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Abstract We succeed in the creation of a new cluster state type which has not been dealt with so far. After that we use it as quantum channel for quantum communication. Namely, in this paper one sender can simultaneously transmit two (three) different single-qubit states for two (three) different receivers under the control of a supervisor in only one protocol by using a five (seven)-qubit cluster state shared beforehand among the participants as quantum nonlocal resources. The merit of our work is that we can also generalize for sending N different single-qubit states to N receivers with also a controller by using a $(2N + 1)$-qubit cluster state. We follow the adaptive measurement strategy in order to achieve unit success probability.

Keywords Controlled simultaneously state preparation · Cluster states · Adaptive measurement · Unit success probability

1 Introduction

Quantum entanglement [1], the non-classical correlations exhibited among the parts of a composite quantum system, plays a key role as a potential resource for quantum communication and quantum information processing. Quantum teleportation (QT) fistly proposed by Bennett et al. [2] and remote state preparation (RSP) fistly proposed by Lo [3] are two quantum communication methods which are two important applications of shared entanglement. The goal of RSP is the same as that of QT. In these protocols, a sender (Alice) wishes to securely and faithfully transmit a quantum state for a spatially distant receiver (Bob) via a previously shared entanglement as quantum channel, without directly physical sending that state but only by means of local operation and classical communication. However, the main differences between RSP and QT are that: Firstly, in QT, Alice does not know the

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information of the state to be transmitted but in RSP, she is allowed to know it. Secondly, in RSP no in QT, the required resources can be traded off or there is a balance between the amount of entanglement and the classical communication cost [5, 6]. After the first appearance of the RSP protocol in [3], RSP has attracted much attention in both theory and experiment using many kinds of methods by using different entangled states. For examples, in theoretical protocols such as low-entanglement RSP [7], higher-dimension RSP[8], optimal RSP [9], etc. and in experimental protocols such as experimental RSP by using the technique of nuclear magnetic resonance [10], experimental RSP of pure and mixed states via dephasing entanglement by using spontaneous parametric down-conversion and linear optical elements [11], etc. As we know, in original versions of QT and RSP the entanglement is shared only among two parties (Alice and Bob). As an obvious extension, many modied versions have been established using various different non-local resources which are shared among more than two parties to perform global multiparty tasks. For example, a novel aspect of RSP has been put forward, which is referred to as multiparty RSP [12, 13] or joint RSP [14–24]. One more example, recently, some researchers have approached to an aspect of RSP called controlled remote state preparation (CRSP) [25]. The principle of CRSP is that the task can not completed without the permission of a person called the supervisor (Charlie) although he is not required to know the details of the state to be prepared. Our general task here is to answer the following question: How can Alice simultaneously prepare N different single-qubit states $|\psi_1\rangle$, $|\psi_2\rangle$, ..., $|\psi_N\rangle$ and send them for N Bob (Bob1, Bob 2,..., Bob N) under the control of the supervisor (Charlie) in only one protocol? In Ref. [26], authors have generalized RSP protocol for multiparties in an exact manner by using dark states. In that work, it seems that the sender can simultaneously prepare a qubit (qutrit) state of her choice at two locations by using a four (six)-qubit dark state as quantum channel in only protocol. However, in fact, we can easily realize that the four (six)-qubit dark state to be used is correct tensor product of two Einstein-Podolsky-Rosen pairs [27] (Greenberger-Horne-Zeilinger trios [28]) or their protocol is obvious two independent RSP protocols. Thus, the quantum channel used in that protocol was not work in the task. So, which quantum channel type can be used for this task in a useful way?

There are many quantum resources types which are useful in various intriguing tasks of quantum information processing and quantum computation such as Einstein-Podolsky-Rosen pair state, Greenberger-Horne-Zeilinger state, W state [29], Parametric down-conversion, Genuine multiqubit entanglement, etc. Namely, they are really useful for dense coding [30], QT, controlled QT [31–33], RSP, etc. However, unfortunately, they are not work for the question as mentioned above. Next we mention another quantum channel in the hope that it can effectively solve our task. Its name is graph state [34], a special type of multi-qubit state that can be represented by a graph. One important type of multiparty graph states is linear cluster state which was introduced by Briegel et al. [35]. Although it has also attracted much promising usefulness attention in quantum information theory, it can not help to solve our problem. Fortunately, we design successfully a new type of cluster state which can become the quantum channel for our task.

In this work, in Section 2 we present the creation of the new cluster state type which is quantum channel for our main task. Next, in Section 3 we use a five-qubit cluster state for the purpose that Alice simultaneously prepare two different single-qubit states for Bob 1 and Bob 2 under the control of Charlie. Similarly, in Section 4 Alice can simultaneously transmit three different single-qubit states for Bob 1, Bob 2 and Bob 3 under the control of Charlie by using a seven-qubit cluster states. In also this section, we generalize for sending N different single-qubit states to N Bob (Bob 1, Bob 2,..., Bob N) with also a controller by using a $(2N + 1)$-qubit cluster state. We use adaptive measurements to obtain unit success probability for the task. In the fifth Section, we shall summarize and draw some conclusions. Finally, we also write in details a set of equations in Section 4 by a Appendix.

2 Quantum Channel for Our Task

Graph state, a multipartite entangled state, is an essential ingredient for complex quantum information processing tasks. In mathematics, it is modeled in a graph by using vertices to represent qubits and edges to represent the control phase shift interactions that have taken place between two qubits. Looking at the graph form, we can see visually the entanglement of qubits by the presence of edges. The class of graph states includes many well studied states. For example, a linear cluster state $|C_N\rangle_{12\ldots N}$ of N qubits itself is a graph state which can be represented compactly in the form

$$|C_N\rangle_{12\ldots N} = (\otimes_j^{N-1} CZ_{j,j+1})(\otimes_{n=1}^{N} |+\rangle_n), \tag{1}$$

where $|+\rangle_n = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_n$ and the control phase shift, $CZ_{j,j'} |p\rangle_j |q\rangle_{j'} = (-1)^{pq} |p\rangle_j |q\rangle_{j'}$ with $p, q \in \{0, 1\}$. It not only provides an efficient model to study multiparticle entanglement but also is the resource used in quantum error correcting codes [36, 37], one-way quantum computing [38, 39] and so on. However, it can not use efficiently to complete our task. Now, we will offer a new cluster state form which will be used as the quantum resource for our task. We build the cluster state $|Q_{2N+1}\rangle_{12\ldots 2N+1}$ of $2N+1$ qubits which is simulated in Fig. 1a and has compact form as follows

$$|Q_{2N+1}\rangle_{12\ldots 2N+1} = (\otimes_{n=1}^{N} H_{2n+1} CZ_{1,2n} CZ_{2n,2n+1})(\otimes_{m=1}^{2N+1} |+\rangle_m), \tag{2}$$

where H is Hardamar operator which actions as follows $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$. Namely, in order to create the state in (2), we will start from tensor product of $2N + 1$

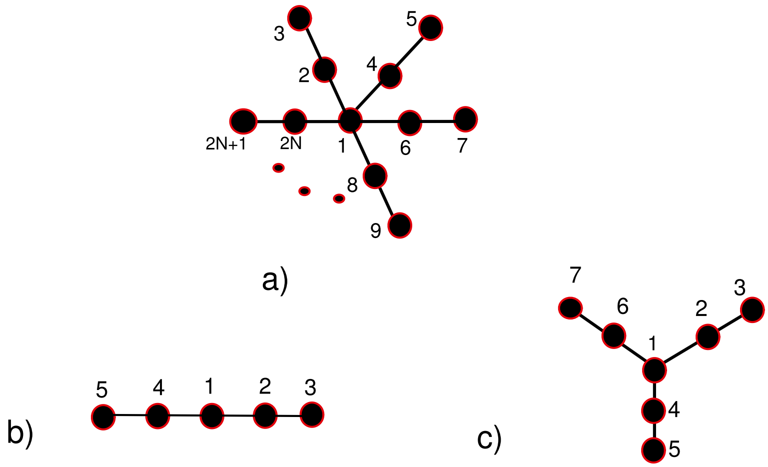

Fig. 1 Graph state is modeled in a graph by using vertices to represent qubits and edges to represent the control phase shift interactions that have taken place between two qubits. (a), (b) and (c) are the figures of the cluster state of (2N+1) qubits, five qubits and seven qubits, respectively. Qubits are represented by dots and the control phase shift interaction between two qubits are connected by solid lines

states, $|+\rangle_1 \otimes |+\rangle_2 \otimes ... \otimes |+\rangle_{2N+1}$. Fistly, we apply $2N+1$ $CZ_{1,2n}$, $CZ_{2n,2n+1}$ gates with $n = 1, 2, ..., N$ on $2N+1$ qubits $|+\rangle_1$, $|+\rangle_2$, ...and $|+\rangle_{2N+1}$. Lastly, we use N Hadamard operators and apply them on N qubits 3, 5, ..., $2N+1$.
For $N = 2$ and $N = 3$, the (2) becomes

$$|Q_5\rangle_{12345} = \frac{1}{2}(|00000\rangle + |11100\rangle + |10011\rangle + |01111\rangle)_{12345} \tag{3}$$

and

$$\begin{aligned}|Q_7\rangle &= \frac{1}{2\sqrt{2}}(|0000000\rangle + |1110000\rangle + |1001100\rangle + |0111100\rangle \\ &+ |1000011\rangle + |0110011\rangle + |0001111\rangle + |1111111\rangle)_{1234567},\end{aligned} \tag{4}$$

respectively. They are the shared nonlocal resources in our protocol and will be described in details in two next sections. We also draw them in Fig. 1b and c.

