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Neutrino mixing with nonzero θ_{13} and CP violation in the 3-3-1 model based on A_4 flavor symmetry

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We propose a 3-3-1 model with neutral fermions based on A_4 flavor symmetry responsible for fermion masses and mixings with nonzero θ_{13}. To get realistic neutrino mixing, we just add a new $SU(3)_L$ triplet being in $\underline{3}$ under A_4. The neutrinos get small masses from two $SU(3)_L$ antisextets and one $SU(3)_L$ triplet. The model can fit the present data on neutrino masses and mixing as well as the effective mass governing neutrinoless double beta decay. Our results show that the neutrino masses are naturally small and a little deviation from the tri-bimaximal neutrino mixing form can be realized. The Dirac CP violation phase δ is predicted to either 5.41° or 354.59° with $\theta_{23} \neq \frac{\pi}{4}$.

Keywords: Neutrino mass and mixing; nonstandard model neutrinos; right-handed neutrinos; charge conjugation; discrete symmetries.

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1. Introduction

Despite the great success of the Standard Model (SM) of the elementary particle physics, the origin of flavor structure, masses and mixings between generations of matter particles are still open questions. The neutrino mass and mixing are one of the most important evidence of beyond SM physics. Many experiments show that neutrinos have tiny masses and their mixing is still mysterious.[1,2] The tri-bimaximal form for explaining the lepton mixing scheme was first proposed by

Harrison–Perkins–Scott (HPS), which apart from the phase redefinitions, is given by[3–6]

$$U_{\mathrm{HPS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{1}$$

can be considered as a good approximation for the recent neutrino experimental data. In fact, the absolute values of the entries of the lepton mixing matrix U_{PMNS} approximately are given by[7–10]

$$|U_{\mathrm{PMNS}}| = \begin{pmatrix} 0.795\text{–}0.846 & 0.513\text{–}0.585 & 0.126\text{–}0.178 \\ 0.205\text{–}0.543 & 0.416\text{–}0.730 & 0.579\text{–}0.808 \\ 0.215\text{–}0.548 & 0.409\text{–}0.725 & 0.567\text{–}0.800 \end{pmatrix}. \tag{2}$$

The data in Refs. 11–15 imply

$$\begin{aligned} \sin^2(2\theta_{12}) &= 0.857 \pm 0.024\,, \\ \sin^2(2\theta_{23}) > 0.95\,, &\quad \sin^2(2\theta_{13}) = 0.098 \pm 0.013\,, \\ \Delta m^2_{21} &= (7.50 \pm 0.20) \times 10^{-5}\ \mathrm{eV}^2\,, \\ \Delta m^2_{32} &= (2.32^{+0.12}_{-0.08}) \times 10^{-3}\ \mathrm{eV}^2\,, \end{aligned} \tag{3}$$

whereas, the best fit values of neutrino mass squared differences and the leptonic mixing angles in Refs. 16 and 17 have been given to be slightly modified from (3), as shown in Tables 1 and 2. These large neutrino mixing angles are completely different from the quark mixing ones defined by the Cabibbo–Kobayashi–Maskawa (CKM) matrix.[18,19] This has stimulated works on flavor symmetries and non-Abelian discrete symmetries, which are considered to be the most attractive candidate to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and leptons. There are many recent models based on the non-Abelian discrete symmetries, see for example Refs. 34–41 and the references therein.

Table 1. The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from Refs. 16 and 17 for normal hierarchy.

Parameter	Best fit	1σ range	2σ range
Δm^2_{21} (10^{-5} eV^2)	7.62	7.43–7.81	7.27–8.01
Δm^2_{31} (10^{-3} eV^2)	2.55	2.64–2.61	2.38–2.68
$\sin^2\theta_{12}$	0.320	0.303–0.336	0.29–0.35
$\sin^2\theta_{23}$	0.613	0.573–0.635	0.38–0.66
$\sin^2\theta_{13}$	0.0246	0.0218–0.0275	0.019–0.03

Table 2. The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from Refs. 16 and 17 for inverted hierarchy.

Parameter	Best fit	1σ range	2σ range
Δm^2_{21} (10^{-5} eV^2)	7.62	7.43–7.81	7.27–8.01
Δm^2_{13} (10^{-3} eV^2)	2.43	2.37–2.50	2.29–2.58
$\sin^2\theta_{12}$	0.32	0.303–0.336	0.29–0.35
$\sin^2\theta_{23}$	0.60	0.569–0.626	0.39–0.65
$\sin^2\theta_{13}$	0.025	0.0223–0.0276	0.02–0.03

An alternative extension of the SM is the 3-3-1 models, in which the SM gauge group $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ is extended to $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$, whose phenomenology has been studied in great detail from various particle physics standpoints.[20–33] The anomaly cancellation and the QCD asymptotic freedom in the models require that the number of families is equal to the number of quark colors, and one family of quarks has to transform under $\mathrm{SU}(3)_L$ differently from the two others. In our previous works,[34–41] the discrete symmetries have been explored to the 3-3-1 models. The simplest explanation is probably due to a S_3 flavor symmetry which is the smallest non-Abelian discrete group, has been explored in our previous work.[36] In Refs. 34 and 35, we have studied the 3-3-1 model with neutral leptons based on A_4 and S_4 groups, in which the exact tri-bimaximal form is obtained, where $\theta_{13} = 0$. As we know, the recent considerations have implied $\theta_{13} \neq 0$, but relatively small as given in (3) or Tables 1 and 2. This problem has been improved in Ref. 36 by adding a new triplet ρ and another antisextet s', in which s' is regarded as a small perturbation. Therefore, the model contains up to eight Higgs multiplets, and the scalar potential of the model is quite complicated. In Ref. 37 we have studied the 3-3-1 model with neutral fermions based on D_4 group, in which the fermion fields are in singlets and doublets under D_4. Our aim in this paper is to construct the 3-3-1 model combined with A_4 to adapt nonzero θ_{13}. For this purpose a $\mathrm{SU}(3)_L$ triplet is added and the result follows without perturbation. We will work on a basis where $\underline{3}$ is a real representation.

There are two typical variants of the 3-3-1 models as far as lepton sectors are concerned. In the minimal version, three $\mathrm{SU}(3)_L$ lepton triplets are (ν_L, l_L, l^c_R), where l_R are ordinary right-handed charged leptons.[20] In the second version, the third components of lepton triplets are the right-handed neutrinos,[25] (ν_L, l_L, ν^c_R). To have a model with the realistic neutrino mixing matrix, we should consider another variant of the form (ν_L, l_L, N^c_R) where N_R are three new fermion singlets under SM symmetry with vanishing lepton-numbers.[34,35]

The contents of the paper are as follows. In Secs. 2 and 3, we present the necessary elements of the 3-3-1 model with A_4 flavor symmetry as in the above choice and introduce necessary Higgs fields responsible for the charged-lepton masses. Section 4 is devoted for the neutrino masses and mixings. In Sec. 5, we discuss the quark sector. We summarize our results and make conclusions in Sec. 6. Appendix A presents a brief summary of the A_4 group. Appendix B provides the lepton number (L) and lepton parity (P_l) of the particles in the model. Appendices C to J give the detailed solutions corresponding to special cases in the normal and inverted spectrum.

2. Fermion Content

The gauge symmetry is based on $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$, where the electroweak factor $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ is extended from those of the SM while the strong interaction sector is retained. Each lepton family includes a new electrically- and leptonically-neutral fermion (N_R) and is arranged under the $\mathrm{SU}(3)_L$ symmetry as a triplet (ν_L, l_L, N_R^c) and a singlet l_R. The residual electric charge operator Q is related to the generators of the gauge symmetry by

$$Q = T_3 - \frac{1}{\sqrt{3}} T_8 + X \,, \tag{4}$$

where T_a $(a = 1, 2, \ldots, 8)$ are $\mathrm{SU}(3)_L$ charges with $\mathrm{Tr}\, T_a T_b = \frac{1}{2}\delta_{ab}$ and X is the $\mathrm{U}(1)_X$ charge. The model under consideration does not contain exotic electric charges in the fundamental fermion, scalar and adjoint gauge boson representations.

Since the particles in the lepton triplet have different lepton number (1 and 0), so the lepton number in the model does not commute with the gauge symmetry unlike the SM. Therefore, it is better to work with a new conserved charge $\mathcal{L}$ (Refs. 42–45) commuting with the gauge symmetry and related to the ordinary lepton number by diagonal matrices[34,35]

$$L = \frac{2}{\sqrt{3}} T_8 + \mathcal{L} \,. \tag{5}$$

The lepton charge arranged in this way (i.e. $L(N_R) = 0$ as assumed) is in order to prevent unwanted interactions due to $\mathrm{U}(1)_{\mathcal{L}}$ symmetry and breaking to obtain the consistent lepton and quark spectra. By this embedding, exotic quarks U, D as well as new non-Hermitian gauge bosons X^0, $Y^{\pm}$ possess lepton charges as of the ordinary leptons: $L(D) = -L(U) = L(X^0) = L(Y^-) = 1$.

In the model under consideration, the fermion contents is same as in Ref. 34. However, this work is distinguished by a new $\mathrm{SU}(3)_L$ triplet (ρ) which is put in $\underline{3}$ under A_4. Under the $[\mathrm{SU}(3)_L, \mathrm{U}(1)_X, \mathrm{U}(1)_{\mathcal{L}}, \underline{A}_4]$ symmetries, the fermions of the model transform as follows[34]

$$\begin{gathered} \psi_L \equiv \psi_{1,2,3L} = (\nu_L \quad l_L \quad N_R^c)^T \sim [3, -1/3, 2/3, \underline{3}] \,, \\ l_{1R} \sim [1, -1, 1, \underline{1}] \,, \quad l_{2R} \sim [1, -1, 1, \underline{1}'] \,, \quad l_{3R} \sim [1, -1, 1, \underline{1}''] \,, \end{gathered} \tag{6}$$

$$\begin{aligned}
Q_{3L} &= \begin{pmatrix} u_{3L} \\ d_{3L} \\ U_L \end{pmatrix} \sim [3, 1/3, -1/3, \underline{1}]\,, \qquad U_R \sim [1, 2/3, -1, \underline{1}]\,, \\
Q_{1L} &= \begin{pmatrix} d_{1L} \\ -u_{1L} \\ D_{1L} \end{pmatrix} \sim [3^*, 0, 1/3, \underline{1}']\,, \quad D_{1R} \sim [1, -1/3, 1, \underline{1}'']\,, \\
Q_{2L} &= \begin{pmatrix} d_{2L} \\ -u_{2L} \\ D_{2L} \end{pmatrix} \sim [3^*, 0, 1/3, \underline{1}'']\,, \quad D_{2R} \sim [1, -1/3, 1, \underline{1}']\,, \\
u_R &\sim [1, 2/3, 0, \underline{3}]\,, \qquad\qquad d_R \sim [1, -1/3, 0, \underline{3}]\,,
\end{aligned} \tag{7}$$

where the subscript numbers on field indicate to respective families which also define components of their A_4 multiplets. In what follows, we consider possibilities of generating the masses for the fermions. The scalar multiplets needed for the purpose are also introduced.

