0^e & G^h &= G_0^h. \end{aligned}$$

First-order expansion

$$\begin{aligned} [\mathcal{G}]^1 &= \begin{pmatrix} G^e & 0 \\ 0 & G^h \end{pmatrix} \begin{pmatrix} \Phi^{ee} & \mathcal{F}^{eh} \\ \mathcal{F}^{he} & \Phi^{hh} \end{pmatrix} \\ &= \begin{pmatrix} G^e \Phi^{ee} & G^e \mathcal{F}^{eh} \\ G^h \mathcal{F}^{he} & G^h \Phi^{hh} \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} G^e \Phi^{ee} &= \begin{pmatrix} [G^e]_{\uparrow\uparrow} & 0 \\ 0 & [G^e]_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} \Phi_{\uparrow\uparrow}^{ee} & \Phi_{\uparrow\downarrow}^{ee} \\ \Phi_{\downarrow\uparrow}^{ee} & \Phi_{\downarrow\downarrow}^{ee} \end{pmatrix} \\ &= \begin{pmatrix} [G^e]_{\uparrow\uparrow} \Phi_{\uparrow\uparrow}^{ee} & [G^e]_{\uparrow\uparrow} \Phi_{\uparrow\downarrow}^{ee} \\ [G^e]_{\downarrow\downarrow} \Phi_{\downarrow\uparrow}^{ee} & [G^e]_{\downarrow\downarrow} \Phi_{\downarrow\downarrow}^{ee} \end{pmatrix}; \end{aligned}$$

$$\begin{aligned}
G^h \Phi^{hh} &= \begin{pmatrix} [G^h]_{\uparrow\uparrow} & 0 \\ 0 & [G^h]_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} \Phi_{\uparrow\uparrow}^{hh} & \Phi_{\uparrow\downarrow}^{hh} \\ \Phi_{\downarrow\uparrow}^{hh} & \Phi_{\downarrow\downarrow}^{hh} \end{pmatrix} \\
&= \begin{pmatrix} [G^h]_{\uparrow\uparrow} & \Phi_{\uparrow\uparrow}^{hh} & [G^h]_{\uparrow\uparrow} & \Phi_{\uparrow\downarrow}^{hh} \\ [G^h]_{\downarrow\downarrow} & \Phi_{\downarrow\uparrow}^{hh} & [G^h]_{\downarrow\downarrow} & \Phi_{\downarrow\downarrow}^{hh} \end{pmatrix}.
\end{aligned}$$

Tracing matrix we obtain

$$\begin{aligned}
\text{Tr} [\mathcal{G}]^1 &= [G^e]_{\uparrow\uparrow} \Phi_{\uparrow\uparrow}^{ee} + [G^e]_{\downarrow\downarrow} \Phi_{\downarrow\downarrow}^{ee} + [G^h]_{\uparrow\uparrow} \Phi_{\uparrow\uparrow}^{hh} + [G^h]_{\downarrow\downarrow} \Phi_{\downarrow\downarrow}^{hh} \\
&= \sum_{\sigma_1} G_{\sigma_1\sigma_1}^e \Phi_{\sigma_1\sigma_1}^{ee} + \sum_{\sigma_1} G_{\sigma_1\sigma_1}^h \Phi_{\sigma_1\sigma_1}^{hh} = \sum_{\alpha_1} \sum_{\sigma_1} G_{\sigma_1\sigma_1}^{\alpha_1} \Phi_{\sigma_1\sigma_1}^{\alpha_1\alpha_1} = \sum_{\alpha_1} \sum_{\sigma_1} G_{\sigma_1\sigma_1}^{\alpha_1} \Delta_{\sigma_1\sigma_1}^{\alpha_1\alpha_1}. \quad (\text{A.2})
\end{aligned}$$