3 Controlled Simultaneously State Preparation at Two Remote Locations

We suppose that Alice wishes to simultaneously transmit two single-qubit states

$$|\psi_1\rangle = a_1 |0\rangle + b_1 e^{i\varphi_1} |1\rangle \tag{5}$$

and

$$|\psi_2\rangle = a_2 |0\rangle + b_2 e^{i\varphi_2} |1\rangle \tag{6}$$

for two receivers Bob 1 and Bob 2, respectively, under the control of Charlie with $a_1, a_2, b_1, b_2, \varphi_1, \varphi_2 \in R$ and $a_1^2 + b_1^2 = 1, a_2^2 + b_2^2 = 1$. The parameters of $a_1, a_2, b_1, b_2, \varphi_1, \varphi_2$ is only known to Alice but nothing to Bob 1, Bob 2 and Charlie.
The quantum channel to be used is a five-qubit cluster state which is shown in (3). In this entangled state, Alice holds two qubits 2 and 4, Bob holds two qubits 3 and 5, qubit 1 belongs to Charlie. In order to achieve unit success probability, we perform five steps as follows:
In the first step, Alice first takes two ancillary qubit $|0\rangle_{2'}$ and $|0\rangle_{4'}$, then applies two $CNOT_{22'}$ and $CNOT_{44'}$ on qubits 2, 2′ and 4, 4′. The quantum channel becomes to

$$\begin{aligned}\left|Q_5'\right\rangle_{242'4'135} &= CNOT_{22'}CNOT_{44'}|Q_5\rangle_{12345} \\ &= \frac{1}{2}(|0000000\rangle + |0101101\rangle + |1010110\rangle + |1111011\rangle)_{242'4'135}.\end{aligned} \tag{7}$$

In the second step, Alice independently measures qubit 2 in the basis $\{|v_k\rangle_2, k = \{0, 1\}\}$ determined by $\{a_1, b_1\}$ as

$$\begin{pmatrix}|v_0\rangle_2 \\ |v_1\rangle_2\end{pmatrix} = \begin{pmatrix}a_1 & b_1 \\ b_1 & -a_1\end{pmatrix}\begin{pmatrix}|0\rangle_2 \\ |1\rangle_2\end{pmatrix} \tag{8}$$

and qubit 4 in the basis $\{|u_l\rangle_4, l = \{0, 1\}\}$ determined by $\{a_2, b_2\}$ as

$$\begin{pmatrix}|u_0\rangle_4 \\ |u_1\rangle_4\end{pmatrix} = \begin{pmatrix}a_2 & b_2 \\ b_2 & -a_2\end{pmatrix}\begin{pmatrix}|0\rangle_4 \\ |1\rangle_4\end{pmatrix}. \tag{9}$$

After measuring, Alice publicly announces the outcomes of the measurements by two classical bits k, l. The state in (7) can now be rewritten as follows

$$\left|Q_5'\right\rangle_{242'4'135} = \frac{1}{2}\sum_{k,l=0}^{1} |v_k\rangle_2 |u_l\rangle_4 |H_{kl}\rangle_{2'4'135}, \tag{10}$$

where

$$|H_{00}\rangle_{2'4'135} = (a_1a_2\,|00000\rangle + a_1b_2\,|01101\rangle + b_1a_2\,|10110\rangle + b_1b_2\,|11011\rangle)_{2'4'135}, \quad (11)$$

$$|H_{01}\rangle_{2'4'135} = (a_1b_2\,|00000\rangle - a_1a_2\,|01101\rangle + b_1b_2\,|10110\rangle - b_1a_2\,|11011\rangle)_{2'4'135}, \quad (12)$$

$$|H_{10}\rangle_{2'4'135} = (b_1a_2\,|00000\rangle + b_1b_2\,|01101\rangle - a_1a_2\,|10110\rangle - a_1b_2\,|11011\rangle)_{2'4'135} \quad (13)$$

and

$$|H_{11}\rangle_{2'4'135} = (b_1b_2\,|00000\rangle - b_1a_2\,|01101\rangle - a_1b_2\,|10110\rangle + a_1a_2\,|11011\rangle)_{2'4'135}. \quad (14)$$

In the third step, Alice use adaptive strategy [4] or depending on the outcomes k, l she independently measures two qubits $2'$ and $4'$ in the basis $\{|\epsilon_m\rangle_{2'}\}$ and $\{|\theta_n\rangle_{4'}\}$, respectively. There are two choices for the measurement basis $\{|\epsilon_m\rangle_{2'}\}$ which depend on φ_1, k as follows:
For $k = 0$, the chosen basis is

$$\begin{pmatrix} |\epsilon_0\rangle_{2'} \\ |\epsilon_1\rangle_{2'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\varphi_1} \\ 1 & -e^{-i\varphi_1} \end{pmatrix} \begin{pmatrix} |0\rangle_{2'} \\ |1\rangle_{2'} \end{pmatrix} \quad (15)$$

and for $k = 1$, the chosen basis is

$$\begin{pmatrix} |\epsilon_0\rangle_{2'} \\ |\epsilon_1\rangle_{2'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi_1} & 1 \\ -e^{-i\varphi_1} & 1 \end{pmatrix} \begin{pmatrix} |0\rangle_{2'} \\ |1\rangle_{2'} \end{pmatrix}. \quad (16)$$

Similarly, there are two choices for the measurement basis $|\theta_n\rangle_{4'}$ which depend on φ_2, l as follows:
For $l = 0$, the chosen basis is

$$\begin{pmatrix} |\theta_0\rangle_{4'} \\ |\theta_1\rangle_{4'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\varphi_2} \\ 1 & -e^{-i\varphi_2} \end{pmatrix} \begin{pmatrix} |0\rangle_{4'} \\ |1\rangle_{4'} \end{pmatrix} \quad (17)$$

and for $l = 1$, the chosen basis is

$$\begin{pmatrix} |\theta_0\rangle_{4'} \\ |\theta_1\rangle_{4'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi_2} & 1 \\ -e^{-i\varphi_2} & 1 \end{pmatrix} \begin{pmatrix} |0\rangle_{4'} \\ |1\rangle_{4'} \end{pmatrix}. \quad (18)$$

After measuring, she also publicly announces the outcomes m, n of the measurements. And, the state in (7) can be rewritten as

$$|Q'_5\rangle_{242'4'135} = \frac{1}{4} \sum_{k,l,m,n=0}^{1} |v_k\rangle_2\,|u_l\rangle_4\,|\epsilon_m\rangle_{2'}\,|\theta_n\rangle_{4'}\,|T_{klmn}\rangle_{135}\,, \quad (19)$$

where

$$|T_{0000}\rangle_{135} = (a_1a_2\,|000\rangle + a_1b_2e^{i\varphi_2}\,|101\rangle + b_1a_2e^{i\varphi_1}\,|110\rangle + b_1b_2e^{i(\varphi_1+\varphi_2)}\,|011\rangle)_{135}, \quad (20)$$

$$|T_{0001}\rangle_{135} = (a_1a_2\,|000\rangle - a_1b_2e^{i\varphi_2}\,|101\rangle + b_1a_2e^{i\varphi_1}\,|110\rangle - b_1b_2e^{i(\varphi_1+\varphi_2)}\,|011\rangle)_{135}, \quad (21)$$

$$|T_{0010}\rangle_{135} = (a_1a_2\,|000\rangle + a_1b_2e^{i\varphi_2}\,|101\rangle - b_1a_2e^{i\varphi_1}\,|110\rangle - b_1b_2e^{i(\varphi_1+\varphi_2)}\,|011\rangle)_{135}, \quad (22)$$

$$|T_{0011}\rangle_{135} = (a_1a_2\,|000\rangle - a_1b_2e^{i\varphi_2}\,|101\rangle - b_1a_2e^{i\varphi_1}\,|110\rangle + b_1b_2e^{i(\varphi_1+\varphi_2)}\,|011\rangle)_{135}, \quad (23)$$

$$|T_{0100}\rangle_{135} = (a_1b_2e^{i\varphi_2}\,|000\rangle - a_1a_2\,|101\rangle + b_1b_2e^{i(\varphi_1+\varphi_2)}\,|110\rangle - b_1a_2e^{i\varphi_1}\,|011\rangle)_{135}, \quad (24)$$

$$|T_{0101}\rangle_{135} = (-a_1b_2e^{i\varphi_2}\,|000\rangle - a_1a_2\,|101\rangle - b_1b_2e^{i(\varphi_1+\varphi_2)}\,|110\rangle - b_1a_2e^{i\varphi_1}\,|011\rangle)_{135}, \quad (25)$$

$$|T_{0110}\rangle_{135} = (a_1b_2e^{i\varphi_2}\,|000\rangle - a_1a_2\,|101\rangle - b_1b_2e^{i(\varphi_1+\varphi_2)}\,|110\rangle + b_1a_2e^{i\varphi_1}\,|011\rangle)_{135}, \quad (26)$$

$$|T_{0111}\rangle_{135} = (-a_1b_2e^{i\varphi_2}\,|000\rangle - a_1a_2\,|101\rangle + b_1b_2e^{i(\varphi_1+\varphi_2)}\,|110\rangle + b_1a_2e^{i\varphi_1}\,|011\rangle)_{135}, \quad (27)$$