3. Charged Lepton Mass

The fermion content of the model is the same as that in Ref. 34 under all symmetries. However, in this work, the breaking of A_4 in charged lepton sector is different from that in Ref. 34. Namely, to generate masses for the charged leptons, we need only one scalar multiplet:

$$\phi = \left(\phi_1^+, \phi_2^0, \phi_3^+\right)^T \sim [3, 2/3, -1/3, \underline{3}]\,. \tag{8}$$

The Yukawa terms are

$$-\mathcal{L}_l = h_1(\bar{\psi}_L\phi)_{\underline{1}}\, l_{1R} + h_2(\bar{\psi}_L\phi)_{\underline{1}''} l_{2R} + h_3(\bar{\psi}_L\phi)_{\underline{1}'} l_{3R} + \text{H.c.} \tag{9}$$

From the potential minimization conditions, we have the followings alignments:

(1) The first alignment: $\langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_3\rangle$ then A_4 is broken into Z_3 consisting of the elements $\{e, T, T^2\}$.
(2) The second alignment: $\langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle$ or $\langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle$ or $\langle\phi_2\rangle \neq \langle\phi_1\rangle = \langle\phi_3\rangle$ or $\langle\phi_3\rangle \neq \langle\phi_1\rangle = \langle\phi_2\rangle$ then A_4 is broken into {Identity}.[a]
(3) The third alignment: $0 = \langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_2\rangle \neq \langle\phi_3\rangle = \langle\phi_1\rangle \neq 0$ or $0 = \langle\phi_3\rangle \neq \langle\phi_1\rangle = \langle\phi_2\rangle \neq 0$ then A_4 is broken into {Identity}.
(4) The fourth alignment: $0 = \langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_2\rangle \neq \langle\phi_1\rangle \neq \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_3\rangle \neq \langle\phi_1\rangle \neq \langle\phi_2\rangle \neq 0$ then A_4 is broken into {Identity}.
(5) The fifth alignment: $0 = \langle\phi_2\rangle = \langle\phi_3\rangle \neq \langle\phi_1\rangle \neq 0$ then A_4 is broken into Z_2 consisting of the elements $\{e, S\}$.

[a] This means A_4 is completely broken.

(6) The sixth alignment: $0 = \langle\phi_1\rangle = \langle\phi_3\rangle \neq \langle\phi_2\rangle \neq 0$ then A_4 is broken into Z_2 consisting of the elements $\{e, T^2ST\}$.
(7) The seventh alignment: $0 = \langle\phi_1\rangle = \langle\phi_2\rangle \neq \langle\phi_3\rangle \neq 0$ then A_4 is broken into Z_2 consisting of the elements $\{e, TST^2\}$.

To obtain a realistic lepton spectrum, we suppose that in charged lepton sector, A_4 is broken down to {Identity}. This breaking is different from Ref. 34 in charged lepton sector, and it can be achieved with the VEV alignment $\langle\phi\rangle = (\langle\phi_1\rangle, \langle\phi_2\rangle, \langle\phi_3\rangle)$ under A_4 where $\langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle$, and

$$\langle\phi_i\rangle = (0 \quad v_i \quad 0)^T \quad (i = 1, 2, 3)\,.$$

The mass Lagrangian for the charged leptons reads

$$\mathcal{L}_l^{\text{mass}} = -(\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L}) M_l (l_{1R}, l_{2R}, l_{3R})^T + \text{H.c.}\,,$$

where

$$M_l = \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 \omega v_2 & h_3 \omega^2 v_2 \\ h_1 v_3 & h_2 \omega^2 v_3 & h_3 \omega v_3 \end{pmatrix}. \tag{10}$$

As we will see in Sec. 4, in the case $A_4 \to Z_3$ consisting of the elements $\{e, T, T^2\}$, i.e, $\langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_3\rangle$ or $v_1 = v_2 = v_3 = v$, the charged lepton matrix M_l in Eq. (10) is diagonalized by the matrix

$$U_{0L} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{11}$$

and the exact tri-bimaximal mixing form is obtained if $A_4 \to Z_3$ in both charged lepton and neutrino sectors. A detailed study on this problem, the reader can see in Ref. 34.

As we know, the realistic lepton mixing form is a small deviation from tri-bimaximal form.[11] The realistic lepton mixing can be achieved with a small value. Hence, we can separate v_2, v_3 into two parts, the first is equal to $v_1 \equiv v$, the second is responsible for that deviation,

$$v_1 = v\,, \quad v_2 = v(1 + \varepsilon_2)\,, \quad v_3 = v(1 + \varepsilon_3)\,, \quad \varepsilon_{2,3} \ll 1 \tag{12}$$

and the matrix M_l in (10) becomes

$$\begin{aligned} M_l &= \begin{pmatrix} h_1 v & h_2 v & h_3 v \\ h_1 v(1+\varepsilon_2) & h_2 \omega v(1+\varepsilon_2) & h_3 \omega^2 v(1+\varepsilon_2) \\ h_1 v(1+\varepsilon_3) & h_2 \omega^2 v(1+\varepsilon_3) & h_3 \omega v(1+\varepsilon_3) \end{pmatrix} \\ &\equiv v \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\varepsilon_2 & 0 \\ 0 & 0 & 1+\varepsilon_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}. \end{aligned} \tag{13}$$

The matrix M_l in Eq. (13) can be diagonalized as follows with the help of notation

$$M'_l = U^+_{0L} M_l = \frac{v}{\sqrt{3}} \begin{pmatrix} (3+\varepsilon_1+\varepsilon_2)h_1 & (\omega\varepsilon_1+\omega^2\varepsilon_2)h_2 & (\omega^2\varepsilon_1+\omega\varepsilon_2)h_3 \\ (\omega^2\varepsilon_1+\omega\varepsilon_2)h_1 & (3+\varepsilon_1+\varepsilon_2)h_2 & (\omega\varepsilon_1+\omega^2\varepsilon_2)h_3 \\ (\omega\varepsilon_1+\omega^2\varepsilon_2)h_1 & (\omega^2\varepsilon_1+\omega\varepsilon_2)h_2 & (3+\varepsilon_1+\varepsilon_2)h_3 \end{pmatrix}, \quad (14)$$

then the matrix M'_l in (14) is diagonalized by

$$U^+_L M'_l \equiv U^+_L U^+_{0L} M_l = \mathrm{diag}(m_e, m_\mu, m_\tau)\,, \quad (15)$$

where

$$m_e = Y_l h_1 v\,, \quad m_\mu = Y_l h_2 v\,, \quad m_\tau = Y_l h_3 v\,, \quad (16)$$

with

$$Y_l = \frac{3\sqrt{3}(1+\varepsilon_3)\left[-4+\varepsilon_3(-4+\varepsilon_3+\sqrt{(\varepsilon_3-12)\varepsilon_3-12})\right]}{(2+\varepsilon_3)\left[-6+\varepsilon_3(-6+\varepsilon_3+\sqrt{(\varepsilon_3-12)\varepsilon_3-12})\right]}\,. \quad (17)$$

The matrix that diagonalize M'_l in (14) takes the form:

$$U_L = \begin{pmatrix} 1 & U^l_{12} & U^l_{13} \\ U^l_{13} & 1 & U^l_{12} \\ U^l_{12} & U^l_{13} & 1 \end{pmatrix}, \quad U_R = 1\,, \quad (18)$$

where

$$\begin{aligned} U^l_{12} &= \frac{\varepsilon_3\left\{6-2i\sqrt{3}-(1+i\sqrt{3})\varepsilon+\varepsilon_3[7-i\sqrt{3}-(1-i\sqrt{3})\varepsilon_3+(1-i\sqrt{3})\varepsilon]\right\}}{2(2+\varepsilon_3)[-6+\varepsilon_3^2-\varepsilon_3(6+\varepsilon)]}\,, \\ U^l_{13} &= \frac{\varepsilon_3\left\{6+2i\sqrt{3}-(1-i\sqrt{3})\varepsilon+\varepsilon_3[7+i\sqrt{3}-(1+i\sqrt{3})\varepsilon_3+(1+i\sqrt{3})\varepsilon]\right\}}{2(2+\varepsilon_3)[-6+\varepsilon_3^2-\varepsilon_3(6+\varepsilon)]}\,, \end{aligned} \quad (19)$$

with

$$\varepsilon = \sqrt{\varepsilon_3^2-12(\varepsilon_3+1)}\,. \quad (20)$$

To get the results in Eqs. (19), we have used the following relations

$$\varepsilon_2 = \frac{2\varepsilon_3-\varepsilon_3^2-\varepsilon_3\varepsilon}{2(\varepsilon_3+2)}\,, \quad \varepsilon_2^* = \frac{\varepsilon_3(-2-3\varepsilon_3+\varepsilon)}{2(\varepsilon_3+1)(\varepsilon_3+2)}\,, \quad \varepsilon_3^* = -1+\frac{1}{1+\varepsilon_3}\,, \quad (21)$$

which are obtained from the unitary condition of U_L.

