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ONE LOOP CORRECTIONS TO DECAY $\tau \to \mu\gamma$ IN ECONOMICAL 3-3-1 MODEL

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Abstract. *Lepton flavor violating (cLFV) decays of charged leptons such as $\tau \to \mu\gamma$, $\tau \to e\gamma$, $\mu \to e\gamma$,..., are now the subjects of experiments as signals of new Physics beyond the Standard Model (SM). In the limit of the unitary gauge, we prove that contributions from one loop corrections to the above decays are very small in the framework of the economical 3-3-1 model.*

Keywords: ???.

I. INTRODUCTION

Recently the upper bound for lepton flavor violating decays of charged leptons $\tau \to \mu\gamma$ are set as [1,2]

$$\tau \to \mu\gamma < 4.4 \times 10^{-8}, \tag{1}$$

and in the near future, the sensitive will reach the value of 10^{-9}. This topic is very interesting for recent experimental physics because the lepton flavor (LF) numbers are experimentally shown to be violated in the neutral lepton sector and it is reasonable to hope that the same situation will be right for charged one. The economical 3-3-1 (E331) model [3,4] contains many new particles involving with one loop contributions to the cLFV decays such as new charged gauge bosons and Higgses. But this model includes only neutrinos with small masses, leading to the prediction that the one loop contribution to above cLFV decay may be very small, as that shown in the SM. It is emphasized that this prediction has not been proved before. Therefore, the aim of this paper is to prove in detail this prediction. The one loop contributions to cLFV decays are very important because of the following reasons: i) the very small branching ratio result (1) suggests that the cLFV decays should be come from the loop rather than tree level arising from the mixing among charged leptons in different families; ii) if the cLFV decays are discovered, interesting information of new particles, i.e beyond the SM, may be found from the loop contributions of these particles.

Our work concentrates only on the E331 model and is arranged as follows. After the introduction, section **II** will review the E331 model and list all needed vertices to draw cLFV diagrams. Section **III** will investigate the branching ratio and amplitudes of the $\tau \to \mu\gamma$ decay at the one loop level. Section **IV** will estimate one-loop contributions to this cLFV decay. The final section is our conclusion.

II. cLFV VERTICES IN THE E331 MODEL

The E331 is one of various versions of the 3-3-1 models including right-handed neutrinos (ν331), which are only new leptons beyond the SM. It is also the model that needs only two Higgs triplets to generate masses for all fermions, although the loop corrections have to be considered for generating consistent quark and neutrino masses. Three fields of each lepton family, namely two left-handed neutrinos ν_{aL}, ν^c_{aL} and a left-handed charged lepton l_a, are included in a triplet of the $\mathrm{SU}(3)_L$ group, $\psi_{aL} = (\nu_a,\ l_a, \nu^c_a)^T_L \sim (3,\ -\frac{1}{3})$. Here we only pay attention to quantum numbers of the $SU(3)_L \times U(1)_X$ group and ignore the color one. Each $\nu^c_{aL} = (\nu_{aR})^c$ corresponds to a new right-handed neutrino which is absent in the SM content. All right-handed charged leptons transform as singlets of the $SU(3)_L$, $l_{aR} \sim (1,-1)$. Two Higgs triplets are constructed as $\chi = (\chi^0_1,\ \chi^+_2, \chi^0_3)^T \sim (3,\ \frac{1}{3})$ and $\phi = (\phi^+_1,\ \phi^0_2,\ \phi^+_3)^T \sim (3, \frac{2}{3})$. The vacuum expectation values (VEVs) of neutral Higgs components are $\langle\chi^0_1\rangle = u/\sqrt{2}$, $\langle\chi^0_3\rangle = w/\sqrt{2}$ and $\langle\phi^0_2\rangle = v/\sqrt{2}$. The VEV v contributes mainly to the masses of SM-like particles, including normal charged leptons, W and Z bosons and its value is equal to the SM spontaneous symmetry breaking scale $v = 246$ GeV. The VEV u is the LFV parameter with the lepton number two. The constraint from rare decays shows that its value is extremely small comparing with v, $u/v \leq 0.1$. This parameter does not contribute to the above cLFV decay where the LF number is violated by only one unit for each L_τ or L_μ. The w value is considered as the $\mathrm{SU}(3)_L$ symmetry breaking scale because it is the source generating masses for all new particles beyond the SM.

Masses of leptons come from the Yukawa part of the Lagrangian that includes only couplings between leptons and Higgs ϕ, namely

$$\mathscr{L}^{\mathrm{Y}}_l = h^l_{ab}\overline{\psi_{aL}}\phi l_{bR} + h^\nu_{ab}\varepsilon^{mnp}(\overline{\psi}^c_a)_n(\psi_b)_n(\phi)_p + \mathrm{H.c.}. \tag{2}$$

Here we only focuss on the largest Yukawa couplings between neutrinos to Higgses and charged lepton. For the E331, masses of neutrinos and charged leptons come from two distinguish mass matrices and in general, they are not simultaneous diagonal. Hence, we will work in the basis where mass matrix of the charged leptons is diagonal, i.e $h^l_{ab} = \delta_{ab}\frac{\sqrt{2}gm_{l_a}}{2m_W}$, while that of the neutrinos is not. This naturally reflects recent results of experiments for the LFV effects in lepton sector. At the tree level, all masses of the charged leptons are consistently generated by the Lagrangian (2) while those of neutrinos include one massless and two degenerate ones, being not suitable with current experimental data [5]. But it was shown that the neutrino masses including loop corrections are consistent and all three new (sterlie) neutrinos are very light [4,6]. The mixing of neutrinos in this model is caused by the term $h^\nu_{ab}\varepsilon^{mnp}(\overline{\psi}^c_a)_n(\psi_b)_n(\phi)_p$ generating masses for neutrinos. Because the neutrinos are oscillated, the relation between flavor ν_a and mass ν_α