$$|T_{1000}\rangle_{135} = (b_1a_2e^{i\varphi_1}\,|000\rangle + b_1b_2e^{i(\varphi_1+\varphi_2)}\,|101\rangle - a_1a_2\,|110\rangle - a_1b_2e^{i\varphi_2}\,|011\rangle)_{135}, \quad (28)$$

$$|T_{1001}\rangle_{135} = (b_1a_2e^{i\varphi_1}\,|000\rangle - b_1b_2e^{i(\varphi_1+\varphi_2)}\,|101\rangle - a_1a_2\,|110\rangle + a_1b_2e^{i\varphi_2}\,|011\rangle)_{135}, \quad (29)$$

$$|T_{1010}\rangle_{135} = (-b_1a_2e^{i\varphi_1}|000\rangle - b_1b_2e^{i(\varphi_1+\varphi_2)}|101\rangle - a_1a_2|110\rangle - a_1b_2e^{i\varphi_2}|011\rangle)_{135}, \quad (30)$$

$$|T_{1011}\rangle_{135} = (-b_1a_2e^{i\varphi_1}|000\rangle + b_1b_2e^{i(\varphi_1+\varphi_2)}|101\rangle - a_1a_2|110\rangle + a_1b_2e^{i\varphi_2}|011\rangle)_{135}, \quad (31)$$

$$|T_{1100}\rangle_{135} = (b_1b_2e^{i(\varphi_1+\varphi_2)}|000\rangle - b_1a_2e^{i\varphi_1}|101\rangle - a_1b_2e^{i\varphi_2}|110\rangle + a_1a_2|011\rangle)_{135}, \quad (32)$$

$$|T_{1101}\rangle_{135} = (-b_1b_2e^{i(\varphi_1+\varphi_2)}|000\rangle - b_1a_2e^{i\varphi_1}|101\rangle + a_1b_2e^{i\varphi_2}|110\rangle + a_1a_2|011\rangle)_{135}, \quad (33)$$

$$|T_{1110}\rangle_{135} = (-b_1b_2e^{i(\varphi_1+\varphi_2)}|000\rangle + b_1a_2e^{i\varphi_1}|101\rangle - a_1b_2e^{i\varphi_2}|110\rangle + a_1a_2|011\rangle)_{135} \quad (34)$$

and

$$|T_{1111}\rangle_{135} = (b_1b_2e^{i(\varphi_1+\varphi_2)}|000\rangle + b_1a_2e^{i\varphi_1}|101\rangle + a_1b_2e^{i\varphi_2}|110\rangle + a_1a_2|011\rangle)_{135}. \quad (35)$$

In the fourth step, Charlie measures qubit 1 in the basis

$$\begin{pmatrix} |-\rangle_1 \\ |+\rangle_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |0\rangle_1 \\ |1\rangle_1 \end{pmatrix}, \quad (36)$$

then publicly announces the outcome $t = 0$ if qubit 1 is found in the state $|+\rangle_1$ and $t = 1$ if it is found in the state $|-\rangle_1$. After Alice and Charlie perform the measurements on their own qubits, qubits 3 and 5 automatic collapse into the state $|\Psi_{klmnt}\rangle_{35}$, with an equal probability of $1/32$ for possible each case of k, l, m, n, t. The states $|\Psi_{klmnt}\rangle_{35}$ have the following general form

$$|\Psi_{klmnt}\rangle_{35} = (R_3^+|\psi_1\rangle_3) \otimes (R_5^+|\psi_2\rangle_5), \quad (37)$$

where R_3 and R_5 are recovery operators which have forms

$$R_3 = \sigma_x^k \sigma_z^{k\oplus m\oplus t} \quad (38)$$

and

$$R_5 = \sigma_x^l \sigma_z^{l\oplus n\oplus t}. \quad (39)$$

In (38) and (39), $\sigma_x = \{\{0, 1\}, \{1, 0\}\}$ is the bit-flip operator and $\sigma_z = \{\{1, 0\}, \{0, -1\}\}$ is the phase-flip one.

In the fifth step, after hearing the outcomes of Alice and Charlie, Bob 1 and Bob 2 always obtain two original states $|\psi_1\rangle$ and $|\psi_2\rangle$ by applying two recover operators R_3 and R_5 on two qubits 3 and 5, respectively. Because Bob 1 and Bob 2 always obtain the desired states so the total success probability is 1. However, although the measurement of Charlie is independent with the measurement of Alice or the fifth step can be performed before the first step, without participation of Charlie the sender and the receivers can not complete the task alone. This is a merit of CRSP protocol.

4 Controlled Simultaneously State Preparation at Three and Many Remote Locations

In this section, we present the case with N receivers (Bob 1, Bob 2,..., Bob N). To clearly, we firstly present for case which has three receivers (Bob 1, Bob 2 and Bob 3). Namely, Alice want to simultaneously send three single-qubit states

$$|\psi_1\rangle = a_1|0\rangle + b_1e^{i\varphi_1}|1\rangle, \quad (40)$$

$$|\psi_2\rangle = a_2|0\rangle + b_2e^{i\varphi_2}|1\rangle \quad (41)$$

and

$$|\psi_3\rangle = a_3|0\rangle + b_3e^{i\varphi_3}|1\rangle \quad (42)$$

for Bob 1, Bob 2 and Bob 3, respectively, under the control of Charlie with $a_1, a_2, a_3, b_1, b_2, b_3, \varphi_1, \varphi_2, \varphi_3 \in R$ and $a_1^2 + b_1^2 = 1, a_2^2 + b_2^2 = 1, a_2^3 + b_2^3 = 1$. The parameters of $a_1, a_2, a_3, b_1, b_2, b_3, \varphi_1, \varphi_2, \varphi_3$ is only known to Alice but nothing to Bob 1, Bob 2, Bob 3 and Charlie.

The quantum channel to be used is a seven-qubit cluster state which is shown in (4). In this entangled state, Alice holds three qubits 2, 4 and 6, Bob holds three qubits 3, 5 and 7, qubit 1 belongs to Charlie. In order to achieve unit success probability, we perform five steps as follows:

In the first step, Alice first takes three ancillary qubit $|0\rangle_{2'}$, $|0\rangle_{4'}$ and $|0\rangle_{6'}$, then applies $CNOT_{22'}$ on two qubits $2, 2'$, $CNOT_{44'}$ on two qubits $4, 4'$ and $CNOT_{66'}$ on two qubits $6, 6'$. The quantum channel in (4) now becomes to

$$\begin{aligned} |Q_7'\rangle_{2462'4'6'1357} &= CNOT_{22'}CNOT_{44'}CNOT_{66'}\,|Q_7\rangle \\ &= \frac{1}{2\sqrt{2}}(|0000000000\rangle + |0010011001\rangle + |0100101010\rangle + |0110110011\rangle \\ &\quad + |1001001100\rangle + |1011010101\rangle + |1101100110\rangle + |1111111111\rangle)_{2462'4'6'1357}. \end{aligned} \tag{43}$$

In the second step, Alice independently measures qubit 2 in the basis $\{|u_k\rangle_2\,, k = \{0,1\}\}$ as

$$\begin{pmatrix} |u_0\rangle_2 \\ |u_1\rangle_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{pmatrix} \begin{pmatrix} |0\rangle_2 \\ |1\rangle_2 \end{pmatrix}, \tag{44}$$

qubit 4 in the basis $\{|e_l\rangle_4\,, l = \{0,1\}\}$ as

$$\begin{pmatrix} |e_0\rangle_4 \\ |e_1\rangle_4 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ b_2 & -a_2 \end{pmatrix} \begin{pmatrix} |0\rangle_4 \\ |1\rangle_4 \end{pmatrix} \tag{45}$$

and qubit 6 in the basis $\{|i_m\rangle_6\,, m = \{0,1\}\}$ as

$$\begin{pmatrix} |i_0\rangle_6 \\ |i_1\rangle_6 \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ b_3 & -a_3 \end{pmatrix} \begin{pmatrix} |0\rangle_6 \\ |1\rangle_6 \end{pmatrix}. \tag{46}$$

