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Neutrino mixing with nonzero θ_{13} and CP violation in the 3-3-1 model based on S_4 flavor symmetry

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The 3-3-1 model proposed in 2011 based on discrete symmetry S_4 responsible for the neutrino and quark masses is updated, in which the nonzero θ_{13} is focused. Neutrino masses and mixings are consistent with the most recent data on neutrino oscillations without perturbation. The new feature is adding a new $SU(3)_L$ anti-sextet lying in doublet under S_4 which can result the nonzero θ_{13} without perturbation, and consequently, the number of Higgs multiplets required is less than those of other models based on non-Abelian discrete symmetries and the 3-3-1 models. The exact tribimaximal form obtained with the breaking $S_4 \to Z_3$ in charged lepton sector and $S_4 \to \mathcal{K}$ in neutrino sector. If both breakings $S_4 \to \mathcal{K}$ and $\mathcal{K} \to Z_2$ are taken place in neutrino sector, the realistic neutrino spectrum is obtained without perturbation. The upper bound on neutrino mass and the effective mass governing neutrinoless double beta decay at the tree level are presented. The model predicts the Dirac CP violation phase $\delta = 292.45^\circ$ in the normal spectrum (with $\theta_{23} \neq \frac{\pi}{4}$) and $\delta = 303.14^\circ$ in the inverted spectrum.

Keywords: Neutrino mass and mixing; nonstandard-model neutrinos, right-handed neutrinos; flavor symmetries; discrete symmetries; models beyond the standard model.

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1. Introduction

Nowadays, particle physicists are attracted by two exciting subjects: Higgs and neutrino physics. The neutrino mass and mixing are the first evidence of beyond Standard Model physics. Many experiments show that neutrinos have tiny masses and their mixing is still mysterious.[1,2] The tribimaximal form for explaining the lepton mixing scheme was first proposed by Harrison–Perkins–Scott (HPS), which apart from the phase redefinitions, is given by[3–6]

$$U_{\mathrm{HPS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{1}$$

can be considered as a good approximation for the recent neutrino experimental data.

The most recent data are a clear sign of rather large value θ_{13}.[7] The data in the Particle Data Group PDG2014[8] imply:

$$\begin{aligned} &\sin^2(2\theta_{12}) = 0.846 \pm 0.021\,, && \sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}\,, \\ &\sin^2(2\theta_{13}) = (9.3 \pm 0.8) \times 10^{-2}\,, && \Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5}\ \mathrm{eV}^2\,, \\ &\Delta m^2_{32} = (2.44 \pm 0.06) \times 10^{-3}\ \mathrm{eV}^2\,, && (\text{Normal hierarchy})\,, \end{aligned} \tag{2}$$

$$\begin{aligned} &\sin^2(2\theta_{12}) = 0.846 \pm 0.021\,, && \sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}\,, \\ &\sin^2(2\theta_{13}) = (9.3 \pm 0.8) \times 10^{-2}\,, && \Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5}\ \mathrm{eV}^2\,, \\ &\Delta m^2_{32} = (2.52 \pm 0.07) \times 10^{-3}\ \mathrm{eV}^2\,, && (\text{Inverted hierarchy})\,. \end{aligned} \tag{3}$$

These large neutrino mixing angles are completely different from the quark mixing ones defined by the Cabibbo–Kobayashi–Maskawa (CKM) matrix,[9,10] and they cannot be explained by the Standard Model. It is an interesting challenge to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and leptons given in a complete natural way as first approximations. This has stimulated work on flavor symmetries and non-Abelian discrete symmetries are considered to be the most attractive candidate to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and lepton. There are many recent models based on the non-Abelian discrete symmetries, such as A_4 (Refs. 11–28), A_5 (Refs. 29–41), S_3 (Refs. 42–83), S_4 (Refs. 84–112), D_4 (Refs. 113–124), D_5 (Refs. 125 and 126), T' (Refs. 127–131) and so forth. In our previous works,[132–140] the discrete symmetries have been explored to the 3-3-1 models. In Ref. 133, we have studied the 3-3-1 model with neutral fermions based on S_4 group, in which most of the Higgs multiplets are in triplets under S_4 except χ lying in a singlet, and the exact tribimaximal form[3–6] is obtained, where $\theta_{13} = 0$.

As we know, the recent considerations have implied $\theta_{13} \neq 0$,[11–28,42–112] but small as given in Eqs. (2) and (3). This problem has been improved in Ref. 134 by adding a new triplet ρ put in $\underline{1}'$ under S_3 and another antisextet s' put in $\underline{2}$ under S_3, in which s' is regarded as a small perturbation, or a new triplet ρ put in $\underline{1}''$ under D_4 regarded as a small perturbation.[135] Therefore, the models contain up to eight Higgs multiplets, and the scalar potential of the model is quite complicated.

In this paper, we introduce another $SU(3)_L$ antisextet lying in $\underline{2}$ under S_4 which can result the nonzero θ_{13} without perturbation. The rest of this work is organized as follows. In Sec. 2, we review some main results from Ref. 133. Section 3 is devoted for the neutrino mass and mixing. Section 4 presents the remark on the vacuum alignments and ρ parameter. We summarize our results in Sec. 5. Appendix A is devoted to S_4 group with its Clebsch–Gordan coefficients. Appendix B presents the lepton numbers and lepton parities of model particles. Appendix C provides the breakings of S_4 group by triplets $\underline{3}$ and $\underline{3}'$.

2. The Model

The fermions in this model under $[\mathrm{SU}(3)_L, \mathrm{U}(1)_X, \mathrm{U}(1)_{\mathcal{L}}, \underline{S}_4]$ symmetries, respectively, transform as[133]

$$\begin{gathered}
\psi_L \equiv \psi_{1,2,3L} = \begin{pmatrix} \nu_{1,2,3L} \\ l_{1,2,3L} \\ N^c_{1,2,3R} \end{pmatrix} \sim [3, -1/3, 2/3, \underline{3}]\,, \\
l_{1R} \sim [1, -1, 1, \underline{1}]\,, \quad l_R \equiv l_{2,3R} \sim [1, -1, 1, \underline{2}]\,, \\
Q_{3L} = \begin{pmatrix} u_{3L} \\ d_{3L} \\ U_L \end{pmatrix} \sim [3, 1/3, -1/3, \underline{1}]\,, \\
Q_L \equiv Q_{1,2L} = \begin{pmatrix} d_{1,2L} \\ -u_{1,2L} \\ D_{1,2L} \end{pmatrix} \sim [3^*, 0, 1/3, \underline{2}]\,, \\
u_R \equiv u_{1,2,3R} \sim [1, 2/3, 0, \underline{3}]\,, \quad d_R \equiv d_{1,2,3R} \sim [1, -1/3, 0, \underline{3}]\,, \\
U_R \sim [1, 2/3, -1, \underline{1}]\,, \quad D_R \equiv D_{1,2R} \sim [1, -1/3, 1, \underline{2}]\,,
\end{gathered} \tag{4}$$

where the numbered subscripts on field indicate respective families and define components of their S_4 multiplet representation. Note that the $\underline{2}$ for quarks meets the requirement of anomaly cancellation where the last two left-quark families are in 3^* while the first one as well as the leptons are in 3 under $SU(3)_L$. All the $\mathcal{L}$ charges of the model multiplets are listed in the square brackets.

