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THE D_4 FLAVOR SYMMETRY IN 3-3-1 MODEL WITH NEUTRAL LEPTONS

V. V. VIEN

Department of Physics, Tay Nguyen University,
567 Le Duan, Buon Ma Thuot, Vietnam
wvienk16@gmail.com

H. N. LONG

Institute of Physics, VAST, 10 Dao Tan, Ba Dinh, Hanoi, Vietnam
hnlong@iop.vast.ac.vn

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We construct a D_4 flavor model based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ gauge symmetry responsible for fermion masses and mixings. The neutrinos get small masses from antisextets which are in a singlet and a doublet under D_4. If the D_4 symmetry is violated as perturbation by a Higgs triplet under $SU(3)_L$ and lying in $\underline{1}'''$ of D_4, the corresponding neutrino mass mixing matrix gets the most general form. In this case, the model can fit the experimental data in 2012 on neutrino masses and mixing. Our results show that the neutrino masses are naturally small and a little deviation from the tribimaximal neutrino mixing form can be realized. The quark masses and mixing matrix are also discussed. In the model under consideration, the CKM matrix can be different from the unit matrix. The scalar potential of the model is more simpler than those of the model based on S_3 and S_4. Assignation of VEVs to antisextets leads to the mixing of the new gauge bosons and those in the Standard Model. The mixing in the charged gauge bosons as well as the neutral gauge boson is considered.

Keywords: Neutrino mass and mixing; nonstandard-model neutrinos; right-handed neutrinos; extensions of electroweak Higgs sector; charge conjugation; parity; time reversal; other discrete symmetries.

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1. Introduction

Following the discovery of neutrino oscillations, there has been a considerable progress in determining values for neutrino mass square differences $m_i^2 - m_j^2$ and the mixing angles relating mass eigenstates to flavor eigenstates. The most recent fits suggest that one of the mixing angles is approximately zero and another has a value that implies a mass eigenstate that is nearly an equal mixture of ν_μ and ν_τ. The data in PDG2012[1–5] imply:

$$\begin{aligned}
\sin^2(2\theta_{12}) &= 0.857 \pm 0.024 \quad (t_{12} \simeq 0.6717)\,,\\
\sin^2(2\theta_{13}) &= 0.098 \pm 0.013 \quad (s_{13} \simeq 0.1585)\,,\\
\sin^2(2\theta_{23}) &> 0.95\,,\\
\Delta m^2_{21} &= (7.50 \pm 0.20) \times 10^{-5}\ \mathrm{eV}^2\,,\\
\Delta m^2_{32} &= (2.32^{+0.12}_{-0.08}) \times 10^{-3}\ \mathrm{eV}^2\,.
\end{aligned} \tag{1}$$

These large neutrino mixing angles are completely different from the quark mixing ones defined by the Cabibbo–Kobayashi–Maskawa (CKM) matrix.[6,7] This has stimulated work on flavor symmetries and non-Abelian discrete symmetries are considered to be the most attractive candidate to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and lepton. There are many recent models based on the non-Abelian discrete symmetries, such as A_4,[8–25] A_5,[26–38] S_3,[39–80] S_4,[81–109] D_4,[110–121] D_5,[122,123] T',[124–128] and so forth. An alternative extension of the Standard Model (SM) is the 3-3-1 models, in which the SM gauge group $SU(2)_L \otimes U(1)_Y$ is extended to $SU(3)_L \otimes U(1)_X$, has been investigated in Refs. 129–142. The anomaly cancelation and the QCD asymptotic freedom in the models require that the number of fermion families is 3, and one family of quarks has to transform under $SU(3)_L$ differently from the other two. In our previous works,[143–145] the discrete symmetries have been explored to the 3-3-1 models. The simplest explanation is probably due to a S_3 flavor symmetry which is the smallest non-Abelian discrete group, has been explored in our previous work.[145] In Ref. 144, we have studied the 3-3-1 model with neutral fermions based on S_4 group, in which most of the Higgs multiplets are in triplets under S_4 except χ lying in a singlet, and the exact tribimaximal form[146–149] is obtained, where $\theta_{13} = 0$.

As we know, the recent considerations have implied $\theta_{13} \neq 0$,[8–25,39–109] but small as given in (1). This problem has been improved in Ref. 145 by adding a new triplet ρ and another antisextet s', in which s' is regarded as a small perturbation. Therefore, the model contains up to eight Higgs multiplets, and the scalar potential of the model is quite complicated.

In this paper, we investigate another choice with D_4, the group of a square, which is the second smallest non-Abelian discrete symmetry. D_4 contains one doublet irreducible representation and four singlets. This feature is useful to separate the third family of fermions from the others. The group contains a $\underline{2}$ irreducible representation which can connect two maximally mixed generations. Besides the facilitating maximal mixing through $\underline{2}$, it provides four inequivalent singlet representations $\underline{1}$, $\underline{1}'$, $\underline{1}''$ and $\underline{1}'''$ which play a crucial role in consistently reproducing fermion masses and mixing as a perturbation. We will point out that this model is simpler than the S_3 one, since fewer Higgs multiplets are needed in order to allow the fermions to gain masses and to break symmetries and the physics we will see is different from the former. On the other hand, the neutrino sector is more simpler than that of S_3 one. The boson masses and mixings are considered more generally and more detail than those in Ref. 150.

There are two typical variants of the 3-3-1 models as far as lepton sectors are concerned. In the minimal version, three $SU(3)_L$ lepton triplets are (ν_L, l_L, l^c_R), where l_R are ordinary right-handed charged-leptons.[129–133] In the second version, the third components of lepton triplets are the right-handed neutrinos, (ν_L, l_L, ν^c_R).[134–138] To have a model with the realistic neutrino mixing matrix, we should consider another variant of the form (ν_L, l_L, N^c_R) where N_R are three new fermion singlets under standard model symmetry with vanishing lepton-numbers.[143,144]

The rest of this paper is organized as follows. In Secs. 2 and 3, we present the necessary elements of the 3-3-1 model with the D_4 symmetry as well as introducing necessary Higgs fields responsible for the charged lepton masses. In Sec. 4, we discuss on quark sector. Section 5 is devoted for the neutrino mass and mixing. In Sec. 6, we consider the Higgs potential and minimization conditions. Section 7 is devoted for the gauge boson mass and mixing. We summarize our results and make conclusions in Sec. 8. Appendix A presents a brief of the D_4 theory. Appendix B provides the lepton number (L) and lepton parity (P_l) of particles in the model.

2. Fermion Content

The gauge symmetry is based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$, where the electroweak factor $SU(3)_L \otimes U(1)_X$ is extended from those of the SM where the strong interaction sector is retained. Each lepton family includes a new electrically- and leptonically-neutral fermion (N_R) and is arranged under the $SU(3)_L$ symmetry as a triplet (ν_L, l_L, N^c_R) and a singlet l_R. The residual electric charge operator Q is therefore related to the generators of the gauge symmetry by

$$Q = T_3 - \frac{1}{\sqrt{3}} T_8 + X\,,$$

where T_a $(a = 1, 2, \ldots, 8)$ are $SU(3)_L$ charges with $\mathrm{Tr}\, T_a T_b = \frac{1}{2}\delta_{ab}$ and X is the $U(1)_X$ charge. This means that the model under consideration does not contain exotic electric charges in the fundamental fermion, scalar and adjoint gauge boson representations.

Since the particles in the lepton triplet have different lepton number (1 and 0), so the lepton number in the model does not commute with the gauge symmetry unlike the SM. Therefore, it is better to work with a new conserved charge $\mathcal{L}$ commuting with the gauge symmetry and related to the ordinary lepton number by diagonal matrices[143,144,151]

$$L = \frac{2}{\sqrt{3}} T_8 + \mathcal{L}\,.$$

The lepton charge arranged in this way (i.e. $L(N_R)=0$ as assumed) is in order to prevent unwanted interactions due to $U(1)_\mathcal{L}$ symmetry and breaking (due to the lepton parity as shown below) to obtain the consistent lepton and quark spectra. By this embedding, exotic quarks U, D as well as new non-Hermitian gauge bosons X^0, $Y^\pm$ possess lepton charges as of the ordinary leptons: $L(D)=-L(U)=L(X^0)=L(Y^-)=1$. A brief of the theory of D_4 group is given in App. A. The D_4 contains one doublet irreducible representation $\underline{2}$ and four singlets $\underline{1}$, $\underline{1}'$, $\underline{1}''$ and $\underline{1}'''$. In this paper we work in real basic, in which the two-dimensional representation $\underline{2}$ of D_4 is real, $2^*(1^*,2^*)=2(1^*,2^*)$, and

$$\underline{2}\,(1,2)\otimes\underline{2}\,(1,2)=\underline{1}(11+22)\oplus\underline{1}'(11-22)\oplus\underline{1}''(12+21)\oplus\underline{1}'''(12-21)\,. \quad (2)$$

In the model under consideration, we put the first family of leptons in singlets $\underline{1}$ of D_4, while the two other families are in the doublets $\underline{2}$. Under the $[SU(3)_L, U(1)_X, U(1)_\mathcal{L}, \underline{D}_4]$ symmetries as proposed, the fermions of the model transform as follows

$$\begin{aligned}
&\psi_{1L}\equiv(\nu_{1L}\quad l_{1L}\quad N^c_{1R})^T\sim[3,-1/3,2/3,\underline{1}]\,,\\
&\psi_{iL}=(\nu_{iL}\quad l_{iL}\quad N^c_{iR})^T\sim[3,-1/3,2/3,\underline{2}]\,,\\
&l_{1R}\sim[1,-1,1,\underline{1}]\,,\quad l_{iR}\sim[1,-1,1,\underline{2}]\quad(i=2,3)\,,\\
&Q_{3L}=(u_{3L}\quad d_{3L}\quad U_L)^T\sim[3,1/3,-1/3,\underline{1}]\,,\\
&Q_{\alpha L}\equiv(d_{\alpha L}\quad -u_{\alpha L}\quad D_{\alpha L})^T\sim[3^*,0,1/3,\underline{2}]\,,\\
&u_{3R}\sim[1,2/3,0,\underline{1}]\,,\quad u_{\alpha R}\sim[1,2/3,0,\underline{2}]\,,\\
&d_{3R}\sim[1,-1/3,0,\underline{1}]\,,\quad d_{\alpha R}\sim[1,-1/3,0,\underline{2}]\,,\\
&U_R\sim[1,2/3,-1,\underline{1}]\,,\quad D_{\alpha R}\sim[1,-1/3,1,\underline{2}]\quad(\alpha=1,2)\,,
\end{aligned}$$

where the subscript numbers on field indicate to respective families which also in order define components of their D_4 multiplets. In the following, we consider possibilities of generating the masses for the fermions. The scalar multiplets needed for the purpose are also introduced.