After measuring, she announces the outcomes k, l, m of the measurements. The state in (43) can now be rewritten as follows

$$|Q_7'\rangle_{22'44'66'1357} = \frac{1}{2\sqrt{2}} \sum_{k,l,m=0}^{1} |u_k\rangle_2\,|e_l\rangle_4\,|i_m\rangle_6\,|H_{klm}\rangle_{2'4'6'1357}\,, \tag{47}$$

where

$$\begin{aligned} |H_{000}\rangle_{2'4'6'1357} &= (a_1a_2a_3\,|0000000\rangle + a_1a_2b_3\,|0011001\rangle + a_1b_2a_3\,|0101010\rangle + a_1b_2b_3\,|0110011\rangle \\ &+\ b_1a_2a_3\,|1001100\rangle + b_1a_2b_3\,|1010101\rangle + b_1b_2a_3\,|1100110\rangle + b_1b_2b_3\,|1111111\rangle)_{2'4'6'1357}, \end{aligned} \tag{48}$$

$$\begin{aligned} |H_{001}\rangle_{2'4'6'1357} &= (a_1a_2b_3\,|0000000\rangle - a_1a_2a_3\,|0011001\rangle + a_1b_2b_3\,|0101010\rangle - a_1b_2a_3|0110011\rangle \\ &+\ b_1a_2b_3\,|1001100\rangle - b_1a_2a_3\,|1010101\rangle + b_1b_2b_3\,|1100110\rangle - b_1b_2a_3\,|1111111\rangle)_{2'4'6'1357}, \end{aligned} \tag{49}$$

$$\begin{aligned} |H_{010}\rangle_{2'4'6'1357} &= (a_1b_2a_3\,|0000000\rangle + a_1b_2b_3\,|0011001\rangle - a_1a_2a_3\,|0101010\rangle - a_1a_2b_3\,|0110011\rangle \\ &+\ b_1b_2a_3\,|1001100\rangle + b_1b_2b_3\,|1010101\rangle - b_1a_2a_3\,|1100110\rangle - b_1a_2b_3\,|1111111\rangle)_{2'4'6'1357}, \end{aligned} \tag{50}$$

$$\begin{aligned} |H_{011}\rangle_{2'4'6'1357} &= (a_1b_2b_3\,|0000000\rangle - a_1b_2a_3\,|0011001\rangle - a_1a_2b_3\,|0101010\rangle + a_1a_2a_3\,|0110011\rangle \\ &+\ b_1b_2b_3\,|1001100\rangle - b_1b_2a_3\,|1010101\rangle - b_1a_2b_3\,|1100110\rangle + b_1a_2a_3\,|1111111\rangle)_{2'4'6'1357}, \end{aligned} \tag{51}$$

$$\begin{aligned} |H_{100}\rangle_{2'4'6'1357} &= (b_1a_2a_3\,|0000000\rangle + b_1a_2b_3\,|0011001\rangle + b_1b_2a_3\,|0101010\rangle + b_1b_2b_3\,|0110011\rangle \\ &-\ a_1a_2a_3\,|1001100\rangle - a_1a_2b_3\,|1010101\rangle - a_1b_2a_3\,|1100110\rangle - a_1b_2b_3\,|1111111\rangle)_{2'4'6'1357}, \end{aligned} \tag{52}$$

$$\begin{aligned}|H_{101}\rangle_{2'4'6'1357} &= (b_1a_2b_3\,|0000000\rangle - b_1a_2a_3\,|0011001\rangle + b_1b_2b_3\,|0101010\rangle - b_1b_2a_3\,|0110011\rangle \\ &- a_1a_2b_3\,|1001100\rangle + a_1a_2a_3\,|1010101\rangle - a_1b_2b_3\,|1100110\rangle + a_1b_2a_3\,|1111111\rangle)_{2'4'6'1357}, \quad (53)\end{aligned}$$

$$\begin{aligned}|H_{110}\rangle_{2'4'6'1357} &= (b_1b_2a_3\,|0000000\rangle + b_1b_2b_3\,|0011001\rangle - b_1a_2a_3\,|0101010\rangle - b_1a_2b_3\,|0110011\rangle \\ &- a_1b_2a_3\,|1001100\rangle - a_1b_2b_3\,|1010101\rangle + a_1a_2a_3\,|1100110\rangle + a_1a_2b_3\,|1111111\rangle)_{2'4'6'1357} \quad (54)\end{aligned}$$

and

$$\begin{aligned}|H_{111}\rangle_{2'4'6'1357} &= (b_1b_2b_3\,|0000000\rangle - b_1b_2a_3\,|0011001\rangle - b_1a_2b_3\,|0101010\rangle + b_1a_2a_3\,|0110011\rangle \\ &- a_1b_2b_3\,|1001100\rangle + a_1b_2a_3\,|1010101\rangle + a_1a_2b_3\,|1100110\rangle - a_1a_2a_3\,|1111111\rangle)_{2'4'6'1357}. \quad (55)\end{aligned}$$

In the third step, Alice independently measures three qubits $2'$, $4'$ and $6'$ in the basis $\{|\lambda_{k'}\rangle_{2'}\}$, $\{|\pi_{l'}\rangle_{4'}\}$ and $\{|\varsigma_{m'}\rangle_{6'}\}$, respectively. Those basis depend not only on $\varphi_1, \varphi_2, \varphi_3$ but also on the outcomes k, l, m of the above measurementts. There are two choices for the measurement basis $\{|\lambda_{k'}\rangle_{2'}\}$ which depend on φ_1 and k:
If $k = 0$ the chosen basis is

$$\begin{pmatrix}|\lambda_0\rangle_{2'}\\|\lambda_1\rangle_{2'}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & e^{-i\varphi_1}\\1 & -e^{-i\varphi_1}\end{pmatrix}\begin{pmatrix}|0\rangle_{2'}\\|1\rangle_{2'}\end{pmatrix} \quad (56)$$

and if $k = 1$ the chosen basis is

$$\begin{pmatrix}|\lambda_0\rangle_{2'}\\|\lambda_1\rangle_{2'}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\varphi_1} & 1\\-e^{-i\varphi_1} & 1\end{pmatrix}\begin{pmatrix}|0\rangle_{2'}\\|1\rangle_{2'}\end{pmatrix}. \quad (57)$$

There are two choices for the measurement basis$\{|\pi_{l'}\rangle_{4'}\}$ which depend on φ_2 and l:
If $l = 0$ the chosen basis is

$$\begin{pmatrix}|\pi_0\rangle_{4'}\\|\pi_1\rangle_{4'}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & e^{-i\varphi_2}\\1 & -e^{-i\varphi_2}\end{pmatrix}\begin{pmatrix}|0\rangle_{4'}\\|1\rangle_{4'}\end{pmatrix} \quad (58)$$

and if $l = 1$ the chosen basis is

$$\begin{pmatrix}|\pi_0\rangle_{4'}\\|\pi_1\rangle_{4'}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\varphi_2} & 1\\-e^{-i\varphi_2} & 1\end{pmatrix}\begin{pmatrix}|0\rangle_{4'}\\|1\rangle_{4'}\end{pmatrix}. \quad (59)$$

There are two choices for the measurement basis $\{|\varsigma_{m'}\rangle_{6'}\}$ which depend on φ_3 and m:
If $m = 0$ the chosen basis is

$$\begin{pmatrix}|\varsigma_0\rangle_{6'}\\|\varsigma_1\rangle_{6'}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & e^{-i\varphi_3}\\1 & -e^{-i\varphi_3}\end{pmatrix}\begin{pmatrix}|0\rangle_{6'}\\|1\rangle_{6'}\end{pmatrix} \quad (60)$$

and if $m = 1$ the chosen basis is

$$\begin{pmatrix}|\varsigma_0\rangle_{6'}\\|\varsigma_1\rangle_{6'}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\varphi_3} & 1\\-e^{-i\varphi_3} & 1\end{pmatrix}\begin{pmatrix}|0\rangle_{6'}\\|1\rangle_{6'}\end{pmatrix}. \quad (61)$$

After measuring, she also publicly announces the outcomes of the measurements by three classical bits k', l', m'. The state in (43) can be rewritten as follows

$$\left|Q_7'\right\rangle_{22'44'66'1357} = \frac{1}{8}\sum_{k,l,m,k'l',m'=0}^{1} |u_k\rangle_2\,|e_l\rangle_4\,|i_m\rangle_6\,|\lambda_{k'}\rangle_{2'}\,|\pi_{l'}\rangle_{4'}\,|\varsigma_{m'}\rangle_{6'}\,|L_{klmk'l'm'}\rangle_{1357}\,, \quad (62)$$

with a probability $1/64$ of state $|L_{klmk'l'm'}\rangle_{1357}$ for any possible cases of k, l, m, k', l', m'. 64 possible cases of $|L_{klmk'l'm'}\rangle_{1357}$ are witten in the Appendix clearly.
In the fourth step, Charlie measures qubit 1 in the basis

$$\begin{pmatrix}|-\rangle_1\\|+\rangle_1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}|0\rangle_1\\|1\rangle_1\end{pmatrix}, \quad (63)$$

then publicly announces the outcome $t = 0$ if qubit 1 is found in the state $|+\rangle_1$ and $t = 1$ if it is found in the state $|-\rangle_1$. After performing the measurements of Alice and Charlie on qubits their own qubits, qubits 3, 5 and 7 automatic collapse into the state $|\Psi_{klmk'l'm't}\rangle_{357}$ with an equal probability of $1/128$ for possible each case of k, l, m, k', l', m', t. The states $|\Psi_{klmk'l'm't}\rangle_{357}$ have general form as follows

$$|\Psi_{klmk'l'm't}\rangle_{357} = (r_3^+ |\psi_1\rangle_3) \otimes (r_5^+ |\psi_2\rangle_5) \otimes (r_7^+ |\psi_3\rangle_7), \tag{64}$$

where r_3, r_5 and r_7 are recovery operators which have forms

$$r_3 = \sigma_x^k \sigma_z^{k \oplus k' \oplus t}, \tag{65}$$

$$r_5 = \sigma_x^l \sigma_z^{l \oplus l' \oplus t} \tag{66}$$

and

$$r_7 = \sigma_x^m \sigma_z^{m \oplus m' \oplus t}. \tag{67}$$

In the fifth step, depending on the outcomes k, l, m, k', l', m', t of the measurements of Alice and Charlie, Bob 1, Bob 2 and Bob 3 apply the recovery operators r_3, r_5 and r_7 on qubits 3, 5 and 7, respectively, in order obtain the desired state $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ with unit success probability for all receivers.

Note that, Charlie is the controller and he has the decision role whether the task should be completed or not. So, he can permit to stop the task if the sender and the receivers are unreliable.