The left- and right-handed mixing matrices in charged lepton sector are given by:

$$U'_L = U_{0L}\cdot U_L = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_1 \\ \alpha_2 & \omega^2\alpha_2 & \omega\alpha_2 \\ \alpha_3 & \omega\alpha_3 & \omega^2\alpha_3 \end{pmatrix}, \quad U'_R = 1\,, \quad (22)$$

where

$$\alpha_1 = \frac{\sqrt{3}[-4+\varepsilon_3^2-\varepsilon_3(4+\varepsilon)]}{(2+\varepsilon_3)[-6+\varepsilon_3^2-\varepsilon_3(6+\varepsilon)]}\,,$$
$$\alpha_2 = \frac{2\sqrt{3}(1+\varepsilon_3)}{6-\varepsilon_3^2+\varepsilon_3(6+\varepsilon)}\,, \tag{23}$$
$$\alpha_3 = (1+\varepsilon_3)\alpha_1\,.$$

In general, $\varepsilon_{2,3} \neq 0$, so α_i $(i = 1,2,3)$ in Eq. (23) are different to each other and different from $\frac{1}{\sqrt{3}}$, and lead to the realistic lepton mixing with nonzero θ_{13} as represented in Sec. 4. This is one of the striking results of the model under consideration.

Taking into account of the discovery of the long-awaited Higgs boson at around 125 GeV by ATLAS[46] and CMS,[47] we can choose the VEVs $v = 100$ GeV. From (16), the charged lepton Yukawa couplings $h_{1,2,3}$ relate to their masses as follows:

$$h_1 = \frac{m_e}{Y_l v}\,, \quad h_2 = \frac{m_\mu}{Y_l v}\,, \quad h_3 = \frac{m_\tau}{Y_l v}\,. \tag{24}$$

The experimental mass values for the charged leptons at the weak scale are given as:[11]

$$m_e \simeq 0.511 \text{ MeV}\,, \quad m_\mu \simeq 105.66 \text{ MeV}\,, \quad m_\tau \simeq 1776.82 \text{ GeV}\,. \tag{25}$$

With the help of (25) we have $\frac{h_1}{h_2} \simeq 0.0048$, $\frac{h_1}{h_3} \simeq 0.0003$ and $\frac{h_2}{h_3} = 0.0595$, i.e. $h_1 \ll h_2 \ll h_3$ for any ε_3. As will be shown in Sec. 4, from experimental constrains on lepton mixing, we obtain two solutions in Eqs. (46) and (47). With ε_3 given in Eq. (46), we get

$$h_1 \simeq 3.0045 \times 10^{-6}\,, \quad h_2 \simeq 6.2124 \times 10^{-4}\,, \quad h_3 \simeq 1.045 \times 10^{-2}\,. \tag{26}$$

We note that the mass hierarchy of the charged leptons are well separated by only one Higgs triplet ϕ, and this is a good feature of the A_4 group. To conclude this section, we remind that the situation here is different from all our previous version presented in Refs. 34–37 that can lead to nonzero θ_{13} which is studied in Sec. 4.

4. Neutrino Mass and Mixing

The neutrino mass arise from the couplings of $\bar{\psi}^c_L \psi_L$ to scalars, where $\bar{\psi}^c_L \psi_L$ transforms as $3^* \oplus 6$ under $\mathrm{SU}(3)_L$ and $\underline{1} \oplus \underline{1}' \oplus \underline{1}'' \oplus \underline{3}_s \oplus \underline{3}_a$ under A_4. For the known scalar triplets, there is no interactions invariant under all subgroups of $G = \mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X \otimes A_4$. We will therefore propose new $\mathrm{SU}(3)_L$ anti-sextets, lying in either $\underline{1}$, $\underline{1}'$, $\underline{1}''$ or $\underline{3}$ under A_4 interacting with $\bar{\psi}^c_L \psi_L$ to produce mass for the neutrinos. Therefore, new $SU(3)_L$ antisextets are proposed. The antisextets transform as follows:

$$\sigma = \begin{pmatrix} \sigma^0_{11} & \sigma^+_{12} & \sigma^0_{13} \\ \sigma^+_{12} & \sigma^{++}_{22} & \sigma^+_{23} \\ \sigma^0_{13} & \sigma^+_{23} & \sigma^0_{33} \end{pmatrix} \sim \left[6^*, \frac{2}{3}, -\frac{4}{3}, \underline{1}\right],$$

$$s_i = \begin{pmatrix} s^0_{11} & s^+_{12} & s^0_{13} \\ s^+_{12} & s^{++}_{22} & s^+_{23} \\ s^0_{13} & s^+_{23} & s^0_{33} \end{pmatrix}_i \sim \left[6^*, \frac{2}{3}, -\frac{4}{3}, \underline{3}\right] \quad (i = 1, 2, 3)\,. \tag{27}$$

Following the potential minimization conditions, we have the followings alignments:

(1) The first alignment: $\langle s_1 \rangle = \langle s_2 \rangle = \langle s_3 \rangle$ then A_4 is broken into Z_3 consisting of the elements $\{e, T, T^2\}$.
(2) The second alignment: $\langle s_1 \rangle \neq \langle s_2 \rangle \neq \langle s_3 \rangle$ or $\langle s_1 \rangle \neq \langle s_2 \rangle = \langle s_3 \rangle$ or $\langle s_2 \rangle \neq \langle s_1 \rangle = \langle s_3 \rangle$ or $\langle s_3 \rangle \neq \langle s_1 \rangle = \langle s_2 \rangle$ then A_4 is broken into {Identity}.
(3) The third alignment: $0 = \langle s_1 \rangle \neq \langle s_2 \rangle = \langle s_3 \rangle \neq 0$ or $0 = \langle s_2 \rangle \neq \langle s_3 \rangle = \langle s_1 \rangle \neq 0$ or $0 = \langle s_3 \rangle \neq \langle s_1 \rangle = \langle s_2 \rangle \neq 0$ then A_4 is broken into {Identity}.
(4) The fourth alignment: $0 = \langle s_1 \rangle \neq \langle s_2 \rangle \neq \langle s_3 \rangle \neq 0$ or $0 = \langle s_2 \rangle \neq \langle s_1 \rangle \neq \langle s_3 \rangle \neq 0$ or $0 = \langle s_3 \rangle \neq \langle s_1 \rangle \neq \langle s_2 \rangle \neq 0$ then A_4 is broken into {Identity}.
(5) The fifth alignment: $0 = \langle s_2 \rangle = \langle s_3 \rangle \neq \langle s_1 \rangle \neq 0$ then A_4 is broken into Z_2 consisting of the elements $\{e, S\}$.
(6) The sixth alignment: $0 = \langle s_1 \rangle = \langle s_3 \rangle \neq \langle s_2 \rangle \neq 0$ then A_4 is broken into Z_2 consisting of the elements $\{e, T^2ST\}$.
(7) The seventh alignment: $0 = \langle s_1 \rangle = \langle s_2 \rangle \neq \langle s_3 \rangle \neq 0$ then A_4 is broken into Z_2 consisting of the elements $\{e, TST^2\}$.

To obtain a realistic neutrino spectrum, we argue that the breaking $A_4 \to Z_2$ must be taken place. This can be achieved within each case below.

- A new $SU(3)_L$ anti-sextet s given in (27), with the VEVs chosen by $\langle s \rangle = (\langle s_1 \rangle, 0, 0)$ under A_4, where

$$\langle s_1 \rangle = \begin{pmatrix} \lambda_s & 0 & v_s \\ 0 & 0 & 0 \\ v_s & 0 & \Lambda_s \end{pmatrix}. \tag{28}$$

- Another $SU(3)_L$ triplet ρ which is also put in the $\underline{3}$ under A_4:

$$\rho_i = \left(\rho^+_1, \rho^0_2, \rho^+_3\right)^T_i \sim [3, 2/3, -4/3, \underline{3}] \quad (i = 1, 2, 3)$$

with the VEV chosen by

$$\langle \rho \rangle = (\langle \rho_1 \rangle, 0, 0)\,, \quad \langle \rho_1 \rangle = (0, v_\rho, 0)^T\,. \tag{29}$$

In this work, we additionally introduce a new $SU(3)_L$ triplet ρ lying in $\underline{3}$ under A_4 to obtain nonzero θ_{13}, which is different from that in Refs. 34 and 35.

The neutrino Yukawa interactions are

$$\begin{aligned}
-\mathcal{L}_\nu &= \frac{x}{2}\left(\bar{\psi}^c_L\psi_L\right)_{\underline{1}}\sigma + \frac{y}{2}\left(\bar{\psi}^c_L\psi_L\right)_{\underline{3}}s + \frac{z}{2}\left(\bar{\psi}^c_L\psi_L\right)_{\underline{3}}\rho + \text{H.c.} \\
&= \frac{x}{2}\left(\bar{\psi}^c_{1L}\psi_{1L} + \bar{\psi}^c_{2L}\psi_{2L} + \bar{\psi}^c_{3L}\psi_{3L}\right)\sigma \\
&\quad + y\left(\bar{\psi}^c_{2L}\psi_{3L}s_1 + \bar{\psi}^c_{3L}\psi_{1L}s_2 + \bar{\psi}^c_{1L}\psi_{2L}s_3\right) \\
&\quad + \frac{z}{2}\Big[\left(\bar{\psi}^c_{2L}\rho_3 - \bar{\psi}^c_{3L}\rho_2\right)\psi_{1L} + \left(\bar{\psi}^c_{3L}\rho_1 - \bar{\psi}^c_{1L}\rho_3\right)\psi_{2L} \\
&\quad + \left(\bar{\psi}^c_{1L}\rho_2 - \bar{\psi}^c_{2L}\rho_1\right)\psi_{3L}\Big] + \text{H.c.}
\end{aligned} \tag{30}$$