3. Charged Lepton Masses

To generate masses for charged leptons, we need a minimum of five $SU(3)_L$ Higgs triplets lying in $\underline{1}$, $\underline{1}'$, $\underline{1}''$, $\underline{1}'''$ and $\underline{2}$. In decomposing of $\underline{2}\otimes\underline{2}$ into irreducible representations, there is no $\underline{2}$ one. So, it is required two Higgs scalars

$$\phi=\begin{pmatrix}\phi_1^+\\ \phi_2^0\\ \phi_3^+\end{pmatrix}\sim[3,2/3,-1/3,\underline{1}]\,,\quad \phi'=\begin{pmatrix}\phi_1'^+\\ \phi_2'^0\\ \phi_3'^+\end{pmatrix}\sim[3,2/3,-1/3,\underline{1}']\,, \quad (3)$$

with VEVs as follows:

$$\langle\phi\rangle = (0 \quad v \quad 0)^T\,, \quad \langle\phi'\rangle = (0 \quad v' \quad 0)^T\,. \tag{4}$$

The Yukawa interactions are

$$\begin{aligned}-\mathcal{L}_l &= h_1(\bar{\psi}_{1L}\phi)_{\underline{1}}\,l_{1R} + h_2(\bar{\psi}_{iL}\phi)_{\underline{2}}\,l_{iR} + h_3(\bar{\psi}_{iL}\phi')_{\underline{2}}\,l_{iR} + \text{h.c.}\\ &= h_1(\bar{\psi}_{1L}\phi)_{\underline{1}}\,l_{1R} + h_2(\bar{\psi}_{2L}\phi l_{2R} + \bar{\psi}_{3L}\phi l_{3R})\\ &\quad + h_3(\bar{\psi}_{2L}\phi' l_{2R} - \bar{\psi}_{3L}\phi' l_{3R}) + \text{h.c.}\end{aligned}$$

The mass Lagrangian of the charged leptons reads

$$-\mathcal{L}_l^{\text{mass}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L})M_l(l_{1R}, l_{2R}, l_{3R})^T + \text{h.c.}\,,$$

$$M_l = \begin{pmatrix} h_1 v & 0 & 0\\ 0 & h_2 v + h_3 v' & 0\\ 0 & 0 & h_2 v - h_3 v'\end{pmatrix} = \begin{pmatrix} m_e & 0 & 0\\ 0 & m_\mu & 0\\ 0 & 0 & m_\tau\end{pmatrix}.$$

It is then diagonalized, and

$$U_{eL}^+ = U_{eR} = I\,.$$

This means that the charged leptons $l_{1,2,3}$ by themselves are the physical mass eigenstates, and the lepton mixing matrix depends on only that of the neutrinos that will be studied in Sec. 5.

We see that the masses of muon and tauon are separated by the ϕ' triplet. This is the reason why we introduce ϕ' in addition to ϕ. The charged lepton Yukawa couplings $h_{1,2,3}$ relate to their masses as follows:

$$h_1 v = m_e\,, \quad 2h_2 v = m_\tau + m_\mu\,, \quad 2h_3 v' = m_\mu - m_\tau\,. \tag{5}$$

The experimental values for masses of the charged leptons at the weak scale are given as:[152,153]

$$m_e = 0.511\text{ MeV}\,, \quad m_\mu = 106.0\text{ MeV}\,, \quad m_\tau = 1.77\text{ GeV}\,. \tag{6}$$

Thus, we get

$$h_1 v = 0.511\text{ MeV}\,, \quad h_2 v = 938\text{ MeV}\,, \quad |h_3 v'| = 832\text{ MeV}\,. \tag{7}$$

It follows that if v' and v are of the same order of magnitude, $h_1 \ll h_2$ and $h_2 \sim |h_3|$.

4. Quark Masses

To generate masses for quarks with a minimal Higgs content, we additionally introduce the following Higgs triplets

$$\begin{aligned}\chi &= \left(\chi_1^0 \quad \chi_2^- \quad \chi_3^0\right)^T \sim [3, -1/3, 2/3, \underline{1}]\,,\\ \eta &= \left(\eta_1^0 \quad \eta_2^- \quad \eta_3^0\right)^T \sim [3, -1/3, -1/3, \underline{1}]\,,\\ \eta' &= \left(\eta_1'^0 \quad \eta_2'^- \quad \eta_3'^0\right)^T \sim [3, -1/3, -1/3, \underline{1}']\,.\end{aligned} \tag{8}$$

The Yukawa interactions are:

$$
\begin{aligned}
-\mathcal{L}_q &= f_3(\bar{Q}_{3L}U_R)_{\underline{1}}\chi + f(\bar{Q}_{\alpha L}D_{\alpha R})_{\underline{1}}\chi^* \\
&\quad + h_3^d(\bar{Q}_{3L}d_{3R})_{\underline{1}}\phi + h^d(\bar{Q}_{\alpha L}d_{\alpha R})_{\underline{1}}\eta^* \\
&\quad + h'^d(\bar{Q}_{\alpha L}d_{\alpha R})_{\underline{1}'}\eta'^* + h_3^u(\bar{Q}_{3L}u_{3R})_{\underline{1}}\eta \\
&\quad + h^u(\bar{Q}_{\alpha L}u_{\alpha R})_{\underline{1}}\phi^* + h'^u(\bar{Q}_{\alpha L}u_{\alpha R})_{\underline{1}'}\phi'^* + \text{h.c.} \\
&= f_3(\bar{Q}_{3L}U_R)_{\underline{1}}\chi + f\left(\bar{Q}_{1L}D_{1R} + \bar{Q}_{2L}D_{2R}\right)\chi^* \\
&\quad + h_3^d(\bar{Q}_{3L}d_{3R})_{\underline{1}}\phi + h^d\left(\bar{Q}_{1L}d_{1R} + \bar{Q}_{2L}d_{2R}\right)\eta^* \\
&\quad + h'^d\left(\bar{Q}_{1L}d_{1R} - \bar{Q}_{2L}d_{2R}\right)\eta'^* + h_3^u(\bar{Q}_{3L}u_{3R})_{\underline{1}}\eta \\
&\quad + h^u\left(\bar{Q}_{1L}u_{1R} + \bar{Q}_{2L}u_{2R}\right)\phi^* \\
&\quad + h'^u(\bar{Q}_{1L}u_{1R} - \bar{Q}_{2L}u_{2R})\phi'^* + \text{h.c.}
\end{aligned} \tag{9}
$$

We now introduce a residual symmetry of lepton number $P_l \equiv (-1)^L$, called "lepton parity,"[143,145] in order to suppress the mixing between ordinary quarks and exotic quarks (for lepton number of the model particles, see App. B). The particles with even parity ($P_l = 1$) have $L = 0, \pm 2$ and the particles with odd parity ($P_l = -1$) have $L = \pm 1$. In this framework we assume that the lepton parity is an exact symmetry, not spontaneously broken. This means that due to the lepton parity conservation, the fields carrying lepton number ($L = \pm 1$) η_3, η'_3 and χ_1 cannot develop VEV. Suppose that the VEVs of χ, η and η' are

$$
\langle\chi\rangle = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \quad \langle\eta\rangle = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}, \quad \langle\eta'\rangle = \begin{pmatrix} u' \\ 0 \\ 0 \end{pmatrix}, \tag{10}
$$

then the exotic quarks get masses

$$
m_U = f_3 w\,, \quad m_{D_{1,2}} = f w\,, \tag{11}
$$

and the mass Lagrangian of the ordinary quarks reads:

$$
\begin{aligned}
-\mathcal{L}_q^{\text{mass}} &= h_3^d v \bar{d}_{3L}d_{3R} + h^d u\left(\bar{d}_{1L}d_{1R} + \bar{d}_{2L}d_{2R}\right) \\
&\quad + h'^d u'\left(\bar{d}_{1L}d_{1R} - \bar{d}_{2L}d_{2R}\right) \\
&\quad + h_3^u u \bar{u}_{3L}u_{3R} - h^u v\left(\bar{u}_{1L}u_{1R} + \bar{u}_{2L}u_{2R}\right) \\
&\quad - h'^u v'\left(\bar{u}_{1L}u_{1R} - \bar{u}_{2L}u_{2R}\right) + \text{h.c.} \\
&= (\bar{u}_{1L}, \bar{u}_{2L}, \bar{u}_{3L}) M_u (u_{1R}, u_{2R}, u_{3R})^T \\
&\quad + (\bar{d}_{1L}, \bar{d}_{2L}, \bar{d}_{3L}) M_d (d_{1R}, d_{2R}, d_{3R})^T + \text{h.c.}
\end{aligned} \tag{12}
$$

From (12), the mass matrices for the ordinary up-quarks and down-quarks are, respectively, obtained as follows:

$$\begin{aligned} M_u &= \begin{pmatrix} -h^u v - h'^u v' & 0 & 0 \\ 0 & -h^u v + h'^u v' & 0 \\ 0 & 0 & h_3^u u \end{pmatrix} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \\ M_d &= \begin{pmatrix} h^d u + h'^d u' & 0 & 0 \\ 0 & h^d u - h'^d u' & 0 \\ 0 & 0 & h_3^d v \end{pmatrix} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \end{aligned} \tag{13}$$

In similarity to the charged leptons, the masses of u and c quarks are also separated by the ϕ' scalar. We see also that the introduction of η' in addition to η is necessary to provide the different masses for d and s quarks. The expression (13) leads to the relations:

$$\begin{aligned} h_3^u u &= m_t\,, \quad -2h^u v = m_u + m_c\,, \quad -2h'^u v' = m_u - m_c\,, \\ h_3^d v &= m_b\,, \quad\ \ 2h^d u = m_d + m_s\,, \quad\ \ 2h'^d u' = m_d - m_s\,. \end{aligned}$$

The current mass values for the quarks are given by[152,153]

$$\begin{aligned} m_u &= (1.8 \div 3.0)\ \text{MeV}\,, & m_d &= (4.5 \div 5.5)\ \text{MeV}\,, \\ m_c &= (1.25 \div 1.30)\ \text{GeV}\,, & m_s &= (90.0 \div 100.0)\ \text{MeV}\,, \\ m_t &= (172.1 \div 174.9)\ \text{GeV}\,, & m_b &= (4.13 \div 4.37)\ \text{GeV}\,. \end{aligned} \tag{14}$$

Hence,

$$\begin{aligned} h_3^u u &= (172.1 \div 174.9)\ \text{GeV}\,, & h_3^d v &= (4.13 \div 4.37)\ \text{GeV}\,, \\ |h^u v| &= (625.9 \div 651.5)\ \text{MeV}\,, & h^d u &= (47.25 \div 52.75)\ \text{MeV}\,, \\ |h'^d u'| &= (42.75 \div 47.25)\ \text{MeV}\,, & h'^u v' &= (624.1 \div 648.5)\ \text{MeV}\,. \end{aligned} \tag{15}$$

It is obvious that if $|u| \sim |v| \sim |v'| \sim |u'|$, the Yukawa coupling hierarchies are $|h'^d| \sim h^d \ll h^u \sim h'^u \ll h_3^u, h_3^d$, and the couplings between up-quarks (h^u, h'^u, h_3^u) and Higgs scalar multiplets are slightly heavier than those of down-quarks (h^d, h'^d, h_3^d), respectively.