From the cases of two and three receivers, we now infer the case of N receivers. For this case, the quantum channel is a $(2N + 1)$-qubit state which was shown in (2). Alice is the sender who wishes to prepare N states $\left|\psi_j\right\rangle = a_j |0\rangle + b_j e^{i\varphi_j} |1\rangle$ to N Bob j with j=1,2,...,N. Alice holds N even qubits 2, 4, ..., $2N$, each Bob j holds one odd qubit 2j+1 and Charlie holds qubit 1. Alice knows a_j, b_j, φ_j but all Bob j and Charlie know nothing. Fistly, Alice uses N ancillary qubits $|0\rangle_{2'}$, $|0\rangle_{4'}$, ..., $|0\rangle_{2N'}$ and applies $CNOT_{22'}$ on two qubits $2, 2'$, $CNOT_{44'}$ on $4, 4'$,..., $CNOT_{2N2N'}$ on $2N, 2N'$. Secondly, Alice independently measures qubits 2, 4, ... and $2N$ in the basis which depend on $\{a_1 b_1\}$, $\{a_2 b_2\}$, ... and $\{a_N b_N\}$, respectively, then announces N outcomes k_j of these measurements (k_1 for the measurements of qubit 2, k_2 for the measurements of qubit 4,..., k_N for the measurements of qubit $2N$). Thirdly, Alice independently measures qubits $2', 4'$, ..., $2N'$ in the basis which depend not only on φ_j, $(j = 1, 2, ..., N)$ but also on the outcomes k_j of the above measurements then publicly N announces N outcomes k'_j (k'_1 for the measurements of qubit 2', k'_2 for the measurements of qubit 4',..., k'_N for the measurements of qubit $2N'$). Fourthly, Charlie measures qubit 1 in the basis $\{|+\rangle, |-\rangle\}$ and announces the outcome s of his measurement if he allows to complete the task. Fifthly, Bob j can obtain the desired state $\left|\psi_j\right\rangle$ with unit success probability by applying the recover operator $r_{2j+1} = \sigma_x^{k_j} \sigma_z^{k_j \oplus k_{j'} \oplus s}$ on qubit $2j + 1$.

5 Conclusion

To summarize, we have built a new cluster state type and have put forward an idea of how one sender can simultaneously transmit two (three) single-qubit states for two (three) receivers securely, deterministically and under the control of one supervisor by using only local operations and classical communication. The original quantum channel whose qubits should be a prior distributed through space among the participants consists only of five (seven)-qubit entangled state which are here assumed to be provided off-line in the new cluster state type. However, the actually working quantum channel, though it consumes two (three) more ancillary qubits and two (three) $CNOT$ gates can be made locally in the preliminary step (i.e. the first step). The reason for extending the original to the working quantum channel is to make room for adaptive measurements as described in the third step, thanks to which unit success probability is achieved. The total classical communication is 5 (7) bits for two (three) receivers. We also describe the situation when the sender simultaneously transmit N single-qubit states for N receivers with a shared $(2N+1)$-qubit entangled state and $2N+1$ classical bits.

Appendix

64 cases of the state $|L_{klmk'l'm'}\rangle_{1357}$:

$$\begin{aligned}|L_{000000}\rangle_{1357} =& (a_1a_2a_3|0000\rangle + a_1a_2b_3e^{i\varphi_3}|1001\rangle + a_1b_2a_3e^{i\varphi_2}|1010\rangle + a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &+ b_1a_2a_3e^{i\varphi_1}|1100\rangle + b_1a_2b_3e^{i\varphi_{13}}|0101\rangle + b_1b_2a_3e^{i\varphi_{12}}|0110\rangle + b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (68)\end{aligned}$$

$$\begin{aligned}|L_{000001}\rangle_{1357} =& (a_1a_2a_3|0000\rangle - a_1a_2b_3e^{i\varphi_3}|1001\rangle + a_1b_2a_3e^{i\varphi_2}|1010\rangle - a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &+ b_1a_2a_3e^{i\varphi_1}|1100\rangle - b_1a_2b_3e^{i\varphi_{13}}|0101\rangle + b_1b_2a_3e^{i\varphi_{12}}|0110\rangle - b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (69)\end{aligned}$$

$$\begin{aligned}|L_{000010}\rangle_{1357} =& (a_1a_2a_3|0000\rangle + a_1a_2b_3e^{i\varphi_3}|1001\rangle - a_1b_2a_3e^{i\varphi_2}|1010\rangle - a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &+ b_1a_2a_3e^{i\varphi_1}|1100\rangle + b_1a_2b_3e^{i\varphi_{13}}|0101\rangle - b_1b_2a_3e^{i\varphi_{12}}|0110\rangle - b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (70)\end{aligned}$$

$$\begin{aligned}|L_{000011}\rangle_{1357} =& (a_1a_2a_3|0000\rangle - a_1a_2b_3e^{i\varphi_3}|1001\rangle - a_1b_2a_3e^{i\varphi_2}|1010\rangle + a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &+ b_1a_2a_3e^{i\varphi_1}|1100\rangle - b_1a_2b_3e^{i\varphi_{13}}|0101\rangle - b_1b_2a_3e^{i\varphi_{12}}|0110\rangle + b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (71)\end{aligned}$$

$$\begin{aligned}|L_{000100}\rangle_{1357} =& (a_1a_2a_3|0000\rangle + a_1a_2b_3e^{i\varphi_3}|1001\rangle + a_1b_2a_3e^{i\varphi_2}|1010\rangle + a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &- b_1a_2a_3e^{i\varphi_1}|1100\rangle - b_1a_2b_3e^{i\varphi_{13}}|0101\rangle - b_1b_2a_3e^{i\varphi_{12}}|0110\rangle - b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (72)\end{aligned}$$

$$\begin{aligned}|L_{000101}\rangle_{1357} =& (a_1a_2a_3|0000\rangle - a_1a_2b_3e^{i\varphi_3}|1001\rangle + a_1b_2a_3e^{i\varphi_2}|1010\rangle - a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &- b_1a_2a_3e^{i\varphi_1}|1100\rangle + b_1a_2b_3e^{i\varphi_{13}}|0101\rangle - b_1b_2a_3e^{i\varphi_{12}}|0110\rangle + b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (73)\end{aligned}$$

$$\begin{aligned}|L_{000110}\rangle_{1357} =& (a_1a_2a_3|0000\rangle + a_1a_2b_3e^{i\varphi_3}|1001\rangle - a_1b_2a_3e^{i\varphi_2}|1010\rangle - a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &- b_1a_2a_3e^{i\varphi_1}|1100\rangle - b_1a_2b_3e^{i\varphi_{13}}|0101\rangle + b_1b_2a_3e^{i\varphi_{12}}|0110\rangle + b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (74)\end{aligned}$$

$$\begin{aligned}|L_{000111}\rangle_{1357} =& (a_1a_2a_3|0000\rangle - a_1a_2b_3e^{i\varphi_3}|1001\rangle - a_1b_2a_3e^{i\varphi_2}|1010\rangle + a_1b_2b_3e^{i\varphi_{23}}|0011\rangle \\ &- b_1a_2a_3e^{i\varphi_1}|1100\rangle + b_1a_2b_3e^{i\varphi_{13}}|0101\rangle + b_1b_2a_3e^{i\varphi_{12}}|0110\rangle - b_1b_2b_3e^{i\varphi_{123}}|1111\rangle)_{1357}, \quad (75)\end{aligned}$$

$$\begin{aligned}|L_{001000}\rangle_{1357} =& (a_1a_2b_3e^{i\varphi_3}|0000\rangle - a_1a_2a_3|1001\rangle + a_1b_2b_3e^{i\varphi_{23}}|1010\rangle - a_1b_2a_3e^{i\varphi_2}|0011\rangle \\ &+ b_1a_2b_3e^{i\varphi_{13}}|1100\rangle - b_1a_2a_3e^{i\varphi_1}|0101\rangle + b_1b_2b_3e^{i\varphi_{123}}|0110\rangle - b_1b_2a_3e^{i\varphi_{12}}|1111\rangle)_{1357}, \quad (76)\end{aligned}$$

$$\begin{aligned}|L_{001001}\rangle_{1357} =& (-a_1a_2b_3e^{i\varphi_3}|0000\rangle - a_1a_2a_3|1001\rangle - a_1b_2b_3e^{i\varphi_{23}}|1010\rangle - a_1b_2a_3e^{i\varphi_2}|0011\rangle \\ &- b_1a_2b_3e^{i\varphi_{13}}|1100\rangle - b_1a_2a_3e^{i\varphi_1}|0101\rangle - b_1b_2b_3e^{i\varphi_{123}}|0110\rangle - b_1b_2a_3e^{i\varphi_{12}}|1111\rangle)_{1357}, \quad (77)\end{aligned}$$

$$\begin{aligned}|L_{001010}\rangle_{1357} =& (a_1a_2b_3e^{i\varphi_3}|0000\rangle - a_1a_2a_3|1001\rangle - a_1b_2b_3e^{i\varphi_{23}}|1010\rangle + a_1b_2a_3e^{i\varphi_2}|0011\rangle \\ &+ b_1a_2b_3e^{i\varphi_{13}}|1100\rangle - b_1a_2a_3e^{i\varphi_1}|0101\rangle - b_1b_2b_3e^{i\varphi_{123}}|0110\rangle + b_1b_2a_3e^{i\varphi_{12}}|1111\rangle)_{1357}, \quad (78)\end{aligned}$$

