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Sum Squeezing, Difference Squeezing, Higher-Order Antibunching and Entanglement of Two-Mode Photon-Added Displaced Squeezed States

Duc Minh Truong · Hoai Thi Xuan Nguyen · An Ba Nguyen

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Abstract In this paper, we study some higher-order nonclassical properties and intermodal entanglement that may arise in the so called two-mode photon-added displaced squeezed state. We derive analytical expressions for the degree of sum squeezing and difference squeezing, which are interesting kinds of two-mode squeezing, as well as for the degree of antibunching to any orders. We also examine the degree of entanglement between the two modes using the existing Hillery-Zubairy criterion. Based on the derived expressions we analyze in detail the behavior of these nonclassical effects and entanglement depending on the parameters involved.

Keywords Photon-added states · Sum squeezing · Difference squeezing · Higher-order antibunching · Entanglement

1 Introduction

The single-mode photon-added coherent state was for the first time introduced by Agarwal and Tara [1] more than two decades ago. This state was considered as a nonlinear coherent state (NCS) [2, 3] and its various nonclassical properties were studied in detail [4–8]. The NCS embraces a wide class of states. As was commonly recognized, adding photons to a nonclassical state (such as K-quantum nonlinear coherent state [9], negative binomial state [10], etc.) makes it become a NCS characterized by its own specific nonlinear function. Experimental schemes for adding photons were realized in 2004 [11, 12], making a remarkable impact on potential application to quantum optics, quantum information and quantum computation [13–15]. The extension to two-mode states known as two-mode photon-added

D.M. Truong (✉) · H.T.X. Nguyen
Center for Theoretical and Computational Physics, Hue University of Pedagogy, 32 Le Loi, Hue, Vietnam
e-mail: tmduc2009@gmail.com

A.B. Nguyen
Center for Theoretical Physics, Institute of Physics, 10 Dao Tan, Hanoi, Vietnam

squeezed vacuum states [16] and two-mode photon-added displaced squeezed states [17] was also done. In principle, the two-mode photon-added squeezed vacuum state is a special case of the general two-mode photon-added displaced squeezed state, which in term of Fock states reads

$$|\zeta,\eta_a,\eta_b,m,n\rangle = \frac{a^{\dagger m}b^{\dagger n}D_{ab}S_{ab}|0_a,0_b\rangle}{\sqrt{\langle 0_a,0_b|S_{ab}^{\dagger}D_{ab}^{\dagger}b^n a^m a^{\dagger m}b^{\dagger n}D_{ab}S_{ab}|0_a,0_b\rangle}}, \tag{1}$$

where a, b ($a^\dagger, b^\dagger$) are the bosonic annihilation (creation) operators of two independent boson modes, $|0_a, 0_b\rangle$ is the two-mode vacuum state, while D_{ab} and S_{ab} are the two-mode displacement and squeeze operators defined respectively by [17]

$$D_{ab} = D_a D_b, \qquad D_x = \exp\bigl(\eta_x x^\dagger - \eta_x^* x\bigr), \tag{2}$$

$$S_{ab} = \exp\bigl(\zeta^* ab - \zeta a^\dagger b^\dagger\bigr), \tag{3}$$

where $\zeta = re^{i\theta}$, $\eta_a = |\eta_a|e^{i\varphi_a}$, $\eta_b = |\eta_b|e^{i\varphi_b}$, with r, θ, φ_a and φ_b being real numbers. In what follows, for definiteness, we shall study the case with $n = 0$ (i.e., only photons of mode a are added), for which the state (1) has the explicit form

$$|\zeta,\eta_a,\eta_b,m,0\rangle = \frac{a^{\dagger m}D_{ab}S_{ab}|0_a,0_b\rangle}{\sqrt{m!\cosh^{2m} r\, L_m^0(-|\eta_a|^2\mathrm{sech}^2 r)}}, \tag{4}$$

which we still call the two-mode photon-added displaced squeezed (TMPADS) state, and $L_m^k(x)$ is the associated Laguerre polynomial:

$$L_m^k(x) = \sum_{j=0}^{m}\frac{(-x)^j(m+k)!}{j!(j+k)!(m-j)!}. \tag{5}$$

From Eqs. (2) and (3), it is simple to verify the following operatoric identities

$$S_{ab}^\dagger D_{ab}^\dagger a^\dagger D_{ab}S_{ab} = a^\dagger\cosh r - be^{-i\theta}\sinh r + \eta_a^*, \tag{6}$$

$$S_{ab}^\dagger D_{ab}^\dagger b^\dagger D_{ab}S_{ab} = b^\dagger\cosh r - ae^{-i\theta}\sinh r + \eta_b^*. \tag{7}$$