To generate masses for the charged leptons, we have introduced two $SU(3)_L$ scalar triplets ϕ and ϕ' lying in $\underline{3}$ and $\underline{3}'$ under S_4, respectively, with the VEVs $\langle\phi\rangle = (v, v, v)$ and $\langle\phi'\rangle = (v', v', v')$ written as those of S_4 components,[133] i.e. S_4 is broken into Z_3 that consists of the elements;[a] $\{1, T, T^2\}$. From the invariant Yukawa interactions for the charged leptons, we obtain $m_e = \sqrt{3}h_1 v$, $m_\mu = \sqrt{3}(h_2 v - h_3 v')$, $m_\tau = \sqrt{3}(h_2 v + h_3 v')$, and the left- and right-handed charged leptons mixing matrices are given[133]

$$U_L = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_R = 1\,. \tag{5}$$

In similarity to the charged lepton sector, to generate the quark masses, we have additionally introduced three scalar Higgs triplets χ, η, η' lying in $\underline{1}$, $\underline{3}$ and $\underline{3}'$ under S_4, respectively. Quark masses can be derived from the invariant Yukawa interactions for quarks with supposing that the VEVs of η, η' and χ are u, u' and v_χ, respectively, where $u = \langle\eta_1^0\rangle$, $u' = \langle\eta_1'^0\rangle$, $v_\chi = \langle\chi_3^0\rangle$ and the other VEVs $\langle\eta_3^0\rangle$, $\langle\eta_3'^0\rangle$, and $\langle\chi_1^0\rangle$ vanish due to the lepton parity conservation. The exotic quarks get masses $m_U = f_3 v_\chi$ and $m_{D_{1,2}} = f v_\chi$. The masses of ordinary up-quarks and down-quarks are:

$$\begin{aligned} m_u &= -\sqrt{3}(h^u v + h'^u v')\,, \quad m_c = -\sqrt{3}(h^u v - h'^u v')\,, \quad m_t = \sqrt{3}h_3^u u\,, \\ m_d &= \sqrt{3}(h^d u + h'^d u')\,, \qquad m_s = \sqrt{3}(h^d u - h'^d u')\,, \qquad m_b = \sqrt{3}h_3^d v\,. \end{aligned} \tag{6}$$

The unitary matrices, which couple the left-handed up- and down-quarks to those in the mass bases, are $U_L^u = 1$ and $U_L^d = 1$, respectively. Therefore, we get the quark mixing matrix $U_{\mathrm{CKM}} = U_L^{d\dagger} U_L^u = 1$. For a detailed study on charged lepton and quark mass, the reader is referred to Ref. 133. In this work, we add a new $SU(3)_L$ anti-sextet lying in $\underline{2}$ under S_4 responsible for the nonzero θ_{13} without perturbation which is different from those in Refs. 133–135. The vacuum alignments and the gauge boson masses and mixings are similar to those in Refs. 135 and 141 so we will not discuss it further in this work.

3. Neutrino Mass and Mixing

In this type of the models, the neutrino masses arise from the couplings of $\bar{\psi}_L^c \psi_L$ to scalars, where $\bar{\psi}_L^c \psi_L$ transforms as $3^* \oplus 6$ under $\mathrm{SU}(3)_L$ and $\underline{1} \oplus \underline{2} \oplus \underline{3} \oplus \underline{3}'$ under S_4. For the known scalar triplets $(\phi, \phi', \chi, \eta, \eta')$, the available interactions are only $(\bar{\psi}_L^c \psi_L)\phi$ and $(\bar{\psi}_L^c \psi_L)\phi'$, but explicitly suppressed because of the $\mathcal{L}$-symmetry. We will therefore propose new $\mathrm{SU}(3)_L$ antisextets, lying in either $\underline{1}$, $\underline{2}$, $\underline{3}$, or $\underline{3}'$ under

[a]With the VEV alignment: $\langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$, S_4 group is broken into S_3 which consist of the elements $\{1, T, T^2, TSTS^2, STS^2, S^2TS\}$; with the VEV alignment: $\langle\phi_1'\rangle = \langle\phi_2'\rangle = \langle\phi_3'\rangle \neq 0$, S_4 is broken into Z_3 that consists of the elements $\{1, T, T^2\}$ as presented in App. C.

S_4, which interact with $\bar{\psi}^c_L \psi_L$ to produce masses for the neutrinos. In Ref. 133, we have introduced two $\mathrm{SU}(3)_L$ antisextets σ, s transform as follows

$$\sigma = \begin{pmatrix} \sigma^0_{11} & \sigma^+_{12} & \sigma^0_{13} \\ \sigma^+_{12} & \sigma^{++}_{22} & \sigma^+_{23} \\ \sigma^0_{13} & \sigma^+_{23} & \sigma^0_{33} \end{pmatrix} \sim [6^*, 2/3, -4/3, \underline{1}]\,,$$
$$s = \begin{pmatrix} s^0_{11} & s^+_{12} & s^0_{13} \\ s^+_{12} & s^{++}_{22} & s^+_{23} \\ s^0_{13} & s^+_{23} & s^0_{33} \end{pmatrix} \sim [6^*, 2/3, -4/3, \underline{3}]\,, \tag{7}$$

with the VEV of s is set as $(\langle s_1 \rangle, 0, 0)$ under S_4, where

$$\langle s_1 \rangle = \begin{pmatrix} \lambda_s & 0 & v_s \\ 0 & 0 & 0 \\ v_s & 0 & \Lambda_s \end{pmatrix}, \tag{8}$$

and the VEV of σ is

$$\langle \sigma \rangle = \begin{pmatrix} \lambda_\sigma & 0 & v_\sigma \\ 0 & 0 & 0 \\ v_\sigma & 0 & \Lambda_\sigma \end{pmatrix}. \tag{9}$$

With these $SU(3)_L$ anti-sextets, the exact tribimaximal form was obtained, in which $\theta_{13} = 0$.[133] However, the recent experimental data have implied $\theta_{13} \neq 0$ as given in Eqs. (2) and (3). So that we need to modify the neutrino mass matrix to fit the recent data.

Notice that the VEV alignment as in (8), S_4 is broken into a group which is isomorphic to Klein four group[93] that consists of the elements $\mathcal{K} = \{1, S^2, TSTS^2, TST\}$. To obtain a realistic neutrino spectrum, in this work we additionally introduce another $SU(3)_L$ anti-sextet (s') which lies in $\underline{2}$ under S_4 and responsible for the breaking $\mathcal{K} \to Z_2$. This happens in any case below: $\langle s' \rangle = (\langle s'_1 \rangle, 0)$, with

$$\langle s'_1 \rangle = \begin{pmatrix} \lambda'_s & 0 & v'_s \\ 0 & 0 & 0 \\ v'_s & 0 & \Lambda'_s \end{pmatrix}. \tag{10}$$

The VEV alignment of s' as in (10) will break $\mathcal{K}$ into Z_2 that consists of the elements $\{1, A^2\}$ (instead of S_4 is broken into another Klein four group[93] that consists of the elements $\{1, S^2, TS^2T^2, T^2S^2T\}$).

In calculation, combining both cases we have the Yukawa interactions responsible for neutrino mass:

$$\begin{aligned} -\mathcal{L}_\nu &= \frac{1}{2} x \left(\bar{\psi}^c_L \psi_L\right)_{\underline{1}} \sigma + \frac{1}{2} y \left(\bar{\psi}^c_L \psi_L\right)_{\underline{3}} s + \frac{1}{2} z \left(\bar{\psi}^c_L \psi_L\right)_{\underline{2}} s' + \mathrm{H.c.} \\ &= \frac{x}{2} \left(\bar{\psi}^c_{1L} \psi_{1L} + \bar{\psi}^c_{2L} \psi_{2L} + \bar{\psi}^c_{3L} \psi_{3L}\right) \sigma \end{aligned}$$

$$
\begin{aligned}
&+\frac{y}{2}\big[\big(\bar{\psi}^c_{2L}\psi_{3L}+\bar{\psi}^c_{3L}\psi_{2L}\big)s_1\\
&+\big(\bar{\psi}^c_{3L}\psi_{1L}+\bar{\psi}^c_{1L}\psi_{3L}\big)s_2+\big(\bar{\psi}^c_{1L}\psi_{2L}+\bar{\psi}^c_{2L}\psi_{1L}\big)s_3\big]\\
&+\frac{z}{2}\big[\big(\bar{\psi}^c_{1L}\psi_{1L}+\omega^2\bar{\psi}^c_{2L}\psi_{2L}+\omega\bar{\psi}^c_{3L}\psi_{3L}\big)s'_2\\
&+\big(\bar{\psi}^c_{1L}\psi_{1L}+\omega\bar{\psi}^c_{2L}\psi_{2L}+\omega^2\bar{\psi}^c_{3L}\psi_{3L}\big)s'_1\big]+\text{H.c.}
\end{aligned}
\tag{11}
$$