$$\begin{aligned}|L_{001011}\rangle_{1357} &= (-a_1a_2b_3e^{i\varphi_3}\,|0000\rangle - a_1a_2a_3\,|1001\rangle + a_1b_2b_3e^{i\varphi_{23}}\,|1010\rangle + a_1b_2a_3e^{i\varphi_2}\,|0011\rangle \\ &- b_1a_2b_3e^{i\varphi_{13}}\,|1100\rangle - b_1a_2a_3e^{i\varphi_1}\,|0101\rangle + b_1b_2b_3e^{i\varphi_{123}}\,|0110\rangle + b_1b_2a_3e^{i\varphi_{12}}\,|1111\rangle)_{1357}, \quad (79)\end{aligned}$$

$$\begin{aligned}|L_{001100}\rangle_{1357} &= (a_1a_2b_3e^{i\varphi_3}\,|0000\rangle - a_1a_2a_3\,|1001\rangle + a_1b_2b_3e^{i\varphi_{23}}\,|1010\rangle - a_1b_2a_3e^{i\varphi_2}\,|0011\rangle \\ &- b_1a_2b_3e^{i\varphi_{13}}\,|1100\rangle + b_1a_2a_3e^{i\varphi_1}\,|0101\rangle - b_1b_2b_3e^{i\varphi_{123}}\,|0110\rangle + b_1b_2a_3e^{i\varphi_{12}}\,|1111\rangle)_{1357}, \quad (80)\end{aligned}$$

$$\begin{aligned}|L_{001101}\rangle_{1357} &= (-a_1a_2b_3e^{i\varphi_3}\,|0000\rangle - a_1a_2a_3\,|1001\rangle - a_1b_2b_3e^{i\varphi_{23}}\,|1010\rangle - a_1b_2a_3e^{i\varphi_2}\,|0011\rangle \\ &+ b_1a_2b_3e^{i\varphi_{13}}\,|1100\rangle + b_1a_2a_3e^{i\varphi_1}\,|0101\rangle + b_1b_2b_3e^{i\varphi_{123}}\,|0110\rangle + b_1b_2a_3e^{i\varphi_{12}}\,|1111\rangle)_{1357}, \quad (81)\end{aligned}$$

$$\begin{aligned}|L_{001011}\rangle_{1357} &= (a_1a_2b_3e^{i\varphi_3}\,|0000\rangle - a_1a_2a_3\,|1001\rangle - a_1b_2b_3e^{i\varphi_{23}}\,|1010\rangle + a_1b_2a_3e^{i\varphi_2}\,|0011\rangle \\ &- b_1a_2b_3e^{i\varphi_{13}}\,|1100\rangle + b_1a_2a_3e^{i\varphi_1}\,|0101\rangle + b_1b_2b_3e^{i\varphi_{123}}\,|0110\rangle - b_1b_2a_3e^{i\varphi_{12}}\,|1111\rangle)_{1357}, \quad (82)\end{aligned}$$

$$\begin{aligned}|L_{001111}\rangle_{1357} &= (-a_1a_2b_3e^{i\varphi_3}\,|0000\rangle + a_1a_2a_3\,|1001\rangle + a_1b_2b_3e^{i\varphi_{23}}\,|1010\rangle - a_1b_2a_3e^{i\varphi_2}\,|0011\rangle \\ &+ b_1a_2b_3e^{i\varphi_{13}}\,|1100\rangle + b_1a_2a_3e^{i\varphi_1}\,|0101\rangle - b_1b_2b_3e^{i\varphi_{123}}\,|0110\rangle - b_1b_2a_3e^{i\varphi_{12}}\,|1111\rangle)_{1357}, \quad (83)\end{aligned}$$

$$\begin{aligned}|L_{010000}\rangle_{1357} &= (a_1b_2a_3e^{i\varphi_2}\,|0000\rangle + a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle - a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \\ &+ b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle + b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle - b_1a_2a_3e^{i\varphi_1}\,|0110\rangle - b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (84)\end{aligned}$$

$$\begin{aligned}|L_{010001}\rangle_{1357} &= (a_1b_2a_3e^{i\varphi_2}\,|0000\rangle - a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle + a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \\ &+ b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle - b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle - b_1a_2a_3e^{i\varphi_1}\,|0110\rangle + b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (85)\end{aligned}$$

$$\begin{aligned}|L_{010010}\rangle_{1357} &= (-a_1b_2a_3e^{i\varphi_2}\,|0000\rangle - a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle - a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \\ &- b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle - b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle - b_1a_2a_3e^{i\varphi_1}\,|0110\rangle - b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (86)\end{aligned}$$

$$\begin{aligned}|L_{010011}\rangle_{1357} &= (-a_1b_2a_3e^{i\varphi_2}\,|0000\rangle + a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle + a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \\ &- b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle + b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle - b_1a_2a_3e^{i\varphi_1}\,|0110\rangle + b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (87)\end{aligned}$$

$$\begin{aligned}|L_{010100}\rangle_{1357} &= (a_1b_2a_3e^{i\varphi_2}\,|0000\rangle + a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle - a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \quad (88) \\ &-b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle - b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle + b_1a_2a_3e^{i\varphi_1}\,|0110\rangle + b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (89)\end{aligned}$$

$$\begin{aligned}|L_{010101}\rangle_{1357} &= (a_1b_2a_3e^{i\varphi_2}\,|0000\rangle - a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle + a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \\ &- b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle + b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle + b_1a_2a_3e^{i\varphi_1}\,|0110\rangle - b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (90)\end{aligned}$$

$$\begin{aligned}|L_{010110}\rangle_{1357} &= (-a_1b_2a_3e^{i\varphi_2}\,|0000\rangle - a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle - a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \\ &+ b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle + b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle + b_1a_2a_3e^{i\varphi_1}\,|0110\rangle + b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (91)\end{aligned}$$

$$\begin{aligned}|L_{010111}\rangle_{1357} &= (-a_1b_2a_3e^{i\varphi_2}\,|0000\rangle + a_1b_2b_3e^{i\varphi_{23}}\,|1001\rangle - a_1a_2a_3\,|1010\rangle + a_1a_2b_3e^{i\varphi_3}\,|0011\rangle \\ &+ b_1b_2a_3e^{i\varphi_{12}}\,|1100\rangle - b_1b_2b_3e^{i\varphi_{123}}\,|0101\rangle + b_1a_2a_3e^{i\varphi_1}\,|0110\rangle - b_1a_2b_3e^{i\varphi_{13}}\,|1111\rangle)_{1357}, \quad (92)\end{aligned}$$

$$\begin{aligned}|L_{011000}\rangle_{1357} &= (a_1b_2b_3e^{i\varphi_{23}}\,|0000\rangle - a_1b_2a_3e^{i\varphi_2}\,|1001\rangle - a_1a_2b_3e^{i\varphi_3}\,|1010\rangle + a_1a_2a_3\,|0011\rangle \\ &+ b_1b_2b_3e^{i\varphi_{123}}\,|1100\rangle - b_1b_2a_3e^{i\varphi_{12}}\,|0101\rangle - b_1a_2b_3e^{i\varphi_{13}}\,|0110\rangle + b_1a_2a_3e^{i\varphi_1}\,|1111\rangle)_{1357}, \quad (93)\end{aligned}$$

$$\begin{aligned}|L_{011001}\rangle_{1357} &= (-a_1b_2b_3e^{i\varphi_{23}}\,|0000\rangle - a_1b_2a_3e^{i\varphi_2}\,|1001\rangle + a_1a_2b_3e^{i\varphi_3}\,|1010\rangle + a_1a_2a_3\,|0011\rangle \\ &- b_1b_2b_3e^{i\varphi_{123}}\,|1100\rangle - b_1b_2a_3e^{i\varphi_{12}}\,|0101\rangle + b_1a_2b_3e^{i\varphi_{13}}\,|0110\rangle + b_1a_2a_3e^{i\varphi_1}\,|1111\rangle)_{1357}, \quad (94)\end{aligned}$$

$$\begin{aligned}|L_{011010}\rangle_{1357} &= (-a_1b_2b_3e^{i\varphi_{23}}\,|0000\rangle + a_1b_2a_3e^{i\varphi_2}\,|1001\rangle - a_1a_2b_3e^{i\varphi_3}\,|1010\rangle + a_1a_2a_3\,|0011\rangle \\ &- b_1b_2b_3e^{i\varphi_{123}}\,|1100\rangle + b_1b_2a_3e^{i\varphi_{12}}\,|0101\rangle - b_1a_2b_3e^{i\varphi_{13}}\,|0110\rangle + b_1a_2a_3e^{i\varphi_1}\,|1111\rangle)_{1357}, \quad (95)\end{aligned}$$

$$\begin{aligned}|L_{011011}\rangle_{1357} &= (a_1b_2b_3e^{i\varphi_{23}}|0000\rangle + a_1b_2a_3e^{i\varphi_2}|1001\rangle + a_1a_2b_3e^{i\varphi_3}|1010\rangle + a_1a_2a_3|0011\rangle \\ &+ b_1b_2b_3e^{i\varphi_{123}}|1100\rangle + b_1b_2a_3e^{i\varphi_{12}}|0101\rangle + b_1a_2b_3e^{i\varphi_{13}}|0110\rangle + b_1a_2a_3e^{i\varphi_1}|1111\rangle)_{1357}, \quad (96)\end{aligned}$$