For the TMPADS state given by Eq. (4) we have obtained the mean value of a general product of operators $a^l b^k b^{\dagger t} a^{\dagger v}$ as

$$\begin{aligned}\bigl\langle a^l b^k b^{\dagger t} a^{\dagger v}\bigr\rangle &= \sum_{i=0}^{m+l}\sum_{p=\max[0,k-t]}^{k}\sum_{q=0}^{\min[i,p]}\frac{(m+l)!(m+v)!k!t!}{m!(m+l-i)!(i-q)!(k-p)!(p-q)!q!}\\ &\quad\times\sum_{\Delta}\frac{(\cosh r)^{2(i+k-p-m)-\Delta}(-\sinh r)^{2q-\Delta}}{(m+v-i+\Delta)!(p+t-k-q+\Delta)!(q-\Delta)!}\\ &\quad\times\frac{|\eta_a|^{2m-2i+l+v+\Delta}|\eta_b|^{2p-2q+t-k+\Delta}e^{i(l-v-\Delta)\varphi_a}e^{i(k-t-\Delta)\varphi_b}e^{i\Delta\theta}}{L_m^0(\chi)},\end{aligned} \tag{8}$$

where Δ in the sum $\sum_\Delta$ runs from $\Delta = \max[i-m-v, q-p-t+k]$ to $\Delta = q$ and $\chi = -|\eta_a|^2\mathrm{sech}^2 r$. Several usual nonclassical effects of the TMPADS state were previously studied in [17]. In this paper, we show that these states can also possess higher-order nonclassical properties such as sum squeezing, difference squeezing and antibunching. We derive the general analytic expressions for such sum and difference squeezing as well as for higher-order antibunching. Using these expressions we determine the parameters' domain in which the sum squeezing and the difference squeezing are possible and how they behave compared to each other. We study the characteristic properties of the higher-order antibunching effect and also prove that the TMPADS state is two-mode entangled.

2 Sum squeezing

Sum squeezing is a multi-mode property of a nonclassical state [18, 19]. For two arbitrary modes a and b, the sum squeezing is associated with a so-called two-mode quadrature operator V_ϕ of the form

$$V_\phi = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\phi} a^\dagger b^\dagger + \mathrm{e}^{-\mathrm{i}\phi} ab\right), \tag{9}$$

where ϕ is an angle made by V_ϕ with the real axis in the complex plane. A state is said to be sum squeezed for a ϕ if

$$\left\langle (\Delta V_\phi)^2 \right\rangle < \frac{1}{4}\langle N_a + N_b + 1\rangle, \tag{10}$$

where $\langle(\Delta V_\phi)^2\rangle = \langle V_\phi^2\rangle - \langle V_\phi\rangle^2$, $N_a = a^\dagger a$ and $N_b = b^\dagger b$. From Eq. (10), we can define the degree of sum squeezing S in the following manner

$$S = \frac{4\langle(\Delta V_\phi)^2\rangle - \langle N_a + N_b + 1\rangle}{\langle N_a + N_b + 1\rangle}. \tag{11}$$

It is clear that sum squeezing only occurs if $S < 0$, but S has a lower bound equal to -1. Hence, the closer the value of S to -1 the higher the degree of sum squeezing. By substituting V_ϕ in Eq. (9) into Eq. (11), we obtain S in the form of antinormally ordered operators as

$$S = \frac{2[\Re(\mathrm{e}^{-2\mathrm{i}\phi}\langle a^2 b^2\rangle) - 2\Re^2(\mathrm{e}^{-\mathrm{i}\phi}\langle ab\rangle) + \langle aa^\dagger bb^\dagger\rangle - \langle aa^\dagger\rangle - \langle bb^\dagger\rangle + 1]}{\langle aa^\dagger\rangle + \langle bb^\dagger\rangle - 1}, \tag{12}$$

with $\Re(z)$ is the real part of a complex number z. Using Eq. (8) in Eq. (12) we readily have S for the TMPADS state of Eq. (4) as

$$\begin{aligned} S = {} & 2\big[(m+1)\cosh^2 r\big(\sinh^2 r + |\eta_b|^2\big)L_{m+1}^0(\chi) + |\eta_a\eta_b|^2\cos(2\varphi_2)L_m^2(\chi) \\ & - \big((m+2)\cos(\varphi_1+\varphi_2) + (m+1)\cos(\varphi_1-\varphi_2)\big)|\eta_a\eta_b|\sinh(2r)L_m^1(\chi) \\ & + \sinh^2 r\big((m+1)(m+2)\cos(2\varphi_1)\cosh^2 r + (m+1)^2\cosh^2 r - 1\big)L_m^0(\chi) \\ & - |\eta_b|^2 L_m^0(\chi) + 2|\eta_a\eta_b|\cos(\varphi_1-\varphi_2)\tanh r L_{m-1}^1(\chi) - m\sinh^2 r L_{m-1}^0(\chi) \\ & - 2\big((m+1)\cos\varphi_1\sinh(2r)L_m^0(\chi)/2 - |\eta_a\eta_b|\cos\varphi_2 L_m^1(\chi)\big)^2/L_m^0(\chi)\big] \\ & \times\big[(m+1)\cosh^2 r L_{m+1}^0(\chi) + \big(\sinh^2 + |\eta_b|^2\big)L_m^0(\chi) + m\sinh^2 r L_{m-1}^0(\chi) \\ & - 2|\eta_a\eta_b|\cos(\varphi_1-\varphi_2)\tanh r L_{m-1}^1(\chi)\big]^{-1}, \end{aligned} \tag{13}$$

where $\varphi_1 = \phi - \theta$ and $\varphi_2 = \phi - \varphi_a - \varphi_b$.