base are denoted as $\nu_a = \sum_\alpha U_{a\alpha}\nu_\alpha$ with $a = e, \mu, \tau$ being flavor indices while $\alpha = 1,2,3$ being indices of three physical mass eigenvectors. Similarly for ν^c, we have $\nu^c_a = \sum_\alpha U'_{a\alpha}\nu^c_\alpha$. The mixing parameter must satisfy the following inequalities

$$|U_{a\alpha}| \leq 1, \quad |U'_{a\alpha}| \leq 1. \tag{3}$$

The Lagrangian part relating with singly charged Higgs mediations in the loop can be modified from [3,4]. In the flavor basis this part is written by

$$\mathscr{L}^{\mathrm{lep}}_{\mathrm{Y}} = \frac{gm_{l_a}}{\sqrt{2}m_W}\left(\bar{\nu}_a P_R l_a \phi_1^+ + \bar{\nu}^c_a P_R l_a \phi_3^+ + \mathrm{H.c.}\right), \tag{4}$$

where $P_R = \frac{1}{2}(1+\gamma^5)$ is the right-handed helicity-projection operator. It is emphasized that (4) is LF conserved because it is deduced from the first term of (2).

As usual, the E331 model contains nine electroweak gauge bosons corresponding to nine generators of the $SU(3)_L \times U(1)_X$ gauge group. After symmetry breaking, only photon remains massless. The other are massive including four singly charged $W^\pm, Y^\pm$; two non-hermitian neutral X^0, X^{0*}; and two hermitian neutral bosons Z, Z'. The gauge boson sector was investigated in [3,4]. Because X^0 (X^{0*}) only couples with neutrinos, it does not involve with cLFV decays.

Lagrangian relating to llV ($V = W^\pm$, $Y^\pm$) vertices is

$$\begin{aligned}\mathscr{L}^{llV} &= \frac{g}{\sqrt{2}}\left(c_\theta \bar{\nu}_a\gamma^\mu P_L \bar{l}_a W^+_\mu - s_\theta \bar{\nu}^c_a \gamma^\mu P_L l_a W^+_\mu\right. \\ &+ \left. c_\theta \nu^c_a \gamma^\mu P_L l_a Y^+_\mu + s_\theta \bar{\nu}_a \gamma^\mu P_L l_a Y^+_\mu\right) + \mathrm{H.c.}.\end{aligned} \tag{5}$$

The Higgs self-couplings are given by the Lagrangian part

$$\mathscr{L}_{\mathrm{H}} = \mu_1^2\chi^\dagger\chi + \mu_2^2\phi^\dagger\phi + \lambda_1(\chi^\dagger\chi)^2 + \lambda_2(\phi^\dagger\phi)^2 + \lambda_3(\chi^\dagger\chi)(\phi^\dagger\phi) + \lambda_4(\chi^\dagger\phi)(\phi^\dagger\chi).$$

The Higgs sector of this model was studied in [4,7], where the original Higgses
$\phi^T = \left(\phi_1^\dagger, \frac{1}{\sqrt{2}}(v + S_2 + iA_2), \phi_3^\dagger\right)^T$ and $\chi^T = \left(\frac{1}{\sqrt{2}}(u + S_1 + iA_1), \chi_2^-, \frac{1}{\sqrt{2}}(w + S_3 + iA_3)\right)^T$.
The physical Higgses include two real neutral and two singly charged ones, denoted by H^0, H^0_1 and $H^\pm_2$, respectively. The cLFV decays are contributed from only charged Higgses. The mass eigenstates of charged scalars are given as

$$\begin{pmatrix}\phi_1^+ \\ \chi_2^+ \\ \phi_3^+\end{pmatrix} = \frac{1}{\sqrt{w^2 + v^2 c_\theta^2}}\begin{pmatrix} ws_\theta & c_\theta\sqrt{w^2+v^2c_\theta^2} & \frac{1}{2}vs_{2\theta} \\ vc_\theta & 0 & -w \\ wc_\theta & -s_\theta\sqrt{w^2+v^2c_\theta^2} & vc_\theta^2\end{pmatrix}\begin{pmatrix}H_2^+ \\ G_5^+ \\ G_6^+\end{pmatrix}, \tag{6}$$

where $G^\pm_5$ and $G^\pm_6$ are respective goldstone bosons of $W^\pm$ and $Y^\pm$; θ is the angle defined by $\tan\theta = u/w$. New notations appearing in (6) is $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$, and $s_{2\theta} \equiv \sin 2\theta$.

From of interactions discussed above, we obtain the couplings needed for investigating cLFV decays of $\tau \to \mu\gamma$ in the E331 model. They are all listed in the Table 1.

The respective Feynman diagrams for cLFV decays are drawn in Fig. 1.

Table 1. Couplings relating with $l_i \to l_j\gamma$ decays in the E331 model. All momenta in the Feynman rules corresponding to these vertices are incoming. In addition $\Gamma_{\mu\nu\sigma} = \Gamma_{\mu\nu\sigma}(k_0, k_+ k_-) = (k_0 - k_+)_\sigma g_{\mu\nu} + (k_+ - k_-)_\mu g_{\nu\sigma} + (k_- - k_0)_\nu g_{\mu\sigma}$.