The mass Lagrangian for the neutrinos is given by

$$
\begin{aligned}
-\mathcal{L}^{\text{mass}}_\nu=\frac{1}{2}x\big(&\lambda_\sigma\bar{\nu}^c_{1L}\nu_{1L}+v_\sigma\bar{\nu}^c_{1L}N^c_{1R}+v_\sigma\bar{N}_{1R}\nu_{1L}+\Lambda_\sigma\bar{N}_{1R}N^c_{1R}\\
&+\lambda_\sigma\bar{\nu}^c_{2L}\nu_{2L}+v_\sigma\bar{\nu}^c_{2L}N^c_{2R}+v_\sigma\bar{N}_{2R}\nu_{2L}+\Lambda_\sigma\bar{N}_{2R}N^c_{2R}\\
&+\lambda_\sigma\bar{\nu}^c_{3L}\nu_{3L}+v_\sigma\bar{\nu}^c_{3L}N^c_{3R}+v_\sigma\bar{N}_{3R}\nu_{3L}+\Lambda_\sigma\bar{N}_{3R}N^c_{3R}\big)\\
&+\frac{y}{2}\big[\lambda_s\big(\bar{\nu}^c_{2L}\nu_{3L}+\bar{\nu}^c_{3L}\nu_{2L}\big)+v_s\big(\bar{\nu}^c_{2L}N^c_{3R}+\bar{\nu}^c_{3L}N^c_{2R}\big)\\
&+v_s\big(\bar{N}_{2R}\nu_{3L}+\bar{N}_{3R}\nu_{2L}\big)+\Lambda_s\big(\bar{N}_{2R}N^c_{3R}+\bar{N}_{2R}N^c_{3R}\big)\big]\\
&+\frac{z}{2}\big[\big(\lambda'_s\bar{\nu}^c_{1L}\nu_{1L}+v'_s\bar{\nu}^c_{1L}N^c_{1R}+v'_s\bar{N}_{1R}\nu_{1L}+\Lambda'_s\bar{N}_{1R}N^c_{1R}\big)\\
&+\omega\big(\lambda'_s\bar{\nu}^c_{2L}\nu_{2L}+v'_s\bar{\nu}^c_{2L}N^c_{2R}+v'_s\bar{N}_{2R}\nu_{2L}+\Lambda'_s\bar{N}_{2R}N^c_{2R}\big)\\
&+\omega^2\big(\lambda'_s\bar{\nu}^c_{3L}\nu_{3L}+v'_s\bar{\nu}^c_{3L}N^c_{3R}+v'_s\bar{N}_{3R}\nu_{3L}+\Lambda'_s\bar{N}_{3R}N^c_{3R}\big)\big]+\text{H.c.}
\end{aligned}
\tag{12}
$$

We can rewrite the mass Lagrangian for the neutrinos in the matrix form:

$$
\begin{aligned}
&-\mathcal{L}^{\text{mass}}_\nu=\frac{1}{2}\bar{\chi}^c_L M_\nu\chi_L+\text{H.c.}\,,\\
&\chi_L\equiv\begin{pmatrix}\nu_L\\ N^c_R\end{pmatrix},\quad M_\nu\equiv\begin{pmatrix}M_L & M_D^T\\ M_D & M_R\end{pmatrix},
\end{aligned}
\tag{13}
$$

where $\nu=(\nu_1,\nu_2,\nu_3)^T$ and $N=(N_1,N_2,N_3)^T$. The mass matrices are then obtained by

$$
M_{L,D,R}=\begin{pmatrix}a_{L,D,R}+d_{L,D,R} & 0 & 0\\ 0 & a_{L,D,R}+\omega d_{L,D,R} & b_{L,D,R}\\ 0 & b_{L,D,R} & a_{L,D,R}+\omega^2 d_{L,D,R}\end{pmatrix},
$$

where

$$
\begin{aligned}
&a_L=\lambda_\sigma x\,,\quad a_D=v_\sigma x\,,\quad a_R=\Lambda_\sigma x\,,\\
&b_L=\lambda_s y\,,\quad b_D=v_s y\,,\quad b_R=\Lambda_s y\,,\\
&d_L=\lambda'_s z\,,\quad d_D=v'_s z\,,\quad d_R=\Lambda'_s z\,.
\end{aligned}
\tag{14}
$$

The VEVs $\Lambda_{\sigma,s}$ break the 3-3-1 gauge symmetry down to that of the standard model, and provide the masses for the neutral fermions N_R and the new gauge bosons: the neutral Z' and the charged $Y^{\pm}$ and $X^{0,0*}$. The $\lambda_{\sigma,s}$ and $v_{\sigma,s}$ belong to the second stage of the symmetry breaking from the standard model down to the $SU(3)_C \otimes U(1)_Q$ symmetry, and contribute the masses to the neutrinos. Hence, to keep a consistency, we assume that $\Lambda_{\sigma,s} \gg v_{\sigma,s},\ \lambda_{\sigma,s}$.[133] The natural smallness of the lepton number violating VEVs $\lambda_{\sigma,s}$ and $v_{\sigma,s}$ was explained in Ref. 133. Three active neutrinos therefore gain masses via a combination of type I and type II seesaw mechanisms derived from (13) as

$$M_{\rm eff} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & D \\ 0 & D & B_2 \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} A &= a_L + d_L - \frac{(a_D + d_D)^2}{a_R + d_R}, \qquad D = \frac{a_2 - b_2}{a_R^2 + d_R^2 - a_R d_R - b_R^2}, \\ B_1 &= -\frac{a_1 + b_1\omega^2 + c_1\omega}{a_R^2 + d_R^2 - a_R d_R - b_R^2}, \quad B_2 = \frac{a_1 + b_1\omega + c_1\omega^2}{a_R^2 + d_R^2 - a_R d_R - b_R^2}, \end{aligned} \tag{16}$$

with

$$\begin{aligned} a_1 &= a_D^2 a_R + 2a_D\big(d_D d_R - b_D b_R\big) + a_R\big(b_D^2 - d_L d_R\big) \\ &\quad - a_L\big(a_R^2 - b_R^2 + d_R^2\big) + a_L a_R d_R, \\ b_1 &= a_D^2 d_R + a_R\big(d_D^2 - d_L d_R\big), \\ c_1 &= 2d_D\big(a_D a_R - b_D b_R\big) + \big(b_D^2 + d_D^2\big) d_R \\ &\quad - d_L\big(a_R^2 - b_R^2 + d_R^2\big), \\ a_2 &= a_D^2 b_R - 2b_D\big(a_D a_R + d_D d_R\big) + a_R^2 b_L \\ &\quad + b_R\big(b_D^2 - b_L b_R + d_D^2\big) + b_L d_R^2, \\ b_2 &= -a_R b_D d_D + a_D b_R d_D - a_D b_D d_R + a_R b_L d_R. \end{aligned} \tag{17}$$

We can diagonalize the mass matrix (15) as follows:

$$U_\nu^T M_{\rm eff} U_\nu = {\rm diag}(m_1, m_2, m_3),$$

where

$$\begin{aligned} m_1 &= \frac{1}{2}\Big(B_1 + B_2 + \sqrt{4D^2 + (B_1 - B_2)^2}\Big), \quad m_2 = A, \\ m_3 &= \frac{1}{2}\Big(B_1 + B_2 - \sqrt{4D^2 + (B_1 - B_2)^2}\Big), \end{aligned} \tag{18}$$

and the corresponding eigenstates put in the lepton mixing matrix:

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{1}{\sqrt{K^2+1}} & 0 & \dfrac{K}{\sqrt{K^2+1}} \\ -\dfrac{K}{\sqrt{K^2+1}} & 0 & \dfrac{1}{\sqrt{K^2+1}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}, \tag{19}$$

where

$$K = \frac{B_1 - B_2 - \sqrt{4D^2 + (B_1 - B_2)^2}}{2D}. \tag{20}$$

The lepton mixing matrix is defined as

$$U_{\rm lep} \equiv U_L^\dagger U_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} \dfrac{1-K}{\sqrt{K^2+1}} & 1 & \dfrac{1+K}{\sqrt{K^2+1}} \\ \dfrac{\omega(\omega-K)}{\sqrt{K^2+1}} & 1 & \dfrac{\omega(K\omega+1)}{\sqrt{K^2+1}} \\ \dfrac{\omega(1-K\omega)}{\sqrt{K^2+1}} & 1 & \dfrac{\omega(\omega+K)}{\sqrt{K^2+1}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{21}$$

It is easy to check that U_L in (5) is a unitary matrix. So, if U_ν in (19) is unitary then $U_{\rm lep}$ in (21) is unitary. Here, we will only consider real values for K since the unitary condition of $U_{\rm lep}$. Furthermore, it is worth noting that in the case of the subgroup $\mathcal{K}$ is unbroken, i.e. without contribution of s' (or $\lambda'_s = v'_s = \Lambda'_s = 0$), the lepton mixing matrix (21) being equal to $U_{\rm HPS}$ as given in (1).