With the VEV of σ is

$$\langle\sigma\rangle = \begin{pmatrix} \lambda_\sigma & 0 & v_\sigma \\ 0 & 0 & 0 \\ v_\sigma & 0 & \Lambda_\sigma \end{pmatrix}, \tag{31}$$

the mass Lagrangian for the neutrinos can be written in matrix form:

$$-\mathcal{L}^{\text{mass}}_\nu = \frac{1}{2}\bar{\chi}^c_L M_\nu \chi_L + \text{H.c.}, \tag{32}$$

where

$$\begin{aligned}
&\chi_L \equiv (\nu_L \quad N^c_R)^T, \qquad M_\nu \equiv \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}, \\
&\nu_L = (\nu_{1L}, \nu_{2L}, \nu_{3L})^T, \quad N_R = (N_{1R}, N_{2R}, N_{3R})^T,
\end{aligned} \tag{33}$$

and the mass matrices are then obtained by

$$M_{L,R,D} = \begin{pmatrix} a_{L,R,D} & 0 & 0 \\ 0 & a_{L,R,D} & b_{L,R,D} + d_{L,R,D} \\ 0 & b_{L,R,D} - d_{L,R,D} & a_{L,R,D} \end{pmatrix}, \tag{34}$$

with

$$\begin{aligned}
&a_L = \lambda_\sigma x, \qquad a_D = v_\sigma x, \quad a_R = \Lambda_\sigma x, \\
&b_L = \lambda_s y, \qquad b_D = v_s y, \quad b_R = \Lambda_s y, \\
&d_L = d_R = 0, \quad d_D = v_\rho z.
\end{aligned} \tag{35}$$

Three observed neutrinos gain masses via a combination of type I and type II seesaw mechanisms derived from (33) and (34) as

$$M_{\text{eff}} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & C \\ 0 & C & B_2 \end{pmatrix}, \tag{36}$$

where

$$\begin{aligned} A &= a_L - \frac{a_D^2}{a_R}\,, \\ B_1 &= a_L - \frac{a_D^2 a_R + a_R(b_D - d_D)^2 - 2a_D b_R(b_D - d_D)}{a_R^2 - b_R^2}\,, \\ B_2 &= B_1 + \frac{4(a_D b_R - a_R b_D)d_D}{a_R^2 - b_R^2}\,, \\ C &= b_L + \frac{b_R(a_D^2 + b_D^2 - d_D^2) - 2a_D a_R b_D}{a_R^2 - b_R^2}\,. \end{aligned} \tag{37}$$

We can diagonalize the mass matrix (36) as follows

$$U_\nu^T M_{\text{eff}} U_\nu = \text{diag}(m_1, m_2, m_3)\,,$$

with

$$\begin{aligned} m_1 &= \frac{1}{2}\left(B_1 + B_2 + \sqrt{(B_1 - B_2)^2 + 4C^2}\right), \quad m_2 = A\,, \\ m_3 &= \frac{1}{2}\left(B_1 + B_2 - \sqrt{(B_1 - B_2)^2 + 4C^2}\right) \end{aligned} \tag{38}$$

and the corresponding neutrino mixing matrix:

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{K^2+1}} & 0 & \frac{K}{\sqrt{K^2+1}} \\ -\frac{K}{\sqrt{K^2+1}} & 0 & \frac{1}{\sqrt{K^2+1}} \end{pmatrix} \times P\,, \tag{39}$$

where $P = \text{diag}(1, 1, i)$, and

$$K = \frac{B_1 - B_2 - \sqrt{(B_1 - B_2)^2 + 4C^2}}{2C}\,. \tag{40}$$

Note that K in Eq. (40) must be a real number since the unitary condition of U_ν. Combined with (22) and (39), the lepton mixing matrix yields the form:

$$U_{\text{lep}} = U_L'^{+} U_\nu = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \times P\,, \tag{41}$$

where

$$
\begin{aligned}
U_{11} &= -\frac{\sqrt{3}\{4(1-K)+\varepsilon_3[6+(\varepsilon-4)K+\varepsilon_3(K+2)]\}}{(2+\varepsilon_3)[-6+\varepsilon_3(-6+\varepsilon_3+\varepsilon)]\sqrt{K^2+1}}\,,\\
U_{12} = U_{22} = U_{32} &= \frac{\sqrt{3}(1+\varepsilon_3)[-4+\varepsilon_3(-4+\varepsilon_3+\varepsilon)]}{(2+\varepsilon_3)[-6+\varepsilon_3(-6+\varepsilon_3+\varepsilon)]}\,,\\
U_{13} &= -\frac{\sqrt{3}\{4(1+K)+\varepsilon_3[4-\varepsilon+6K+\varepsilon_3(2K-1)]\}}{(2+\varepsilon_3)[-6+\varepsilon_3(-6+\varepsilon_3+\varepsilon)]\sqrt{K^2+1}}\,,\\
U_{21} &= \frac{2(-3i+\sqrt{3})(1+\varepsilon_3)+\frac{(3i+\sqrt{3})[-4+\varepsilon_3(-4+\varepsilon_3+\varepsilon)]K}{2+\varepsilon_3}}{2[-6+\varepsilon_3(-6+\varepsilon_3+\varepsilon)]\sqrt{K^2+1}}\,,\\
U_{23} &= -\frac{\frac{(3i+\sqrt{3})[-4+\varepsilon_3(-4+\varepsilon_3+\varepsilon)]}{2+\varepsilon_3}-2(-3i+\sqrt{3})(1+\varepsilon_3)K}{2[-6+\varepsilon_3(-6+\varepsilon_3+\varepsilon)]\sqrt{K^2+1}}\,,\\
U_{31} &= \frac{2(3i+\sqrt{3})(1+\varepsilon_3)+\frac{(-3i+\sqrt{3})[-4+\varepsilon_3(-4+\varepsilon_3+\varepsilon)]K}{2+\varepsilon_3}}{2[-6+\varepsilon_3(-6+\varepsilon_3+\varepsilon)]\sqrt{K^2+1}}\,,\\
U_{33} &= -\frac{\frac{(-3i+\sqrt{3})[-4+\varepsilon_3(-4+\varepsilon_3+\varepsilon)]}{2+\varepsilon_3}-2(3i+\sqrt{3})(1+\varepsilon_3)K}{2[-6+\varepsilon_3(-6+\varepsilon_3+\varepsilon)]\sqrt{K^2+1}}\,,
\end{aligned}
\tag{42}
$$

with ε is defined in Eq. (20). We see that all the elements of the matrix $U_{\rm lep}$ in Eq. (42) depend only on one parameter ε_3. From experimental constraints on the elements of the lepton mixing matrix given in Eq. (2), we can find out the regions of K and ε_3 that satisfy experimental data on lepton mixing matrix. The good value of K is in one of the following regions:

$$
\begin{aligned}
&K \in (-1.45, -1.4)\,, \quad K \in (1.4, 1.45)\,,\\
&K \in (-0.75, -0.65)\,, \quad K \in (0.65, 0.75)\,.
\end{aligned}
\tag{43}
$$

At present, the values of the absolute neutrino masses as well as the mass ordering of neutrinos are still open problems. An upper bound on the absolute value of neutrino mass was found from the analysis of the latest cosmological data[48]

$$m_i \leq 0.6 \text{ eV}\,. \tag{44}$$

The 95% upper limit on the sum of neutrino mass is given in Ref. 49

$$\sum_{i=1}^{3} |m_i| \leq 0.66 \text{ eV}\,. \tag{45}$$

The mass ordering of neutrino depends on the sign of Δm^2_{13} which is currently unknown. In the case of 3-neutrino mixing, the two possible signs of Δm^2_{13}

corresponding to two types of neutrino mass spectrum can be provided as follows

(1) Normal hierarchy (NH): $|m_1| \simeq |m_2| < |m_3|$, $\Delta m^2_{31} = m^2_3 - m^2_1 > 0$.
(2) Inverted hierarchy (IH): $|m_3| < |m_1| \simeq |m_2|$, $\Delta m^2_{31} = m^2_3 - m^2_1 < 0$.

As will be discussed below, the neutrino mass matrix in (36) can provide both normal and inverted mass hierarchies.

In this work, to have explicit values of the model parameters, the values of K: $K = -1.43$, $K = 1.43$, $K = -0.7$ and $K = 0.7$ [which all satisfy (43)] are used. The corresponding expressions of $B_{1,2}$, C and $m_{1,2,3}$ are given in Apps. C–F for normal hierarchy and in Apps. G–J for inverted hierarchy. However, the corresponding physical results such as the values of the absolute neutrino masses are the same. So, here we only consider in detail the case $K = -1.43$ for both normal and inverted spectrum.

Combining with the constraint values on the element U_{11} of lepton mixing matrix,[7] $U_{11} = 0.812$, we obtain two solutions on ε_3:

$$\varepsilon_3 = -0.0318467 - 0.00695743i \tag{46}$$

and

$$\varepsilon_3 = -0.0318467 + 0.00695743i\,. \tag{47}$$

With the solution (46), it follows:

$$U_{\text{lep}} \simeq \begin{pmatrix} 0.812 & 0.567 + 0.003i & -0.139 + 0.013i \\ -0.395 - 0.128i & 0.567 + 0.003i & 0.067 - 0.708i \\ -0.417 + 0.128i & 0.567 + 0.003i & 0.074 + 0.695i \end{pmatrix} \times P \tag{48}$$

or

$$|U_{\text{lep}}| = \begin{pmatrix} 0.812 & 0.567 & 0.140 \\ 0.415 & 0.567 & 0.711 \\ 0.436 & 0.567 & 0.699 \end{pmatrix}, \tag{49}$$

and $\varepsilon_2 = -0.0224354 + 0.0238703i$.

With the solution (47), we get:

$$U_{\text{lep}} = \begin{pmatrix} 0.812 & 0.567 - 0.003i & -0.139 - 0.013i \\ -0.417 - 0.128i & 0.567 - 0.003i & 0.074 - 0.695i \\ -0.395 + 0.128i & 0.567 - 0.003i & 0.067 + 0.708i \end{pmatrix} \times P \tag{50}$$

or

$$|U_{\text{lep}}| = \begin{pmatrix} 0.812 & 0.567 & 0.140 \\ 0.436 & 0.567 & 0.699 \\ 0.415 & 0.567 & 0.711 \end{pmatrix} \tag{51}$$

and $\varepsilon_2 = -0.0224354 - 0.0238703i$.