Vertex	Coupling	Vertex	Coupling
$\overline{\nu}_\alpha l_a H_2^+$	$\frac{igP_R}{\sqrt{2}}\frac{m_{l_a}}{m_W}\frac{\omega s_\theta}{\sqrt{\omega^2+v^2c_\theta^2}}U_{a\alpha}$	$H_2^-\overline{l_a}\nu_\alpha$	$\frac{igP_L}{\sqrt{2}}\frac{m_{l_a}}{m_W}\frac{\omega s_\theta}{\sqrt{\omega^2+v^2c_\theta^2}}U^*_{a\alpha}$
$\overline{\nu}^c_\alpha l_a H_2^+$	$\frac{igP_R}{\sqrt{2}}\frac{m_{l_a}}{m_W}\frac{\omega s_\theta}{\sqrt{\omega^2+v^2c_\theta^2}}U'_{a\alpha}$	$H_2^-\overline{l_a}\nu^c_\alpha$	$\frac{igP_L}{\sqrt{2}}\frac{m_{l_a}}{m_W}\frac{\omega s_\theta}{\sqrt{\omega^2+v^2c_\theta^2}}U'^*_{a\alpha}$
$\overline{\nu}_\alpha l_a W_\mu^+$	$\frac{ig}{\sqrt{2}}c_\theta\gamma^\mu U_{a\alpha}P_L$	$\overline{\nu_\alpha}^c l_a W_\mu^+$	$-\frac{ig}{\sqrt{2}}s_\theta\gamma^\mu U'_{a\alpha}P_L$
$\overline{\nu_\alpha} l_a Y_\mu^+$	$\frac{ig}{\sqrt{2}}s_\theta\gamma^\mu U_{a\alpha}P_L$	$\overline{\nu_\alpha}^c l_a Y_\mu^+$	$\frac{ig}{\sqrt{2}}c_\theta\gamma^\mu U'_{a\alpha}P_L$
$A^\mu(k_0)W^{\nu+}(k_+)W^{\sigma-}(k_-)$	$-ie\Gamma_{\mu\nu\sigma}$	$A^\mu(k_0)Y^{\nu+}(k_+)Y^{\sigma-}(k_-)$	$-ie\Gamma_{\mu\nu\sigma}$
$A^\mu H_2^+ H_2^-$	$ie(p_+ - p_-)_\mu$	$A^\mu \overline{l}_a l_a$	$ie\gamma_\mu$

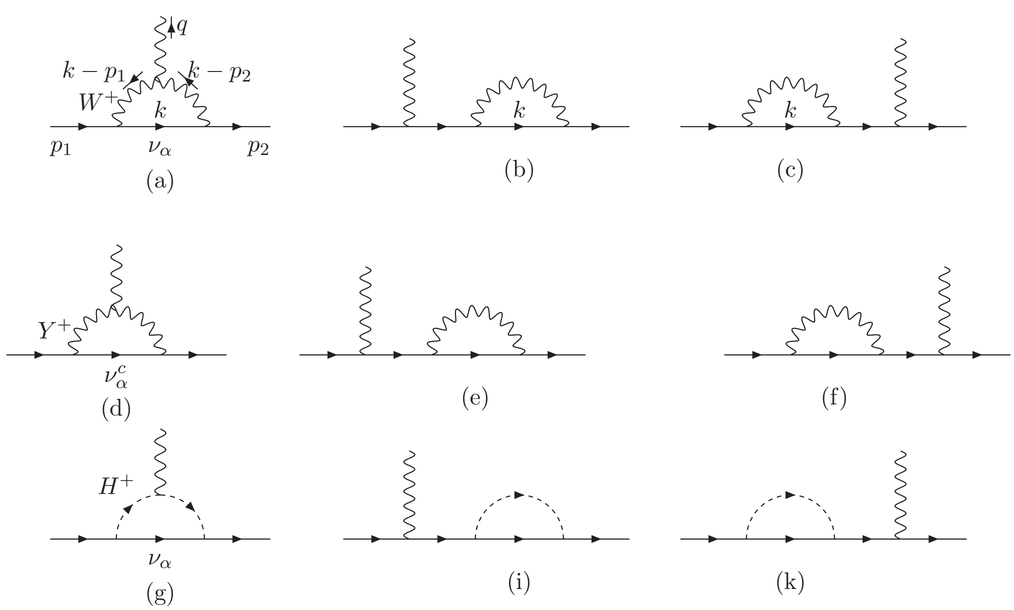

Fig. 1. One-loop Feynman diagrams contribute to $l_1 \to l_2\gamma$ in the unitary gauge

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III. BRANCHING RATIO

III.1. Master integrals and branching ratio

In this section we list some one-loop integrals used in this work. A complete library for one-loop integrals is given in [8]. We will consider the special case where p_1 and p_2 are fixed as momenta of incoming and outcoming leptons, respectively. Furthermore, we consider only LFV cases $m_1 \neq m_2$, where m_1 and m_2 are masses of leptons. Here $p_1^2 = m_1^2,\ p_2^2 = m_2^2$. First, some general forms of useful master integrals defined in literature are listed as follows,

$$\begin{aligned} & C^{\mu} \equiv C^{\mu}(p_1,p_2;M_1,M_2,M_2) \\ = & \ \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^Dk \times k^{\mu}}{(k^2-M_1^2+i\delta)\left[(k-p_1)^2-M_2^2+i\delta\right]\left[(k-p_2)^2-M_2^2+i\delta\right]}, \\ & C^{\mu\nu} \equiv C^{\mu\nu}(p_1,p_2;M_1,M_2,M_2) \\ = & \ \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^Dk \times k^{\mu}k^{\nu}}{(k^2-M_1^2+i\delta)\left[(k-p_1)^2-M_2^2+i\delta\right]\left[(k-p_2)^2-M_2^2+i\delta\right]}, \end{aligned} \tag{7}$$

where M_1 and M_2 are masses of virtual particles in loops. All tensors in (7) can be written in terms of sum of scalar integrals times external momenta, namely

$$\begin{aligned} C^{\mu} &= C_1 p_1^{\mu} + C_2 p_2^{\mu}, \\ C^{\mu\nu} &= C_{00} g^{\mu\nu} + C_{11} p_1^{\mu} p_1^{\nu} + C_{12}(p_1^{\mu} p_2^{\nu} + p_1^{\nu} p_2^{\mu}) + C_{22} p_2^{\mu} p_2^{\nu}, \end{aligned} \tag{8}$$

where factors $C_1,\ C_2,\ C_{00},\ C_{11},\ C_{12}$ and C_{22} are scalar and may be determined analytically in particular cases. Recall that all of these factors are convergent in four dimensions, except C_{00}. For convenience, we will also use new notations for particular values of mass parameters, for example $C_1(p_1,p_2;m_{\nu_\alpha},m_W,m_W) = C_1|_{(M_1,M_2\to m_{\nu_\alpha},m_W)}$.