The unitary matrices, which couple the left-handed up and down-quarks to those in the mass bases, are $U_L^u = 1$ and $U_L^d = 1$, respectively. Therefore, we get the CKM matrix

$$U_{\text{CKM}} = U_L^{d\dagger} U_L^u = 1\,. \tag{16}$$

This is a good approximation for the realistic quark mixing matrix, which implies that the mixings among the quarks are dynamically small. The small permutations such as a breaking of the lepton parity due to the odd VEVs $\langle \eta_3^0 \rangle$, $\langle \eta_3'^0 \rangle$, $\langle \chi_1^0 \rangle$, or a violation of $\mathcal{L}$ and/or D_4 symmetry due to unnormal Yukawa interactions, namely $\bar{Q}_{3L}\chi u_{3R}$, $\bar{Q}_{\alpha L}\chi^* d_{\alpha R}$, $\bar{Q}_{3L}\chi u_{\alpha R}$, $\bar{Q}_{\alpha L}\chi^* d_{3R}$ and so forth, will disturb the tree level matrix resulting in mixing between ordinary and exotic quarks and possibly providing the desirable quark mixing pattern. This also leads to the flavor changing neutral current at the tree level but strongly suppressed.[143,144]

Note that $\bar{Q}_{\alpha L}d_{\alpha R}$ and $\bar{Q}_{\alpha L}u_{\alpha R}$ transform as $1 \oplus 1' \oplus 1'' \oplus 1'''$ under D_4. All terms of the Yukawa interactions responsible for quarks masses in (9) are invariant under the $[SU(3)_L, U(1)_X, U(1)_{\mathcal{L}}, \underline{D}_4]$ symmetries. If $\bar{Q}_{\alpha L}d_{\alpha R}$ and $\bar{Q}_{\alpha L}u_{\alpha R}$ lying in $1''$ and/or $1'''$, the 1–2 mixing of ordinary quarks will take place. In this work, we add soft terms which violate D_4 symmetry with $1''$. Hence, the total Lagrangian of the ordinary quarks is added two extra terms $-\Delta\mathcal{L}_q^d$ and $-\Delta\mathcal{L}_q^u$, given by

$$\begin{aligned} -\Delta\mathcal{L}_q^d &= k^d(\bar{Q}_{\alpha L}d_{\alpha R})_{\underline{1}''}\eta^* + \text{h.c.} \\ &= k^d u\bar{d}_{1L}d_{2R} + k^d u\bar{d}_{2L}d_{1R} + \text{h.c.}\,, \end{aligned} \tag{17}$$

$$\begin{aligned} -\Delta\mathcal{L}_q^u &= k^u(\bar{Q}_{\alpha L}u_{\alpha R})_{\underline{1}''}\phi^* \\ &= -k^u v\bar{u}_{1L}u_{2R} - k^u v\bar{u}_{2L}d_{1R} + \text{h.c.} \end{aligned} \tag{18}$$

The total mass matrices for the ordinary up-quarks and down-quarks then take the form:

$$M'_u = M_u + \Delta M_u = \begin{pmatrix} -h^u v - h'^u v' & -k^u v & 0 \\ -k^u v & -h^u v + h'^u v' & 0 \\ 0 & 0 & h_3^u u \end{pmatrix}, \tag{19}$$

$$M'_d = M_d + \Delta M_d = \begin{pmatrix} h^d u + h'^d u' & k^d u & 0 \\ k^d u & h^d u - h'^d u' & 0 \\ 0 & 0 & h_3^d v \end{pmatrix}. \tag{20}$$

The M'_u in (19) is diagonalized as

$$V_L^{u+}M'_u V_R^u = \text{diag}\big(m'_u, m'_c, m'_t\big)\,,$$

where

$$\begin{aligned} m'_u &= \frac{[(k^u)^2 - (h^u)^2]v^2 + (h'^u)^2 v'^2}{\sqrt{(k^u)^2v^2 + (h^u v - h'^u v')^2}}\,, \\ m'_c &= \frac{[(k^u)^2 - (h^u)^2]v^2 + (h'^u)^2 v'^2}{\sqrt{(k^u)^2v^2 + (h^u v + h'^u v')^2}}\,, \\ m'_t &\equiv m_t = h_3^u u \end{aligned} \tag{21}$$

and

$$V_R^u = 1\,, \quad V_L^u = \begin{pmatrix} \frac{h^u v - h'^u v'}{\sqrt{(k^u)^2v^2 + (h^u v - h'^u v')^2}} & -\frac{k^u v}{\sqrt{(k^u)^2v^2 + (h^u v + h'^u v')^2}} & 0 \\ -\frac{k^u v}{\sqrt{(k^u)^2v^2 + (h^u v + h'^u v')^2}} & \frac{h^u v + h'^u v'}{\sqrt{(k^u)^2v^2 + (h^u v + h'^u v')^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The terms in (18) and (17) violate the D_4 symmetry, therefore they should be much weaker than those in (9). This means that

$$k^u \ll h^u,\ h'^u\,, \tag{22}$$

$$k^d \ll h^d,\ h'^d\,. \tag{23}$$

From condition (22), it follows that

$$\mathcal{P} \equiv \frac{2h^u k^u v^2}{\sqrt{(k^u)^2 v^2 + (h^u v - h'^u v')^2}\sqrt{(k^u)^2 v^2 + (h^u v + h'^u v')^2}}$$

is very small, and

$$V_L^{u+} V_L^u = \begin{pmatrix} 1 & -\mathcal{P} & 0 \\ -\mathcal{P} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq I\,.$$

Similarly, the M'_d in (20) is diagonalized as

$$V_L^{d+} M'_d V_R^d = \mathrm{diag}\big(m'_d, m'_s, m'_b\big)\,,$$

where

$$\begin{aligned} m'_d &= \frac{[(k^d)^2 - (h^d)^2]u^2 + (h'^d)^2 u'^2}{\sqrt{(k^d)^2 u^2 + (h^d u - h'^d u')^2}}\,, \\ m'_s &= \frac{[(k^d)^2 - (h^d)^2]u^2 + (h'^d)^2 u'^2}{\sqrt{(k^d)^2 u^2 + (h^d u + h'^d u')^2}}\,, \\ m'_b &\equiv m_b = h_3^d v\,, \end{aligned} \tag{24}$$

and

$$V_R^d = 1\,, \quad V_L^d = \begin{pmatrix} \frac{-(h^d u - h'^d u')}{\sqrt{(k^d)^2 u^2 + (h^d u - h'^d u')^2}} & -\frac{k^d u}{\sqrt{(k^d)^2 u^2 + (h^d u + h'^d u')^2}} & 0 \\ \frac{k^d u}{\sqrt{(k^d)^2 u^2 + (h^d u - h'^d u')^2}} & \frac{h^d u + h'^d u'}{\sqrt{(k^d)^2 u^2 + (h^d u + h'^d u')^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

Analogously, from condition (23), we see that the value defined as

$$\mathcal{K} \equiv \frac{2h^d k^d u^2}{\sqrt{(k^d)^2 u^2 + (h^d u - h'^d u')^2}\sqrt{(k^d)^2 u^2 + (h^d u + h'^d u')^2}}$$

is very small, and

$$V_L^{d+} V_L^d = \begin{pmatrix} 1 & \mathcal{K} & 0 \\ \mathcal{K} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq I\,.$$

The CKM matrix then takes the form:

$$V_{\rm CKM} = V_L^{u+} V_L^d = \begin{pmatrix} V_{11} & V_{12} & 0 \\ V_{21} & V_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{26}$$

where

$$V_{11} = \frac{-(h^u h^d + k^u k^d) uv + h^u h'^d u' v + h'^u h^d u v' - h'^u h'^d u' v'}{\sqrt{(k^d)^2 u^2 + (h^d u - h'^d u')^2}\sqrt{(k^u)^2 v^2 + (h^u v - h'^u v')^2}},$$

$$V_{12} = \frac{-(h^u k^d + k^u h^d) uv - k^u h'^d u' v + h'^u k^d u v'}{\sqrt{(k^d)^2 u^2 + (h^d u + h'^d u')^2}\sqrt{(k^u)^2 v^2 + (h^u v - h'^u v')^2}},$$

$$V_{21} = \frac{(h^u k^d + k^u h^d) uv - k^u h'^d u' v + h'^u k^d u v'}{\sqrt{(k^d)^2 u^2 + (h^d u - h'^d u')^2}\sqrt{(k^u)^2 v^2 + (h^u v + h'^u v')^2}},$$

$$V_{22} = \frac{k^u k^d uv + h^d u (h^u v + h'^u v') + h'^d u' (h^u v + h'^u v')}{\sqrt{(k^d)^2 u^2 + (h^d u + h'^d u')^2}\sqrt{(k^u)^2 v^2 + (h^u v + h'^u v')^2}}.$$

With the help of conditions (22) and (23) we have:

$$V_{11} \simeq 1, \quad V_{12} \simeq 0, \quad V_{21} \simeq 0, \quad V_{22} \simeq 1,$$

and the $V_{\rm CKM}$ in (26) becomes

$$V_{\rm CKM} \simeq I.$$

If $SU(3)_L$ Higgs triplet ϕ in (3) lying in $\underline{2}$ under D_4, the 1–3 and 2–3 mixings of the ordinary quarks will take place. A detailed study on these problems are out of the scope of this work and should be skipped.

5. Neutrino Mass and Mixing

The neutrino masses arise from the couplings of $\bar{\psi}^c_{iL}\psi_{iL}$, $\bar{\psi}^c_{1L}\psi_{1L}$ and $\bar{\psi}^c_{1L}\psi_{iL}$ to scalars, where $\bar{\psi}^c_{iL}\psi_{iL}$ transforms as $3^* \oplus 6$ under $SU(3)_L$ and $\underline{1} \oplus \underline{1}' \oplus \underline{1}'' \oplus \underline{1}'''$ under D_4; $\bar{\psi}^c_{1L}\psi_{1L}$ transforms as $3^* \oplus 6$ under $SU(3)_L$ and $\underline{1}$ under D_4, and $\bar{\psi}^c_{1L}\psi_{iL}$ transforms as $3^* \oplus 6$ under $SU(3)_L$ and $\underline{2}$ under D_4. For the known scalar triplets $(\phi, \phi', \chi, \eta, \eta')$, the available interactions are only $(\bar{\psi}^c_{iL}\psi_{iL})\phi$ and $(\bar{\psi}^c_{iL}\psi_{iL})\phi'$, but explicitly suppressed because of the $\mathcal{L}$-symmetry. We will therefore propose new SU$(3)_L$ antisextets, lying in either $\underline{1}$, $\underline{1}'$, $\underline{1}''$ or $\underline{1}'''$ under D_4, interact with $\bar{\psi}^c_L\psi_L$ to produce masses for the neutrino. To obtain a realistic neutrino spectrum, the antisextets transform as follows

$$\sigma = \begin{pmatrix} \sigma^0_{11} & \sigma^+_{12} & \sigma^0_{13} \\ \sigma^+_{12} & \sigma^{++}_{22} & \sigma^+_{23} \\ \sigma^0_{13} & \sigma^+_{23} & \sigma^0_{33} \end{pmatrix} \sim [6^*, 2/3, -4/3, \underline{1}],$$

$$s_k = \begin{pmatrix} s^0_{11} & s^+_{12} & s^0_{13} \\ s^+_{12} & s^{++}_{22} & s^+_{23} \\ s^0_{13} & s^+_{23} & s^0_{33} \end{pmatrix}_k \sim [6^*, 2/3, -4/3, \underline{2}]\,,$$

where the numbered subscripts on the component scalars are the $SU(3)_L$ indices, whereas $k = 1, 2$ is that of D_4. The VEV of s and σ is set as $(\langle s_1\rangle, \langle s_2\rangle)$ under D_4, in which

$$\langle\sigma\rangle = \begin{pmatrix} \lambda_\sigma & 0 & v_\sigma \\ 0 & 0 & 0 \\ v_\sigma & 0 & \Lambda_\sigma \end{pmatrix}, \tag{27}$$

$$\langle s_k\rangle = \begin{pmatrix} \lambda_k & 0 & v_k \\ 0 & 0 & 0 \\ v_k & 0 & \Lambda_k \end{pmatrix}. \tag{28}$$

Following the potential minimization conditions, we have several VEV alignments. The first alignment is that $\langle s_1\rangle = \langle s_2\rangle$ or $\langle s_1\rangle \neq 0 = \langle s_2\rangle$ or $\langle s_1\rangle = 0 \neq \langle s_2\rangle$ then the D_4 *is broken into* Z_2 *that consists of the elements* $\{e, a^3b\}$ *or* $\{e, b\}$ *or* $\{e, a^2b\}$, *respectively.* The second one is that $0 \neq \langle s_1\rangle \neq \langle s_2\rangle \neq 0$, then the D_4 is broken into {identity} (or $Z_2 \to$ {identity}). In this work, we impose the first case in the first alignment of D_4 breaking, i.e.

$$\lambda_1 = \lambda_2 \equiv \lambda_s\,, \quad v_1 = v_2 \equiv v_s\,, \quad \Lambda_1 = \Lambda_2 \equiv \Lambda_s\,.$$