Fig. 1 The sum squeezing degree S as a function of φ_1 and φ_2 with $|\eta_a| = 2$, $|\eta_b| = 5$ and $r = 0.5$ for $m = 1$ photon added

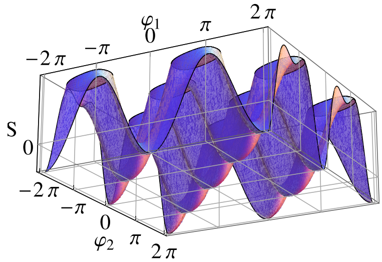


Fig. 2 The sum squeezing degree S as a function of φ_2 with $|\eta_a| = 2$, $|\eta_b| = 5$, $r = 0.5$ when $\varphi_1 = 0$ for $m = 1$ (*the solid line*), $m = 5$ (*the dashed curve*) and $m = 10$ (*the dashed-dotted curve*)

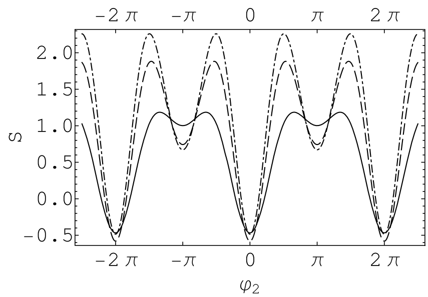
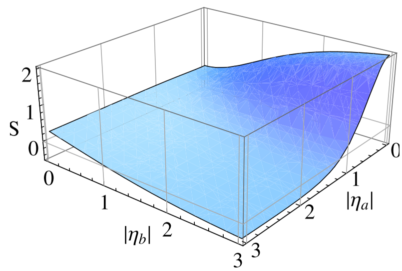


Fig. 3 The sum squeezing degree S as a function of $|\eta_a|$ and $|\eta_b|$ with $\varphi_1 = \varphi_2 = 0$, $r = 0.35$ for $m = 1$



Let us first consider the angle dependence of the sum squeezing. We plot in Fig. 1 the sum squeezing degree S, Eq. (13), as a function of the angles φ_1 and φ_2 with $|\eta_a| = 2$, $|\eta_b| = 5$ and $r = 0.5$ for $m = 1$. We can see that the TMPADS state exhibits a maximum degree of sum squeezing under simultaneous conditions

$$\varphi_1 = 2k_1\pi \quad \text{and} \quad \varphi_2 = 2k_2\pi, \tag{14}$$

with $k_1, k_2 \in \{0, \pm 1, \pm 2, \ldots\}$. These conditions for maximum degree of sum squeezing hold also for higher values of m as depicted in Fig. 2 which uses the same values of $|\eta_a|$, $|\eta_b|$ and r as in Fig. 1 but φ_1 is fixed at 0 and $m = 1, 5, 10$. Visually, S becomes most negative at $\varphi_2 = \ldots, -2\pi, 0, 2\pi, \ldots$, for any m. We next examine the dependence on the modal displacement parameters $|\eta_a|$ and $|\eta_b|$. In order to best visualize the effect of sum squeezing we concentrate on the situation when φ_1 and φ_2 satisfy the conditions (14). Thus, we set $\varphi_1 = \varphi_2 = 0$ and plot S in Fig. 3 as a function of $|\eta_a|$ and $|\eta_b|$ for $r = 0.35$ and $m = 1$. As seen from Fig. 3, no sum squeezing arises at all when either $|\eta_a| < 1$ or $|\eta_b| < 1$. However, the dependence on $|\eta_a|$ and $|\eta_b|$ appears to be more delicate. If mode b is displaced by a fixed value of $|\eta_b|$, then it is possible for the TMPADS state to be sum squeezed only in

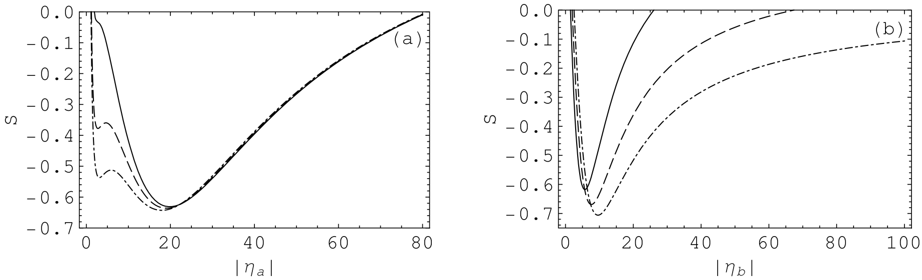
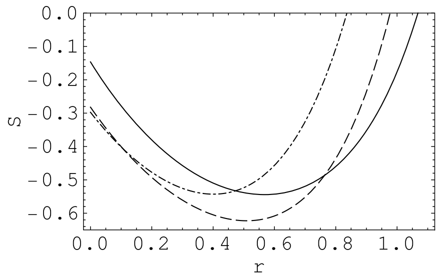