The value of the Jarlskog invariant $J_{\rm CP}$, which gives a convention-independent measure of CP violation, is defined from (21) as

$$J_{\rm CP} = {\rm Im}[U_{21} U_{31}^* U_{22}^* U_{32}] = \frac{1-K^2}{6\sqrt{3}(1+K^2)}. \tag{22}$$

Until now the values of neutrino masses (or the absolute neutrino masses) as well as the mass ordering of neutrinos are unknown. The neutrino mass spectrum can be the normal hierarchy ($|m_1| \simeq |m_2| < |m_3|$), the inverted hierarchy ($|m_3| < |m_1| \simeq |m_2|$) or nearly degenerate ($|m_1| \simeq |m_2| \simeq |m_3|$). An upper bound on the absolute value of neutrino mass was found from the analysis of the cosmological data[142]

$$m_i \leq 0.6 \text{ eV}, \tag{23}$$

while the upper limit on the sum of neutrino masses given in[143]

$$\sum_{i=1}^{3} m_i < 0.23 \text{ eV}. \tag{24}$$

In the case of three-neutrino mixing, the two possible signs of Δm^2_{23} corresponding to two types of neutrino mass spectrum can be provided as follows:

- ∘ Normal hierarchy (NH): $|m_1| \simeq |m_2| < |m_3|$, $\Delta m^2_{32} = m^2_3 - m^2_2 > 0$.
- ∘ Inverted hierarchy (IH): $|m_3| < |m_1| \simeq |m_2|$, $\Delta m^2_{32} = m^2_3 - m^2_2 < 0$.

As will be discussed below, the model under consideration can provide both normal and inverted mass hierarchy.

3.1. *Normal case* ($\Delta m^2_{32} > 0$)

In the Normal Hierarchy, combining (22) with the data in Ref. 8, $J_{\rm CP} = -0.032$, we get

$$K = -1.41297\,, \tag{25}$$

and the lepton mixing matrices are obtained as

$$U_{\rm lep} = \begin{pmatrix} 0.805 & \frac{1}{\sqrt{3}} & 0.138 \\ -0.402 + 0.119i & \frac{1}{\sqrt{3}} & 0.069 + 0.697i \\ -0.402 - 0.119i & \frac{1}{\sqrt{3}} & 0.069 - 0.697i \end{pmatrix} \times P\,, \tag{26}$$

or

$$|U_{\rm lep}| = \begin{pmatrix} 0.805 & 0.577 & 0.138 \\ 0.420 & 0.577 & 0.700 \\ 0.420 & 0.577 & 0.700 \end{pmatrix}. \tag{27}$$

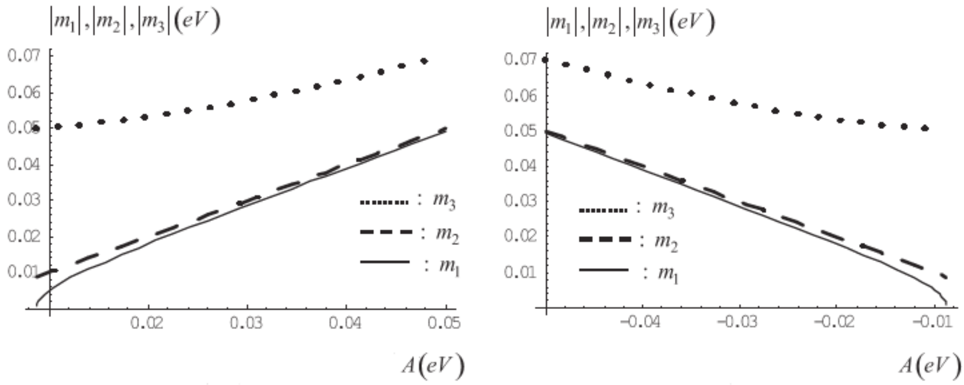
In the standard parametrization, the lepton mixing matrix can be parametrized as

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \cdot \mathcal{P}\,, \tag{28}$$

where $\mathcal{P} = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$, and $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with θ_{12}, θ_{23} and θ_{13} being the solar, atmospheric and reactor angles, respectively. $\delta = [0, 2\pi]$ is the Dirac CP violation phase while α and β are two Majorana CP violation phases. Using the parametrization in Eq. (28), we get

$$J_{\rm CP} = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\sin\delta\,. \tag{29}$$

With the help of (2), (25) and (29) we have $\sin\delta_{\rm CP} = -0.9242$, i.e. $\delta_{\rm CP} = -67.55^\circ$ or $\delta_{\rm CP} = 292.45^\circ$.



(a) (b)

Fig. 1. $|m_{1,2,3}|$ as functions of A in the case of $\Delta m^2_{32} > 0$ with (a) $A \in (0.00867, 0.05)$ eV and (b) $A \in (-0.05, -0.00867)$ eV.

From Eqs. (20) and (25) we get

$$B_1 = B_2 - 0.705241D. \tag{30}$$

In the normal case, i.e. $\Delta m^2_{32} = m^2_3 - m^2_2 > 0$, taking the central values of neutrino mass squared difference from the data in Ref. 8 as shown in (2): $\Delta m^2_{21} = 7.53 \times 10^{-5}$ eV^2 and $\Delta m^2_{32} = 2.44 \times 10^{-3}$ eV^2, with $m_{1,2,3}$ given in Eq. (18), we get a solution[b] (in [eV])

$$\begin{aligned} B_2 &= -0.5\sqrt{4A^2 - 0.0003} - 0.707729D\,, \\ D &= 0.471543\left(\sqrt{A^2 + 2.44 \times 10^{-3}} - \sqrt{A^2 - 7.53 \times 10^{-5}}\right). \end{aligned} \tag{31}$$

With $B_{1,2}$ and D in Eqs. (30) and (31), $m_{1,2,3}$ depends only on one parameter A, so we will consider $m_{1,2,3}$ as functions of A. By using the upper bound on the absolute value of neutrino mass in (23) we can restrict the values of A: $|A| \le 0.6$ eV. However, in this case, $A \in (0.0087, 0.05)$ eV or $A \in (-0.05, -0.0087)$ eV are good regions of A that can reach the realistic neutrino mass hierarchy.

In Fig. 1, we have plotted the absolute value $|m_{1,2,3}|$ as functions of A with $A \in (0.0087, 0.05)$ eV and $A \in (-0.05, -0.0087)$ eV, respectively. This figure shows that there exist allowed regions for values of A where either normal or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained if $|A| \in [0.05\text{ eV}, +\infty)$. However, $|A|$ must be small enough because of the scale of $|m_{1,2,3}|$). The normal mass hierarchy will be obtained if A takes the values around $(0.0087, 0.05)$ eV or $(-0.05, -0.0087)$ eV. The sum of

[b]In fact, this system of equations has four solutions, however, these equations differ only by the sign of $m_{1,2,3}$ that it will not appear in the neutrino oscillation experiments. So, here we only consider in detail the solution in (31).

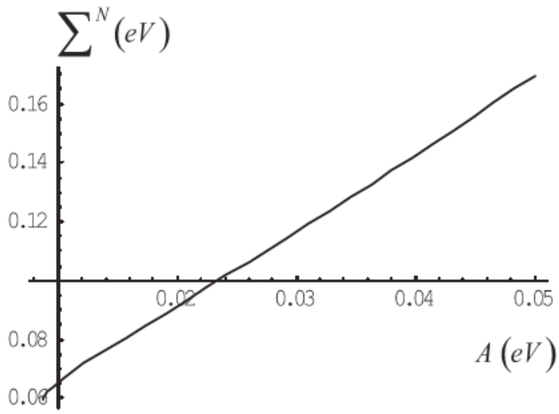


Fig. 2. $\sum^N$ as a function of A with $A \in (0.00867, 0.05)$ eV in the case of $\Delta m_{32}^2 > 0$.

neutrino masses in the normal case $\sum^N = \sum_{i=1}^3 |m_i|$ with $A \in (0.0087, 0.05)$ eV is depicted in Fig. 2 which is consistent with the upper limit given in Eq. (24).