And, we additionally introduce another scalar triplet lying in either $\underline{1}', \underline{1}''$ or $\underline{1}'''$ responsible for breaking the Z_2 subgroup as the second stage of D_4 breaking. This can be achieved by introducing a new $SU(3)_L$ triplet, ρ lying in $\underline{1}'''$ as follows

$$\rho = \begin{pmatrix} \rho^+_1 \\ \rho^0_2 \\ \rho^+_3 \end{pmatrix} \sim [3, 2/3, -4/3, \underline{1}''']\,, \tag{29}$$

with the VEV given by

$$\langle\rho\rangle = (0, v_\rho, 0)^T\,. \tag{30}$$

The Yukawa interactions are:

$$\begin{aligned} -\mathcal{L}_\nu &= \frac{1}{2}x\big(\bar{\psi}^c_{1L}\psi_{1L}\big)_{\underline{1}}\sigma + \frac{1}{2}y\big(\bar{\psi}^c_{2L}\psi_{2L} + \bar{\psi}^c_{3L}\psi_{3L}\big)\sigma \\ &\quad + \frac{1}{2}z\big[\big(\bar{\psi}^c_{1L}\psi_{iL}\big)s + \big(\bar{\psi}^c_{iL}\psi_{1L}\big)s\big] + \frac{1}{2}\tau\big(\bar{\psi}^c_{iL}\psi_{iL}\big)_{1'''}\rho + \text{h.c.} \end{aligned}$$

$$\begin{aligned}&= \frac{1}{2}x\big(\bar{\psi}^c_{1L}\psi_{1L}\big)_{\underline{1}}\sigma + \frac{1}{2}y\big(\bar{\psi}^c_{2L}\psi_{2L} + \bar{\psi}^c_{3L}\psi_{3L}\big)\sigma\\
&\quad + \frac{1}{2}z\big(\bar{\psi}^c_{1L}\psi_{2L}s_1 + \bar{\psi}^c_{1L}\psi_{3L}s_2 + \bar{\psi}^c_{2L}\psi_{1L}s_1 + \bar{\psi}^c_{3L}\psi_{1L}s_2\big)\\
&\quad + \frac{1}{2}\tau\big(v_\rho\bar{N}_{2R}\nu_{3L} - v_\rho\bar{\nu}^c_{2L}N^c_{3R} - v_\rho\bar{N}_{3R}\nu_{2L} + v_\rho\bar{\nu}^c_{3L}N^c_{2R}\big) + \text{h.c.} \qquad (31)\end{aligned}$$

The neutrino mass Lagrangian can be written in matrix form as follows

$$-\mathcal{L}^{\text{mass}}_\nu = \frac{1}{2}\bar{\chi}^c_L M_\nu \chi_L + \text{h.c.}\,, \qquad (32)$$

where

$$\chi_L \equiv \begin{pmatrix}\nu_L & N^c_R\end{pmatrix}^T, \qquad M_\nu \equiv \begin{pmatrix} M_L & M_D^T \\ M_D & M_R\end{pmatrix},$$

$$\nu_L = (\nu_{1L}, \nu_{2L}, \nu_{3L})^T, \quad N_R = (N_{1R}, N_{2R}, N_{3R})^T,$$

and the mass matrices are then obtained by

$$M_{L,R,D} = \begin{pmatrix} a_{L,R,D} & b_{L,R,D} & b_{L,R,D} \\ b_{L,R,D} & c_{L,R,D} & d_{L,R,D} \\ b_{L,R,D} & -d_{L,R,D} & c_{L,R,D}\end{pmatrix}, \qquad (33)$$

with

$$\begin{aligned} &a_L = \lambda_\sigma x\,, \quad a_D = v_\sigma x\,, \quad a_R = \Lambda_\sigma x\,,\\
&b_L = \lambda_s z\,, \quad b_D = v_s z\,, \quad b_R = \Lambda_s z\,,\\
&c_L = \lambda_\sigma y\,, \quad c_D = v_\sigma y\,, \quad c_R = \Lambda_\sigma y\,,\\
&d_L = 0\,, \quad d_D = v_\rho\tau \equiv d\,, \quad d_R = 0\,. \end{aligned} \qquad (34)$$

Three observed neutrinos gain masses via a combination of type I and type II seesaw mechanisms derived from (32) and (33) as

$$M_{\text{eff}} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} A & B_1 & B_2 \\ B_1 & C_1 & D_1 \\ B_2 & D_1 & C_2\end{pmatrix}, \qquad (35)$$

where

$$A = \frac{\big(2a_R b_D^2 - 4a_D b_D b_R + 2a_L b_R^2 + a_D^2 c_R - a_L a_R c_R\big)}{\big(2b_R^2 - a_R c_R\big)}\,,$$

$$B_1 = \frac{\big[-2b_D^2 b_R + b_L(2b_R^2 - a_R c_R) + b_D(a_D c_R + a_R c_D - a_R d_D) - a_D b_R(c_D - d_D)\big]}{\big(2b_R^2 - a_R c_R\big)}\,,$$

$$B_2 = \frac{\big[-2b_D^2 b_R + b_L(2b_R^2 - a_R c_R) + b_D(a_D c_R + a_R c_D + a_R d_D) - a_D b_R(c_D + d_D)\big]}{\big(2b_R^2 - a_R c_R\big)}\,,$$

$$C_1 = \frac{\begin{array}{c}\big[-2b_D b_R c_R(c_D - d_D) - b_R^2[(c_D + d_D)^2 - 2c_L c_R] \\ + c_R(b_D^2 c_R + a_R c_D^2 - a_R c_L c_R + a_R d_D^2)\big]\end{array}}{[c_R(2b_R^2 - a_R c_R)]},$$

$$C_2 = \frac{\begin{array}{c}\big[-2b_D b_R c_R(c_D + d_D) - b_R^2[(c_D - d_D)^2 - 2c_L c_R] \\ + c_R(b_D^2 c_R + a_R c_D^2 - a_R c_L c_R + a_R d_D^2)\big]\end{array}}{[c_R(2b_R^2 - a_R c_R)]},$$

$$D_1 = \frac{\big[(b_R c_D - b_D c_R)^2 - b_R^2 d_D^2\big]}{[c_R(2b_R^2 - a_R c_R)]}. \tag{36}$$

5.1. *Experimental constraints in the case without the ρ triplet*

In the case without the ρ contribution ($v_\rho = 0$), $\lambda_1 = \lambda_2 = \lambda_s$, $v_1 = v_2 = v_s$, $\Lambda_1 = \Lambda_2 = \Lambda_s$, we have $B_1 = B_2 \equiv B$, $C_1 = C_2 \equiv C$ and M_{eff} in (35) becomes

$$M_{\text{eff}}^0 = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix}, \tag{37}$$

where

$$A = \frac{x\big[\Lambda_\sigma(\Lambda_\sigma\lambda_\sigma - v_\sigma^2)xy - 2(\Lambda_s^2\lambda_\sigma + \Lambda_\sigma v_s^2 - 2\Lambda_s v_s v_\sigma)z^2\big]}{\Lambda_\sigma^2 xy - 2\Lambda_s^2 z^2},$$

$$B = \frac{\big[\Lambda_\sigma^2\lambda_s + v_\sigma(\Lambda_s v_\sigma - 2\Lambda_\sigma v_s)\big]xyz + 2\Lambda_s(v_\sigma^2 - \lambda_s\Lambda_s)z^3}{\Lambda_\sigma^2 xy - 2\Lambda_s^2 z^2},$$

$$C = \frac{y\big\{\Lambda_\sigma^2(\lambda_\sigma\Lambda_\sigma - v_\sigma^2)xy - \big[\Lambda_\sigma^2 v_s^2 - 2\Lambda_s\Lambda_\sigma v_s v_\sigma + \Lambda_s^2(2\lambda_\sigma\Lambda_s - v_\sigma^2)\big]z^2\big\}}{\Lambda_\sigma^3 xy - 2\Lambda_s^2\Lambda_\sigma z^2}, \tag{38}$$

$$D = -\frac{(\Lambda_\sigma v_s - \Lambda_s v_\sigma)^2 yz^2}{\Lambda_\sigma^3 xy - 2\Lambda_s^2\Lambda_\sigma z^2} = -\frac{\Lambda_s}{\Lambda_\sigma}\frac{\Lambda_s yz^2}{\left(xy - 2\frac{\Lambda_s^2}{\Lambda_\sigma^2}z^2\right)}\left(\frac{v_s}{\Lambda_s} - \frac{v_\sigma}{\Lambda_\sigma}\right)^2 \ll 1.$$

This mass matrix takes the form similar to that of unbroken Z_2 (i.e. $v_\rho = 0$). However, the breaking of Z_2 ($v_\rho \neq 0$) in this case is necessary to fit the data (see below). Indeed, we can diagonalize M_{eff}^0 in (37) as follows:

$$U^T M_{\text{eff}} U = \text{diag}(m_1, m_2, m_3),$$

where

$$m_1 = \frac{1}{2}\left(A + C + D - \sqrt{8B^2 + (A - C - D)^2}\right),$$

$$m_2 = \frac{1}{2}\left(A + C + D + \sqrt{8B^2 + (A - C - D)^2}\right), \tag{39}$$

$$m_3 = C - D,$$

and the corresponding eigenstates put in the lepton mixing matrix:

$$U = \begin{pmatrix} \frac{K}{\sqrt{K^2+2}} & -\frac{\sqrt{2}}{\sqrt{K^2+2}} & 0 \\ \frac{1}{\sqrt{K^2+2}} & \frac{1}{\sqrt{2}}\frac{K}{\sqrt{K^2+2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{K^2+2}} & \frac{1}{\sqrt{2}}\frac{K}{\sqrt{K^2+2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{A-m_2}{\sqrt{(A-m_2)^2+2B^2}} & -\frac{\sqrt{2}B}{\sqrt{(A-m_2)^2+2B^2}} & 0 \\ \frac{1}{\sqrt{2}}\frac{\sqrt{2}B}{\sqrt{(A-m_2)^2+2B^2}} & \frac{1}{\sqrt{2}}\frac{A-m_2}{\sqrt{(A-m_2)^2+2B^2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\frac{\sqrt{2}B}{\sqrt{(A-m_2)^2+2B^2}} & \frac{1}{\sqrt{2}}\frac{A-m_2}{\sqrt{(A-m_2)^2+2B^2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (40)$$

with

$$K = \frac{A-C-D-\sqrt{8B^2+(A-C-D)^2}}{2B}.$$

Relations between K and m_1, m_2, m_3 take the forms:

$$\begin{aligned} m_1 = KB + C + D\,, \quad m_2 = -KB + A\,, \quad m_3 = C - D\,, \\ m_1 + m_2 + m_3 = A + 2C\,, \quad m_1 m_2 = -2B^2 + A(C+D)\,. \end{aligned} \quad (41)$$

The U matrix in (40) can be parametrized in three Euler's angles, which implies:

$$\theta_{13} = 0\,, \quad \theta_{23} = \frac{\pi}{4}\,, \quad \tan\theta_{12} = \frac{\sqrt{2}B}{A-m_2} \equiv \frac{\sqrt{2}}{K}\,. \quad (42)$$

The recent data imply that $\theta_{13} \neq 0$.[1–5] If it is correct, this case will fail. However, the following case improves this.