III.2. Amplitude

Considering the LFV of charged lepton decay $l_1 \to l_2\gamma$, where $(l_1,\ l_2) = \{(\tau,\mu),\ (\tau,e),\ (\mu,\ e)\}$. The amplitude of the decay can be written in the following form [9, 10],

$$\begin{aligned} iM &= 2(p.\varepsilon)\left[C_L \bar{u}_2(p_2) P_L u_1(p_1) + C_R \bar{u}_2(p_2) P_R u_1(p_1)\right] \\ &+ \ D_L \bar{u}_2(p_2) \not{\varepsilon} P_L u_1(p_1) + D_R \bar{u}_2(p_2) \not{\varepsilon} P_R u_1(p_1), \end{aligned} \tag{9}$$

where ε^{μ} is the polarized vector of the photon. The decay width of the decay $l_1 \to l_2\gamma$ is

$$\Gamma(l_1 \to l_2\gamma) = \frac{(m_1^2 - m_2^2)^3}{16\pi m_1^3}\left(|C_L|^2 + |C_R|^2\right), \tag{10}$$

where m_1 and m_2 ($m_2 < m_1$) are masses of the initial and final leptons l_1 and l_2, respectively. We can calculate the branching ratio of decay $\tau \to \mu\gamma$ from the partial decay widths (10) by formula

$$\mathrm{Br}(\tau \to \mu\gamma) = \left(1 - \frac{m_\mu^2}{m_\tau^2}\right)^3 \times \frac{12\pi^2}{G_F^2 m_\tau^2}\left(|C_L|^2 + |C_R|^2\right) \times \mathrm{Br}(\tau \to \mu\bar{\nu}_\mu \nu_\tau) \tag{11}$$

with $G_F = \frac{g^2\sqrt{2}}{8m_W^2}$, $\mathrm{Br}(\tau \to \mu\bar{\nu}_\mu\nu_\tau) \simeq 17.41\%$. Note that for this particular case, we have $m_1, m_2 \to m_\tau, m_\mu$ and u_1, u_2 can be replaced with u_τ, u_μ.

It is needed to recall that C_L and C_R are convergent because of the Ward Identity. Furthermore, the terms relating with these two factors come from only three-point diagrams shown in Fig. 1. It was shown in many previous works that these contributions were really suppressed in the case of the SM [11], due to the very tiny masses of these neutrinos or the very small mixing angles among them. While the case of the 3-3-1 models may be different because the contributions from new particles, namely the $SU(3)_L$ particles, to the cLFV decays $l_1 \to l_2\gamma$ are hoped to be very significant.

In the next section we will show two important results: i) From the gauge invariance condition we get

$$D_L = -m_1 C_R - m_2 C_L \quad \text{and} \quad D_R = -m_1 C_L - m_2 C_R; \tag{12}$$

ii) Contributions to $C_{L,R}$ come from only diagrams a), d) and g) in the Fig. 1. We just pay attention to these diagrams. Because $\frac{m_{l_a}}{m_W} \leq \frac{1,76}{80.4} \sim 10^{-2}$, $s_\theta \ll 1$ and $c_\theta \to 1$, the couplings involving charged Higgs are extremely smaller than those involving charged gauge bosons. So, we can ignore the diagram g). The analytic formulas of contributions to $C_{L,R}$ from the diagrams a) are

$$\begin{aligned} C_L^{(a)}(m_{\nu_\alpha}, m_W, m_W) &= -\frac{eg^2 c_\theta^2 U_{1\alpha}^* U_{2\alpha}}{2} \times \frac{m_\mu}{16\pi^2} \\ &\times \left\{ 2\left(C_1 + C_{12} + C_{22}\right) + \frac{1}{m_W^2} \times \left[m_\tau^2 \left(C_{11} + C_{12} - C_1\right) \right.\right. \\ &+ \left.\left. m_{\nu_\alpha}^2 \left(C_0 + C_{12} + C_{22} - C_1 - 2C_2\right) \right] \right\}\Big|_{(M_1, M_2 \to m_{\nu_\alpha}, m_W)}, \end{aligned} \tag{13}$$

$$\begin{aligned} C_R^{(a)}(m_{\nu_\alpha}, m_W, m_W) &= -\frac{eg^2 c_\theta^2 U_{1\alpha}^* U_{2\alpha}}{2} \times \frac{m_\tau}{16\pi^2} \\ &\times \left\{ 2\left(C_2 + C_{11} + C_{12}\right) + \frac{1}{m_W^2} \times \left[m_\mu^2 \left(C_{12} + C_{22} - C_2\right) \right.\right. \\ &+ \left.\left. m_{\nu_\alpha}^2 \left(C_0 + C_{11} + C_{12} - 2C_1 - C_2\right) \right] \right\}\Big|_{(M_1, M_2 \to m_{\nu_\alpha}, m_W)}, \end{aligned} \tag{14}$$

where $C_0 = C_0(p_1, p_2; M_1, M_2, M_2)$ is also convergent [8]. For the diagram d) we have

$$\begin{aligned} C_L^{(d)}(m_{\nu_\alpha^c}, m_Y, m_Y) &= -\frac{eg^2 c_\theta^2 U_{1\alpha}'^* U_{2\alpha}'}{2} \times \frac{m_\mu}{16\pi^2} \\ &\times \left\{ 2\left(C_1 + C_{12} + C_{22}\right) + \frac{1}{m_Y^2} \times \left[m_\tau^2 \left(C_{11} + C_{12} - C_1\right) \right.\right. \\ &+ \left.\left. m_{\nu_\alpha^c}^2 \left(C_0 + C_{12} + C_{22} - C_1 - 2C_2\right) \right] \right\}\Big|_{(M_1, M_2 \to m_{\nu_\alpha^c}, m_Y)}, \\ C_R^{(d)}(m_{\nu_\alpha^c}, m_Y, m_Y) &= -\frac{eg^2 c_\theta^2 U_{1\alpha}'^* U_{2\alpha}'}{2} \times \frac{m_\tau}{16\pi^2} \\ &\times \left\{ 2\left(C_2 + C_{11} + C_{12}\right) + \frac{1}{m_Y^2} \times \left[m_\mu^2 \left(C_{12} + C_{22} - C_2\right) \right.\right. \\ &+ \left.\left. m_{\nu_\alpha^c}^2 \left(C_0 + C_{11} + C_{12} - 2C_1 - C_2\right) \right] \right\}\Big|_{(M_1, M_2 \to m_{\nu_\alpha^c}, m_Y)}. \end{aligned} \tag{15}$$

Two above formulas of $C_{L,R}$ also indicate a very interesting consequence that $C_{L,R}$ and $D_{L,R}$ are all convergent, because both C_L and C_R are functions of convergent integrals $C_{0,1,2}$ and C_{ij} with $i,j = 1,2$.