In the same way, we also obtain:
quadratic expansion

$$\begin{aligned}
\text{Tr} [\mathcal{G}]^2 &= -\frac{1}{2} \left\{ \sum_{\sigma_1\sigma_2} (G_{\sigma_1\sigma_1}^e \Phi_{\sigma_1\sigma_2}^{ee} G_{\sigma_2\sigma_2}^e \Phi_{\sigma_2\sigma_1}^{ee}) + \sum_{\sigma_1\sigma_2} (G_{\sigma_1\sigma_1}^h \Phi_{\sigma_1\sigma_2}^{hh} G_{\sigma_2\sigma_2}^h \Phi_{\sigma_2\sigma_1}^{hh}) \right. \\
&\quad \left. + \sum_{\sigma_1\sigma_2} (G_{\sigma_1\sigma_1}^e \mathcal{F}_{\sigma_1\sigma_2}^{eh} G_{\sigma_2\sigma_2}^h \mathcal{F}_{\sigma_2\sigma_1}^{he}) + \sum_{\sigma_1\sigma_2} (G_{\sigma_1\sigma_1}^h \mathcal{F}_{\sigma_1\sigma_2}^{he} G_{\sigma_2\sigma_2}^e \mathcal{F}_{\sigma_2\sigma_1}^{eh}) \right\} \\
&= -\frac{1}{2} \left\{ \sum_{\alpha_1} \sum_{\sigma_1\sigma_2} (G_{\sigma_1\sigma_1}^{\alpha_1} \Phi_{\sigma_1\sigma_2}^{\alpha_1\alpha_1} G_{\sigma_2\sigma_2}^{\alpha_1} \Phi_{\sigma_2\sigma_1}^{\alpha_1\alpha_1}) + \sum_{\substack{\alpha_1, \alpha_2 \\ \alpha_1 \neq \alpha_2}} \sum_{\sigma_1\sigma_2} (G_{\sigma_1\sigma_1}^{\alpha_1} \mathcal{F}_{\sigma_1\sigma_2}^{\alpha_1\alpha_2} G_{\sigma_2\sigma_2}^{\alpha_2} \mathcal{F}_{\sigma_2\sigma_1}^{\alpha_2\alpha_1}) \right\}, \quad (\text{A.3})
\end{aligned}$$

cubic expansion

$$\begin{aligned}
\text{Tr} [\mathcal{G}]^3 &= \frac{1}{3} \left\{ \sum_{\sigma_i} G_{\sigma_1\sigma_1}^e \Phi_{\sigma_1\sigma_2}^{ee} G_{\sigma_2\sigma_2}^e \Phi_{\sigma_2\sigma_3}^{ee} G_{\sigma_3\sigma_3}^e \Phi_{\sigma_3\sigma_1}^{ee} + \sum_{\sigma_i} G_{\sigma_1\sigma_1}^h \Phi_{\sigma_1\sigma_2}^{hh} G_{\sigma_2\sigma_2}^h \Phi_{\sigma_2\sigma_3}^{hh} G_{\sigma_3\sigma_3}^h \Phi_{\sigma_3\sigma_1}^{hh} \right. \\
&\quad \left. + 3 \sum_{\sigma_i} G_{\sigma_1\sigma_1}^e \Phi_{\sigma_1\sigma_2}^{ee} G_{\sigma_2\sigma_2}^e \mathcal{F}_{\sigma_2\sigma_3}^{eh} G_{\sigma_3\sigma_3}^h \mathcal{F}_{\sigma_3\sigma_1}^{he} + 3 \sum_{\sigma_i} G_{\sigma_1\sigma_1}^h \Phi_{\sigma_1\sigma_2}^{hh} G_{\sigma_2\sigma_2}^h \mathcal{F}_{\sigma_2\sigma_3}^{he} G_{\sigma_3\sigma_3}^e \mathcal{F}_{\sigma_3\sigma_1}^{eh} \right\} \\
&= \frac{1}{3} \sum_{\sigma_i=\uparrow,\downarrow} \sum_{\alpha_i=e,h} G_{\sigma_1\sigma_1}^{\alpha_1} \Delta_{\sigma_1\sigma_2}^{\alpha_1\alpha_2} G_{\sigma_2\sigma_2}^{\alpha_2} \Delta_{\sigma_2\sigma_3}^{\alpha_2\alpha_3} G_{\sigma_3\sigma_3}^{\alpha_3} \Delta_{\sigma_3\sigma_1}^{\alpha_3\alpha_1}, \quad (\text{A.4})
\end{aligned}$$