5.2. *Experimental constraints in the case with the ρ triplet*

In this case with the ρ contribution, $v_\rho \neq 0$, the general neutrino mass matrix in (35) can be rewritten in the form:

$$M_{\rm eff} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix} + \begin{pmatrix} 0 & p_1 & -p_1 \\ p_1 & q_1 & r \\ -p_1 & r & q_2 \end{pmatrix}, \quad (43)$$

where A, B, C and D are given by (38) due to the contribution from the scalar antisextets s and σ only. The second matrix in (43) is a deviation arising from the contribution due to the scalar triplet ρ, namely $p_1 = B_1 - B = -(B_2 - B)$, $q_{1,2} = C_{1,2} - C$ and $r = D_1 - D$, with the A, $B_{1,2}$, $C_{1,2}$ and D_1 being defined in

(36). Indeed, if the ρ contribution is neglected, the deviations p_1, $q_{1,2}$, r will vanish. Hence the mass matrix M_{eff} in (35) reduces to its form in (37). The first term, as shown in Subsec. 5.1 can approximately fit the Particle Data Group 2010[152] with a small deviation for θ_{13}. The second term is proportional to p_1, $q_{1,2}$, r due to contribution of the triplet ρ, will take the role for such a deviation of θ_{13}. So, in this work we consider the ρ contribution as a small perturbation and terminating the theory at the first-order.

Assuming that $\lambda_s \ll v_s \ll \Lambda_s$, $\lambda_\sigma \ll v_\sigma \ll \Lambda_\sigma$ or $\frac{\lambda_s}{\Lambda_s} \simeq \frac{\lambda_\sigma}{\Lambda_\sigma} \ll 1$, $\frac{v_s}{\Lambda_s} \simeq \frac{v_\sigma}{\Lambda_\sigma} \ll 1$, $v_\rho \ll v_s, v_\sigma$, $\Lambda_\sigma \sim \Lambda_s$ and x, y, z, τ being in the same order then we get

$$p_1 = \frac{\tau v_\rho(\Lambda_\sigma v_s - \Lambda_s v_\sigma)xz}{\Lambda_\sigma^2 xy - 2\Lambda_s^2 z^2} \simeq -\tau v_\rho\left(\frac{v_s}{\Lambda_s} - \frac{v_\sigma}{\Lambda_\sigma}\right), \tag{44}$$

$$\begin{aligned} q_1 &= \frac{\tau v_\rho\left[-\Lambda_\sigma^2 \tau v_\rho xy + \Lambda_s(\Lambda_s \tau v_\rho - 2\Lambda_\sigma v_s y + 2\Lambda_s v_\sigma y)z^2\right]}{\Lambda_\sigma y(\Lambda_\sigma^2 xy - 2\Lambda_s^2 z^2)} \\ &\simeq 2\tau v_\rho\left(\frac{v_s}{\Lambda_s} - \frac{v_\sigma}{\Lambda_\sigma}\right), \end{aligned} \tag{45}$$

$$q_2 = \frac{\tau v_\rho\left[-\Lambda_\sigma^2 \tau v_\rho xy + \Lambda_s(\Lambda_s t v_\rho + 2\Lambda_\sigma v_s y - 2\Lambda_s v_\sigma y)z^2\right]}{\Lambda_\sigma y(\Lambda_\sigma^2 xy - 2\Lambda_s^2 z^2)} \simeq -q_1\,, \tag{46}$$

which all start from the first-order of the perturbation

$$r = \frac{\Lambda_s^2 \tau^2 v_\rho^2 z^2}{\Lambda_\sigma^3 xy^2 - 2\Lambda_s^2 \Lambda_\sigma y z^2} \simeq -\tau v_\rho\left(\frac{v_\rho}{\Lambda_\sigma}\right). \tag{47}$$

Because of $v_s \ll \Lambda_s$, $v_\sigma \ll \Lambda_\sigma$ and $v_\rho \ll v_s$, v_σ so r in (47) is the second-order of the perturbation. Consequently, it can be ignored. The last matrix in (43) now takes the form:

$$\begin{pmatrix} 0 & p_1 & -p_1 \\ p_1 & q_1 & r \\ -p_1 & r & q_2 \end{pmatrix} \approx \epsilon v_\rho \tau \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}, \tag{48}$$

where $\epsilon = \frac{v_s}{\Lambda_s} - \frac{v_\sigma}{\Lambda_\sigma}$ is very small and plays the role of the perturbation parameter. The explicit form of the mass matrix (35) is thus given by

$$M_{\text{eff}} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix} + \epsilon M^{(1)}, \tag{49}$$

where $M^{(1)}$ is the perturbation contribution at the first-order:

$$M^{(1)} \equiv v_\rho \tau \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

It is clear that the first term in (49) can approximately fit the data with a "small" deviation as shown in Subsec. 5.1. So, in this case we consider ϵ being small as a perturbation parameter.

At the first-order of perturbation theory, the matrix $M^{(1)}$ does not give contribution to eigenvalues. However, it changes the eigenvectors. The physical neutrino masses are thus obtained as:

$$m'_1 = m_1\,, \quad m'_2 = m_2\,, \quad m'_3 = m_3\,,$$

where $m_{1,2,3}$ are the masses in the case without contribution of ρ given by (39). For the corresponding perturbed eigenstates, we put:

$$U \to U' = U + \Delta U\,,$$

where U is defined by (40), and

$$\Delta U = \begin{pmatrix} 0 & 0 & -\epsilon\frac{\sqrt{2}K(K-2)\tau v_\rho}{(K^2+2)(m_1-m_3)} \\ -\epsilon\frac{(K-2)\tau v_\rho}{\sqrt{K^2+2}(m_1-m_3)} & \epsilon\frac{\sqrt{2}(K+1)\tau v_\rho}{\sqrt{K^2+2}(m_2-m_3)} & -\epsilon\frac{\sqrt{2}(K-2)\tau v_\rho}{(K^2+2)(m_1-m_3)} \\ \epsilon\frac{(K-2)\tau v_\rho}{\sqrt{K^2+2}(m_1-m_3)} & -\epsilon\frac{\sqrt{2}(K+1)\tau v_\rho}{\sqrt{K^2+2}(m_2-m_3)} & -\epsilon\frac{\sqrt{2}(K-2)\tau v_\rho}{(K^2+2)(m_1-m_3)} \end{pmatrix}. \tag{50}$$

The lepton mixing matrix in this case can still be parametrized in three new Euler's angles θ'_{ij}, which are also a perturbation from the θ_{ij} (without contribution from the ρ triplet), defined by

$$\begin{aligned} s'_{13} &= -U'_{13} = \epsilon\frac{\sqrt{2}K(K-2)\tau v_\rho}{(K^2+2)(m_1-m_3)}\,, \\ t'_{12} &= -\frac{U'_{12}}{U'_{11}} = \frac{\sqrt{2}}{K} \equiv t_{12}\,, \\ t'_{23} &= -\frac{U'_{23}}{U'_{33}} = 1 + \frac{4\epsilon(K-2)v_\rho\tau}{2\epsilon(K-2)v_\rho\tau + (K^2+2)(m_1-m_3)}\,. \end{aligned}$$

It is easy to show that our model is consistent since the five experimental constraints on the mixing angles and squared mass differences of neutrinos can be respectively fitted with four Yukawa coupling parameters x, y, z and τ of the s, σ antisextets and ρ triplet scalars, with the given VEVs. To see this, let us take the data in 2012 as shown in (1). It follows $K \simeq 2.1054$, and $t'_{23} = 1.2383$ [$\theta'_{23} \simeq 51.08°$, $\sin^2(2\theta'_{23}) = 0.9556$ satisfying the condition $\sin^2(2\theta'_{23}) > 0.95$].

Until now values of neutrino masses (or the absolute neutrino masses) as well as the mass ordering of neutrinos is unknown. The tritium experiment[154,155] provides an upper bound on the absolute value of neutrino mass

$$m_i \leq 2.2 \text{ eV}\,.$$

A more stringent bound was found from the analysis of the latest cosmological data[156]

$$m_i \leq 0.6 \text{ eV}\,,$$

while arguments from the growth of large-scale structure in the early Universe yield the upper bound[157]

$$\sum_{i=1}^{3} m_i \leq 0.5 \text{ eV}.$$

The neutrino mass spectrum can be the normal mass hierarchy ($m_1 \simeq m_2 < m_3$), the inverted hierarchy ($m_3 < m_1 \simeq m_2$) or nearly degenerate ($m_1 \simeq m_2 \simeq m_3$). The mass ordering of neutrino depends on the sign of Δm_{23}^2 which is currently unknown. In the case of 3-neutrino mixing, in the model under consideration, the two possible signs of Δm_{23}^2 corresponding to two types of neutrino mass spectrum can be provided as shown below.

5.2.1. *Normal case* ($\Delta m_{23}^2 > 0$)

In this case, the neutrino masses are functions of $\delta = \epsilon v_\rho \tau$ as follows

$$m_1 = -\frac{0.00388981}{\delta} + 0.153928\delta, \tag{51}$$

$$m_2 = \pm 8 \times 10^{-3}\sqrt{-17.5391 + \frac{0.236416}{\delta^2} + 370.216\delta^2}, \tag{52}$$

$$m_3 = -\frac{0.00388981}{\delta} - 0.153928\delta. \tag{53}$$

In Fig. 1, we have plotted the absolute value $|m_i|$ ($i = 1, 2, 3$) as a function of δ with the values of $\delta \in (-0.5, 0.5)$ eV. This figure shows that there exist allowed regions for value of δ where either normal or quasi-degenerate neutrino masses spectrum achieved. The quasi-degenerate mass hierarchy obtained when $\delta \to 0$ or $\delta \to \pm\infty$ ($|\delta|$ increase but must be small enough because of the scale of ϵ, v_ρ, τ). The normal mass hierarchy will be obtained if δ takes the values around $(-0.20, -0.15)$ eV or $(0.15, 0.20)$ eV as shown in Figs. 2(a) and 2(b), respectively. Figures 3(a) and 3(b) give three absolute neutrino masses m_i with $\delta \in (-0.2, -0.15)$ and $\delta \in (0.15, 0.20)$, respectively. The values $\sum_{i=1}^{3} m_i$ as well as $\sum_{i=1}^{3} |m_i|$ as a functions of δ are plotted in Figs. 4 and 5, respectively.

To get explicit values of the model parameters, we assume $\delta \equiv \epsilon v_\rho \tau = 0.15$ eV, which is safely small. Then the neutrino masses are explicitly given as $m_1 \simeq -0.00284$ eV, $m_2 \simeq \pm 0.00911$ eV and $m_3 \simeq -0.04902$ eV. It follows that $A \simeq 8.75 \times 10^{-4}$ eV, $B \simeq -3.913 \times 10^{-3}$ eV, $C \simeq -2.18 \times 10^{-2}$ eV and $D \simeq 2.72 \times 10^{-2}$ eV (equivalently to $m_2 = 0.00911$ eV), or $A \simeq -7.16 \times 10^{-3}$ eV, $B \simeq -2.05 \times 10^{-3}$ eV, $C \simeq -2.69 \times 10^{-2}$ eV and $D \simeq 2.21 \times 10^{-2}$ eV (equivalently to $m_2 = -0.00911$ eV). This solution means a normal neutrino mass spectrum as mentioned above. Furthermore, if $\lambda_s = \lambda_\sigma = 1$ eV, $v_s = v_\sigma$, $\Lambda_s = -\Lambda_\sigma = -v_s^2$ eV, we obtain $x \simeq -1.57 \times 10^{-4}$, $y \simeq -7.16 \times 10^{-3}$, $z \simeq 1.09 \times 10^{-3}$ (equivalently to $m_2 = 0.00911$ eV), or $x \simeq -3.68 \times 10^{-3}$, $y \simeq -1.12 \times 10^{-2}$, $z \simeq -3.69 \times 10^{-2}$ (equivalently to $m_2 = -0.00911$ eV) and $\tau \simeq \frac{150}{v_\rho}$. If $v_\rho \sim 1.5 \times 10^{-3}$ GeV then $\tau \sim 1.5 \times 10^{-4}$ which is on the same order in magnitude with x, y, z.