Fig. 4 The sum squeezing degree S as a function of the displacement parameter in (**a**) mode a ($|\eta_b|$ is fixed at 20); (**b**) mode b ($|\eta_a|$ is fixed at 5) with $\varphi_1 = \varphi_2 = 0$, $r = 0.5$ for $m = 1$ (*the solid line*), $m = 5$ (*the dashed curve*) and $m = 10$ (*the dashed-dotted curve*)

Fig. 5 The sum squeezing degree S as a function of r with $\varphi_1 = \varphi_2 = 0$, and $|\eta_a| = 2.5$, $|\eta_b| = 5$ for $m = 1$ (*the solid line*), $m = 5$ (*the dashed curve*) and $m = 10$ (*the dashed-dotted curve*)



a certain interval of $|\eta_a|$, which is almost independent of the number of added photons m (see Fig. 4(a)). Contrastingly, if we fix $|\eta_a|$, the displacement parameter in mode a, and vary $|\eta_b|$, then S, as a function of $|\eta_b|$, is quite sensitive to m. This sensitivity is evident from Fig. 4(b): with increasing m the interval of $|\eta_b|$, within which S is negative, widens and the value of $|\eta_b|$, at which S gets minimum, moves to the high-value side.

Finally, we study the dependence on the squeeze parameter r with $|\eta_a|$ and $|\eta_b|$ kept constant in the case of $\varphi_1 = \varphi_2 = 0$. Figure 5 illustrates such a dependence for several values of m. As observed from Fig. 5, when r is increasing the degree of sum squeezing is first increasing (i.e., S is getting more negative) until a critical value r_1, but afterwards it is becoming smaller and smaller (i.e., S is getting less negative) and, eventually disappears (i.e., S turns out to be positive) if r is beyond another critical value r_2. Of interest is the property that the actual values of both r_1 and r_2 decrease with increasing m.

3 Difference squeezing

Another interesting kind of multi-mode squeezing is the so-called difference squeezing [18, 20, 21]. In the two-mode situation it is associated with a "collective" operator W_ϕ of the form

$$W_\phi = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\phi} a b^\dagger + \mathrm{e}^{-\mathrm{i}\phi} a^\dagger b\right), \tag{15}$$

where ϕ is an angle made by W_ϕ with the real axis in the complex plane. A state is said to be difference squeezed for a ϕ if

$$\left\langle (\Delta W_\phi)^2 \right\rangle < \frac{1}{4} \left| \langle N_a - N_b \rangle \right|. \tag{16}$$

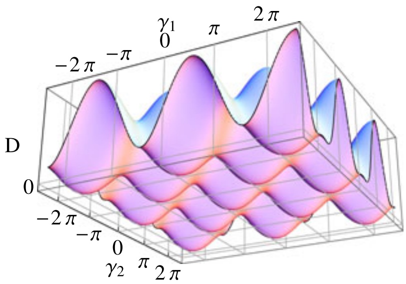


Fig. 6 The difference squeezing degree D as a function of γ_1 and γ_2 with $|\eta_a| = 2$, $|\eta_b| = 5$ and $r = 0.5$ for $m = 1$ photon added

Like in the case of sum squeezing, we can make use of Eq. (16) to define degree of difference squeezing as

$$D = \frac{4\langle(\Delta W_\phi)^2\rangle - |\langle N_a - N_b\rangle|}{|\langle N_a - N_b\rangle|}. \tag{17}$$

Hence, a state is difference squeezed if $-1 \le D < 0$. Taking again Eq. (8) into account in the definition (17) we readily obtain for the TMPADS state

$$\begin{aligned} D = \Big[&(m+1)\cosh^2 r\big(2\big(\cosh^2 r + |\eta_b|^2\big) - 1\big)L^0_{m+1}(\chi) \\ &+ (m+1)^2\sinh^2(2r)L^0_m(\chi)/2 - \big(|\eta_b|^2 + \cosh^2 r\big)L^0_m(\chi) \\ &+ 2|\eta_a\eta_b|\cos(\gamma_1 - \gamma_2)\tanh r L^1_{m-1}(\chi) - m\sinh^2 r L^0_{m-1}(\chi) \\ &- 2|\eta_a\eta_b|\big((m+1)\cos(\gamma_1 - \gamma_2)\sinh(2r)L^1_m(\chi) - |\eta_a\eta_b|\cos(2\gamma_2)L^2_m(\chi)\big) \\ &- 2|\eta_a|^3\big(2|\eta_b|\cos(\gamma_1 + \gamma_2)\tanh r L^3_{m-1}(\chi) - |\eta_a|\cos(2\gamma_1)\tanh^2 r L^4_{m-2}(\chi)\big) \\ &- 4|\eta_a|^2\big(|\eta_a|\cos\gamma_1\tanh r L^2_{m-1}(\chi) - |\eta_b|\cos\gamma_2 L^1_m(\chi)\big)^2/L^0_m(\chi)\Big] \\ &\times\Big|(m+1)\cosh^2 r L^0_{m+1}(\chi) - \big(\cosh^2 r + |\eta_b|^2\big)L^0_m(\chi) \\ &- m\sinh^2 r L^0_{m-1}(\chi) + 2|\eta_a\eta_b|\cos(\gamma_1 - \gamma_2)\tanh r L^1_{m-1}(\chi)\Big|^{-1} - 1, \end{aligned} \tag{18}$$