IV. APPROXIMATION CALCULATION IN CASE OF VERY SMALL NEUTRINO MASS

IV.1. Calculating $C_{L,R}$ and $D_{L,R}$

First we will prove the equalities in (12). The total amplitude M in (9) can be written as $M = \varepsilon^\mu M_\mu$. The gauge invariant condition, or the Ward Identity, means that $q^\mu M_\mu = 0$, where q is the photon momentum, $q = p_1 - p_2$. By using the following equalities

$$\begin{aligned} q^2 &= 0, \quad 2p_1.q = p_1^2 - p_2^2 = m_\tau^2 - m_\mu^2, \\ u_2 \not{q} P_L u_1 &= m_\tau u_2 P_L u_1 + m_\mu u_2 P_R u_1; \quad u_2 \not{q} P_R u_1 = m_\tau u_2 P_R u_1 + m_\mu u_2 P_L u_1, \end{aligned} \tag{16}$$

as well as M_μ derived from (9), we can prove that both factors of $[u_2 P_R u_1]$ and $[u_2 P_L u_1]$ in $(q^\mu M_\mu)$ must be zeros, resulting a series of two equations $m_\mu D_L - m_\tau D_R = (m_\tau^2 - m_\mu^2) C_L$ and $-m_\tau D_L + m_\mu D_R = (m_\tau^2 - m_\mu^2) C_R$. Solving them we will get the results mentioned in (12). So having $C_{L,R}$, formulas of $D_{L,R}$ are easily determined.

To calculate $C_{L,R}$, we should look in diagrams drawn in the Fig. 1. As illustration, we will represent in detail the analytic formulas of three diagrams in the first row and show that the contributions to $C_{L,R}$ come from only the first one. For diagrams in the second and third rows, we obtain the same results.

Contribution to the total amplitude from the diagram 1 a) is

$$\begin{aligned} i\mathscr{M}^{(a)}(m_{\nu_\alpha}, m_W, m_W) &= \int \frac{d^4k}{(2\pi)^4} \bar{u}_2 \left(\frac{ig}{\sqrt{2}} U_{2\alpha} \gamma_\mu P_L \right) \times \frac{i(\not{k} + m_{\nu_\alpha})}{k^2 - m_{\nu_\alpha}^2} \times \left(\frac{ig}{\sqrt{2}} U_{1\alpha}^* \gamma_\nu P_L \right) u_1 \\ &\times \frac{-i}{(k-p_1)^2 - m_W^2} \left[g^{\nu\beta} - \frac{(k-p_1)^\nu (k-p_1)^\beta}{m_W^2} \right] \\ &\times (-ie)\varepsilon^\lambda \left[(-q-k+p_2)_\beta g_{\lambda\alpha} + (k - p_2 + k - p_1)_\lambda g_{\alpha\beta} \right. \\ &+ \left. (-k + p_1 + q)_\alpha g_{\beta\lambda} \right] \\ &\times \frac{-i}{(k-p_2)^2 - m_W^2} \left[g^{\alpha\mu} - \frac{(k-p_2)^\alpha (k-p_2)^\mu}{m_W^2} \right] \\ &= \frac{eg^2 U_{1\alpha}^* U_{2\alpha}}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}_2 \gamma_\mu \not{k} \gamma_\nu P_L u_1}{\left[k^2 - m_{\nu_\alpha}^2\right] \left[(k-p_1)^2 - m_W^2\right] \left[(k-p_2)^2 - m_W^2\right]} \\ &\times \left[g^{\nu\beta} - \frac{(k-p_1)^\nu (k-p_1)^\beta}{m_W^2} \right] \left[g^{\alpha\mu} - \frac{(k-p_2)^\alpha (k-p_2)^\mu}{m_W^2} \right] \\ &\times (-1) \left[(k + p_1 - 2p_2)_\beta \varepsilon_\alpha + 2(p_1.\varepsilon - k.\varepsilon) g_{\alpha\beta} + (k - 2p_1 + p_2)_\alpha \varepsilon_\beta \right] \\ &\equiv -\frac{eg^2 U_{1\alpha}^* U_{2\alpha}}{2} (N_1 + N_2) \end{aligned} \tag{17}$$

where

$$\begin{aligned}
N_1 &= \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}_2\gamma_\mu \not{k}\gamma_\nu P_L u_1}{D_0D_1D_2} \\
&\times\; g^{\nu\beta} g^{\mu\alpha} \left[(k+p_1-2p_2)_\beta \varepsilon_\alpha + 2(p_1.\varepsilon - k.\varepsilon) g_{\alpha\beta} + (k-2p_1+p_2)_\alpha \varepsilon_\beta\right] \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}_2\gamma_\mu \not{k}\gamma_\nu P_L u_1}{D_0D_1D_2} \\
&\times\; \left[(k+p_1-2p_2)^\nu \varepsilon^\mu + 2(p_1.\varepsilon - k.\varepsilon) g^{\mu\nu} + (k-2p_1+p_2)^\mu \varepsilon^\nu\right] \qquad (18)
\end{aligned}$$