quartic expansion

$$\begin{aligned}
\text{Tr} [\mathcal{G}]^4 &= -\frac{1}{4} \left\{ (G^e \Phi^{ee})^4 + 4 (G^e \Phi^{ee})^2 (G^e \mathcal{F}^{eh} G^h \mathcal{F}^{he}) + 4 (G^e \Phi^{ee}) (G^h \Phi^{hh}) (G^e \mathcal{F}^{eh} G^h \mathcal{F}^{he}) \right. \\
&\quad \left. + (G^h \Phi^{hh})^4 + 4 (G^h \Phi^{hh})^2 (G^h \mathcal{F}^{he} G^e \mathcal{F}^{eh}) + 2 (G^h \mathcal{F}^{he} G^e \mathcal{F}^{eh})^2 \right\} \\
&= -\frac{1}{4} \left\{ \sum_{\sigma_i} \sum_{\alpha=e,h} G_{\sigma_1\sigma_1}^{\alpha} \Phi_{\sigma_1\sigma_2}^{\alpha\alpha} G_{\sigma_2\sigma_2}^{\alpha} \Phi_{\sigma_2\sigma_3}^{\alpha\alpha} G_{\sigma_3\sigma_3}^{\alpha} \Phi_{\sigma_3\sigma_4}^{\alpha\alpha} G_{\sigma_4\sigma_4}^{\alpha} \Phi_{\sigma_4\sigma_1}^{\alpha\alpha} \right. \\
&\quad \left. + 4 \sum_{\sigma_i} \sum_{\substack{\alpha, \beta=e,h \\ \alpha \neq \beta}} G_{\sigma_1\sigma_1}^{\alpha} \Phi_{\sigma_1\sigma_2}^{\alpha\alpha} G_{\sigma_2\sigma_2}^{\alpha} \Phi_{\sigma_2\sigma_3}^{\alpha\alpha} G_{\sigma_3\sigma_3}^{\alpha} \mathcal{F}_{\sigma_3\sigma_4}^{\alpha\beta} G_{\sigma_4\sigma_4}^{\beta} \mathcal{F}_{\sigma_4\sigma_1}^{\beta\alpha} \right\}
\end{aligned}$$

$$\begin{aligned}
 & +2 \sum_{\sigma_i} \sum_{\substack{\alpha, \beta = e, h \\ \alpha \neq \beta}} G_{\sigma_1 \sigma_1}^\alpha \Phi_{\sigma_1 \sigma_2}^{\alpha\alpha} G_{\sigma_2 \sigma_2}^\alpha \mathcal{F}_{\sigma_2 \sigma_3}^{\alpha\beta} G_{\sigma_3 \sigma_3}^\beta \Phi_{\sigma_3 \sigma_4}^{\beta\beta} G_{\sigma_4 \sigma_4}^\beta \mathcal{F}_{\sigma_4 \sigma_1}^{\beta\alpha} \\
 & + \sum_{\sigma_i} \sum_{\substack{\alpha, \beta = e, h \\ \alpha \neq \beta}} \left. G_{\sigma_1 \sigma_1}^\alpha \mathcal{F}_{\sigma_1 \sigma_2}^{\alpha\beta} G_{\sigma_2 \sigma_2}^\beta \mathcal{F}_{\sigma_2 \sigma_3}^{\beta\alpha} G_{\sigma_3 \sigma_3}^\alpha \mathcal{F}_{\sigma_3 \sigma_4}^{\alpha\beta} G_{\sigma_4 \sigma_4}^\beta \mathcal{F}_{\sigma_4 \sigma_1}^{\beta\alpha} \right\} \\
 & = -\frac{1}{4} \sum_{\sigma_i=\uparrow, \downarrow} \sum_{\alpha_i=e, h} G_{\sigma_1 \sigma_1}^{\alpha_1} \Delta_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2} G_{\sigma_2 \sigma_2}^{\alpha_2} \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3} G_{\sigma_3 \sigma_3}^{\alpha_3} \Delta_{\sigma_3 \sigma_4}^{\alpha_3 \alpha_4} G_{\sigma_4 \sigma_4}^{\alpha_4} \Delta_{\sigma_4 \sigma_1}^{\alpha_4 \alpha_1} \quad (\text{A.5})
 \end{aligned}$$

By using inductive method, we get

$$\begin{aligned}
 \text{Tr} [\mathcal{G}]^N &= (-1)^{N-1} \frac{1}{N} \sum_{\sigma_i=\uparrow, \downarrow} \sum_{\alpha_i=e, h} \sum_{\mathbf{p}_i, \omega_{n_i}} G_{\sigma_1 \sigma_1}^{\alpha_1}(\mathbf{p}_1, \omega_{n_1}) \Delta_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}(\mathbf{p}_1, \omega_{n_1}; \mathbf{p}_2, \omega_{n_2}) \\
 & G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \dots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 &= (-1)^{N-1} \frac{1}{N} \sum_{\sigma_i=\uparrow, \downarrow} \sum_{\alpha_i=e, h} \sum_{\substack{\omega_n, \omega_{\nu_i} \\ \mathbf{k}, \mathbf{q}}} G_{\sigma_1 \sigma_1}^{\alpha_1}(\mathbf{k}, \omega_n) \Delta_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}(\mathbf{q}_1, \omega_{\nu_1}) G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{k} - \mathbf{q}_1, \omega_n - \omega_{\nu_1}) \\
 & \quad \times \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{q}_2, \omega_{\nu_2}) \dots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{k} - \mathbf{q}_1 - \dots - \mathbf{q}_{N-1}, \omega_n - \omega_{\nu_1} - \dots - \omega_{\nu_{N-1}}) \\
 & \quad \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(-\mathbf{q}_1 - \dots - \mathbf{q}_{N-1}, -\omega_{\nu_1} - \dots - \omega_{\nu_{N-1}}). \quad (\text{A.6})
 \end{aligned}$$