In the standard Particle Data Group (PDG) parametrization, the lepton mixing matrix can be parametrized as

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{13} & -s_{13}e^{-i\delta} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} + s_{12}s_{23}s_{13}e^{i\delta} & -s_{23}c_{13} \\ s_{12}s_{23} + c_{12}c_{23}s_{13}e^{i\delta} & c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \mathcal{P}\,, \quad (52)$$

where $\mathcal{P} = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$, and $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with θ_{12}, θ_{23} and θ_{13} being the solar, atmospheric and reactor angles, respectively. $\delta = [0, 2\pi]$ is the Dirac CP violation phase while α and β are two Majorana CP violation phases. The observable angles in the standard PMNS parametrization are given by[11]

$$s_{13} = |U_{13}|\,, \quad s_{23} = \frac{|U_{23}|}{\sqrt{1-|U_{13}|^2}}\,, \quad s_{12} = \frac{|U_{12}|}{\sqrt{1-|U_{13}|^2}}\,. \quad (53)$$

Combining Eqs. (48) and (53) yields:

$$\sin\theta_{13} = 0.140\,, \quad \sin\theta_{23} = 0.719\,, \quad \sin\theta_{12} = 0.573$$

or

$$\theta_{13} \simeq 8.055^\circ\,, \quad \theta_{23} \simeq 45.95^\circ\,, \quad \theta_{12} \simeq 34.93^\circ\,,$$

which are all very consistent with the recent data on neutrino mixing angles. On the other hand, comparing Eqs. (42) and (41) yields $\alpha = 0$, $\beta = \frac{\pi}{2}$ and $\delta = 5.41^\circ$ since $e^{i\delta} = -s_{13}/U_{13} = 0.995547 + 0.0942709i$. These results also implies that in the model under consideration, the value of the Jarlskog invariant $J_{\rm CP}$ which determines the magnitude of CP violation in neutrino oscillations is determined:[50]

$$J_{\rm CP} = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\sin\delta = 0.003\,. \quad (54)$$

Similarly, with the solution (47), we get the results given in Table 3.

Table 3. The model parameters with the solution (47) in normal hierarchy.

Parameter	Best fit	1σ range	2σ range
A [eV]	10^{-2}	J	−0.00303931
B_1 [eV]	−0.0356733	$\|m_1\|$ [eV]	0.00487852
B_2 [eV]	−0.0199378	$\|m_2\|$	0.01
C [eV]	0.0215348	$\|m_3\|$ [eV]	0.0507326
θ_{13} [°]	8.05436	$\sum$ [eV]	0.0656112
θ_{12} [°]	34.929	$\|m_{ee}\|$ [eV]	0.0010064
θ_{23} [°]	44.9281	$\|m_\beta\|$ [eV]	0.00991761
δ [°]	354.59		

Now, substituting $K = -1.43$ in Eq. (40) we obtain

$$B_1 = B_2 - 0.730699C\,. \tag{55}$$

4.1. *Normal hierarchy* ($\Delta m^2_{31} > 0$)

Combining (55) and (38) with the two experimental constraints on squared mass differences of neutrinos in normal hierarchy as shown in Table 1, we get the solutions (in [eV]) given in App. C. The solutions from Eqs. (C.1)–(C.4) have the same absolute values of $m_{1,2,3}$, the unique difference is the sign of $m_{1,3}$. Hence, we only consider in detail the case of (C.1). On the other hand, the expressions from (C.1)–(C.4) show that m_i ($i = 1, 2, 3$) depends only on one parameter $A = m_2$, so we will consider $m_{1,3}$ as functions of m_2. However, to have an explicit hierarchy on neutrino masses, in the following figures, m_2 should be included. The use of the upper bound on absolute value of neutrino mass in (44) leads to $A \le 0.6$ eV. Moreover, in this case, $A \in (0.00873, 0.01)$ eV or $A \in (-0.01, -0.00873)$ eV are good regions of A that can reach the realistic neutrino mass hierarchy.

In Fig. 1, we have plotted the absolute values $|m_{1,2,3}|$ as functions of A with $A \in (0.00873, 0.01)$ eV. This figure shows that there exist allowed regions for values A (or m_2) where either normal or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained when $|A|$ lies in a region $[0.05\text{ eV}, +\infty]$ ($|A|$ increases but must be small enough because of the scale of $m_{1,2,3}$). The normal mass hierarchy will be obtained if $|A|$ takes the values around $(0.00873, 0.05)$ eV. The sum $\sum = \sum_{i=1}^{3} |m_i|$ is plotted in Fig. 2 with $m_2 \in (0.00873, 0.01)$ eV.

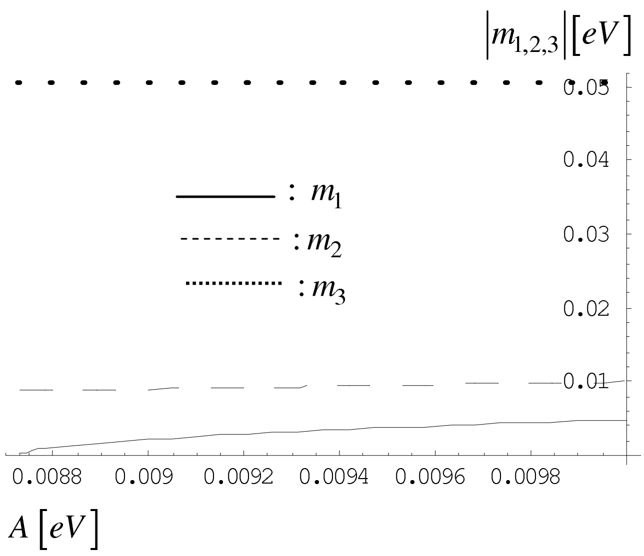

Fig. 1. $|m_{1,2,3}|$ as functions of A in the case of $\Delta m^2_{31} > 0$ with $A \in (0.00873, 0.01)$ eV.

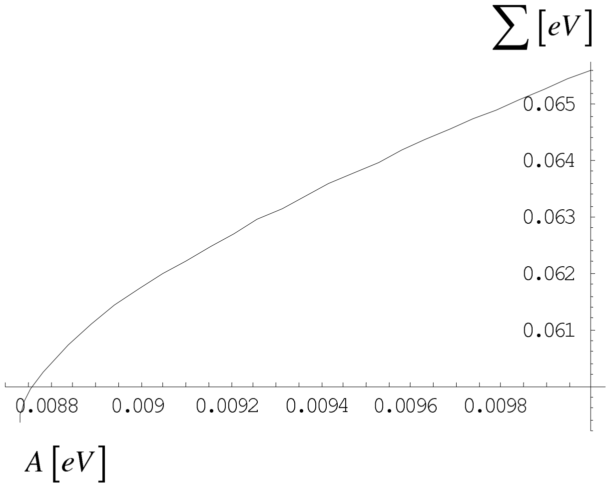

Fig. 2. $\sum$ as a function of A with $A \in (0.00873, 0.01)$ eV in the case of $\Delta m_{31}^2 > 0$.

The effective mass $\langle m_{ee} \rangle$ governing neutrinoless double beta decay[51–55] is then obtained,

$$\begin{aligned}\langle m_{ee} \rangle = \sum_{i=1}^{3} U_{ei}^2 m_i = &(0.321369 + 0.00369042i)A \\ &- (0.339313 - 0.00184244i)\sqrt{4A^2 - 0.0003048} \\ &- (0.0205292 - 0.0039231i)\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,, \end{aligned} \quad (56)$$

and

$$\begin{aligned} m_\beta^2 &= \sum_{i=1}^{3} |U_{ei}|^2 m_i^2 \\ &= -4.66947 \times 10^{-6} + 1.03963A^2 \\ &\quad - 0.0445038\sqrt{\beta_1} + 0.0209007\sqrt{4A^2 - 0.0003048}\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,, \end{aligned} \quad (57)$$

with α_1, β_1 are given in (C.5).

We also note that in the normal spectrum, $|m_1| \approx |m_2| < |m_3|$, so m_1 given in (C.1) is the lightest neutrino mass, therefore, it is denoted as $m_1 \equiv m_{\text{light}}$. In Fig. 3 we have plotted the values of $|m_{ee}|$, $|m_\beta|$ and $|m_{\text{light}}|$ as functions of A with $A \in (0.00875, 0.05)$ eV.

Figure 3 shows that in normal case $\langle m_{ee} \rangle < |m_\beta| < |m_{\text{light}}|$, all of them are consistent with the recent experimental data.[11] By assuming $A \equiv m_2 = 10^{-2}$ eV, which is safely small, then the other neutrino masses are explicitly given as $m_1 = -4.87852 \times 10^{-3}$ eV, $m_3 = -5.07326 \times 10^{-2}$ eV and $\sum = 6.56112 \times 10^{-2}$ eV, $|m_{ee}| \simeq 0.900184 \times 10^{-3}$ eV, $|m_\beta| \simeq 0.991761 \times 10^{-2}$ eV. Three physical neutrino masses are: $|m_1| = 4.87852 \times 10^{-3}$ eV, $|m_2| = 10^{-2}$ eV, $|m_3| = 5.07326 \times 10^{-2}$.

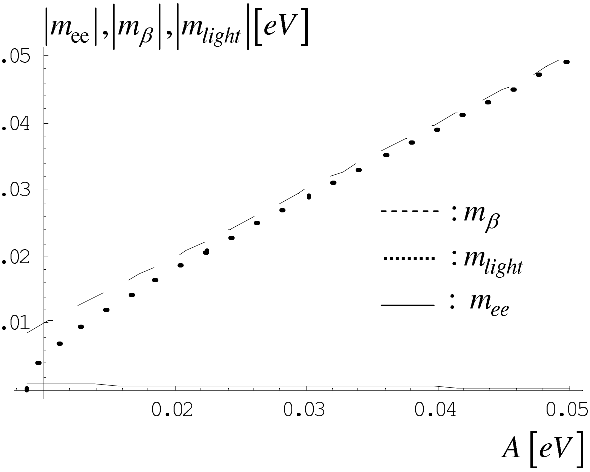

Fig. 3. $|m_{ee}|$, $|m_\beta|$ and $|m_{\rm light}|$ as functions of A from (C.1) in normal hierarchy with $A \in (0.00875, 0.05)$ eV.