with $m_1 = m_\tau,\ m_2 = m_\mu$ and

$$\begin{aligned}
N_2 &= -\int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}_2\gamma_\mu \not{k}\gamma_\nu P_L u_1}{D_0D_1D_2} \left[\frac{g^{\nu\beta}(k-p_2)^\mu (k-p_2)^\alpha + g^{\mu\alpha}(k-p_1)^\nu (k-p_1)^\beta}{m_W^2}\right] \\
&\times\; \left[(k+p_1-2p_2)_\beta \varepsilon_\alpha + 2(p_1.\varepsilon - k.\varepsilon) g_{\alpha\beta} + (k-2p_1+p_2)_\alpha \varepsilon_\beta\right] \\
&= -\frac{1}{m_W^2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}_2\gamma_\mu \not{k}\gamma_\nu P_L u_1}{D_0D_1D_2} \left[(k-p_1)^2 (k-p_2)^\mu \varepsilon^\nu + (k-p_2)^2 \varepsilon^\mu (k-p_1)^\nu\right. \\
&-\; \left. 2(k.\varepsilon - p_1.\varepsilon)(k-p_2)^\mu (k-p_1)^\nu\right], \qquad (19)
\end{aligned}$$

where we have used $q^2 = 0,\ q = p_1 - p_2,\ (k-p_1-q).(k-p_2) = (k-p_1-q).(k-p_1+q) = (k-p_1)^2 - q^2 = (k-p_1)^2$ and $(k-p_1).(k-p_2+q) = (k-p_2)^2$. Apart from C functions shown in (8), we will use notations of Veltman-Passarino functions defined in [8], such as $A_0(m)$, $B_0^{(0)} \equiv B_0(0;M_2,M_2)$, $B_0(p_i,M_1,M_2) = B_i^{(0)} p_i^\mu$ $(i = 1,2)$ and $B^\mu(p_i,M_1,M_2) = B_i p_i^\mu$. Using again useful formulas in (16) for intermediate calculations, the final results of N_1 and N_2 are

$$\begin{aligned}
N_1 &= \frac{i}{16\pi^2}[\bar{u}_2 \not{\varepsilon} P_L u_1] \times \left[2B_0^{(0)} + 2m_{\nu_\alpha}^2 C_0 - m_1^2 C_1 - m_2^2 C_2 - 2(m_1^2+m_2^2)(C_1+C_2) + 2(d-2)C_{00}\right] \\
&-\; \frac{i}{16\pi^2}[\bar{u}_2 \not{\varepsilon} P_R u_1] \times m_1 m_2 3\left[C_1 + C_2\right] \\
&+\; \frac{i}{16\pi^2}[\bar{u}_2 P_L u_1] \times 2(p_1.\varepsilon) \times m_2\left[C_1 + 2(C_{12}+C_{22})\right] \\
&+\; \frac{i}{16\pi^2}[\bar{u}_2 P_R u_1] \times 2(p_1.\varepsilon) \times m_1\left[C_2 + 2(C_{11}+C_{12})\right] \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
N_2 &= \frac{i}{16\pi^2}[\overline{u}_2 \not{\varepsilon} P_L u_1] \times \frac{1}{m_W^2} \times \Big[-2A_0(m_W) - m_{\nu_\alpha}^2 \left(B_0^{(1)} + B_0^{(2)}\right) - 2m_W^2 B_0^{(0)} - 2m_W^2 m_{\nu_\alpha}^2 C_0 \\
&\quad + m_1^2 B_1 + m_2^2 B_2 + m_W^2 (m_1^2 C_1 + m_2^2 C_2) + m_W^2 (B_0^{(0)} + 1) + 2m_{\nu_\alpha}^2 C_{00}\Big] \\
&+ \frac{i}{16\pi^2}[\overline{u}_2 \not{\varepsilon} P_R u_1] \times \frac{m_1 m_2}{m_W^2} \times \left[2C_{00} - m_W^2 (C_1 + C_2)\right] \\
&+ \frac{i}{16\pi^2}[\overline{u}_2 P_L u_1] \times (2p_1.\varepsilon) \times \frac{m_2}{m_W^2} \times \big[m_1^2 (C_{11} + C_{12} - C_1) + m_W^2 C_1 \\
&\quad + m_{\nu_\alpha}^2 (C_0 + C_{12} + C_{22} - C_1 - 2C_2)\big] \\
&+ \frac{i}{16\pi^2}[\overline{u}_2 P_R u_1] \times (2p_1.\varepsilon) \times \frac{m_1}{m_W^2} \times \big[m_2^2 (C_{12} + C_{22} - C_2) + m_W^2 C_2 \\
&\quad + m_{\nu_\alpha}^2 (C_0 + C_{11} + C_{12} - 2C_1 - C_2)\big]. \qquad (21)
\end{aligned}$$

From Fig. 1 b):

$$\begin{aligned}
iM^{(b)}(m_{\nu_\alpha}, m_W, m_W) &= \int \frac{d^4k}{(2\pi)^4} \overline{u}_2 \left(\frac{ig}{\sqrt{2}} U_{2\alpha} \gamma_\mu P_L\right) \times \frac{i(\not{k} + m_{\nu_\alpha})}{k^2 - m_{\nu_\alpha}^2} \times \left(\frac{ig}{\sqrt{2}} U_{1\alpha}^* \gamma_\nu P_L\right) \\
&\times \frac{i(\not{p}_2 + m_1)}{p_2^2 - m_1^2} \times (-ie)\not{\varepsilon} u_1 \times \frac{-i}{(k-p_2)^2 - m_W^2} \left(g^{\mu\nu} - \frac{(k-p_2)^\mu (k-p_2)^\nu}{m_W^2}\right) \\
&= \frac{-eg^2 U_{2\alpha} U_{1\alpha}^*}{2(m_2^2 - m_1^2)} \int \frac{d^4k}{(2\pi)^4} \times \frac{\overline{u}_2 \gamma_\mu \not{k} \gamma_\nu P_L (\not{p}_2 + m_1) \not{\varepsilon} u_1}{(k^2 - m_{\nu_\alpha}^2)\left[(k-p_2)^2 - m_W^2\right]} \\
&\times \left(g^{\mu\nu} - \frac{(k-p_2)^\mu (k-p_2)^\nu}{m_W^2}\right) \\
&= \frac{-eg^2 U_{2\alpha} U_{1\alpha}^*}{2(m_2^2 - m_1^2)} \times \frac{i}{16\pi^2} \times [m_2 \overline{u}_2 \not{\varepsilon} P_L u_1 + m_1 \overline{u}_2 \not{\varepsilon} P_R u_1] \times m_2 \\
&\times \left\{(2-d)B_2 + \frac{1}{m_W^2}\left[A_0(m_W) + 2m_{\nu_\alpha}^2 B_0^{(2)} - (m_{\nu_\alpha}^2 + m_2^2) B_2\right]\right\}. \qquad (22)
\end{aligned}$$