Appendix B. Eliminating parameters θ, ϕ

field equation (13) for φ_1 can be written

$$\begin{aligned}
 & \sum_{\sigma_2} \sum_{\substack{p_2, p'_2 \\ \omega_{n_2}, \omega'_{n_2}}} (\varphi_1)_{\sigma_2 \sigma'_2}^* (p_2, \omega_{n_2}; p'_2, \omega'_{n_2}) (V_1)_{\sigma_2 \sigma'_2 \sigma_1 \sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) = \\
 &= \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \dots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \underbrace{\frac{\partial}{\partial (\varphi_1)_{\sigma_l \sigma_k}} \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}(\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l=\sigma_1, \sigma_k=\sigma_2} \right. \\
 & \quad \times G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \dots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 & \quad \left. + \dots \right. \\
 & + \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \dots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \\
 & \quad \times \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \dots G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \underbrace{\frac{\partial}{\partial (\varphi_1)_{\sigma_l \sigma_k}} \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}(\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l=\sigma_N, \sigma_k=\sigma_1} \Big\} \quad (\text{B.1})
 \end{aligned}$$

Multiplying both sides by $\tanh \theta (\Phi_1)_{\sigma_l \sigma_k}(\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})$, we have:

$$\begin{aligned}
 & \sum_{\substack{\sigma_2 \\ \sigma'_2}} \sum_{\substack{p_2, p'_2 \\ \omega_{n_2}, \omega'_{n_2}}} \left(\begin{array}{c} (\varphi_1)_{\sigma_2 \sigma'_2}^* (p_2, \omega_{n_2}; p'_2, \omega'_{n_2}) (V_1)_{\sigma_2 \sigma'_2 \sigma_1 \sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) \\ \times (\Phi_1)_{\sigma_1 \sigma_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \end{array} \right) \tanh \theta = \\
 & = \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_1 \sigma_l}^{\alpha_l} (\mathbf{p}_1, \omega_{n_l}) \right. \\
 & \times \left[\underbrace{\tanh \theta (\Phi_1)_{\sigma_1 \sigma_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \frac{\partial}{\partial (\varphi_1)_{\sigma_1 \sigma_k}} \Delta_{\sigma_1 \sigma_k}^{\alpha_1 \alpha_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l = \sigma_1, \sigma_k = \sigma_2} \right] \times \\
 & \times G_{\sigma_k \sigma_k}^{\alpha_k} (\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3} (\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_N \sigma_N}^{\alpha_N} (\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1} (\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 & + \cdots \\
 & + \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \cdots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k} (\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2} (\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) G_{\sigma_2 \sigma_2}^{\alpha_2} (\mathbf{p}_2, \omega_{n_2}) \\
 & \times \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3} (\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_1 \sigma_l}^{\alpha_l} (\mathbf{p}_1, \omega_{n_l}) \\
 & \times \left. \left[\underbrace{\tanh \theta (\Phi_1)_{\sigma_1 \sigma_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \frac{\partial}{\partial (\varphi_1)_{\sigma_1 \sigma_k}} \Delta_{\sigma_1 \sigma_k}^{\alpha_1 \alpha_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l = \sigma_N, \sigma_k = \sigma_1} \right] \right\}. \quad (\text{B.2})
 \end{aligned}$$

Similarly, multiplying both sides of the field equation (13) for φ_2 by $\coth \theta (\Phi_2)_{\sigma_1 \sigma_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})$, and for φ_3 by $\tanh \theta (\varphi_3)_{\sigma_l \sigma_k}^{\alpha_l} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})$ we have:

$$\begin{aligned}
 & \sum_{\substack{\sigma_2 \\ \sigma'_1}} \sum_{\substack{p_2, p'_1 \\ \omega_{n_2}, \omega'_{n_1}}} \left(\begin{array}{c} (\varphi_2)_{\sigma_2 \sigma'_1}^* (p_2, \omega_{n_2}; p'_1, \omega'_{n_1}) (V_2)_{\sigma_2 \sigma'_1 \sigma_1 \sigma_k}^{-1} (p'_1, \omega'_{n_1}; p_l, \omega_{n_l}) \\ (\Phi_2)_{\sigma_1 \sigma_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \end{array} \right) \coth \theta = \\
 & = \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_1 \sigma_l}^{\alpha_l} (\mathbf{p}_1, \omega_{n_l}) \right. \\
 & \times \left[\underbrace{\coth \theta (\Phi_2)_{\sigma_1 \sigma_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \frac{\partial}{\partial (\varphi_2)_{\sigma_1 \sigma_k}} \Delta_{\sigma_1 \sigma_k}^{\alpha_1 \alpha_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l = \sigma_1, \sigma_k = \sigma_2} \right] \\
 & \times G_{\sigma_k \sigma_k}^{\alpha_k} (\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3} (\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_N \sigma_N}^{\alpha_N} (\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1} (\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 & + \cdots \\
 & + \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \cdots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k} (\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2} (\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) G_{\sigma_2 \sigma_2}^{\alpha_2} (\mathbf{p}_2, \omega_{n_2})
 \end{aligned}$$

$$\times \Delta_{\sigma_2\sigma_3}^{\alpha_2\alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \dots G_{\sigma_1\sigma_l}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l})$$

$$\times \left[\underbrace{\coth \theta (\Phi_2)_{\sigma_l\sigma_k}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \frac{\partial}{\partial (\varphi_2)_{\sigma_l\sigma_k}} \Delta_{\sigma_l\sigma_k}^{\alpha_l\alpha_k}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l=\sigma_N, \sigma_k=\sigma_1} \right] \Bigg\}, \quad (\text{B.3})$$

and

$$\sum_{\substack{\sigma'_1 \\ \sigma'_2}} \sum_{\substack{p'_1, p'_2 \\ \omega'_{n_1}, \omega'_{n_2}}} \left((\varphi_3)_{\sigma_2\sigma'_2}^* (p'_1, \omega'_{n_1}; p'_2, \omega'_{n_2}) (V_3)_{\sigma'_1\sigma'_2\sigma_l\sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) \right. \\ \left. \times (\varphi_3)_{\sigma_l\sigma_k}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \right) \tanh \theta =$$

$$= \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \dots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_l\sigma_l}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l}) \right.$$

$$\times \left[\underbrace{\tanh \theta (\varphi_3)_{\sigma_l\sigma_k}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \frac{\partial}{\partial (\varphi_3)_{\sigma_l\sigma_k}} \Delta_{\sigma_l\sigma_k}^{\alpha_l\alpha_k}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l=\sigma_1, \sigma_k=\sigma_2} \right] \times$$

$$\times G_{\sigma_k\sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k\sigma_3}^{\alpha_k\alpha_3}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \dots G_{\sigma_N\sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N\sigma_1}^{\alpha_N\alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1})$$

$$+ \dots$$

$$+ \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \dots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k\sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k\sigma_2}^{\alpha_k\alpha_2}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) G_{\sigma_2\sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2})$$

$$\times \Delta_{\sigma_2\sigma_3}^{\alpha_2\alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \dots G_{\sigma_1\sigma_l}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l})$$

$$\times \left[\underbrace{\tanh \theta (\varphi_3)_{\sigma_l\sigma_k}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \frac{\partial}{\partial (\varphi_3)_{\sigma_l\sigma_k}} \Delta_{\sigma_l\sigma_k}^{\alpha_l\alpha_k}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l=\sigma_N, \sigma_k=\sigma_1} \right] \Bigg\}. \quad (\text{B.4})$$

Adding respective sides together for three equations, we obtain:

$$\sum_{\substack{\sigma_2 \\ \sigma'_2}} \sum_{\substack{p_2, p'_2 \\ \omega_{n_2}, \omega'_{n_2}}} \left((\varphi_1)_{\sigma_2\sigma'_2}^* (p_2, \omega_{n_2}; p'_2, \omega'_{n_2}) (V_1)_{\sigma_2\sigma'_2\sigma_l\sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) \right. \\ \left. \times (\Phi_1)_{\sigma_l\sigma_k}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \right) \tanh \theta +$$

$$\sum_{\substack{\sigma_2 \\ \sigma'_1}} \sum_{\substack{p_2, p'_1 \\ \omega_{n_2}, \omega'_{n_1}}} \left((\varphi_2)_{\sigma_2\sigma'_1}^* (p_2, \omega_{n_2}; p'_1, \omega'_{n_1}) (V_2)_{\sigma_2\sigma'_1\sigma_l\sigma_k}^{-1} (p'_1, \omega'_{n_1}; p_l, \omega_{n_l}) \right. \\ \left. \times (\Phi_2)_{\sigma_l\sigma_k}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \right) \coth \theta +$$