From the expressions (20), (21), (30) and (31), it is easy to obtain the effective masses governing neutrinoless double beta decay,[144–149]

$$m_{ee}^N = \sum_{i=1}^{3} U_{ei}^2 |m_i| , \quad m_\beta^N = \left(\sum_{i=1}^{3} |U_{ei}|^2 m_i^2 \right)^{1/2} , \tag{32}$$

which is plotted in Fig. 3 with $A \in (0.0087, 0.05)$ eV in the case of $\Delta m_{32}^2 > 0$. We also note that in the normal spectrum, $|m_1| \approx |m_2| < |m_3|$, so m_1 given in (18) is the lightest neutrino mass, which is denoted as $m_1 \equiv m_{\rm light}^N$.

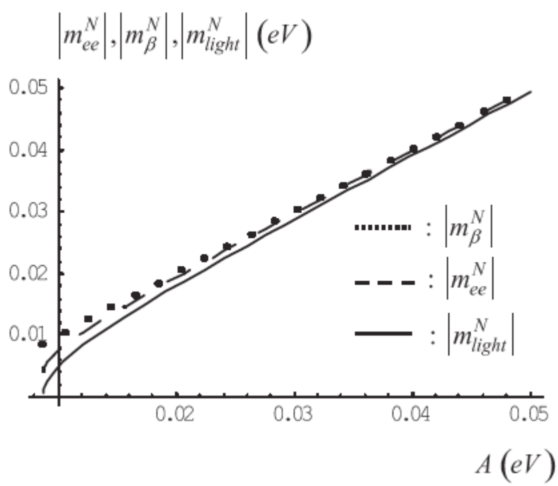


Fig. 3. $|m_{ee}^N|$, $|m_\beta^N|$ and $|m_{\rm light}^N|$ as functions of A with $A \in (0.0087, 0.05)$ eV in the case of $\Delta m_{32}^2 > 0$.

To get explicit values of the model parameters, we set $A = 10^{-2}$ eV, which is safely small. The other physical neutrino masses are explicitly given as

$$|m_1| \simeq 4.97 \times 10^{-3} \text{ eV}\,, \quad |m_2| = 10^{-2} \text{ eV}\,, \quad |m_3| \simeq 5.04 \times 10^{-2} \text{ eV}\,. \tag{33}$$

It follows that

$$|m^N_{ee}| \simeq 7.50 \times 10^{-3} \text{ eV}\,, \quad |m^N_\beta| = 9.87 \times 10^{-3} \text{ eV}\,, \tag{34}$$

$$B_1 = -3.523 \times 10^{-2} \text{ eV}\,, \quad B_2 = -2.013 \times 10^{-2} \text{ eV}\,, \quad D = 2.142 \times 10^{-2} \text{ eV}\,. \tag{35}$$

This solution means a normal mass spectrum as mentioned above. Furthermore, by assuming that[c]

$$\lambda_s = \lambda'_s = \lambda_\sigma = 1 \text{ eV}\,, \quad v_s = v'_s = v_\sigma\,, \quad \Lambda'_s = \Lambda_\sigma = \Lambda_s\,, \quad \Lambda_s = a v_s^2\,, \tag{36}$$

we obtain a solution

$$\begin{aligned} &x \simeq (2.0 + 0.2i) \times 10^{-3}\,, \qquad &&y \simeq -(6.1 + 0.61i) \times 10^{-3}\,, \\ &z = -(4.85 + 0.48) \times 10^{-3}\,, &&a \simeq 0.222 + 0.017i\,. \end{aligned} \tag{37}$$

3.2. *Inverted case* ($\Delta m^2_{32} < 0$)

For inverted hierarchy, the data in Ref. 8 implies $J_{\rm CP} = -0.029$. Hence, we get

$$K = -1.36483\,, \tag{38}$$

and the lepton mixing matrices are obtained as

$$U_{\rm lep} = \begin{pmatrix} 0.807 & \frac{1}{\sqrt{3}} & -0.125 \\ -0.403 + 0.108i & \frac{1}{\sqrt{3}} & 0.062 + 0.699i \\ -0.403 - 0.108i & \frac{1}{\sqrt{3}} & 0.062 - 0.699i \end{pmatrix} \times P\,, \tag{39}$$

or

$$|U_{\rm lep}| = \begin{pmatrix} 0.807 & 0.577 & 0.125 \\ 0.418 & 0.577 & 0.701 \\ 0.418 & 0.577 & 0.701 \end{pmatrix}. \tag{40}$$

Combining (3), (29) and (38) yields $\sin\delta_{\rm CP} = -0.8371$, i.e. $\delta_{\rm CP} = -56.84°$ or $\delta_{\rm CP} = 303.14°$.

From Eqs. (20) and (38) we get

$$B_1 = B_2 - 0.632138 D\,. \tag{41}$$

[c]The values of the parameters λ_s, λ'_s, λ_σ, v_s, v'_s, v_σ, Λ_s, Λ'_s, Λ_σ have not been confirmed by experiment, however, their hierarchies were given in Ref. 134. The parameters in Eqs. (36) and (37) is a set of the model parameters that can fit the experimental data on neutrino given in (2).

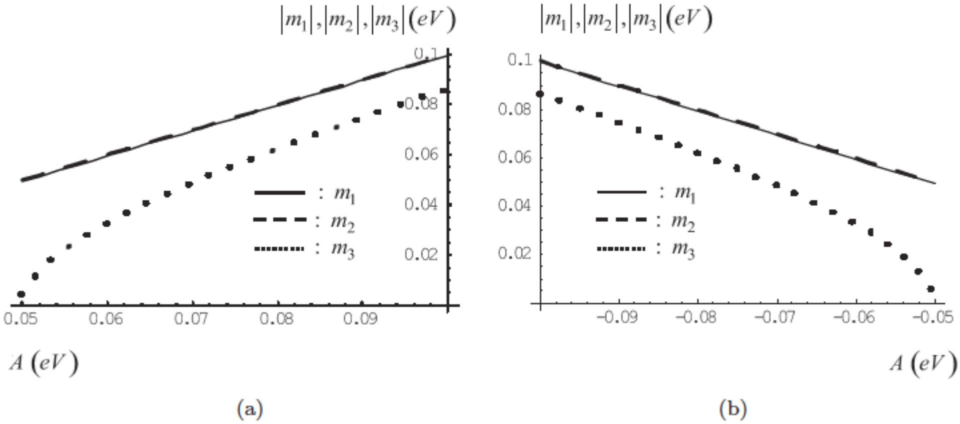


Fig. 4. $|m_{1,2,3}|$ as functions of A in the case of $\Delta m^2_{32} < 0$ with (a) $A \in (0.05, 0.1)$ eV and (b) $A \in (-0.1, -0.05)$ eV.

In the inverted case, $\Delta m^2_{32} = m^2_3 - m^2_2 < 0$, taking the central values of neutrino mass squared difference from the data in Ref. 8 as shown in (3): $\Delta m^2_{21} = 7.53 \times 10^{-5}$ eV^2 and $\Delta m^2_{32} = -2.52 \times 10^{-3}$ eV^2, with $m_{1,2,3}$ given in Eq. (18), we get a solution[d] (in [eV])

$$\begin{aligned} B_2 &= -0.5\sqrt{4A^2 - 0.0003} - 0.732692D\,, \\ D &= -0.476753\left(\sqrt{A^2 + 2.52 \times 10^{-3}} + \sqrt{A^2 - 7.53 \times 10^{-5}}\right). \end{aligned} \tag{42}$$

With $B_{1,2}$ and D in Eqs. (41) and (42), $m_{1,2,3}$ depends only on one parameter A, so we will consider $m_{1,2,3}$ as functions of A. In this case, $A \in (0.05, 0.1)$ eV or $A \in (-0.1, -0.05)$ eV are good regions of A that can reach the realistic neutrino mass hierarchy.

In Fig. 4, we have plotted the absolute value $|m_{1,2,3}|$ as functions of A with $A \in (0.05, 0.1)$ eV and $A \in (-0.1, -0.05)$ eV, respectively. We see that there exist allowed regions for values of A where either inverted or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained if $|A| \in [0.1\ \text{eV}, +\infty)$. However, $|A|$ must be small enough because of the scale of $|m_{1,2,3}|$. The inverted mass hierarchy will be obtained if $|A|$ takes the values around $(0.05, 0.1)$ eV. The sum of neutrino masses in the normal case $\sum^N = \sum_{i=1}^3 |m_i|$ with $A \in (0.0087, 0.05)$ eV is depicted in Fig. 2 which is consistent with the upper limit given in Eq. (24). The effective masses governing neutrinoless double beta decay defined in (32) is plotted in Fig. 6 with $A \in (0.05, 0.1)$ eV in the case of $\Delta m^2_{32} < 0$. We also note that in the inverted spectrum, $|m_3| \approx |m_2| \simeq |m_1|$, so m_3 given in (18) is the lightest neutrino mass, which is denoted as $m_3 \equiv m^I_{\text{light}}$.