From Fig. 1 c):

$$\begin{aligned}
iM^{(c)}(m_{\nu_\alpha}, m_W, m_W) &= \int \frac{d^4k}{(2\pi)^4} \overline{u}_2 \times (-ie)\not{\varepsilon} \times \frac{i(\not{p}_1 + m_1)}{p_1^2 - m_2^2} \left(\frac{ig}{\sqrt{2}} U_{2\alpha} \gamma_\mu P_L\right) \times \frac{i(\not{k} + m_{\nu_\alpha})}{k^2 - m_{\nu_\alpha}^2} \\
&\times \left(\frac{ig}{\sqrt{2}} U_{1\alpha}^* \gamma_\nu P_L\right) u_1 \times \frac{-i}{(k-p_1)^2 - m_W^2} \left(g^{\mu\nu} - \frac{(p_1-k)^\mu (p_1-k)^\nu}{m_W^2}\right) \\
&= \frac{-eg^2 U_{2\alpha} U_{1\alpha}^*}{2(m_1^2 - m_2^2)} \int \frac{d^4k}{(2\pi)^4} \times \frac{\overline{u}_2 \not{\varepsilon} (\not{p}_1 + m_2) \gamma_\mu \not{k} \gamma_\nu P_L u_1}{(k^2 - m_{\nu_\alpha}^2)\left[(k-p_2)^2 - m_W^2\right]} \\
&\times \left(g^{\mu\nu} - \frac{(k-p_1)^\mu (k-p_1)^\nu}{m_W^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-eg^2 U^*_{2\alpha} U_{1\alpha}}{2(m_1^2-m_2^2)} \times \frac{i}{16\pi^2} \times [m_1 \overline{u}_2 \not{\varepsilon} P_L u_1 + m_2 \overline{u}_2 \not{\varepsilon} P_R u_1] \times m_1 \\
&\times \left\{(2-d)B_1 + \frac{1}{m_W^2}\left[A_0(m_W) + 2m^2_{\nu_\alpha} B_0^{(1)} - (m^2_{\nu_\alpha} + m_1^2)B_1\right]\right\}. \quad (23)
\end{aligned}
$$

Two formulas (22) and (23) do not contain any terms being relevant with $\overline{u}_2 P_L \overline{u}_1$ or $\overline{u}_2 P_R \overline{u}_1$, therefore do not contribute to $C_{L,R}$. Collecting terms in (20) and (21) containing factors $\overline{u}_2 P_L \overline{u}_1$ and $\overline{u}_2 P_R \overline{u}_1$ we get the results of $C_{L,R}$ shown in (13) and (14).

IV.2. Estimation in the limit of very small neutrino mass

From now we just focus on the largest contributions of $|C_{L,R}|$ to the value of branching ratio (11). From (13) and (14), we have $|C_L| \ll |C_R|$ because $m_\mu \ll m_\tau$. In addition, in C_R we can ignore terms containing factors $\frac{m_\mu^2}{m_W^2}$ and $\frac{m^2_{\nu_\alpha}}{m_W^2}$, which are extremely smaller than one, and keep those that do not contain these factors. So the total largest contribution from three virtual neutrinos is

$$
C_R^{(a)}(m_{\nu_\alpha}, m_W, m_W) \simeq -eg^2 c_\theta^2 \sum_{\alpha=1}^{3} U^*_{1\alpha} U_{2\alpha} \times \frac{m_\tau}{16\pi^2} \left(C_2 + C_{11} + C_{12}\right)\Big|_{(M_1, M_2 \to m_{\nu_\alpha}, m_W)} \quad (24)
$$

and $C_L(m_{\nu_\alpha}, m_W, m_W) \simeq 0$.

To calculate (24), we use the condition $m_{\nu_\alpha} \ll m_W$ for applying the approximation method shown in [11] (subsection 13.3) to evaluate the approximate formulas of C_2, C_{11} and C_{12} from (7) and (8). Some main steps are as follows. Replacing $\frac{1}{k^2 - m^2_{\nu_\alpha}}$ with the below expansion

$$
\frac{1}{k^2 - m^2_{\nu_\alpha}} = \frac{1}{k^2} + \frac{m^2_{\nu_\alpha}}{(k^2)^2} + \mathscr{O}([m^2_{\nu_\alpha}]^2) \quad (25)
$$

in the integrands of (7) to rewrite $C_{i,ij}$ in new forms. We easily see that the first term in (25) does not depend on m_{ν_α}, $U_{1\alpha}$ and $U_{2\alpha}$. It cancels when all of $C_{i,ij}$ functions are inserted into (24) because $\sum U^*_{1\alpha} U_{2\alpha} = 0$. Now we need only the second term, ignore the higher powers of $m^2_{\nu_\alpha}$. Using the Feynman parametrization trick,

$$
\begin{aligned}
&\frac{1}{(k^2)^2\left[(k-p_1)^2 - m_W^2\right]\left[(k-p_1)^2 - m_W^2\right]} \\
&= 3! \int_0^1 dx\, dy\, dz \frac{\delta(1-x-y-z)x}{\left\{xk^2 + y\left[(k-p_1)^2 - m_W^2\right] + z\left[(k-p_2)^2 - m_W^2\right]\right\}^4} \quad (26)
\end{aligned}
$$

we will write C^μ as

$$
C^\mu(m_{\nu_\alpha}, m_W, m_W) = \frac{3! m^2_{\nu_\alpha}}{i\pi^2} \int_0^1 dy \int_0^{1-y} dz (1-y-z) \int d^4k \frac{y p_1^\mu + z p_2^\mu}{(k^2 - M^2)^4}, \quad (27)
$$

where $M^2 = (yp_1 + zp_2)^2 + (y+z)m_W^2 - (yp_1^2 + zp_2^2)$ and we have changed $k \to k + (yp_1 + zp_2)$. The $C_2(m_{\nu_\alpha}, m_W, m_W)$ relates with p_μ^2 in (27), in particularly

$$
C_2(m_{\nu_\alpha}, m_W, m_W) = \frac{3! m^2_{\nu_\alpha}}{i\pi^2} \int_0^1 dy \int_0^{1-y} z(1-y-z) dz \int \frac{d^4k}{(k^2 - M^2)^4}. \quad (28)
$$