$$\sum_{\substack{\sigma'_1 \\ \sigma'_2}} \sum_{\substack{p'_1, p'_2 \\ \omega'_{n_1}, \omega'_{n_2}}} \left((\varphi_3)_{\sigma_2\sigma'_2}^* (p'_1, \omega'_{n_1}; p'_2, \omega'_{n_2}) (V_3)_{\sigma'_1\sigma'_2\sigma_l\sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) \right. \\ \left. \times (\varphi_3)_{\sigma_l\sigma_k}^{\alpha_l}(\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \right) \tanh \theta =$$

$$\begin{aligned}
 &= \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \right. \\
 &\times \left[\underbrace{\tanh \theta(\Phi_1)_{\sigma_l \sigma_k} \frac{\partial \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}}{\partial (\varphi_1)_{\sigma_l \sigma_k}} + \coth \theta(\Phi_2)_{\sigma_l \sigma_k} \frac{\partial \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}}{\partial (\varphi_2)_{\sigma_l \sigma_k}} + \tanh \theta(\varphi_3)_{\sigma_l \sigma_k} \frac{\partial \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}}{\partial (\varphi_3)_{\sigma_l \sigma_k}}}_{\frac{\partial}{\partial \theta} \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}} \right] \\
 &\times G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 &+ \cdots \\
 &+ \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \cdots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) \\
 &\times G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_1 \sigma_1}^{\alpha_1}(\mathbf{p}_1, \omega_{n_1}) \\
 &\times \left. \left[\underbrace{\tanh \theta(\Phi_1)_{\sigma_l \sigma_k} \frac{\partial \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}}{\partial (\varphi_1)_{\sigma_l \sigma_k}} + \coth \theta(\Phi_2)_{\sigma_l \sigma_k} \frac{\partial \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}}{\partial (\varphi_2)_{\sigma_l \sigma_k}} + \tanh \theta(\varphi_3)_{\sigma_l \sigma_k} \frac{\partial \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}}{\partial (\varphi_3)_{\sigma_l \sigma_k}}}_{\frac{\partial}{\partial \theta} \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}} \right] \right\} \\
 &= \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \sum_{\substack{\sigma_3 \\ \alpha_3}} \sum_{\substack{p_3 \\ \omega_{n_3}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} G_{\sigma_l \sigma_l}^{\alpha_l}(\mathbf{p}_l, \omega_{n_l}) \left[\underbrace{\frac{\partial}{\partial \theta} \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}(\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l = \sigma_1, \sigma_k = \sigma_2} \right] \right. \\
 &\times G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_3}^{\alpha_k \alpha_3}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_N \sigma_N}^{\alpha_N}(\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1}(\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \\
 &+ \cdots \\
 &+ \sum_{\substack{\sigma_2 \\ \alpha_2}} \sum_{\substack{p_2 \\ \omega_{n_2}}} \cdots \sum_{\substack{\sigma_{N-1} \\ \alpha_{N-1}}} \sum_{\substack{p_{N-1} \\ \omega_{n_{N-1}}}} G_{\sigma_k \sigma_k}^{\alpha_k}(\mathbf{p}_k, \omega_{n_k}) \Delta_{\sigma_k \sigma_2}^{\alpha_k \alpha_2}(\mathbf{p}_k, \omega_{n_k}; \mathbf{p}_2, \omega_{n_2}) \\
 &\times G_{\sigma_2 \sigma_2}^{\alpha_2}(\mathbf{p}_2, \omega_{n_2}) \Delta_{\sigma_2 \sigma_3}^{\alpha_2 \alpha_3}(\mathbf{p}_2, \omega_{n_2}; \mathbf{p}_3, \omega_{n_3}) \cdots G_{\sigma_1 \sigma_1}^{\alpha_1}(\mathbf{p}_1, \omega_{n_1}) \left. \left[\underbrace{\frac{\partial}{\partial \theta} \Delta_{\sigma_l \sigma_k}^{\alpha_l \alpha_k}(\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})}_{\sigma_l = \sigma_N, \sigma_k = \sigma_1} \right] \right\}. \tag{B.5}
 \end{aligned}$$