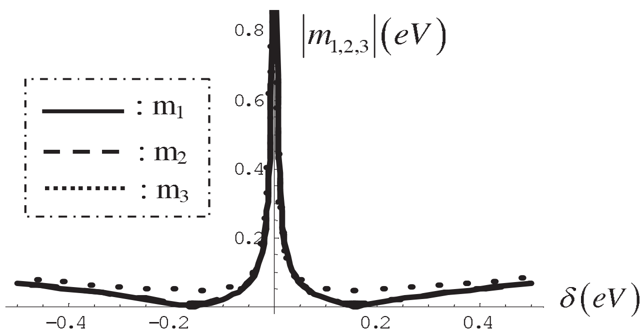

Fig. 1. The absolute values $|m_1|$, $|m_2|$, $|m_3|$ as functions of δ with $\delta \in (-0.5, 0.5)$ eV in the case of $\Delta m_{23}^2 > 0$.

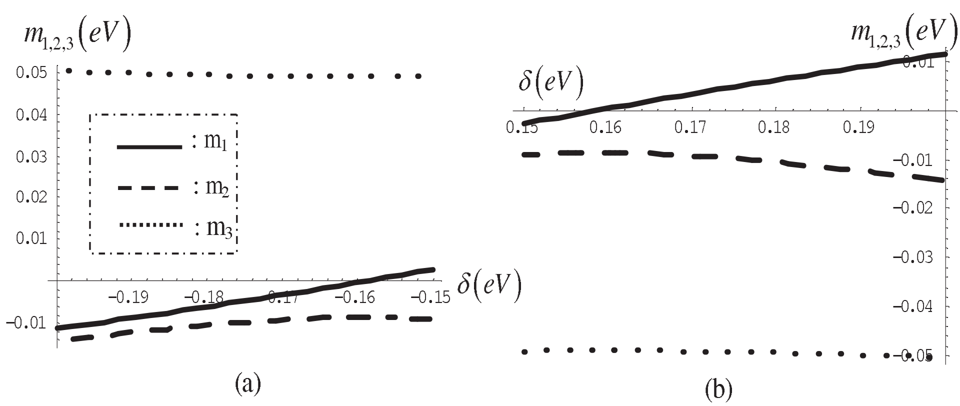

Fig. 2. The values m_i $(i = 1, 2, 3)$ as functions of δ. (a) $\delta \in (-0.20, -0.15)$ eV, (b) $\delta \in (0.15, 0.20)$ eV in the case of $\Delta m_{23}^2 > 0$.

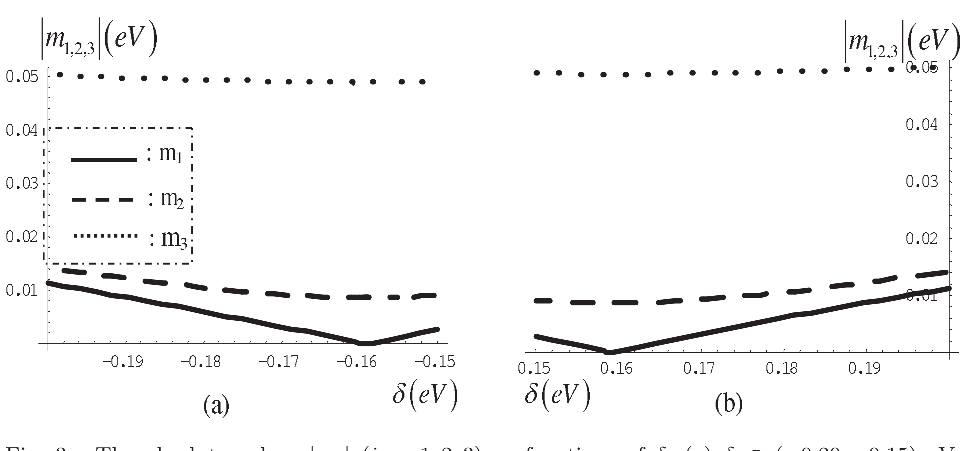

Fig. 3. The absolute values $|m_i|$ $(i = 1, 2, 3)$ as functions of δ. (a) $\delta \in (-0.20, -0.15)$ eV, (b) $\delta \in (0.15, 0.20)$ eV in the case of $\Delta m_{23}^2 > 0$.

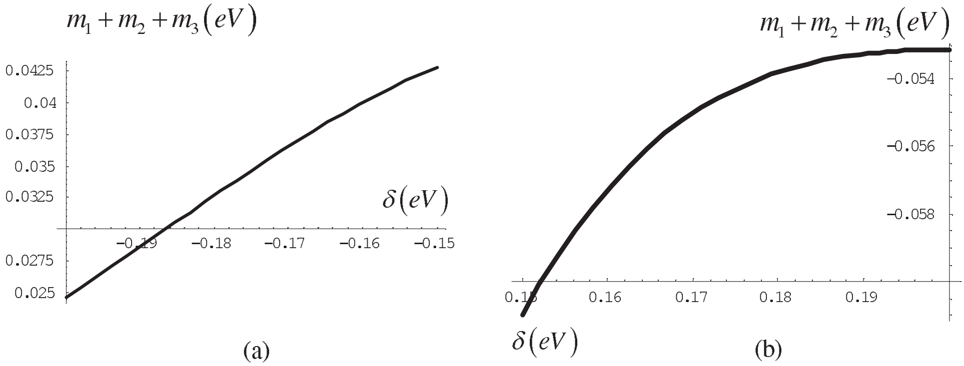

Fig. 4. The value $\sum m_i$ as a function of δ in the case of $\Delta m^2_{23} > 0$. (a) $\delta \in (-0.20, -0.15)$ eV, (b) $\delta \in (0.15, 0.20)$ eV.

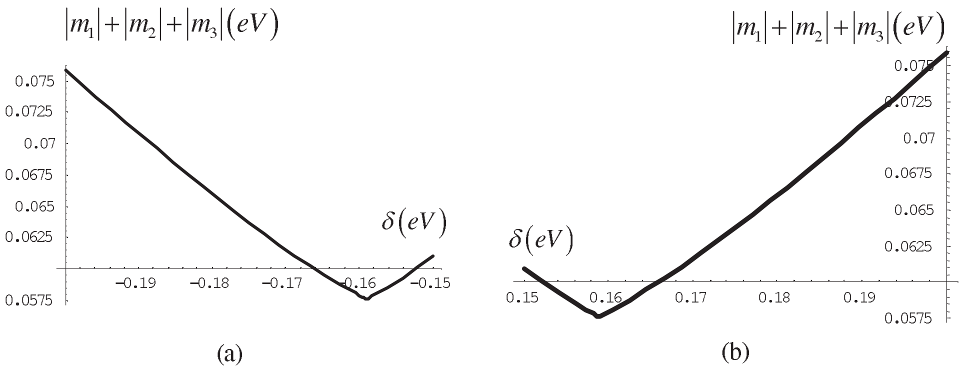

Fig. 5. The value $\sum |m_i|$ as a function of δ in the case of $\Delta m^2_{23} > 0$. (a) $\delta \in (-0.20, -0.15)$ eV, (b) $\delta \in (0.15, 0.20)$ eV.

5.2.2. *Inverted case* ($\Delta m^2_{23} < 0$)

In this case, the neutrino masses are functions of $\delta = \epsilon v_\rho \tau$ as follows

$$m_1 = \frac{0.00364619}{\delta} + 0.153928\delta\,, \tag{54}$$

$$m_2 = \pm 8 \times 10^{-3}\sqrt{18.7109 + \frac{0.207729}{\delta^2} + 370.216\delta^2}\,, \tag{55}$$

$$m_3 = \frac{0.00364619}{\delta} - 0.153928\delta\,. \tag{56}$$

In Fig. 6, we have plotted the values $m_i\,(i = 1, 2, 3)$ as a function of δ with the values of $\delta \in (-0.5, 0.5)$ eV. This figure shows that there exist the allowed regions for value of δ where either inverted ($|m_1| \simeq |m_2| > |m_3|$) or quasi-degenerate neutrino masses spectrum ($|m_1| \simeq |m_2| \simeq |m_3|$) achieved. The quasi-degenerate mass hierarchy obtained when $\delta \to 0$ or $\delta \to \pm\infty$. The inverted mass hierarchy

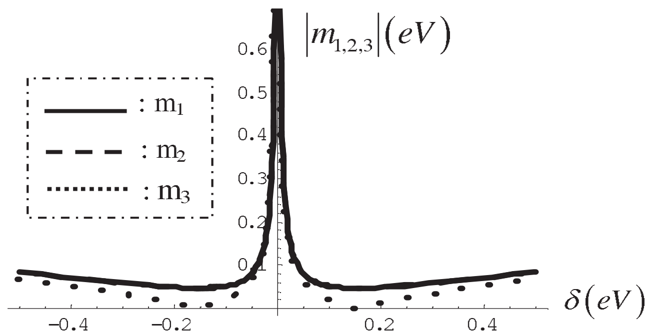

Fig. 6. The absolute values $|m_1|$, $|m_2|$, $|m_3|$ as functions of δ with $\delta \in (-0.5, 0.5)$ eV in the case of $\Delta m^2_{23} < 0$.

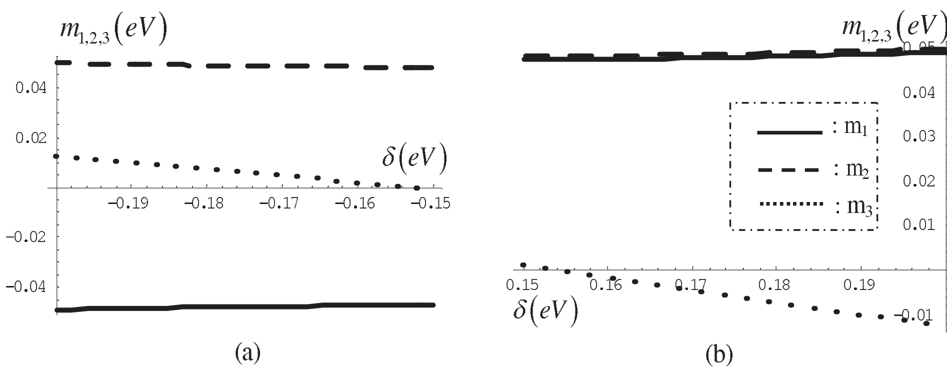

Fig. 7. The values m_i $(i = 1, 2, 3)$ as functions of δ. (a) $\delta \in (-0.20, -0.15)$ eV, (b) $\delta \in (0.15, 0.20)$ eV in the case of $\Delta m^2_{23} < 0$.

is obtained if δ takes the values around $(-0.20, -0.15)$ eV or $(0.15, 0.20)$ eV as shown in Figs. 7(a) and 7(b), respectively. Figures 8(a) and 8(b) give three absolute neutrino masses m_i with $\delta \in (-0.2, -0.15)$ and $\delta \in (0.15, 0.20)$, respectively. The values $\sum_{i=1}^{3} m_i$ as well as $\sum_{i=1}^{3} |m_i|$ as a functions of δ are plotted in Figs. 9 and 10, respectively.