[d]Similar to the normal case, there are four solutions in the inverted hierarchy. Here we only consider in detail the solution in (42).

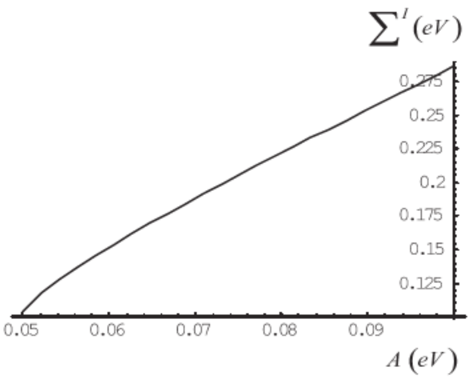


Fig. 5. $\sum^I$ as a function of A with $A \in (0.05, 0.1)$ eV in the case of $\Delta m^2_{32} < 0$.

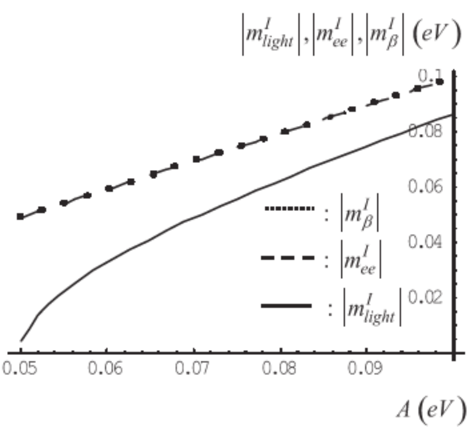


Fig. 6. $|m^I_{ee}|$, $|m^I_\beta|$ and $|m^I_{\rm light}|$ as functions of A with $A \in (0.05, 0.1)$ eV in the case of $\Delta m^2_{32} < 0$.

To get explicit values of the model parameters, we set $A = 5 \times 10^{-2}$ eV, which is safely small. The other physical neutrino masses are explicitly given as

$$\begin{aligned} |m_1| &\simeq 4.924 \times 10^{-2}\ \text{eV}\,, \\ |m_2| &= 5 \times 10^{-2}\ \text{eV}\,, \\ |m_3| &\simeq 4.472 \times 10^{-3}\ \text{eV}\,. \end{aligned} \tag{43}$$

It follows that

$$|m^I_{ee}| \simeq 4.88 \times 10^{-2}\ \text{eV}\,, \quad |m^I_\beta| = 4.91 \times 10^{-2}\ \text{eV}\,, \tag{44}$$

$$\begin{aligned} B_1 &= (1.72 - 0.29i) \times 10^{-2}\ \text{eV}\,, \\ B_2 &= (3.204 - 0.156i) \times 10^{-2}\ \text{eV}\,, \\ D &= (2.348 + 0.213i) \times 10^{-2}\ \text{eV}\,. \end{aligned} \tag{45}$$

This solution means an inverted mass spectrum. Furthermore, by assuming that[e]

$$\begin{aligned} &\lambda_s = \lambda'_s = \lambda_\sigma = a\,, \quad v_s = v'_s = -v_\sigma\,, \quad \Lambda'_s = \Lambda_s = -\Lambda_\sigma\,, \\ &\Lambda_s = v_s^2\,, \qquad\qquad\quad \Lambda'_s = v'^2_s\,, \qquad\qquad \Lambda_\sigma = -v_\sigma^2\,, \end{aligned} \tag{46}$$

we obtain a solution

$$\begin{aligned} &x \simeq (3.192 - 0.452i) \times 10^{-2}\,, \qquad y \simeq (-2.563 + 0.294i) \times 10^{-2}\,, \\ &z = -(1.910 + 0.825i) \times 10^{-2}\,, \quad a \simeq 0.105 - 0.186i\,. \end{aligned} \tag{47}$$

4. Remark on the Vacuum Alignments and ρ Parameter

In the model under consideration, to generate masses for all fermions, we need eight Higgs scalars ϕ, ϕ', χ, η, η', σ, s, s'. It is important to note that χ and s' do not break S_4 since they are put in $\underline{1}$ under S_4 while s', ϕ, η; ϕ', η' can break S_4 into its subgroups since they are put in nontrivial representations $\underline{2}$, $\underline{3}$, $\underline{3}'$ of S_4. The breaking of S_4 group depends on the vacuum alignment of the flavones.

For doublets $\underline{2}$ (s') we have two followings alignments. The first alignment, $0 \neq \langle s'_1\rangle \neq \langle s'_2\rangle = 0$ or $0 \neq \langle s'_2\rangle \neq \langle s'_1\rangle = 0$ or $0 \neq \langle s'_1\rangle \neq \langle s'_2\rangle \neq 0$ then S_4 is broken into a group which is isomorphic to Klein four group[93] that consists of the elements $\{1, TS^2T^2, S^2, T^2S^2T\}$. The second alignment, $\langle s'_1\rangle = \langle s'_2\rangle \neq 0$ then S_4 is broken into D_4 that consists of the elements $\{1, TSTS^2, TST, S, S^3, TS^2T^2, S^2, T^2S^2T\}$. For triplets $\underline{3}$ and $\underline{3}'$ the breakings of S_4 are given in App. C.

To obtain a realistic neutrino spectrum, in this work, we argue that the breaking $S_4 \to Z_3$ has taken place in charged lepton sector while both breakings $S_4 \to \mathcal{K}$ and $\mathcal{K} \to Z_2$ must be taken place in neutrino sector.

Note that Λ_σ, Λ_s, $\Lambda_{\sigma'}$ are needed to the same order and not to be so large that can naturally be taken at TeV scale as the VEV v_χ of χ. This is because v_σ, v_s and $v_{\sigma'}$ carry lepton number, simultaneously breaking the lepton parity which is naturally constrained to be much smaller than the electroweak scale.[132–134,150,151] This is also behind a theoretical fact that v_χ, Λ_σ, Λ_s, $\Lambda_{\sigma'}$ are scales for the gauge symmetry breaking in the first stage from $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X \to \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ in the original form of 3-3-1 models.[150–153] They provide masses for the new gauge bosons Z', X and Y. Also, the exotic quarks gain masses from v_χ while the neutral fermions masses arise from Λ_σ, Λ_s, $\Lambda_{\sigma'}$. The second stage of the gauge symmetry breaking from $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$ is achieved by the electroweak scale VEVs such as u, v responsible for ordinary quark masses. In combination with those of type II seesaw as determined, in this type of the model, the following limit is often taken into account:[132–134,150–153]

$$\begin{aligned} (\mathrm{eV})^2 \sim \lambda_\sigma^2, \lambda_s^2, \lambda_{\sigma'}^2 \ll v_\sigma^2, v_s^2, v_{\sigma'}^2 \ll u^2, u'^2, v^2, v'^2 \\ \ll v_\chi^2 \sim \Lambda_\sigma^2 \sim \Lambda_s^2 \sim \Lambda_{\sigma'}^2 \sim (\mathrm{TeV})^2\,. \end{aligned} \tag{48}$$

[e]The values of the parameters λ_s, λ'_s, λ_σ, v_s, v'_s, v_σ, Λ_s, Λ'_s, Λ_σ have not been confirmed by experiment, however, their hierarchies were given in Ref. 134. The parameters in Eqs. (46) and (47) is a set of the model parameters that can fit the experimental data on neutrino given in (3).