This solution means a normal neutrino mass spectrum as mentioned above and consistent with the recent experimental data.[11,16] It follows that

$$B_1 = -0.0356733 \text{ eV}, \quad B_2 = -0.0199378 \text{ eV}, \quad C = 0.0215348 \text{ eV}. \quad (58)$$

There has not yet been an explicit experimental test of the values of parameters $\lambda_{s,\sigma}$, $v_{s,\sigma}$, $\Lambda_{s,\sigma}$, however, from the original form of the 3-3-1 models they obey the relation[56] $\lambda_{s,\sigma} \sim v^2_{s,\sigma}/\Lambda_{s,\sigma}$. To show that there exist the model parameters that consist with experimental data, the following assumption is used:

$$\lambda_s = \lambda_\sigma = 1 \text{ eV}, \quad v_\rho = v_s = v_\sigma, \quad \Lambda_s = -\Lambda_\sigma = v^2_\sigma, \quad (59)$$

It is then

$$A = 2x, \quad B_1 = \frac{x(2x^2 + 2y^2 - 4yz + z^2)}{x^2 - y^2},$$
$$B_2 = \frac{x(2x^2 + 2y^2 + 4yz + z^2)}{x^2 - y^2}, \quad C = \frac{y(4x^2 - z^2)}{x^2 - y^2}. \quad (60)$$

Combining (58) and (60) yields: $x = 5 \times 10^{-3}$, $y \simeq -7.73 \times 10^{-3}$, $z \simeq 1.77 \times 10^{-3}$.

4.2. *Inverted case* ($\Delta m^2_{31} < 0$)

For inverted hierarchy, by combining (55) and (38) with the two experimental constraints on squared mass differences of neutrinos as shown in Table 2, we get the solutions (in [eV]) given in App. G. The solutions from Eq. (G.1) to Eq. (G.2) have the same absolute values of $m_{1,2,3}$, the unique difference is the sign of $m_{1,3}$. Hence, we only consider in detail the case of (G.1). Because $m_i \quad (i = 1, 2, 3)$ only depends on one parameter $A = m_2$, so we will consider $m_{1,3}$ as functions of A. However, to

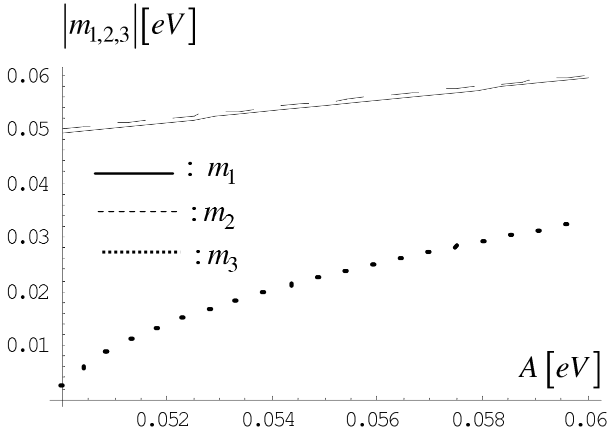

Fig. 4. $|m_{1,2,3}|$ as functions of A with $A \in (0.05, 0.06)$ eV in the case of $\Delta m^2_{31} < 0$.

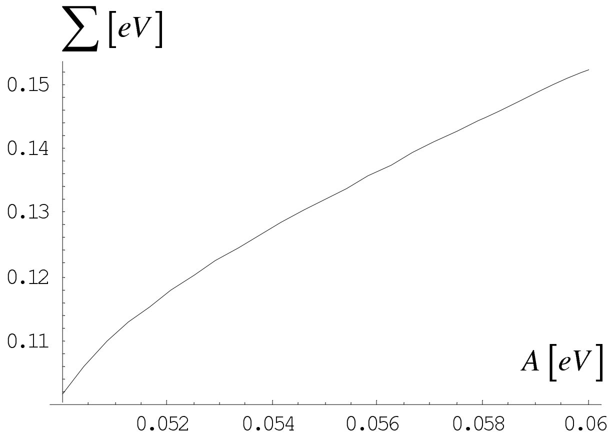

Fig. 5. $\sum$ as a function of A with $A \in (0.05, 0.06)$ eV in the case of $\Delta m^2_{31} < 0$.

have an explicit hierarchy on neutrino masses, in the following figures, m_2 should be included. In this case, $A \in (0.05, 0.06)$ eV is a good region of A that can reach the realistic neutrino mass hierarchy.

In Fig. 4, we have plotted the absolute values $|m_{1,2,3}|$ as functions of A with $A \in (0.05, 0.06)$ eV. This figure shows that there exist allowed regions for values A (or m_2) where either inverted or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained when $|A|$ lies in a region $[0.06\ \text{eV}, +\infty]$ ($|A|$ increases but must be small enough because of the scale of $m_{1,2,3}$). The inverted mass hierarchy will be obtained if $|A|$ takes the values around $(0.05, 0.06)$ eV. The sum $\sum = \sum_{i=1}^{3} |m_i|$ is plotted in Fig. 5 with $m_2 \in (0.05, 0.06)$ eV.

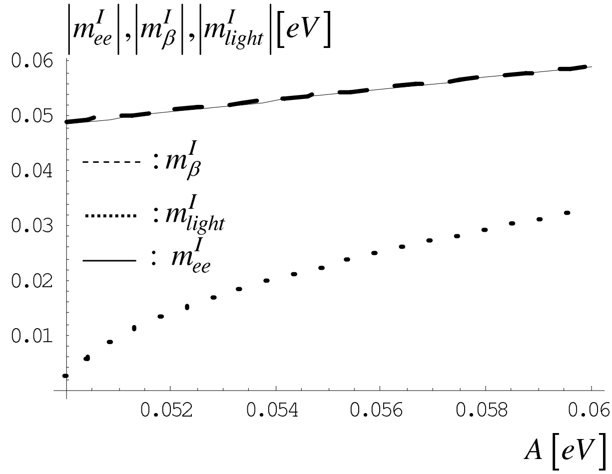

Fig. 6. $|m^I_{ee}|$, $|m^I_\beta|$ and $|m^I_{\rm light}|$ as functions of A from (G.1) in inverted hierarchy with $A \in (0.05, 0.06)$ eV.

The effective mass $\langle m_{ee}\rangle$ governing neutrinoless double beta decay[51–55] in inverted hierarchy is then obtained,

$$\begin{aligned}\langle m^I_{ee}\rangle &= \left|\sum_{i=1}^{3} U^2_{ei} m_i\right| \\ &= \Big|(0.321369 + 0.00369042i)A \\ &\quad + (0.339313 - 0.00184244i)\sqrt{4A^2 - 0.0003048} \\ &\quad - (0.0205292 - 0.0039231i)\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\Big|,\end{aligned} \tag{61}$$

and

$$\begin{aligned}\left(m^I_\beta\right)^2 &= \sum_{i=1}^{3} |U_{ei}|^2 m_i^2 \\ &= -0.000102434 + 1.03963A^2 + 0.0445038\sqrt{\beta} \\ &\quad - 0.0209007\sqrt{4A^2 - 0.0003048}\sqrt{\alpha_3 - 2\sqrt{\beta_3}},\end{aligned} \tag{62}$$

with α_3, β_3 are given in (G.3).

We also note that in the inverted spectrum, $|m_2| \approx |m_1| > |m_3|$, so m_3 given in (G.1) is the lightest neutrino mass, therefore, it is denoted as $m_3 \equiv m^I_{\rm light}$. In Fig. 6, we have plotted the values of $|m^I_{ee}|$, $|m^I_\beta|$ and $|m^I_{\rm light}|$ as functions of A with $A \in (0.05, 0.06)$ eV.

By assuming $A \equiv m_2 = 5 \times 10^{-2}$ eV, which is safely small, then the other physical neutrino masses are explicitly given as $|m_1| = 4.92321 \times 10^{-2}$ eV, $|m_3| = 2.48998 \times 10^{-3}$ eV, and $\sum = 0.101722$ eV, $m^I_{ee} \simeq 4.85391 \times 10^{-2}$ eV,

$m_\beta^I \simeq 4.90048 \times 10^{-2}$ eV. This solution means an inverted neutrino mass spectrum as mentioned above and consistent with the recent experimental data.[11,16] It follows that

$$\begin{aligned} B_1 &= (1.61687 + 0.167223i) \times 10^{-2}\ \text{eV}, \\ B_2 &= (3.30634 + 0.0817754i) \times 10^{-2}\ \text{eV}, \\ C &= (2.31213 - 0.116939i) \times 10^{-2}\ \text{eV}. \end{aligned} \tag{63}$$

Furthermore, by assuming that[56]

$$\lambda_s = \lambda_\sigma = 1\ \text{eV}, \quad v_\rho = v_s = v_\sigma, \quad \Lambda_s = a\Lambda_\sigma, \quad \Lambda_\sigma = -v_\sigma^2, \tag{64}$$

we obtain a solution

$$\begin{aligned} &A = 2x, \quad C = \frac{(a+3)x}{a+1} + \frac{az^2}{x(1-a^2)}, \quad x = y, \\ &B_1 = \frac{(a-1)(a+3)x^2 - 2(a-1)xz - z^2}{x(a^2-1)}, \\ &B_2 = \frac{(a-1)(a+3)x^2 + 2(a-1)xz - z^2}{x(a^2-1)}. \end{aligned} \tag{65}$$

Combining (63) and (65) yields:

$$\begin{aligned} a &\simeq 1.0587 - 0.0097i, \\ x &= y = 2.5 \times 10^{-2}, \\ z &\simeq (8.6933 - 0.4808i) \times 10^{-3}. \end{aligned} \tag{66}$$