In similarity to the normal case, to get explicit values of the model parameters, we also assume $\delta \equiv \epsilon v_\rho \tau = 0.15$ eV, which is safely small. Then the neutrino masses are explicitly given as

$$m_1 \simeq 4.74 \times 10^{-2} \text{ eV}, \quad m_2 \simeq 4.82 \times 10^{-2} \text{ eV}, \quad m_3 \simeq 1.22 \times 10^{-3} \text{ eV}, \tag{57}$$

or

$$m_1 \simeq 4.74 \times 10^{-2} \text{ eV}, \quad m_2 \simeq -4.82 \times 10^{-2} \text{ eV}, \quad m_3 \simeq 1.22 \times 10^{-3} \text{ eV}. \tag{58}$$

From (57) we find out

$$\begin{aligned} &A \simeq 4.76 \times 10^{-2} \text{ eV}, \quad B \simeq 2.57 \times 10^{-4} \text{ eV}, \\ &C \simeq 2.46 \times 10^{-2} \text{ eV}, \quad D \simeq 2.34 \times 10^{-2} \text{ eV}. \end{aligned} \tag{59}$$

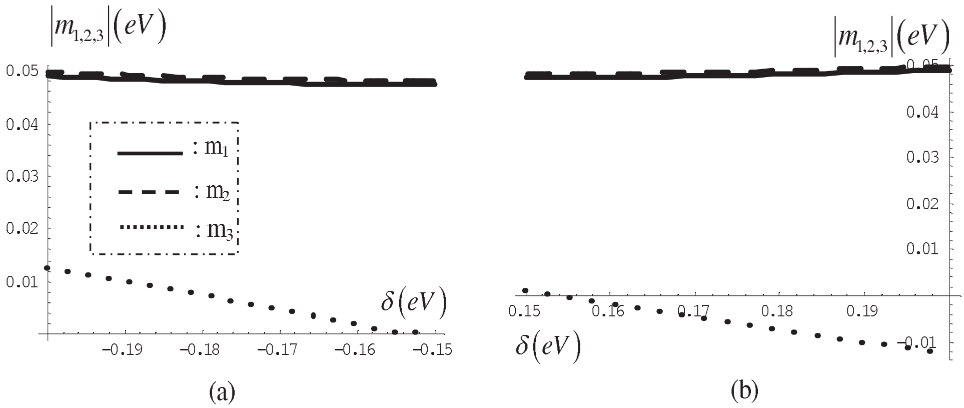

Fig. 8. The absolute values $|m_i|$ $(i = 1, 2, 3)$ as functions of δ. (a) $\delta \in (-0.20, -0.15)$ eV, (b) $\delta \in (0.15, 0.20)$ eV in the case of $\Delta m^2_{23} < 0$.

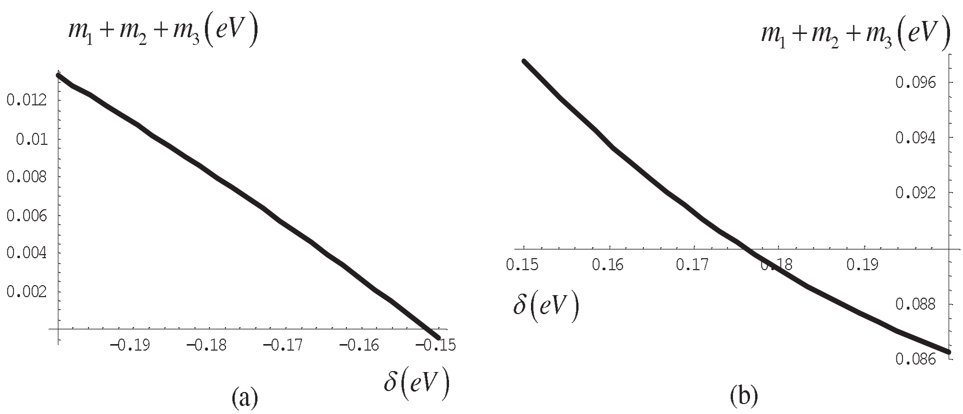

Fig. 9. The value $\sum m_i$ as a function of δ in the case of $\Delta m^2_{23} < 0$. (a) $\delta \in (-0.20, -0.15)$ eV, (b) $\delta \in (0.15, 0.20)$ eV.

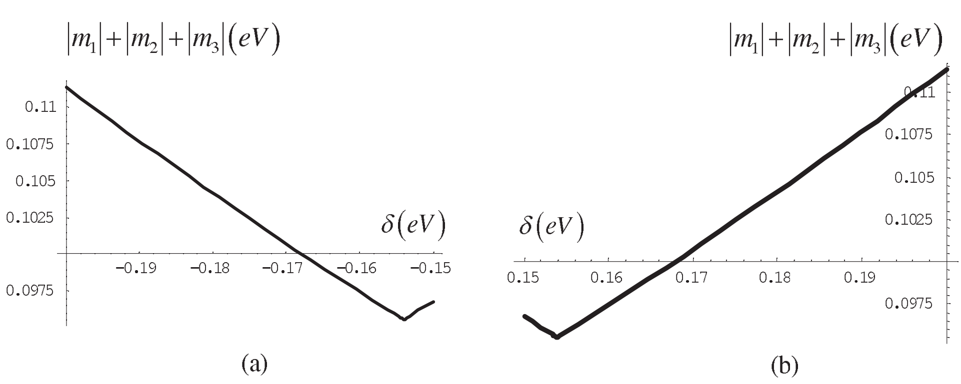

Fig. 10. The values $m_{1,2,3}$ as functions of δ in the case of $\Delta m^2_{23} < 0$.

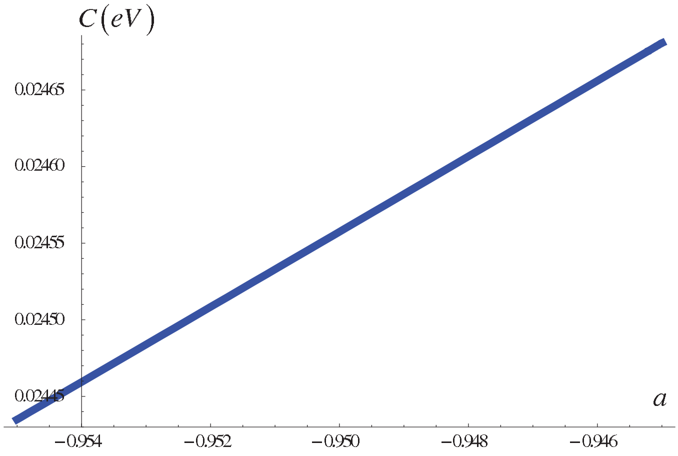

Fig. 11. The value C as a function of a in the case of $\Delta m^2_{23} < 0$.

Furthermore, suppose that

$$\lambda_s = \lambda_\sigma = 1 \text{ eV}\,, \quad v_s = v_\sigma\,, \quad \Lambda_s = \Lambda_\sigma = v_s^2\,, \quad \frac{v_\sigma}{\Lambda_\sigma} = a\frac{v_s}{\Lambda_s}\,, \tag{60}$$

we obtain the relation between C and a as in Fig. 11. Then the satisfied value of a, which can be inferred from this figure, is as follows $a = -0.950$. With this value of a we get $x \simeq 1.22 \times 10^{-2}$, $y \simeq 1.23 \times 10^{-2}$, $z \simeq 1.11$.

In a similar way, from (58) we get $A \simeq -1.85 \times 10^{-2}$ eV, $B \simeq -3.13 \times 10^{-2}$ eV, $C \simeq 9.45 \times 10^{-3}$ eV, $D \simeq 8.23 \times 10^{-3}$ eV. With the assumption in (60) we get $a \simeq -0.9815$, and it follows $x \simeq -3.47 \times 10^{-2}$, $y \simeq 3.32 \times 10^{-2}$, $z \simeq -9.12 \times 10^{-3}$.

In both case, the parameter $\tau \sim 1.5 \times 10^{-4}$ provided that $\epsilon \sim 10^{-3}$ eV and $v_\rho \sim 1.5 \times 10^{-3}$ GeV. The solutions in (57) and (58) mean a inverted neutrino mass spectrum.

6. Vacuum Alignment

In order to make this work complete we write out the scalar potentials of the model. It is to be noted that $(\mathrm{Tr}\,A)(\mathrm{Tr}\,B) = \mathrm{Tr}(A\,\mathrm{Tr}\,B)$ and we have used the following notation: $V(X \to X', Y \to Y', \ldots) \equiv V(X, Y, \ldots)|_{X=X', Y=Y', \ldots}$. The general potential invariant under all subgroups takes the form:

$$V_{\text{total}} = V_{\text{tri}} + V_{\text{sext}} + V_{\text{tri-sext}}\,, \tag{61}$$

where V_{tri} comes from only contributions of $SU(3)_L$ triplets given as a sum of:

$$V(\chi) = \mu_\chi^2 \chi^\dagger \chi + \lambda^\chi (\chi^\dagger \chi)^2\,, \tag{62}$$

$$\begin{gathered} V(\phi) = V(\chi \to \phi)\,, \quad V(\phi') = V(\chi \to \phi')\,, \quad V(\eta) = V(\chi \to \eta)\,, \\ V(\eta') = V(\chi \to \eta')\,, \quad V(\rho) = V(\chi \to \rho)\,, \end{gathered} \tag{63}$$

$$V(\phi, \chi) = \lambda_1^{\phi\chi} (\phi^\dagger \phi)(\chi^\dagger \chi) + \lambda_2^{\phi\chi} (\phi^\dagger \chi)(\chi^\dagger \phi)\,, \tag{64}$$

$$\begin{aligned}
V(\phi,\phi') &= V(\phi,\chi\to\phi') + \lambda_3^{\phi\phi'}(\phi^\dagger\phi')(\phi^\dagger\phi') + \lambda_4^{\phi\phi'}(\phi'^\dagger\phi)(\phi'^\dagger\phi)\,,\\
V(\phi,\eta) &= V(\phi,\chi\to\eta)\,, \qquad V(\phi,\eta') = V(\phi,\chi\to\eta')\,,\\
V(\phi,\rho) &= V(\phi,\chi\to\rho)\,, \qquad V(\phi',\chi) = V(\phi\to\phi',\chi)\,,\\
V(\phi',\eta) &= V(\phi\to\phi',\chi\to\eta)\,, \quad V(\phi',\eta') = V(\phi\to\phi',\chi\to\eta')\,,\\
V(\phi',\rho) &= V(\phi\to\phi',\chi\to\rho)\,, \quad V(\chi,\eta) = V(\phi\to\chi,\chi\to\eta)\,,\\
V(\chi,\eta') &= V(\phi\to\chi,\chi\to\eta')\,, \quad V(\chi,\rho) = V(\phi\to\chi,\chi\to\rho)\,,\\
V(\eta,\eta') &= V(\phi\to\eta,\chi\to\eta') + \lambda_3^{\eta\eta'}(\eta^\dagger\eta')(\eta^\dagger\eta') + \lambda_4^{\eta\eta'}(\eta'^\dagger\eta)(\eta'^\dagger\eta)\,,\\
V(\eta,\rho) &= V(\phi\to\eta,\chi\to\rho)\,, \quad V(\eta',\rho) = V(\phi\to\eta',\chi\to\rho)\,,\\
V_{\chi\phi\phi'\eta\eta'\rho} &= \mu_1\chi\phi\eta + \mu_1'\chi\phi'\eta' + \lambda_1^1(\phi^\dagger\phi')(\eta^\dagger\eta') + \lambda_1^2(\phi^\dagger\phi')(\eta'^\dagger\eta)\\
&\quad + \lambda_1^3(\phi^\dagger\eta)(\eta'^\dagger\phi') + \lambda_1^4(\phi^\dagger\eta')(\eta^\dagger\phi') + \text{h.c.}
\end{aligned} \tag{65}$$