where $\gamma_1 = \phi - \theta + 2\varphi_a$ and $\gamma_2 = \phi + \varphi_a - \varphi_b$. In terms of such γ_1 and γ_2, the degree of difference squeezing is maximal under the simultaneous conditions

$$\gamma_1 = 2k_1\pi \quad \text{and} \quad \gamma_2 = 2k_2\pi, \tag{19}$$

with $k_1, k_2 \in \{0, \pm 1, \pm 2, \ldots\}$. This is seen from Fig. 6 which is a plot of D as a function of γ_1 and γ_2 with $|\eta_a| = 2$, $|\eta_b| = 5$, $r = 0.5$ and $m = 1$. The dependence of D on $|\eta_a|$ ($|\eta_b|$) when $|\eta_b|$ ($|\eta_a|$) is fixed is shown in Fig. 7(a) (Fig. 7(b)) for $\gamma_1 = \gamma_2 = 0$ (to make the squeezing most favorable), $r = 0.5$ and different values of m. As it should be, the roles of $|\eta_a|$ and $|\eta_b|$ are asymmetric in the TMPADS state defined by Eq. (4). Although the minimum of D gets "deeper" for a greater m (i.e., for adding more photons), its position moves to the left in Fig. 7(a) but to the right in Fig. 7(b). Also, for a fixed $|\eta_b|$ the interval of $|\eta_a|$ within which $D < 0$ is quite restricted (see Fig. 7(a)), while such an interval of $|\eta_b|$ for a fixed $|\eta_a|$ is much more spread (see Fig. 7(b)). From Fig. 8, a plot of D as a function of r with other parameters kept constant, we can realize that difference squeezing exists only with small values of r, a feature similar to sum squeezing. The distinction is, however, that the maximum achievable degree of difference squeezing seems increasing with m (namely,

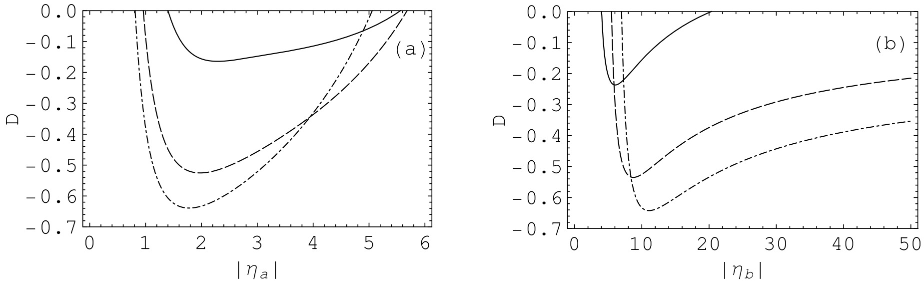


Fig. 7 The difference squeezing degree D as a function of the displacement parameter in (**a**) mode a ($|\eta_b|$ is fixed at 10); (**b**) mode b ($|\eta_a|$ is fixed at 2) with $\gamma_1 = \gamma_2 = 0$, $r = 0.5$ for $m = 1$ (*the solid line*), $m = 5$ (*the dashed curve*) and $m = 10$ (*the dashed-dotted curve*)

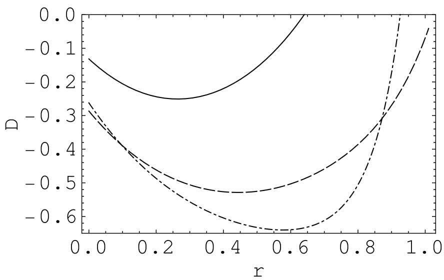


Fig. 8 The difference squeezing degree D as a function of r with $\gamma_1 = \gamma_2 = 0$, and $|\eta_a| = 2$, $|\eta_b| = 10$ for $m = 1$ (*the solid line*), $m = 5$ (*the dashed curve*) and $m = 10$ (*the dashed-dotted curve*)

$\min D(m = 10) < \min D(m = 5) < \min D(m = 1)$ in Fig. 8), whereas this is not the case for sum squeezing for which $\min D(m = 5) < \min D(m = 1)$, $\min D(m = 10)$ as in Fig. 5.