From the well-known integral

$$\int \frac{d^4k}{(k^2-M^2)^4} = \frac{i\pi^2}{6M^4}$$

and taking $M^2 \simeq (y+z)m_W^2$, because $p_1^2 = m_\tau^2, p_2^2 = m_\mu^2 \ll M_W^2$, we get the approximate value of C_2,

$$C_2(p_1,p_2;m_{\nu_\alpha},m_W,m_W) \simeq \frac{3!m_{\nu_\alpha}^2}{i\pi^2} \times \frac{i\pi^2}{6m_W^4}\int_0^1 dy \int_0^{1-y} dz \frac{(1-y-z)z}{(y+z)^2} = \frac{m_{\nu_\alpha}^2}{4m_W^4}. \quad (29)$$

Similarly for calculating $C_{11,12}$, we get

$$\begin{aligned} C_{11}(p_1,p_2;m_{\nu_\alpha},m_W,m_W) &= \frac{m_{\nu_\alpha}^2}{m_W^4}\int_0^1 dy \int_0^{1-y} dz \frac{(1-y-z)y^2}{(y+z)^2} = \frac{m_{\nu_\alpha}^2}{18m_W^4}, \\ C_{12}(p_1,p_2;m_{\nu_\alpha},m_W,m_W) &= \frac{m_{\nu_\alpha}^2}{m_W^4}\int_0^1 dy \int_0^{1-y} dz \frac{(1-y-z)yz}{(y+z)^2} = \frac{m_{\nu_\alpha}^2}{36m_W^4}. \end{aligned} \quad (30)$$

From (29) and (30), we get an approximate result of (24),

$$C_R^{(a)}(p_1,p_2;m_{\nu_\alpha},m_W,m_W) \simeq -m_\tau \times \frac{eg^2c_\theta^2}{48\pi^2 m_W^2} \times \sum_\alpha \frac{U_{1\alpha}^* U_{2\alpha} m_{\nu_\alpha}^2}{m_W^2}. \quad (31)$$

Also, the contribution to C_R from $C_R^{(d)}$ given in (15) can be written approximately in the same way. Now the largest contribution to the branching ratio (11) is

$$\begin{aligned} \mathrm{Br}(\tau \to \mu\gamma) &\simeq \frac{12\pi^2}{G_F^2} \times \left(\frac{eg^2c_\theta^2}{48\pi^2 m_W^2}\right)^2 \left| \sum_\alpha \frac{U_{1\alpha}^* U_{2\alpha} m_{\nu_\alpha}^2}{m_W^2} + \frac{m_W^4}{m_Y^4}\sum_\alpha \frac{U_{1\alpha}'^* U_{2\alpha}' m_{\nu_\alpha^c}^2}{m_Y^2}\right|^2 \mathrm{Br}(\tau \to \mu\bar{\nu}_\mu\nu_\tau) \\ &= \frac{2\alpha}{3\pi}\left| \sum_\alpha \frac{U_{1\alpha}^* U_{2\alpha} m_{\nu_\alpha}^2}{m_W^2} + \frac{m_W^4}{m_Y^4}\sum_\alpha \frac{U_{1\alpha}'^* U_{2\alpha}' m_{\nu_\alpha^c}^2}{m_Y^2}\right|^2 \mathrm{Br}(\tau \to \mu\bar{\nu}_\mu\nu_\tau), \end{aligned} \quad (32)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant. At the electroweak scale, $\alpha = 1/128$.

As shown in [6], ν and ν^c are also very light, so we can set an upper bound $m_{\nu_\alpha^c}^2, m_{\nu_\alpha}^2 < (1eV)^2 = (10^{-9}GeV)^2$. We also have $m_Y > m_W \simeq 80GeV$ and $|U_{a\alpha}|, |U'_{a\alpha}| < 1$, leading to

$$\mathrm{Br}(\tau \to \mu\gamma) < \frac{2\alpha}{3\pi}\left(2\times 3\times\frac{10^{-18}}{80^2}\right)^2 \sim 10^{-45}, \quad (33)$$

which is extremely smaller than the upper bound of experiments (1).

V. CONCLUSION

In the E331 model we have established in detail the analytic formula of the branching ratio of the cLFV decay $\tau \to \mu\gamma$ at the one loop level. From the prediction of this model that all neutrinos are very light, we construct an approximate formula of the branching ratio for this decay and indicate that $\mathrm{Br}(\tau \to \mu\gamma) < 10^{-45}$ in the present experimental limit of neutrino and Y boson masses. This shows that the E331 model predicts the very suppressed values of the cLFV decay branching ratio comparing with the present experiment value, $\mathrm{Br}(\tau \to \mu\gamma) < 4.4 \times 10^{-8}$. Hence if this decay is detected in the near future, it must come from sources beyond the prediction of the E331 model, for example the suppersymmetric version of well-known models such as MSSM, SUSYE331,... One more important result of our work is obtaining the very precise amplitudes of the cLFV decay as functions of the well-known Veltman-Passarino functions before any approximate estimation, so it can be seen exactly the convergence of the $C_{L,R}$ and therefore $D_{L,R}$. This makes our calculation very clear and it can be applied for investigation of cLFV decays in many other models, including 3-3-1 ones.

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