5. Quarks Sector

We note that the scalar triplet ϕ in Eq. (8) is not enough to generate mass for all the quarks. Hence, to generate masses for quarks, two $\mathrm{SU}(3)_L$ triplets, put in $\underline{1}$ and $\underline{3}$ under A_4, are additionally introduced:[34]

$$\eta = \begin{pmatrix} \eta_1^0 & \eta_2^- & \eta_3^0 \end{pmatrix}^T \sim (3, -1/3, -1/3, \underline{3}), \tag{67}$$

$$\chi = \begin{pmatrix} \chi_1^0 & \chi_2^- & \chi_3^0 \end{pmatrix}^T \sim (3, -1/3, 2/3, \underline{1}). \tag{68}$$

It is worth mentioning that the $\mathrm{SU}(3)_L$ triplet ρ does not give new Yukawa terms, so the results in quark sector remain the same. The Yukawa interactions are:

$$\begin{aligned} -\mathcal{L}_q = {} & h_3^d \bar{Q}_{3L}(\phi d_R)_1 + h_1^u \bar{Q}_{1L}(\phi^* u_R)_{1''} + h_2^u \bar{Q}_{2L}(\phi^* u_R)_{1'} \\ & + h_3^u \bar{Q}_{3L}(\eta u_R)_1 + h_1^d \bar{Q}_{1L}(\eta^* d_R)_{1''} + h_2^d \bar{Q}_{2L}(\eta^* d_R)_{1'} \\ & + f_3 \bar{Q}_{3L}\chi U_R + f_1 \bar{Q}_{1L}\chi^* D_{1R} + f_2 \bar{Q}_{2L}\chi^* D_{2R} + \text{H.c.} \end{aligned} \tag{69}$$

The VEVs of η and χ are supposed to be[34]

$$\langle\eta\rangle = (\langle\eta_1\rangle, \langle\eta_1\rangle, \langle\eta_1\rangle) \tag{70}$$

under A_4, where $\langle\eta_1\rangle = (u\ 0\ 0)^T$ and $\langle\chi\rangle = (0\ 0\ v_\chi)^T$. The exotic quarks get masses directly from the VEV of χ:[34] $m_U = f_3 v_\chi$, $m_{D_i} = f_i v_\chi$, $(i = 1, 2)$.

Substituting (8), (12) and (70) into (69), the mass matrices for ordinary up-quarks and down-quarks are, respectively, obtained as follows:

$$M_u = \begin{pmatrix} -h_1^u v & -\omega h_1^u v(1+\varepsilon_2) & -\omega^2 h_1^u v(1+\varepsilon_2) \\ -h_2^u v & -\omega^2 h_2^u v(1+\varepsilon_2) & -\omega h_2^u v(1+\varepsilon_2) \\ h_3^u u & h_3^u u & h_3^u u \end{pmatrix}, \tag{71}$$

$$M_d = \begin{pmatrix} h_1^d u & \omega h_1^d u & \omega^2 h_1^d u \\ h_2^d u & \omega^2 h_2^d u & \omega h_2^d u \\ h_3^d v & h_3^d v(1+\varepsilon_2) & h_3^d v(1+\varepsilon_3) \end{pmatrix}. \tag{72}$$

The matrices M_u and M_d in (71), (72) are, respectively, diagonalized as

$$\begin{aligned} U_L^{u+} M_u U_R^u &= \mathrm{diag}(m_u, m_c, m_t)\,, \\ U_L^{d+} M_d U_R^d &= \mathrm{diag}(m_d, m_s, m_b)\,, \end{aligned} \tag{73}$$

where

$$\begin{aligned} m_u &= -\frac{(1+i\sqrt{3})(3+2\varepsilon_2+2\varepsilon_3+\varepsilon_2\varepsilon_3)h_1^u v}{3i+\sqrt{3}+(i+\sqrt{3})+2i\varepsilon_3}\,, \\ m_c &= \frac{(-1+i\sqrt{3})(3+2\varepsilon_2+2\varepsilon_3+\varepsilon_2\varepsilon_3)h_2^u v}{-3i+\sqrt{3}+(-i+\sqrt{3})\varepsilon_3}\,, \\ m_t &= \frac{(3+2\varepsilon_2+2\varepsilon_3+\varepsilon_2\varepsilon_3)h_3^u u}{\sqrt{3}(1+\varepsilon_2)}\,, \\ m_d &= \frac{(1+i\sqrt{3})(3+2\varepsilon_2+2\varepsilon_3)h_1^d u}{3i+\sqrt{3}+(i+\sqrt{3})\varepsilon_3+2i\varepsilon_2}\,, \\ m_s &= \frac{(1-i\sqrt{3})(3+\varepsilon_2+\varepsilon_3)h_2^d u}{-3i+\sqrt{3}-2i\varepsilon_3}\,, \\ m_b &= \frac{(3+\varepsilon_2+\varepsilon_3)h_3^d v}{\sqrt{3}}\,, \end{aligned} \tag{74}$$

and U_R^u, U_R^d are the right-handed up- and down-quarks mixing matrices; U_L^u, U_L^d are the left-handed up- and down-quarks mixing matrices, respectively,

$$U_R^u = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \frac{\omega^2(1+\varepsilon_2)-(1+\varepsilon_3)}{\omega^2(1+\varepsilon_3)-1} & 1+\varepsilon_3 \\ \frac{\omega(1+\varepsilon_3)-1}{\omega^2(1+\varepsilon_2)-\omega(1+\varepsilon_3)} & \omega & \frac{1+\varepsilon_3}{1+\varepsilon_2} \\ \frac{1-\omega^2(1+\varepsilon_2)}{\omega^2(1+\varepsilon_2)-\omega(1+\varepsilon_3)} & \frac{\omega(1-\omega)-\omega^2\varepsilon_2}{\omega^2(1+\varepsilon_3)-1} & 1 \end{pmatrix},$$

$$U_R^d = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \frac{\omega^2(1+\varepsilon_3)-(1+\varepsilon_2)}{\omega^2-(1+\varepsilon_3)} & 1 \\ \frac{\omega-(1+\varepsilon_3)}{\omega^2(1+\varepsilon_3)-\omega(1+\varepsilon_2)} & \omega & 1 \\ \frac{1+\varepsilon_2-\omega^2}{\omega^2(1+\varepsilon_3)-\omega(1+\varepsilon_2)} & \frac{\omega(1-\omega)+\omega\varepsilon_2}{\omega^2-(1+\varepsilon_3)} & 1 \end{pmatrix}, \quad (75)$$

$$U_L^u = U_L^d = 1\,.$$

The right-handed up- and down-quarks mixing matrices U_R^u, U_R^d given in (75) is one of the different issues of this work compared with Ref. 34. However, the CKM matrix is then given as[11]

$$U_{\rm CKM} = U_L^u U_L^{d\dagger} = 1\,, \quad (76)$$

which is the same as that in Ref. 34. A tree-level CKM matrix obtained equal to the identity matrix is the common property for some models based on the A_4 group.[35]

6. Conclusions

In this paper, we have proposed a 3-3-1 model with neutral fermions based on A_4 flavor symmetry responsible for fermion masses and mixings with nonzero θ_{13}. For this purpose, we additionally introduce a new $\mathrm{SU}(3)_L$ triplet (ρ) lying in $\underline{3}$ under A_4. The neutrinos get small masses from two $\mathrm{SU}(3)_L$ anti-sextets and one $\mathrm{SU}(3)_L$ triplet. The model can fit the present data on neutrino masses and mixing as well as the effective mass governing neutrinoless double beta decay. Our results show that the neutrino masses are naturally small and a little deviation from the tri-bimaximal neutrino mixing form can be realized. The Dirac CP violation phase δ is predicted to either 5.41° or 354.59° with $\theta_{23} \neq \frac{\pi}{4}$. It is emphasized that this consequence does not require $\theta_{23} = \frac{\pi}{4}$.

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Appendix A. A_4 Group and Clebsch–Gordan Coefficients

A_4 is the group of even permutation of four objects, which is also the symmetry group of a regular tetrahedron. It has 12 elements and four equivalence classes with three inequivalent one-dimensional representations and one three-dimensional one. Any element of A_4 can be formed by multiplication of the generators S and T obeying the relations[34] $S^2 = T^3 = (ST)^3 = \mathbf{1}$. Without loss of generality, we could choose $S = (12)\,(34)$, $T = (234)$ where the cycles (12) (34) denote the permutation $(1,2,3,4) \to (2,1,4,3)$, and (234) means $(1,2,3,4) \to (1,3,4,2)$. The conjugacy classes of A_4 generated from S and T are

$$\begin{aligned}
&C_1 : 1\,,\\
&C_2 : S,\ TST^2,\ T^2ST\,,\\
&C_3 : T,\ TS,\ ST,\ STS\,,\\
&C_4 : T^2,\ ST^2,\ T^2S,\ TST\,.
\end{aligned}$$

The character table of A_4 is given in Table A.1, where n is the order of class and h the order of elements within each class. We will work on a basis where $\underline{3}$ is a real representation. One possible choice of generators is given as follows

$$\begin{array}{rll}
\underline{1}: & S = 1\,, & T = 1\,,\\
\underline{1}': & S = 1\,, & T = \omega\,,\\
\underline{1}'': & S = 1\,, & T = \omega^2\,,\\
\underline{3}: & S = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}, & T = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},
\end{array}$$

where $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ is the cube root of unity. Using them we calculate the Clebsch–Gordan coefficients for all the tensor products as given below.

First, let us put $\underline{3}(1,2,3)$ which means some $\underline{3}$ multiplet such as $x = (x_1, x_2, x_3) \sim \underline{3}$ or $y = (y_1, y_2, y_3) \sim \underline{3}$ and so on, and similarly for the other representations. Moreover, the numbered multiplets such as $(\ldots, ij, \ldots)$ mean $(\ldots, x_i y_j, \ldots)$ where x_i and y_j are the multiplet components of different representations x and y, respectively. In the following, the components of representations on the left-hand side will be omitted and should be understood, but they always

Table A.1. The character table of A_4 group.