4 Higher-order antibunching

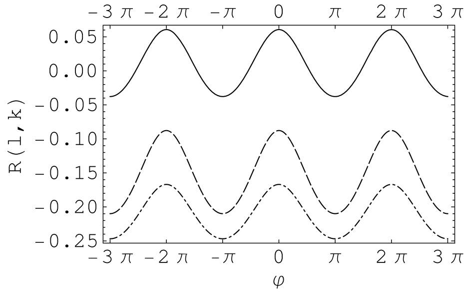
The notion of higher-order antibunching was introduced by Lee [22, 23] and it has been adopted to investigate several nonclassical states [24–26]. Recently, some experiment schemes based on hybrid photodetectors and time multiplexed detection to detect the higher-order antibunching are proposed [27–29]. According to Lee, a criterion for the existence of higher-order antibunching in a two-mode radiation field is defined by the coefficient $R(l, k)$ as

$$R(l,k) \equiv \frac{\langle N_a^{(l+1)} N_b^{(k-1)} \rangle + \langle N_a^{(k-1)} N_b^{(l+1)} \rangle}{\langle N_a^{(l)} N_b^{(k)} \rangle + \langle N_a^{(k)} N_b^{(l)} \rangle} - 1 < 0, \tag{20}$$

with $l \geq k$ and $N_a^{(n)} := a^{\dagger n} a^n$. The quantity $\langle N_a^{(l)} N_b^{(k)} \rangle$ can be expressed in terms of antinormally ordered operators as [30],

$$\langle N_a^{(l)} N_b^{(k)} \rangle = \sum_{j=0}^{l} \frac{[l!]^2 (-1)^j}{j![(l-j)!]^2} \sum_{i=0}^{k} \frac{[k!]^2 (-1)^i}{i![(k-i)!]^2} \langle a^{l-j} \left(a^\dagger\right)^{l-j} b^{k-i} \left(b^\dagger\right)^{k-i} \rangle, \tag{21}$$

Fig. 9 The coefficients $R(1,1)$ (*the solid line*), $R(3,1)$ (*the dashed curve*) and $R(5,2)$ (*the dashed-dotted curve*) as functions of φ with $|\eta_a|=0.1$, $|\eta_b|=0.7$, $r=0.8$ for $m=3$



with $\langle a^{l-j}(a^\dagger)^{l-j}b^{k-i}(b^\dagger)^{k-i}\rangle$ having the explicit form

$$\begin{aligned}\left\langle a^{l-j}b^{k-i}b^{\dagger k-i}a^{\dagger l-j}\right\rangle &= \sum_{t=0}^{m+l-j}\sum_{p=0}^{k-i}\sum_{q=0}^{\min[t,p]}\sum_{\Delta}\frac{(m+l-j)!^2(k-i)!^2}{L_m^0(\chi)(m+l-j-t)!}\\ &\quad\times\frac{(\cosh r)^{2(t+k-i-p-m)-\Delta}(-\sinh r)^{2q-\Delta}}{m!(t-q)!(k-i-p)!(p-q)!q!}\\ &\quad\times\frac{|\eta_a|^{2m-2t+2l-2j+\Delta}|\eta_b|^{2p-2q+\Delta}\mathrm{e}^{\mathrm{i}\Delta\varphi}}{(m+l-j-t+\Delta)!(p-q+\Delta)!(q-\Delta)!},\end{aligned} \tag{22}$$

where the sum over Δ runs from $\Delta=\max[t-m-l+j, q-p]$ to $\Delta=q$ and $\varphi=\theta-\varphi_a-\varphi_b$. Based on the above formulae we can calculate $R(l,k)$ for any orders l, k and m. To observe the antibunching properties easily, we find values of φ at which the effect displays most pronouncedly. In Fig. 9 we show $R(1,1)$ (the solid line), $R(3,1)$ (the dashed curve) and $R(5,2)$ (the dashed-dotted curve) as functions of φ with $|\eta_a|=0.1$, $|\eta_b|=0.7$, $r=0.8$ and $m=3$. It is obvious that the antibunching is strongest along the directions determined by

$$\varphi=(2k+1)\pi, \tag{23}$$

with $k=0,\pm1,\pm2,\ldots$. Then we can choose a fixed value of $\varphi=\pi$ for further consideration. With such a choice of φ, we plot $R(1,1)$ in Fig. 10(a) and $R(4,2)$ in Fig. 10(b) versus r with $|\eta_a|=0.1$, $|\eta_b|=0.7$ and various values of m. For given l and k, the antibunching at fixed values of $|\eta_a|$ and $|\eta_b|$ becomes less pronounced with increasing m. Also, the TMPADS state may get antibunched starting from a certain value of r depending on l, k, m, but it always becomes bunched asymptotically in the limit of large r (i.e., $\lim_{r\to\infty}R(l,k)=0$ for any l, k and m).