On the other hand, our model can modify the precision electroweak parameter such as ρ parameter at the tree-level. To see this, let us approximate the masses of W and Z bosons:[f]

$$M_W^2 \simeq 2g^2(3u^2 - v_\sigma^2)\,, \quad M_Z^2 \simeq \frac{g^2u^2}{c_W^2}\left(6 - \frac{v_\sigma^2}{12}\right), \tag{49}$$

$$M_Y^2 \simeq \frac{g^2}{2}\left(6\Lambda_\sigma^2 + 4\Lambda_{\sigma'}^2 + 2\Lambda_{\sigma'}^2 + v_\chi^2\right). \tag{50}$$

The ρ parameter is defined as

$$\rho = \frac{M_W^2}{c_W^2 M_Z^2} \simeq 1 - \frac{v_s^2}{3u^2}\,. \tag{51}$$

It is easy to see that the ρ parameter in (51) is absolutely close to the unity since $v_s^2 \ll u^2$ and this is in agreement with the data in Ref. 8.

The mixings between the charged gauge bosons $W - Y$ and the neutral ones $Z' - W_4$ are in the same order since they are proportional to $\frac{v_\sigma}{\Lambda_\sigma}$, and in the limit $v_\sigma \ll \lambda_\sigma$ these mixing angles tend to zero. In addition, from (48) and (49), (50), it follows that M_W^2 is much smaller than M_Y^2.

5. Conclusions

In this paper, we have modified the previous 3-3-1 model combined with discrete S_4 symmetry to adapt the most recent neutrino mixing with nonzero θ_{13}. We have shown that the realistic neutrino masses and mixings can be obtained if the two directions of the breakings $S_4 \to \mathcal{K}$ and $\mathcal{K} \to Z_2$ simultaneously take place in neutrino sector and are equivalent in size, i.e. the contributions due to s, σ and s' are comparable. The new feature is adding a new $SU(3)_L$ anti-sextet lying in $\underline{2}$ under S_4 which can result the nonzero θ_{13} without perturbation, and consequently, the number of Higgs multiplets required is less than those of other models based on non-Abelian discrete symmetries and the 3-3-1 models. The exact tribimaximal form obtained with the breaking $S_4 \to Z_3$ in charged lepton sector while $S_4 \to \mathcal{K}$ in neutrino sector. If both the breakings $S_4 \to \mathcal{K}$ and $\mathcal{K} \to Z_2$ are taken place in neutrino sector, the realistic neutrino spectrum is obtained without perturbation. The upper bound on neutrino mass as well as the effective mass governing neutrinoless double beta decay at the level are presented. The model predicts the Dirac CP violation phase $\delta = 292.45^\circ$ in the normal spectrum (with $\theta_{23} \neq \frac{\pi}{4}$) and $\delta = 303.14^\circ$ in the inverted spectrum. We have found some regions of model parameters that can fit the experimental data in 2014 on neutrino masses and mixing without perturbation.

[f] We have used the notation $s_W = \sin\theta_W$, $c_W = \cos\theta_W$, $t_W = \tan\theta_W$, and the continuation of the gauge coupling constant g of the $\mathrm{SU}(3)_L$ at the spontaneous symmetry breaking point[135,141,153] $t = \frac{3\sqrt{2}s_W}{\sqrt{3-4s_W^2}}$ was used.

Acknowledgments

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Appendix A. S_4 Group and Clebsch–Gordan Coefficients

S_4 is the permutation group of four objects, which is also the symmetry group of a cube. It has 24 elements divided into five conjugacy classes, with $\underline{1}$, $\underline{1}'$, $\underline{2}$, $\underline{3}$ and $\underline{3}'$ as its five irreducible representations. Any element of S_4 can be formed by multiplication of the generators S and T obeying the relations $S^4 = T^3 = 1$, $ST^2S = T$. Without loss of generality, we could choose $S = (1234)$, $T = (123)$ where the cycle (1234) denotes the permutation $(1,2,3,4) \to (2,3,4,1)$, and (123) means $(1,2,3,4) \to (2,3,1,4)$. The conjugacy classes generated from S and T are

$$\begin{aligned}
C_1 :\ & 1\,,\\
C_2 :\ & (12)(34) = TS^2T^2\,, \quad (13)(24) = S^2\,, \quad (14)(23) = T^2S^2T\,,\\
C_3 :\ & (123) = T\,, \quad (132) = T^2\,, \quad (124) = T^2S^2\,,\\
& (142) = S^2T\,, \quad (134) = S^2TS^2\,, \quad (143) = STS\,,\\
& (234) = S^2T^2\,, \quad (243) = TS^2\,,\\
C_4 :\ & (1234) = S\,, \quad (1243) = T^2ST\,, \quad (1324) = ST\,,\\
& (1342) = TS\,, \quad (1423) = TST^2\,, \quad (1432) = S^3\,,\\
C_5 :\ & (12) = STS^2\,, \quad (13) = TSTS^2\,, \quad (14) = ST^2\,,\\
& (23) = S^2TS\,, \quad (24) = TST\,, \quad (34) = T^2S\,.
\end{aligned}$$

The character table of S_4 is given as follows

Class	n	h	$\chi_{\underline{1}}$	$\chi_{\underline{1}'}$	$\chi_{\underline{2}}$	$\chi_{\underline{3}}$	$\chi_{\underline{3}'}$
C_1	1	1	1	1	2	3	3
C_2	3	2	1	1	2	-1	-1
C_3	8	3	1	1	-1	0	0
C_4	6	4	1	-1	0	-1	1
C_5	6	2	1	-1	0	1	-1

where n is the order of class and h is the order of elements within each class. Let us note that $C_{1,2,3}$ are even permutations, while $C_{4,5}$ are odd permutations. The two three-dimensional representations differ only in the signs of their C_4 and C_5 matrices. Similarly, the two one-dimensional representations behave the same.

We will work in the basis where $\underline{3}$, $\underline{3}'$ are real representations whereas $\underline{2}$ is complex. One possible choice of generators is given as follows

$$\underline{1}: \quad S = 1\,, \qquad T = 1\,,$$
$$\underline{1}': \quad S = -1\,, \qquad T = 1\,,$$
$$\underline{2}: \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},$$
$$\underline{3}: \quad S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\underline{3}': \quad S = -\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ is the cube root of unity. Using them, we calculate the Clebsch–Gordan coefficients for all the tensor products as given below.

First, let us put $\underline{3}(1,2,3)$ which means some $\underline{3}$ multiplets such as $x = (x_1, x_2, x_3) \sim \underline{3}$ or $y = (y_1, y_2, y_3) \sim \underline{3}$ or so on, and similarly for the other representations. Moreover, the numbered multiplets such as $(\ldots, ij, \ldots)$ mean $(\ldots, x_i y_j, \ldots)$ where x_i and y_j are the multiplet components of different representations x and y, respectively. In the following, the components of representations on the left-hand side will be omitted and should be understood, but they always exist in order in the components of decompositions on the right-hand side:

$$\underline{1} \otimes \underline{1} = \underline{1}(11)\,, \quad \underline{1}' \otimes \underline{1}' = \underline{1}(11)\,, \quad \underline{1} \otimes \underline{1}' = \underline{1}'(11)\,,$$
$$\underline{1} \otimes \underline{2} = \underline{2}(11, 12)\,, \quad \underline{1}' \otimes \underline{2} = \underline{2}(11, -12)\,,$$
$$\underline{1} \otimes \underline{3} = \underline{3}(11, 12, 13)\,, \quad \underline{1}' \otimes \underline{3} = \underline{3}'(11, 12, 13)\,,$$
$$\underline{1} \otimes \underline{3}' = \underline{3}'(11, 12, 13)\,, \quad \underline{1}' \otimes \underline{3}' = \underline{3}(11, 12, 13)\,,$$
$$\underline{2} \otimes \underline{2} = \underline{1}(12 + 21) \oplus \underline{1}'(12 - 21) \oplus \underline{2}(22, 11)\,,$$
$$\underline{2} \otimes \underline{3} = \underline{3}((1+2)1, \omega(1+\omega 2)2, \omega^2(1+\omega^2 2)3)$$
$$\oplus\, \underline{3}'((1-2)1, \omega(1-\omega 2)2, \omega^2(1-\omega^2 2)3)\,,$$
$$\underline{2} \otimes \underline{3}' = \underline{3}'((1+2)1, \omega(1+\omega 2)2, \omega^2(1+\omega^2 2)3)$$
$$\oplus\, \underline{3}((1-2)1, \omega(1-\omega 2)2, \omega^2(1-\omega^2 2)3)\,,$$
$$\underline{3} \otimes \underline{3} = \underline{1}(11 + 22 + 33) \oplus \underline{2}(11 + \omega^2 22 + \omega 33, 11 + \omega 22 + \omega^2 33)$$
$$\oplus\, \underline{3}_s(23 + 32, 31 + 13, 12 + 21) \oplus \underline{3}'_a(23 - 32, 31 - 13, 12 - 21)\,,$$
$$\underline{3}' \otimes \underline{3}' = \underline{1}(11 + 22 + 33) \oplus \underline{2}(11 + \omega^2 22 + \omega 33, 11 + \omega 22 + \omega^2 33)$$
$$\oplus\, \underline{3}_s(23 + 32, 31 + 13, 12 + 21) \oplus \underline{3}'_a(23 - 32, 31 - 13, 12 - 21)\,,$$
$$\underline{3} \otimes \underline{3}' = \underline{1}'(11 + 22 + 33) \oplus \underline{2}(11 + \omega^2 22 + \omega 33, -11 - \omega 22 - \omega^2 33)$$
$$\oplus\, \underline{3}'_s(23 + 32, 31 + 13, 12 + 21) \oplus \underline{3}_a(23 - 32, 31 - 13, 12 - 21)\,,$$

where the subscripts s and a respectively refer to their symmetric and antisymmetric product combinations as explicitly pointed out. We also notice that many group multiplication rules above have similar forms as those of S_3 and A_4 groups.