Class	n	h	$\chi_{\underline{1}}$	$\chi_{\underline{1}'}$	$\chi_{\underline{1}''}$	$\chi_{\underline{3}}$
C_1	1	1	1	1	1	3
C_2	3	2	1	1	1	-1
C_3	4	3	1	ω	ω^2	0
C_4	4	3	1	ω^2	ω	0

exist in order in the components of decompositions on the right-hand side:

$$\underline{1}\otimes\underline{1}=\underline{1}(11)\,,\qquad \underline{1}\otimes\underline{1}'=\underline{1}'(11)\,,\qquad \underline{1}\otimes\underline{1}''=\underline{1}''(11)\,, \tag{A.1}$$

$$\underline{1}'\otimes\underline{1}'=\underline{1}''(11)\,,\quad \underline{1}'\otimes\underline{1}''=\underline{1}(11)\,,\quad \underline{1}''\otimes\underline{1}''=\underline{1}'(11)\,, \tag{A.2}$$

$$\begin{aligned}\underline{1}\otimes\underline{3}&=\underline{3}(11,12,13)\,,\\ \underline{1}'\otimes\underline{3}&=\underline{3}(11,\omega 12,\omega^2 13)\,,\\ \underline{1}''\otimes\underline{3}&=\underline{3}(11,\omega^2 12,\omega 13)\,,\end{aligned} \tag{A.3}$$

$$\begin{aligned}\underline{3}\otimes\underline{3}=&\ \underline{1}(11+22+33)\\ &\oplus\underline{1}'(11+\omega^2 22+\omega 33)\\ &\oplus\underline{1}''(11+\omega 22+\omega^2 33)\\ &\oplus\underline{3}_s(23+32,31+13,12+21)\\ &\oplus\underline{3}_a(23-32,31-13,12-21)\,,\end{aligned} \tag{A.4}$$

where the subscripts "s" and "a" respectively refer to their symmetric and anti-symmetric product combinations as explicitly pointed out.

In the paper, we usually use the following notations, for example, $(xy')_{\underline{3}}=[xy']_{\underline{3}}\equiv(x_2y'_3-x_3y'_2,x_3y'_1-x_1y'_3,x_1y'_2-x_2y'_1)$ which is the Clebsch–Gordan coefficients of $\underline{3}_a$ in the decomposition of $\underline{3}\otimes\underline{3}$, where as mentioned $x=(x_1,x_2,x_3)\sim\underline{3}$ and $y'=(y'_1,y'_2,y'_3)\sim\underline{3}$.

The rules to conjugate the representations $\underline{1}$, $\underline{1}'$, $\underline{1}''$ and $\underline{3}$ are given by

$$\begin{aligned}\underline{1}^*(1^*)&=\underline{1}(1^*)\,,\\ \underline{1}'^*(1^*)&=\underline{1}'(1^*)\,,\\ \underline{1}''^*(1^*)&=\underline{1}''(1^*)\,,\\ \underline{3}^*(1^*,2^*,3^*)&=\underline{3}(1^*,2^*,3^*)\,.\end{aligned} \tag{A.5}$$

Appendix B. Lepton Number and Lepton Parity

The lepton number (L) and lepton parity (P_l) of the model particles are given in Table B.1.

Table B.1. The model particles.

Particles	L	P_l
N_R, u, d, ϕ_1^+, $\phi_1'^+$, ϕ_2^0, $\phi_2'^0$, η_1^0, $\eta_1'^0$, η_2^-, $\eta_2'^-$, χ_3^0, σ_{33}^0, s_{33}^0	0	1
ν_L, l, U, D^*, ϕ_3^+, $\phi_3'^+$, η_3^0, $\eta_3'^0$, χ_1^{0*}, χ_2^+, σ_{13}^0, σ_{23}^+, s_{13}^0, s_{23}^+	-1	-1
σ_{11}^0, σ_{12}^+, σ_{22}^{++}, s_{11}^0, s_{12}^+, s_{22}^{++}	-2	1

Appendix C. The Solutions with $K = -1.43$ in the Normal Case

- The first case:

$$\begin{aligned}
C &= 0.5\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,.
\end{aligned} \tag{C.1}$$

- The second case:

$$\begin{aligned}
C &= 0.5\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,.
\end{aligned} \tag{C.2}$$

- The third case:

$$\begin{aligned}
C &= 0.5\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,.
\end{aligned} \tag{C.3}$$

- The fourth case:

$$\begin{aligned}
C &= 0.5\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,.
\end{aligned} \tag{C.4}$$

where

$$\begin{aligned}
\alpha_1 &= 0.00211525 + 1.76448A^2\,,\\
\beta_1 &= -1.46721\times 10^{-7} + 0.00186616A^2 + 0.778345A^4\,.
\end{aligned} \tag{C.5}$$

Appendix D. The Solutions with $K = 1.43$ in the Normal Case

- The first case:

$$\begin{aligned}
C &= -0.5\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,.
\end{aligned} \tag{D.1}$$

- The second case:

$$\begin{aligned}
C &= -0.5\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,.
\end{aligned} \tag{D.2}$$

- The third case:

$$\begin{aligned}
C &= -0.5\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 - 2\sqrt{\beta_1}}\,.
\end{aligned} \tag{D.3}$$

- The fourth case:

$$\begin{aligned}
C &= -0.5\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_1 + 2\sqrt{\beta_1}}\,,
\end{aligned} \tag{D.4}$$

where α_1, β_1 are given in (C.5).

Appendix E. The Solutions with $K = -0.70$ in the Normal Case

- The first case:

$$\begin{aligned} C &= 0.5\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,, \\ B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,, \\ m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,, \\ m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,. \end{aligned} \tag{E.1}$$

- The second case:

$$\begin{aligned} C &= 0.5\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,, \\ B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,, \\ m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,, \\ m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,. \end{aligned} \tag{E.2}$$

- The third case:

$$\begin{aligned} C &= 0.5\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,, \\ B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,, \\ m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,, \\ m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,. \end{aligned} \tag{E.3}$$

- The fourth case:

$$\begin{aligned} C &= 0.5\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,, \\ B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,, \\ m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,, \\ m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,, \end{aligned} \tag{E.4}$$

where

$$\begin{aligned} \alpha_2 &= 0.0021167 + 1.76569A^2\,, \\ \beta_2 &= -1.46922 \times 10^{-7} + 0.00186872A^2 + 0.779412A^4\,. \end{aligned} \tag{E.5}$$

Appendix F. The Solutions with $K = 0.70$ in the Normal Case

- The first case:

$$
\begin{aligned}
C &= -0.5\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,,\\
B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,,\\
m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,,\\
m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,.
\end{aligned} \tag{F.1}
$$

- The second case:

$$
\begin{aligned}
C &= -0.5\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,,\\
B_2 &= -0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,,\\
m_1 &= -0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,,\\
m_3 &= -0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,.
\end{aligned} \tag{F.2}
$$

- The third case:

$$
\begin{aligned}
C &= -0.5\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 - 2\sqrt{\beta_2}}\,.
\end{aligned} \tag{F.3}
$$

- The fourth case:

$$
\begin{aligned}
C &= -0.5\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_2 + 2\sqrt{\beta_2}}\,,
\end{aligned} \tag{F.4}
$$

where α_2, β_2 are given in (E.5).

Appendix G. The Solutions with $K = -1.43$ in the Inverted Case

- The first case:

$$\begin{aligned}
C &= 0.5\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,,\quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,.
\end{aligned} \tag{G.1}$$

- The second case:

$$\begin{aligned}
C &= 0.5\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,,\quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,,
\end{aligned} \tag{G.2}$$

where

$$\begin{aligned}
\alpha_3 &= -0.00227829 + 1.76448A^2\,,\\
\beta_3 &= 1.48642\times 10^{-7} - 0.00201A^2 + 0.778345A^4\,.
\end{aligned} \tag{G.3}$$

Appendix H. The Solutions with $K = 1.43$ in the Inverted Case

- The first case:

$$\begin{aligned}
C &= -0.5\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,,\quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_3 - 2\sqrt{\beta_3}}\,.
\end{aligned} \tag{H.1}$$

- The second case:

$$
\begin{aligned}
C &= -0.5\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.34965\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\\
&\quad - 1.11022\times 10^{-16}\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06465\sqrt{\alpha_3 + 2\sqrt{\beta_3}}\,,
\end{aligned}
\tag{H.2}
$$

where α_3, β_3 are given in (G.3).

Appendix I. The Solutions with $K = -0.7$ in the Inverted Case

- The first case:

$$
\begin{aligned}
C &= 0.5\sqrt{\alpha_4 - 2\sqrt{\beta_4}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_4 - 2\sqrt{\beta_4}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_4 - 2\sqrt{\beta_4}}\,.
\end{aligned}
\tag{I.1}
$$

- The second case:

$$
\begin{aligned}
C &= 0.5\sqrt{\alpha_4 + 2\sqrt{\beta_4}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_4 + 2\sqrt{\beta_4}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_4 + 2\sqrt{\beta_4}}\,,
\end{aligned}
\tag{I.2}
$$

where

$$
\begin{aligned}
\alpha_4 &= -0.00227985 + 1.76569A^2\,,\\
\beta_4 &= 1.48846\times 10^{-7} - 0.00201275A^2 + 0.779412A^4\,.
\end{aligned}
\tag{I.3}
$$

Appendix J. The Solutions with $K = 0.7$ in the Inverted Case

- The first case:

$$
\begin{aligned}
C &= -0.5\sqrt{\alpha_4 - 2\sqrt{\beta_4}}\,,\\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_4 - 2\sqrt{\beta_4}}\,,\\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,,\\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_4 - 2\sqrt{\beta_4}}\,.
\end{aligned}
\tag{J.1}
$$

- The second case:

$$
\begin{aligned}
C &= -0.5\sqrt{\alpha_4 + 2\sqrt{\beta_4}}\,, \\
B_2 &= 0.5\sqrt{4A^2 - 0.0003048} - 0.714286\sqrt{\alpha_4 + 2\sqrt{\beta_4}}\,, \\
m_1 &= 0.5\sqrt{4A^2 - 0.0003048}\,, \quad m_2 = A\,, \\
m_3 &= 0.5\sqrt{4A^2 - 0.0003048} - 1.06429\sqrt{\alpha_4 + 2\sqrt{\beta_4}}\,,
\end{aligned}
\tag{J.2}
$$

where α_4, β_4 are given in (I.3).

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