Next, to get more insight into the higher-order antibunching effect, we examine $R(l,k)$ by varying the orders of l and k but keeping the same value of m. As an example, Figs. 11 and 12 display different $R(l,k)$ versus r with fixed $|\eta_a|$, $|\eta_b|$ and $\varphi=\pi$, for $m=1$ and $m=3$. It is interesting that neither l nor k alone decides the actual degree of antibunching. For a given k the antibunching gets stronger with increasing l (Fig. 11(a)), but for a given l the antibunching gets weaker with increasing k (Fig. 11(b)). These observations combined with the behavior in Fig. 12 with both l and k variable suggest that the difference $l-k$ plays a role: the TMPADS state would be more antibunched with a greater value of $l-k$ and this feature holds for all m.

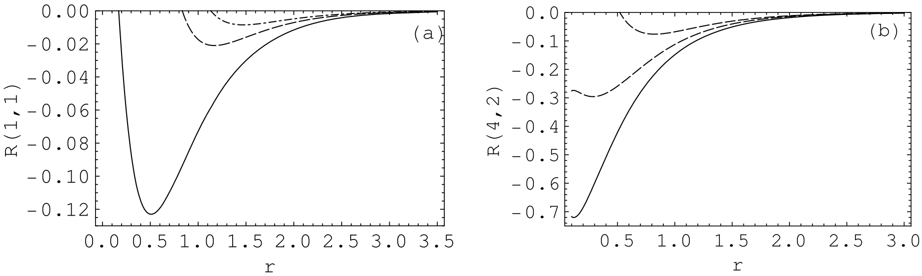


Fig. 10 The coefficients (**a**) $R(1,1)$ and (**b**) $R(4,2)$ as functions of r with $|\eta_a| = 0.1$, $|\eta_b| = 0.7$ and $\varphi = \pi$ for $m = 2$ (*the solid line*), $m = 4$ (*the dashed curve*) and $m = 6$ (*the dashed-dotted curve*)

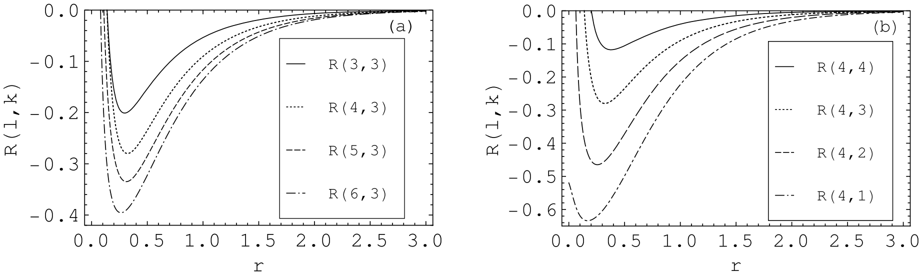
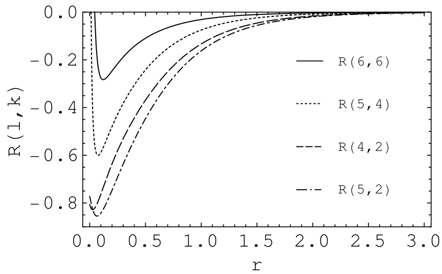


Fig. 11 The coefficient $R(l,k)$ as a function of r with $|\eta_a| = 0.1$, $|\eta_b| = 0.7$, $\varphi = \pi$, $m = 1$ for different values of l and k: (**a**) $k = 3$ and l changes from 3 to 6, (**b**) $l = 4$ and k changes from 1 to 4

Fig. 12 The coefficient $R(l,k)$ as a function of r with $|\eta_a| = 0.1$, $|\eta_b| = 0.7$, $\varphi = \pi$ and $m = 3$ for different values of l and k

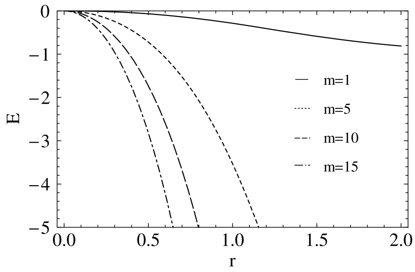


5 Intermodal entanglement

Very interesting in a two-mode quantum state is the possible entanglement between the modes since this can be exploited to perform a nonlocal job just by local operations combined with classical communication. In this section we examine whether our state (4) exhibits intermodal entanglement and, if it does, how much the entanglement is. For that purpose a certain criterion to detect entanglement is needed. Indeed, bipartite entanglement criteria in the form of inequalities were introduced in [31, 32] which are developed further by several authors [33–36] and were applied to detect entanglement of some two-mode systems [13, 37, 38]. Here we employ the criterion given by Hillery-Zubairy in Ref. [34], according to which the two modes of a two-mode state is mutually entangled if the following inequality

$$E = \langle N_a \rangle \langle N_b \rangle - |\langle ab \rangle|^2 < 0 \tag{24}$$