Taking the sum of both sides of equation (B.5) with respect to $\alpha_l, \alpha_k; \sigma_l, \sigma_k; p_l, p_k; \omega_{n_l}, \omega_{n_k}$ (short-hand notation $a \equiv (\alpha_l, \alpha_k; \sigma_l, \sigma_k; p_l, p_k; \omega_{n_l}, \omega_{n_k})$), we obtain:

$$\begin{aligned}
 &\sum_a \sum_{\substack{\sigma_2 \\ \sigma'_2}} \sum_{\substack{p_2, p'_2 \\ \omega_{n_2}, \omega'_{n_2}}} \left((\varphi_1)_{\sigma_2 \sigma'_2}^* (p_2, \omega_{n_2}; p'_2, \omega'_{n_2}) (V_1)_{\sigma_2 \sigma'_2 \sigma_l \sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) \right) \tanh \theta + \\
 &\quad \times (\Phi_1)_{\sigma_l \sigma_k}(\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \\
 &\sum_a \sum_{\substack{\sigma_2 \\ \sigma'_1}} \sum_{\substack{p_2, p'_1 \\ \omega_{n_2}, \omega'_{n_1}}} \left((\varphi_2)_{\sigma_2 \sigma'_1}^* (p_2, \omega_{n_2}; p'_1, \omega'_{n_1}) (V_2)_{\sigma_2 \sigma'_1 \sigma_l \sigma_k}^{-1} (p'_1, \omega'_{n_1}; p_l, \omega_{n_l}) \right) \coth \theta + \\
 &\quad \times (\Phi_2)_{\sigma_l \sigma_k}(\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})
 \end{aligned}$$

$$\begin{aligned}
 \sum_a \sum_{\substack{\sigma'_1 \\ \sigma'_2}} \sum_{\substack{p'_1, p'_2 \\ \omega_{n'_1}, \omega'_{n_2}}} & \left(\begin{array}{c} (\varphi_3)_{\sigma_2 \sigma'_2}^* (p'_1, \omega_{n'_1}; p'_2, \omega'_{n_2}) (V_3)_{\sigma'_1 \sigma'_2 \sigma_1 \sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) \\ \times (\varphi_3)_{\sigma_l \sigma_k}^{\alpha_l} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \end{array} \right) \tanh \theta = \\
 & = \sum_{N \geq 1} (-1)^{N-1} \frac{1}{N} \left\{ \right. \\
 & \quad \sum_{\substack{\sigma_1 \\ \alpha_1}} \sum_{\substack{p_1 \\ \omega_{n_1}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} \left(\begin{array}{c} G_{\sigma_1 \sigma_1}^{\alpha_1} (\mathbf{p}_1, \omega_{n_1}) \frac{\partial}{\partial \theta} \Delta_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2} (\mathbf{p}_1, \omega_{n_1}; \mathbf{p}_2, \omega_{n_2}) \\ \dots \times G_{\sigma_N \sigma_N}^{\alpha_N} (\mathbf{p}_N, \omega_{n_N}) \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1} (\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \end{array} \right) \\
 & \quad + \cdots \\
 & \quad + \sum_{\substack{\sigma_1 \\ \alpha_1}} \sum_{\substack{p_1 \\ \omega_{n_1}}} \cdots \sum_{\substack{\sigma_N \\ \alpha_N}} \sum_{\substack{p_N \\ \omega_{n_N}}} \left(\begin{array}{c} G_{\sigma_1 \sigma_1}^{\alpha_1} (\mathbf{p}_1, \omega_{n_1}) \Delta_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2} (\mathbf{p}_1, \omega_{n_1}; \mathbf{p}_2, \omega_{n_2}) \\ \dots \times G_{\sigma_N \sigma_N}^{\alpha_N} (\mathbf{p}_N, \omega_{n_N}) \frac{\partial}{\partial \theta} \Delta_{\sigma_N \sigma_1}^{\alpha_N \alpha_1} (\mathbf{p}_N, \omega_{n_N}; \mathbf{p}_1, \omega_{n_1}) \end{array} \right) \left. \right\} \\
 & = \frac{\partial}{\partial \theta} \ln \det \left(\frac{1}{2} [\mathcal{G}_0]^{-1} \right). \quad (\text{B.6})
 \end{aligned}$$

Since $\left\langle \frac{\partial}{\partial \theta} \ln \det \left(\frac{1}{2} [\mathcal{G}_0]^{-1} \right) \right\rangle = 0$ and note that $\left\langle (\varphi_i)^* (V_i^{-1}) (\varphi_i)^* \right\rangle = 0$, we derive:

$$\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle \tanh \theta + \left\langle \varphi_2^* V_2^{-1} \varphi_2 \right\rangle \coth \theta + \left\langle \varphi_3^* V_2^{-1} \varphi_3 \right\rangle \tanh \theta = 0. \quad (\text{B.7})$$