$$\begin{aligned}|L_{011100}\rangle_{1357} &= (a_1b_2b_3e^{i\varphi_{23}}|0000\rangle - a_1b_2a_3e^{i\varphi_2}|1001\rangle - a_1a_2b_3e^{i\varphi_3}|1010\rangle + a_1a_2a_3|0011\rangle \\ &- b_1b_2b_3e^{i\varphi_{123}}|1100\rangle + b_1b_2a_3e^{i\varphi_{12}}|0101\rangle + b_1a_2b_3e^{i\varphi_{13}}|0110\rangle - b_1a_2a_3e^{i\varphi_1}|1111\rangle)_{1357}, \quad (97)\end{aligned}$$

$$\begin{aligned}|L_{011101}\rangle_{1357} &= (-a_1b_2b_3e^{i\varphi_{23}}|0000\rangle - a_1b_2a_3e^{i\varphi_2}|1001\rangle + a_1a_2b_3e^{i\varphi_3}|1010\rangle + a_1a_2a_3|0011\rangle \\ &+ b_1b_2b_3e^{i\varphi_{123}}|1100\rangle + b_1b_2a_3e^{i\varphi_{12}}|0101\rangle - b_1a_2b_3e^{i\varphi_{13}}|0110\rangle - b_1a_2a_3e^{i\varphi_1}|1111\rangle)_{1357}, \quad (98)\end{aligned}$$

$$\begin{aligned}|L_{011110}\rangle_{1357} &= (a_1b_2b_3e^{i\varphi_{23}}|0000\rangle - a_1b_2a_3e^{i\varphi_2}|1001\rangle - a_1a_2b_3e^{i\varphi_3}|1010\rangle + a_1a_2a_3|0011\rangle \\ &+ b_1b_2b_3e^{i\varphi_{123}}|1100\rangle - b_1b_2a_3e^{i\varphi_{12}}|0101\rangle - b_1a_2b_3e^{i\varphi_{13}}|0110\rangle + b_1a_2a_3e^{i\varphi_1}|1111\rangle)_{1357}, \quad (99)\end{aligned}$$

$$\begin{aligned}|L_{011111}\rangle_{1357} &= (-a_1b_2b_3e^{i\varphi_{23}}|0000\rangle + a_1b_2a_3e^{i\varphi_2}|1001\rangle - a_1a_2b_3e^{i\varphi_3}|1010\rangle + a_1a_2a_3|0011\rangle \\ &+ b_1b_2b_3e^{i\varphi_{123}}|1100\rangle - b_1b_2a_3e^{i\varphi_{12}}|0101\rangle + b_1a_2b_3e^{i\varphi_{13}}|0110\rangle - b_1a_2a_3e^{i\varphi_1}|1111\rangle)_{1357}, (100)\end{aligned}$$

$$\begin{aligned}|L_{100000}\rangle_{1357} &= (b_1a_2a_3e^{i\varphi_1}|0000\rangle + b_1a_2b_3e^{i\varphi_{13}}|1001\rangle + b_1b_2a_3e^{i\varphi_{12}}|1010\rangle + b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &- a_1a_2a_3|1100\rangle - a_1a_2b_3e^{i\varphi_3}|0101\rangle - a_1b_2a_3e^{i\varphi_2}|0110\rangle - a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (101)\end{aligned}$$

$$\begin{aligned}|L_{100001}\rangle_{1357} &= (b_1a_2a_3e^{i\varphi_1}|0000\rangle - b_1a_2b_3e^{i\varphi_{13}}|1001\rangle + b_1b_2a_3e^{i\varphi_{12}}|1010\rangle - b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &- a_1a_2a_3|1100\rangle + a_1a_2b_3e^{i\varphi_3}|0101\rangle - a_1b_2a_3e^{i\varphi_2}|0110\rangle + a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (102)\end{aligned}$$

$$\begin{aligned}|L_{100010}\rangle_{1357} &= (b_1a_2a_3e^{i\varphi_1}|0000\rangle + b_1a_2b_3e^{i\varphi_{13}}|1001\rangle - b_1b_2a_3e^{i\varphi_{12}}|1010\rangle - b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &-a_1a_2a_3|1100\rangle - a_1a_2b_3e^{i\varphi_3}|0101\rangle + a_1b_2a_3e^{i\varphi_2}|0110\rangle + a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (103)\end{aligned}$$

$$\begin{aligned}|L_{100011}\rangle_{1357} &= (b_1a_2a_3e^{i\varphi_1}|0000\rangle - b_1a_2b_3e^{i\varphi_{13}}|1001\rangle - b_1b_2a_3e^{i\varphi_{12}}|1010\rangle + b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &- a_1a_2a_3|1100\rangle + a_1a_2b_3e^{i\varphi_3}|0101\rangle + a_1b_2a_3e^{i\varphi_2}|0110\rangle - a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (104)\end{aligned}$$

$$\begin{aligned}|L_{100100}\rangle_{1357} &= (-b_1a_2a_3e^{i\varphi_1}|0000\rangle - b_1a_2b_3e^{i\varphi_{13}}|1001\rangle - b_1b_2a_3e^{i\varphi_{12}}|1010\rangle - b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &- a_1a_2a_3|1100\rangle - a_1a_2b_3e^{i\varphi_3}|0101\rangle - a_1b_2a_3e^{i\varphi_2}|0110\rangle - a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (105)\end{aligned}$$

$$\begin{aligned}|L_{100101}\rangle_{1357} &= (-b_1a_2a_3e^{i\varphi_1}|0000\rangle + b_1a_2b_3e^{i\varphi_{13}}|1001\rangle - b_1b_2a_3e^{i\varphi_{12}}|1010\rangle + b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &- a_1a_2a_3|1100\rangle + a_1a_2b_3e^{i\varphi_3}|0101\rangle - a_1b_2a_3e^{i\varphi_2}|0110\rangle + a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (106)\end{aligned}$$

$$\begin{aligned}|L_{100110}\rangle_{1357} &= (-b_1a_2a_3e^{i\varphi_1}|0000\rangle - b_1a_2b_3e^{i\varphi_{13}}|1001\rangle + b_1b_2a_3e^{i\varphi_{12}}|1010\rangle + b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &- a_1a_2a_3|1100\rangle - a_1a_2b_3e^{i\varphi_3}|0101\rangle + a_1b_2a_3e^{i\varphi_2}|0110\rangle + a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (107)\end{aligned}$$

$$\begin{aligned}|L_{100111}\rangle_{1357} &= (-b_1a_2a_3e^{i\varphi_1}|0000\rangle + b_1a_2b_3e^{i\varphi_{13}}|1001\rangle + b_1b_2a_3e^{i\varphi_{12}}|1010\rangle - b_1b_2b_3e^{i\varphi_{123}}|0011\rangle \\ &- a_1a_2a_3|1100\rangle + a_1a_2b_3e^{i\varphi_3}|0101\rangle + a_1b_2a_3e^{i\varphi_2}|0110\rangle - a_1b_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (108)\end{aligned}$$

$$\begin{aligned}|L_{101000}\rangle_{1357} &= (b_1a_2b_3e^{i\varphi_{13}}|0000\rangle - b_1a_2a_3e^{i\varphi_1}|1001\rangle + b_1b_2b_3e^{i\varphi_{123}}|1010\rangle - b_1b_2a_3e^{i\varphi_{12}}|0011\rangle \\ &- a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle - a_1b_2b_3e^{i\varphi_3}|0110\rangle + a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (109)\end{aligned}$$

$$\begin{aligned}|L_{101001}\rangle_{1357} &= (-b_1a_2b_3e^{i\varphi_{13}}|0000\rangle - b_1a_2a_3e^{i\varphi_1}|1001\rangle - b_1b_2b_3e^{i\varphi_{123}}|1010\rangle - b_1b_2a_3e^{i\varphi_{12}}|0011\rangle \\ &+ a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle + a_1b_2b_3e^{i\varphi_3}|0110\rangle + a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (110)\end{aligned}$$

$$\begin{aligned}|L_{101010}\rangle_{1357} &= (b_1a_2b_3e^{i\varphi_{13}}|0000\rangle - b_1a_2a_3e^{i\varphi_1}|1001\rangle - b_1b_2b_3e^{i\varphi_{123}}|1010\rangle + b_1b_2a_3e^{i\varphi_{12}}|0011\rangle \\ &- a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle + a_1b_2b_3e^{i\varphi_3}|0110\rangle - a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (111)\end{aligned}$$

$$\begin{aligned}|L_{101011}\rangle_{1357} &= (-b_1a_2b_3e^{i\varphi_{13}}|0000\rangle - b_1a_2a_3e^{i\varphi_1}|1001\rangle + b_1b_2b_3e^{i\varphi_{123}}|1010\rangle + b_1b_2a_3e^{i\varphi_{12}}|0011\rangle \\ &+ a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle - a_1b_2b_3e^{i\varphi_3}|0110\rangle - a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (112)\end{aligned}$$

$$\begin{aligned}|L_{101100}\rangle_{1357} &= (-b_1a_2b_3e^{i\varphi_{13}}|0000\rangle + b_1a_2a_3e^{i\varphi_1}|1001\rangle - b_1b_2b_3e^{i\varphi_{123}}|1010\rangle + b_1b_2a_3e^{i\varphi_{12}}|0011\rangle \\ &- a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle - a_1b_2b_3e^{i\varphi_3}|0110\rangle + a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (113)\end{aligned}$$