Fig. 13 The entanglement parameter E as a function of r with $|\eta_a| = 1$, $|\eta_b| = 0$, $\varphi_a = \pi/2$ and $\varphi_b = \theta = 0$ for $m = 1, 5, 10, 15$, respectively



is satisfied. For the TMPADS state (4), we have

$$\begin{aligned} E &= (m+1)\cosh^2 r \frac{L^0_{m+1}(\chi)}{L^0_m(\chi)}\left[\sinh^2 r + |\eta_b|^2 + m\sinh^2 r \frac{L^0_{m-1}(\chi)}{L^0_m(\chi)}\right] \\ &\quad - m\sinh^2 r \frac{L^0_{m-1}(\chi)}{L^0_m(\chi)} - |\eta_a\eta_b|^2\left(\frac{L^1_m(\chi)}{L^0_m(\chi)}\right)^2 - \sinh^2 r - |\eta_b|^2 \\ &\quad - \frac{1}{4}(m+1)^2\sinh^2 2r - (m+1)|\eta_a\eta_b|\sinh 2r\cos(\varphi_a+\varphi_b-\theta) \\ &\quad \times \left[\frac{L^0_{m+1}(\chi)L^1_{m-1}(\chi)}{(L^0_m(\chi))^2} - \frac{\mathrm{sech}^2 r}{m+1}\frac{L^1_{m-1}(\chi)}{L^0_m(\chi)} - \frac{L^1_m(\chi)}{L^0_m(\chi)}\right]. \end{aligned} \tag{25}$$

To see whether $E < 0$ we analyze it with regard to the parameters involved. Thus we plot E in Fig. 13 as a function of r with $|\eta_a| = 1$, $|\eta_b| = 0$, $\varphi_a = \pi/2$ and $\varphi_b = \theta = 0$ for several values of m. It is clear from Fig. 13 that E is negative for any values of r and m. More concretely, E gets more negative not only for a larger value of r but also for a greater value of m. The latter implies that adding more photons to a mode of a two-mode state may improve the degree of intermodal entanglement. In this context, we would like to mention a recent preprint [39] in spirit of which we plan to follow for our TMPADS.

6 Conclusion

We have investigated sum squeezing, difference squeezing and higher-order antibunching effects that may arise in the TMPADS state defined by Eq. (4). It has been shown that all these effects can coexist in certain domains of the parameters. The behaviors of sum squeezing and difference squeezing are common in some aspects. Both kinds of squeezing are most favorable under the angular conditions (14) and (19), which turn out identical in terms of actually involved angles, i.e., $\theta - \varphi_a - \varphi_b = 2k\pi$, with $k = 0, \pm 1, \pm 2, \ldots$. Their dependences on $|\eta_a|$ and $|\eta_b|$ are similar qualitatively: when $|\eta_b|$ ($|\eta_a|$) is fixed, both S and D are negative in a restricted (an extended) domain of $|\eta_a|$ ($|\eta_b|$) and the squeezing degree increases with the number of added photons m. For their dependences on r, both kinds of squeezing exist only for small values of r, but sum squeezing degree does not simply increase with m, while difference squeezing degree does. As for higher-order antibunching, it is most favorable under a different angular condition (23), which in terms of actually involved angles reads $\theta - \varphi_a - \varphi_b = 2(k+1)\pi$, with $k = 0, \pm 1, \pm 2, \ldots$. That is to say, when the sum and difference squeezing is most pronounced, the antibunching is least pronounced and vice versa. Also, it is worth noting that the degree of higher-order antibunching decreases with the number of added photons, as opposed to the degree of sum and difference squeezing. An interesting feature we have found out is that the degree of higher-order antibunching is determined not by l and k separately, but rather by their difference $l-k$. Quantitatively, the greater the difference $l-k$ the more antibunched the TMPADS state is. Moreover, the more antibunched the TMPADS state is the more nonclassical the TMPADS state has. Generally, it would mean that such higher-order criteria can be more suitable to detect weak nonclassicalities, a fact to justify the relevance of studying higher-order nonclassical properties [28, 29]. Finally, we used a simple entanglement criterion (24) proposed by Hillery-Zubairy to show that the intermodal entanglement of the TMPADS state exists for the whole range of the parameters r and m: its degree increases with r and/or m. Last but not least, let us briefly describe the important issue of how to generate the quantum state of our concern, i.e., the state (4). While the squeeze operator S_{ab} is available by means of an optical nondegenerate parametric downconverter, the displacement operator $D_{ab}=D_aD_b$ can be realized quite easily by a beam-splitter with high transmittivity and a strong coherent beam. However, the most challenging task is how to add photons to the squeezed displaced state, that is, how to implement the action of $a^{\dagger m}$ on the state $D_{ab}S_{ab}|0_a,0_b\rangle$. Fortunately enough, techniques have been found recently that allow to accomplish in the lab not only addition of photons to but also substraction of photons from a given state by means of beam-splitters and photodetectors (for details see, e.g., a good overview [40] and the references therein).

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