Form above equation, we obtain:

$$\sinh \theta = \left[-\frac{\left\langle \varphi_2^* V_2^{-1} \varphi_2 \right\rangle}{\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle + \left\langle \varphi_2^* V_2^{-1} \varphi_2 \right\rangle + \left\langle \varphi_3^* V_3^{-1} \varphi_3 \right\rangle} \right]^{1/2}, \quad (\text{B.8})$$

$$\coth \theta = \left[\frac{\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle + \left\langle \varphi_3^* V_3^{-1} \varphi_3 \right\rangle}{\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle + \left\langle \varphi_2^* V_2^{-1} \varphi_2 \right\rangle + \left\langle \varphi_3^* V_3^{-1} \varphi_3 \right\rangle} \right]^{1/2}. \quad (\text{B.9})$$

Similar to calculations with parameter θ , for parameter ϕ we also have:

$$\sin \phi = \left[\frac{\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle}{\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle + \left\langle \varphi_3^* V_3^{-1} \varphi_3 \right\rangle} \right]^{1/2}, \quad (\text{B.10})$$

$$\cos \phi = \left[\frac{\left\langle \varphi_3^* V_3^{-1} \varphi_3 \right\rangle}{\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle + \left\langle \varphi_3^* V_3^{-1} \varphi_3 \right\rangle} \right]^{1/2}. \quad (\text{B.11})$$

for convenience the following short-hand notations have been used

$$\left\langle \varphi_1^* V_1^{-1} \varphi_1 \right\rangle = \left\langle \sum_a \sum_{\substack{\sigma_2, p_2, \omega_{n_2} \\ \sigma'_2, p'_2, \omega'_{n_2}}} \left(\begin{array}{c} (\varphi_1)_{\sigma_2 \sigma'_2}^* (p_2, \omega_{n_2}; p'_2, \omega'_{n_2}) (V_1)_{\sigma_2 \sigma'_2 \sigma_1 \sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l}) \\ \times (\varphi_1)_{\sigma_l \sigma_k} (\mathbf{p}_1, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k}) \end{array} \right) \right\rangle,$$

$$\langle \varphi_2^* V_2^{-1} \varphi_2 \rangle = \left\langle \sum_a \sum_{\substack{\sigma_2, p_2, \omega_{n_2} \\ \sigma'_1, p'_1, \omega'_{n_1}}} \left(\frac{(\varphi_2)_{\sigma_2 \sigma'_1}^* (p_2, \omega_{n_2}; p'_1, \omega'_{n_1}) (V_2)_{\sigma_2 \sigma'_1 \sigma_l \sigma_k}^{-1} (p'_1, \omega'_{n_1}; p_l, \omega_{n_l})}{(\varphi_2)_{\sigma_l \sigma_k} (\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})} \right) \right\rangle,$$

$$\langle \varphi_3^* V_2^{-1} \varphi_3 \rangle = \left\langle \sum_a \sum_{\substack{\sigma'_1, p'_1, \omega'_{n_1} \\ \sigma'_2, p'_2, \omega'_{n_2}}} \left(\frac{(\varphi_3)_{\sigma_2 \sigma'_2}^* (p'_1, \omega'_{n_1}; p'_2, \omega'_{n_2}) (V_3)_{\sigma'_1 \sigma'_2 \sigma_l \sigma_k}^{-1} (p'_2, \omega'_{n_2}; p_l, \omega_{n_l})}{\times (\varphi_3)_{\sigma_l \sigma_k} (\mathbf{p}_l, \omega_{n_l}; \mathbf{p}_k, \omega_{n_k})} \right) \right\rangle.$$

Appendix C. Figures and figure captions

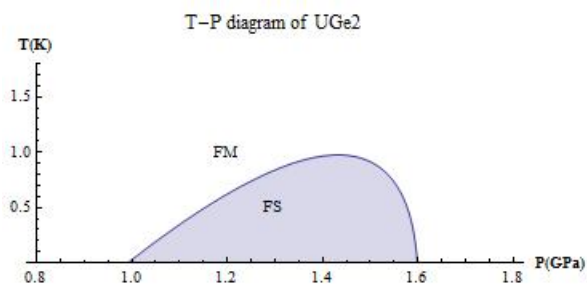


Figure C1. An illustration of T P phase diagram of UGe2 calculated for $T_s = 0, T_{f0} = 52\text{K}, P_c = 1.6\text{GPa}, \gamma/\kappa = 0.10496, \delta/\kappa = 0.17902$. The FS phase domain is shaded. The solid line shows the second order FM-FS phase transition.

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