$$\begin{aligned}|L_{101101}\rangle_{1357} &= (b_1a_2b_3e^{i\varphi_{13}}|0000\rangle + b_1a_2a_3e^{i\varphi_1}|1001\rangle + b_1b_2b_3e^{i\varphi_{123}}|1010\rangle + b_1b_2a_3e^{i\varphi_{12}}|0011\rangle \\ &+ a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle + a_1b_2b_3e^{i\varphi_3}|0110\rangle + a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (114)\end{aligned}$$

$$|L_{101110}\rangle_{1357} = (-b_1a_2b_3e^{i\varphi_{13}}|0000\rangle + b_1a_2a_3e^{i\varphi_1}|1001\rangle + b_1b_2b_3e^{i\varphi_{123}}|1010\rangle - b_1b_2a_3e^{i\varphi_{12}}|0011\rangle - a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle + a_1b_2b_3e^{i\varphi_3}|0110\rangle - a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (115)$$

$$|L_{101111}\rangle_{1357} = (b_1a_2b_3e^{i\varphi_{13}}|0000\rangle + b_1a_2a_3e^{i\varphi_1}|1001\rangle - b_1b_2b_3e^{i\varphi_{123}}|1010\rangle - b_1b_2a_3e^{i\varphi_{12}}|0011\rangle + a_1a_2b_3e^{i\varphi_3}|1100\rangle + a_1a_2a_3|0101\rangle - a_1b_2b_3e^{i\varphi_3}|0110\rangle - a_1b_2a_3e^{i\varphi_2}|1111\rangle)_{1357}, \quad (116)$$

$$|L_{110000}\rangle_{1357} = (b_1b_2a_3e^{i\varphi_{12}}|0000\rangle + b_1b_2b_3e^{i\varphi_{123}}|1001\rangle - b_1a_2a_3e^{i\varphi_1}|1010\rangle - b_1a_2b_3e^{i\varphi_{13}}|0011\rangle - a_1b_2a_3e^{i\varphi_2}|1100\rangle - a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle + a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (117)$$

$$|L_{110001}\rangle_{1357} = (b_1b_2a_3e^{i\varphi_{12}}|0000\rangle - b_1b_2b_3e^{i\varphi_{123}}|1001\rangle - b_1a_2a_3e^{i\varphi_1}|1010\rangle + b_1a_2b_3e^{i\varphi_{13}}|0011\rangle - a_1b_2a_3e^{i\varphi_2}|1100\rangle + a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle - a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (118)$$

$$|L_{110010}\rangle_{1357} = (-b_1b_2a_3e^{i\varphi_{12}}|0000\rangle - b_1b_2b_3e^{i\varphi_{123}}|1001\rangle - b_1a_2a_3e^{i\varphi_1}|1010\rangle - b_1a_2b_3e^{i\varphi_{13}}|0011\rangle + a_1b_2a_3e^{i\varphi_2}|1100\rangle + a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle + a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (119)$$

$$|L_{110011}\rangle_{1357} = (-b_1b_2a_3e^{i\varphi_{12}}|0000\rangle + b_1b_2b_3e^{i\varphi_{123}}|1001\rangle - b_1a_2a_3e^{i\varphi_1}|1010\rangle + b_1a_2b_3e^{i\varphi_{13}}|0011\rangle + a_1b_2a_3e^{i\varphi_2}|1100\rangle - a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle - a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (120)$$

$$|L_{110100}\rangle_{1357} = (-b_1b_2a_3e^{i\varphi_{12}}|0000\rangle - b_1b_2b_3e^{i\varphi_{123}}|1001\rangle + b_1a_2a_3e^{i\varphi_1}|1010\rangle + b_1a_2b_3e^{i\varphi_{13}}|0011\rangle - a_1b_2a_3e^{i\varphi_2}|1100\rangle - a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle + a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (121)$$

$$|L_{110101}\rangle_{1357} = (-b_1b_2a_3e^{i\varphi_{12}}|0000\rangle + b_1b_2b_3e^{i\varphi_{123}}|1001\rangle + b_1a_2a_3e^{i\varphi_1}|1010\rangle - b_1a_2b_3e^{i\varphi_{13}}|0011\rangle - a_1b_2a_3e^{i\varphi_2}|1100\rangle + a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle - a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (122)$$

$$|L_{110110}\rangle_{1357} = (b_1b_2a_3e^{i\varphi_{12}}|0000\rangle + b_1b_2b_3e^{i\varphi_{123}}|1001\rangle + b_1a_2a_3e^{i\varphi_1}|1010\rangle + b_1a_2b_3e^{i\varphi_{13}}|0011\rangle + a_1b_2a_3e^{i\varphi_2}|1100\rangle + a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle + a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (123)$$

$$|L_{110111}\rangle_{1357} = (b_1b_2a_3e^{i\varphi_{12}}|0000\rangle - b_1b_2b_3e^{i\varphi_{123}}|1001\rangle + b_1a_2a_3e^{i\varphi_1}|1010\rangle - b_1a_2b_3e^{i\varphi_{13}}|0011\rangle + a_1b_2a_3e^{i\varphi_2}|1100\rangle - a_1b_2b_3e^{i\varphi_{23}}|0101\rangle + a_1a_2a_3|0110\rangle - a_1a_2b_3e^{i\varphi_3}|1111\rangle)_{1357}, \quad (124)$$

$$|L_{111000}\rangle_{1357} = (b_1b_2b_3e^{i\varphi_{123}}|0000\rangle - b_1b_2a_3e^{i\varphi_{12}}|1001\rangle - b_1a_2b_3e^{i\varphi_{13}}|1010\rangle + b_1a_2a_3e^{i\varphi_1}|0011\rangle - a_1b_2b_3e^{i\varphi_{23}}|1100\rangle + a_1b_2a_3e^{i\varphi_2}|0101\rangle + a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}, \quad (125)$$

$$|L_{111001}\rangle_{1357} = (-b_1b_2b_3e^{i\varphi_{123}}|0000\rangle - b_1b_2a_3e^{i\varphi_{12}}|1001\rangle + b_1a_2b_3e^{i\varphi_{13}}|1010\rangle + b_1a_2a_3e^{i\varphi_1}|0011\rangle + a_1b_2b_3e^{i\varphi_{23}}|1100\rangle + a_1b_2a_3e^{i\varphi_2}|0101\rangle - a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}, \quad (126)$$

$$|L_{111010}\rangle_{1357} = (-b_1b_2b_3e^{i\varphi_{123}}|0000\rangle + b_1b_2a_3e^{i\varphi_{12}}|1001\rangle - b_1a_2b_3e^{i\varphi_{13}}|1010\rangle + b_1a_2a_3e^{i\varphi_1}|0011\rangle + a_1b_2b_3e^{i\varphi_{23}}|1100\rangle - a_1b_2a_3e^{i\varphi_2}|0101\rangle + a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}, \quad (127)$$

$$|L_{111011}\rangle_{1357} = (b_1b_2b_3e^{i\varphi_{123}}|0000\rangle + b_1b_2a_3e^{i\varphi_{12}}|1001\rangle + b_1a_2b_3e^{i\varphi_{13}}|1010\rangle + b_1a_2a_3e^{i\varphi_1}|0011\rangle - a_1b_2b_3e^{i\varphi_{23}}|1100\rangle - a_1b_2a_3e^{i\varphi_2}|0101\rangle - a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}, \quad (128)$$

$$|L_{111100}\rangle_{1357} = (-b_1b_2b_3e^{i\varphi_{123}}|0000\rangle + b_1b_2a_3e^{i\varphi_{12}}|1001\rangle + b_1a_2b_3e^{i\varphi_{13}}|1010\rangle - b_1a_2a_3e^{i\varphi_1}|0011\rangle - a_1b_2b_3e^{i\varphi_{23}}|1100\rangle + a_1b_2a_3e^{i\varphi_2}|0101\rangle + a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}, \quad (129)$$

$$|L_{111101}\rangle_{1357} = (b_1b_2b_3e^{i\varphi_{123}}|0000\rangle + b_1b_2a_3e^{i\varphi_{12}}|1001\rangle - b_1a_2b_3e^{i\varphi_{13}}|1010\rangle - b_1a_2a_3e^{i\varphi_1}|0011\rangle + a_1b_2b_3e^{i\varphi_{23}}|1100\rangle + a_1b_2a_3e^{i\varphi_2}|0101\rangle - a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}, \quad (130)$$

$$|L_{111110}\rangle_{1357} = (b_1b_2b_3e^{i\varphi_{123}}|0000\rangle - b_1b_2a_3e^{i\varphi_{12}}|1001\rangle + b_1a_2b_3e^{i\varphi_{13}}|1010\rangle - b_1a_2a_3e^{i\varphi_1}|0011\rangle + a_1b_2b_3e^{i\varphi_{23}}|1100\rangle - a_1b_2a_3e^{i\varphi_2}|0101\rangle + a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}, \quad (131)$$

$$|L_{111111}\rangle_{1357} = (-b_1b_2b_3e^{i\varphi_{123}}|0000\rangle - b_1b_2a_3e^{i\varphi_{12}}|1001\rangle - b_1a_2b_3e^{i\varphi_{13}}|1010\rangle - b_1a_2a_3e^{i\varphi_1}|0011\rangle - a_1b_2b_3e^{i\varphi_{23}}|1100\rangle - a_1b_2a_3e^{i\varphi_2}|0101\rangle - a_1a_2b_3e^{i\varphi_3}|0110\rangle - a_1a_2a_3|1111\rangle)_{1357}. \quad (132)$$

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