In the text we usually use the following notations, for example, $(xy')_{\underline{3}} = [xy']_{\underline{3}} \equiv (x_2y'_3 - x_3y'_2, x_3y'_1 - x_1y'_3, x_1y'_2 - x_2y'_1)$ which is the Clebsch–Gordan coefficients of $\underline{3}_a$ in the decomposition of $\underline{3} \otimes \underline{3}'$, where as mentioned $x = (x_1, x_2, x_3) \sim \underline{3}$ and $y' = (y'_1, y'_2, y'_3) \sim \underline{3}'$.

The rules to conjugate the representations $\underline{1}$, $\underline{1}'$, $\underline{2}$, $\underline{3}$ and $\underline{3}'$ are given by

$$\underline{2}^*(1^*, 2^*) = \underline{2}(2^*, 1^*)\,, \quad \underline{1}^*(1^*) = \underline{1}(1^*)\,, \quad \underline{1}'^*(1^*) = \underline{1}'(1^*)\,,$$
$$\underline{3}^*(1^*, 2^*, 3^*) = \underline{3}(1^*, 2^*, 3^*)\,, \quad \underline{3}'^*(1^*, 2^*, 3^*) = \underline{3}'(1^*, 2^*, 3^*)\,,$$

where, for example, $\underline{2}^*(1^*, 2^*)$ denotes some $\underline{2}^*$ multiplet of the form $(x_1^*, x_2^*) \sim \underline{2}^*$.

Appendix B. The Numbers

In the following, we will explicitly point out the lepton number (L) and lepton parity (P_l) of the model particles (notice that the family indices are suppressed):

Particles	L	P_l
N_R, u, d, ϕ_1^+, $\phi_1'^+$, ϕ_2^0, $\phi_2'^0$, η_1^0, $\eta_1'^0$, η_2^-, $\eta_2'^-$ χ_3^0, σ_{33}^0, s_{33}^0	0	1
ν_L, l, U, D^*, ϕ_3^+, $\phi_3'^+$, η_3^0, $\eta_3'^0$, χ_1^{0*}, χ_2^+, σ_{13}^0, σ_{23}^+, s_{13}^0, s_{23}^+	-1	-1
σ_{11}^0, σ_{12}^+, σ_{22}^{++}, s_{11}^0, s_{12}^+, s_{22}^{++}	-2	1

Appendix C. The Breakings of S_4 by Triplets $\underline{3}$ and $\underline{3}'$

For triplets $\underline{3}$, we have the followings alignments:

(1) The first alignment: $\langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle$ then S_4 is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. S_4 is completely broken.

(2) The second alignment: $0 \neq \langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$ or $0 \neq \langle\phi_1\rangle = \langle\phi_3\rangle \neq \langle\phi_2\rangle \neq 0$ or $0 \neq \langle\phi_1\rangle = \langle\phi_2\rangle \neq \langle\phi_3\rangle \neq 0$ then S_4 is broken into Z_2 which consisting of the elements $\{1, TSTS^2\}$ or $\{1, TSS^2\}$ or $\{1, S^2TS\}$, respectively.

(3) The third alignment: $\langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$ then S_4 is broken into S_3 which consisting of the elements $\{1, T, T^2, TSTS^2, STS^2, S^2TS\}$.

(4) The fourth alignment: $0 = \langle\phi_2\rangle \neq \langle\phi_1\rangle = \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_3\rangle \neq \langle\phi_1\rangle = \langle\phi_2\rangle \neq 0$ then S_4 is broken into Z_2 which consisting of the elements $\{1, TSTS^2\}$ or $\{1, TSS^2\}$ or $\{1, S^2TS\}$, respectively.

(5) The fifth alignment: $0 = \langle\phi_2\rangle \neq \langle\phi_1\rangle \neq \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle \neq 0$ or $0 \neq \langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle = 0$ then S_4 is completely broken.

(6) The sixth alignment: $0 \neq \langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle = 0$ or $0 \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle = \langle\phi_1\rangle = 0$ or $0 \neq \langle\phi_3\rangle \neq \langle\phi_1\rangle = \langle\phi_1\rangle = 0$ then S_4 is broken into Klein four group $\mathcal{K}$ which consisting of the elements $\{1, S^2, TSTS^2, TST\}$ or $\{1, TS^2T^2, STS^2, T^2S\}$ or $\{1, T^2S^2T, ST^2, S^2TS\}$, respectively.

For triplets $\underline{3}'$ we have the followings alignments:

(1) The first alignment: $\langle\phi'_1\rangle \neq \langle\phi'_2\rangle \neq \langle\phi'_3\rangle$ then S_4 is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. S_4 is completely broken.
(2) The second alignment: $0 \neq \langle\phi'_1\rangle \neq \langle\phi'_2\rangle = \langle\phi'_3\rangle \neq 0$ or $0 \neq \langle\phi'_1\rangle = \langle\phi'_3\rangle \neq \langle\phi'_2\rangle \neq 0$ or $0 \neq \langle\phi'_1\rangle = \langle\phi'_2\rangle \neq \langle\phi'_3\rangle \neq 0$ then S_4 is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. S_4 is completely broken.
(3) The third alignment: $\langle\phi'_1\rangle = \langle\phi'_2\rangle = \langle\phi'_3\rangle \neq 0$ then S_4 is broken into Z_3 that consists of the elements $\{1, T, T^2\}$.
(4) The fourth alignment: $0 = \langle\phi'_2\rangle \neq \langle\phi'_1\rangle = \langle\phi'_3\rangle \neq 0$ or $0 = \langle\phi'_1\rangle \neq \langle\phi'_2\rangle = \langle\phi'_3\rangle \neq 0$ or $0 = \langle\phi'_3\rangle \neq \langle\phi'_1\rangle = \langle\phi'_2\rangle \neq 0$ then S_4 is broken into Z_2 which consisting of the elements $\{1, T^2S\}$ or $\{1, TST\}$ or $\{1, ST^2\}$, respectively.
(5) The fifth alignment: $0 = \langle\phi'_2\rangle \neq \langle\phi'_1\rangle \neq \langle\phi'_3\rangle \neq 0$ or $0 = \langle\phi'_1\rangle \neq \langle\phi'_2\rangle \neq \langle\phi'_3\rangle \neq 0$ or $0 \neq \langle\phi'_1\rangle \neq \langle\phi'_2\rangle \neq \langle\phi'_3\rangle = 0$ then S_4 is completely broken.
(6) The sixth alignment: $0 \neq \langle\phi'_1\rangle \neq \langle\phi'_2\rangle = \langle\phi'_3\rangle = 0$ or $0 \neq \langle\phi'_2\rangle \neq \langle\phi'_3\rangle = \langle\phi'_1\rangle = 0$ or $0 \neq \langle\phi'_3\rangle \neq \langle\phi'_1\rangle = \langle\phi'_1\rangle = 0$ then S_4 is broken into a four-element subgroup generated by a four-cycle, which consisting of the elements $\{1, S, S^2, S^3\}$ or $\{1, TST^2, ST, TS^2T^2\}$ or $\{1, TS, T^2ST, T^2S^2T\}$, respectively.

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