The V_{sext} is summed over only antisextet contributions:

$$\begin{aligned}
V(s) &= \mu_s^2\,\text{Tr}(s^\dagger s) + \lambda_1^s\,\text{Tr}\big[(s^\dagger s)_{\underline{1}}(s^\dagger s)_{\underline{1}}\big] + \lambda_2^s\,\text{Tr}\big[(s^\dagger s)_{\underline{1}'}(s^\dagger s)_{\underline{1}'}\big]\\
&\quad + \lambda_3^s\,\text{Tr}\big[(s^\dagger s)_{\underline{1}''}(s^\dagger s)_{\underline{1}''}\big] + \lambda_4^s\,\text{Tr}\big[(s^\dagger s)_{\underline{1}'''}(s^\dagger s)_{\underline{1}'''}\big]\\
&\quad + \lambda_5^s\,\text{Tr}(s^\dagger s)_{\underline{1}}\,\text{Tr}(s^\dagger s)_{\underline{1}} + \lambda_6^s\,\text{Tr}(s^\dagger s)_{\underline{1}'}\,\text{Tr}(s^\dagger s)_{\underline{1}'}\\
&\quad + \lambda_7^s\,\text{Tr}(s^\dagger s)_{\underline{1}''}\,\text{Tr}(s^\dagger s)_{\underline{1}''} + \lambda_8^s\,\text{Tr}(s^\dagger s)_{\underline{1}'''}\,\text{Tr}(s^\dagger s)_{\underline{1}'''}\,,
\end{aligned} \tag{66}$$

$$\begin{aligned}
V(\sigma) &= \mu_\sigma^2\,\text{Tr}(\sigma^\dagger\sigma) + \lambda_1^\sigma\,\text{Tr}\big[(\sigma^\dagger\sigma)_{\underline{1}}(\sigma^\dagger\sigma)_{\underline{1}}\big] + \lambda_2^\sigma\,\text{Tr}(\sigma^\dagger\sigma)_{\underline{1}}\,\text{Tr}(\sigma^\dagger\sigma)_{\underline{1}}\,,\\
V(s,\sigma) &= \lambda_1^{s\sigma}\,\text{Tr}\big[(s^\dagger s)_{\underline{1}}(\sigma^\dagger\sigma)_{\underline{1}}\big] + \lambda_2^{s\sigma}\,\text{Tr}\big[(s^\dagger s)_{\underline{1}}\big]\,\text{Tr}\big[(\sigma^\dagger\sigma)_{\underline{1}}\big]\\
&\quad + \lambda_3^{s\sigma}\,\text{Tr}\big[(s^\dagger\sigma)_{\underline{2}}(\sigma^\dagger s)_{\underline{2}}\big] + \lambda_4^{s\sigma}\,\text{Tr}\big[(s^\dagger\sigma)_{\underline{2}}\big]\,\text{Tr}\big[(\sigma^\dagger s)_{\underline{2}}\big] + \text{h.c.}
\end{aligned} \tag{67}$$

The $V_{\text{tri-sext}}$ is given as a sum over all the terms connecting both the sectors:

$$\begin{aligned}
V(\phi,s) &= \lambda_1^{\phi s}(\phi^\dagger\phi)\,\text{Tr}(s^\dagger s)_{\underline{1}} + \lambda_2^{\phi s}\big[(\phi^\dagger s^\dagger)(s\phi)\big]_{\underline{1}}\,,\\
V(\phi',s) &= V(\phi\to\phi',s)\,, \qquad V(\chi,s) = V(\phi\to\chi,s)\,,\\
V(\eta,s) &= V(\phi\to\eta,s)\,, \qquad V(\eta',s) = V(\phi\to\eta',s)\,,\\
V(\rho,s) &= V(\phi\to\rho,s) + \big\{\lambda_3^{\rho s}\rho\big[(\rho s^\dagger)s^\dagger\big]_{\underline{1}'} + \text{h.c.}\big\}\,,\\
V(\phi,\sigma) &= V(\phi,s\to\sigma) + \lambda_3^{\phi\sigma}(\phi^\dagger\sigma)(\sigma^\dagger\phi)_{\underline{1}}\,,\\
V(\phi',\sigma) &= V(\phi\to\phi',\sigma)\,, \qquad V(\chi,\sigma) = V(\phi\to\chi,\sigma)\,,\\
V(\eta,\sigma) &= V(\phi\to\eta,\sigma)\,, \qquad V(\eta',\sigma) = V(\phi\to\eta',\sigma)\,,\\
V(\rho,\sigma) &= V(\phi\to\rho,\sigma) + \big\{\lambda_3^{\rho\sigma}\rho\big[(\rho\sigma^\dagger)\sigma^\dagger\big]_{\underline{1}'''} + \text{h.c.}\big\}\,,\\
V(\phi,s,\sigma) &= 0\,, \qquad V(\phi',s,\sigma) = 0\,,\\
V(\chi,s,\sigma) &= 0\,, \qquad V(\eta,s,\sigma) = 0\,,\\
V(\eta',s,\sigma) &= 0\,, \qquad V(\rho,s,\sigma) = 0\,,
\end{aligned} \tag{68}$$

$$V_{s\sigma\chi\phi\phi'\eta\eta'\rho} = (\lambda_1\phi^\dagger\phi' + \lambda_2\eta^\dagger\eta')\,\mathrm{Tr}(s^\dagger s)_{\underline{1}'} + \lambda_3\left[(\phi^\dagger s^\dagger)(s\phi')\right]_{\underline{1}} + \lambda_4\left[(\eta^\dagger s^\dagger)(s\eta')\right]_{\underline{1}} + \mathrm{h.c.} \tag{69}$$

To provide the Majorana masses for the neutrinos, the lepton number must be broken. This can be achieved via the scalar potential violating $U(1)_{\mathcal{L}}$. However, the other symmetries should be conserved. The violating $\mathcal{L}$ potential is given by

$$\begin{aligned}\bar{V} = {} & \big[\bar{\lambda}_1\phi^\dagger\phi + \bar{\lambda}_2\phi'^\dagger\phi' + \bar{\lambda}_3\chi^\dagger\chi + \bar{\lambda}_4\eta^\dagger\eta + \bar{\lambda}_5\eta'^\dagger\eta' + \bar{\lambda}_6\rho^\dagger\rho \\ & + \bar{\lambda}_7\eta^\dagger\chi + \bar{\lambda}_8\,\mathrm{Tr}(s^\dagger s)_{\underline{1}} + \bar{\lambda}_9\,\mathrm{Tr}(\sigma^\dagger\sigma)_{\underline{1}}\big](\eta^\dagger\chi) \\ & + \big[\bar{\lambda}_{10}\phi^\dagger\phi' + \bar{\lambda}_{11}\phi'^\dagger\phi + \bar{\lambda}_{12}\eta^\dagger\eta' + \bar{\lambda}_{13}\eta'^\dagger\eta + \bar{\lambda}_{14}\eta'^\dagger\chi + \bar{\lambda}_{15}\,\mathrm{Tr}(s^\dagger s)_{\underline{1}'}\big](\eta'^\dagger\chi) \\ & + \big[\bar{\lambda}_{16}\eta^\dagger\phi + \bar{\lambda}_{17}\eta'^\dagger\phi' + \bar{\lambda}_{18}\eta'^\dagger\rho\big](\phi^\dagger\chi) + \big[\bar{\lambda}_{19}\eta^\dagger\phi' + \bar{\lambda}_{20}\eta'^\dagger\phi\big](\phi'^\dagger\chi) \\ & + \bar{\lambda}_{21}\,\mathrm{Tr}(s^\dagger s)_{\underline{1}'''}(\phi^\dagger\rho) + \bar{\lambda}_{22}\,\mathrm{Tr}(s^\dagger s)_{\underline{1}''}(\phi'^\dagger\rho) + \bar{\lambda}_{23}(\eta^\dagger s^\dagger)(s\chi)_{\underline{1}} \\ & + \bar{\lambda}_{24}\left[(\eta^\dagger\sigma^\dagger)(\sigma\chi)\right]_{\underline{1}} + \bar{\lambda}_{25}\left[(\eta'^\dagger s^\dagger)(s\chi)\right]_{\underline{1}} + \bar{\lambda}_{26}\left[(\rho^\dagger s^\dagger)(s\phi)\right]_{\underline{1}} \\ & + \bar{\lambda}_{27}\phi\left[(\phi s^\dagger)s^\dagger\right]_{\underline{1}} + \bar{\lambda}_{28}\phi\left[(\phi\sigma^\dagger)\sigma^\dagger\right]_{\underline{1}} + \bar{\lambda}_{29}\phi'\left[(\phi' s^\dagger)s^\dagger\right]_{\underline{1}'} \\ & + \bar{\lambda}_{30}\phi'\left[(\phi'\sigma^\dagger)\sigma^\dagger\right]_{\underline{1}'} + \bar{\lambda}_{31}\phi\left[(\phi' s^\dagger)s^\dagger\right]_{\underline{1}} + \bar{\lambda}_{32}\phi'\left[(\phi s^\dagger)s^\dagger\right]_{\underline{1}'} + \mathrm{h.c.}\end{aligned} \tag{70}$$

In the decomposing of $\underline{2}\otimes\underline{2}$, $\underline{2}\otimes\underline{2} = \underline{1}\oplus\underline{1}'\oplus\underline{1}''\oplus\underline{1}'''$, there is no term which, as shown in (68), (69) and (70), is invariant under combination of one scalar triplet and two different antisixtets; and some couplings between ρ and some other triplets are ruled out. As a consequence, the general scalar potential violating $\mathcal{L}$ and being invariant under D_4, is more simpler than those of S_3 and S_4.

Let us now consider the potential V_{tri}. The flavons χ, ϕ, ϕ', η, η' with their VEVs aligned in the same direction (all of them are singlets) are an automatical solution from the minimization conditions of V_{tri}. To explicitly see this, in the system of equations for minimization, let us put $v^* = v$, $v'^* = v'$, $u^* = u$, $u'^* = u'$, $v_\chi^* = v_\chi$, $v_\rho^* = v_\rho$. Then, the potential minimization conditions for triplets reduces to

$$\begin{aligned}\frac{\partial V_{\mathrm{tri}}}{\partial\omega} = {} & 4\lambda^\chi\omega^3 + 2\Big(\mu_\chi^2 + \lambda_1^{\chi\eta}u^2 + \lambda_1^{\chi\eta'}u'^2 + \lambda_1^{\chi\phi}v^2 + \lambda_1^{\chi\phi'}v'^2 + \lambda_1^{\chi\rho}v_\rho^2\Big)\omega \\ & - \mu_1 uv - \mu_1' u'v',\end{aligned} \tag{71}$$

$$\begin{aligned}\frac{\partial V_{\mathrm{tri}}}{\partial v_\rho} = {} & 4\lambda^\rho v_\rho^3 + 2\Big[\mu_\rho^2 + \lambda_1^{\rho\eta}u^2 + \lambda_1^{\rho\eta'}u'^2 + \Big(\lambda_1^{\rho\phi} + \lambda_2^{\rho\phi}\Big)v^2 \\ & + \Big(\lambda_1^{\rho\phi'} + \lambda_2^{\rho\phi'}\Big)v'^2 + \lambda_1^{\chi\rho}\omega^2\Big]v_\rho,\end{aligned} \tag{72}$$

$$\begin{aligned}\frac{\partial V_{\mathrm{tri}}}{\partial v} = {} & 4\lambda^\phi v^3 + 2\Big[\mu_\phi^2 + \lambda_1^{\phi\eta}u^2 + \lambda_1^{\phi\eta'}u'^2 + \Big(\lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'}\Big)v'^2 \\ & + \Big(\lambda_1^{\rho\phi} + \lambda_2^{\rho\phi}\Big)v_\rho^2 + \omega^2\lambda_1^{\phi\chi}\Big]v + \Big(\lambda_1^1 + \lambda_1^2\Big)uu'v' - \mu_1\omega u,\end{aligned} \tag{73}$$

$$\frac{\partial V_{\rm tri}}{\partial v'} = 4\lambda^{\phi'} v'^3 + 2\Big[\mu^2_{\phi'} + \lambda_1^{\phi'\eta} u^2 + \lambda_1^{\phi'\eta'} u'^{\,2} + \Big(\lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'}\Big) v^2 + \Big(\lambda_1^{\rho\phi'} + \lambda_2^{\rho\phi'}\Big) v_\rho^2 + \omega^2 \lambda_1^{\phi'\chi}\Big] v' + \Big(\lambda_1^1 + \lambda_1^2\Big) u u' v - \mu_1' \omega u' , \tag{74}$$

$$\frac{\partial V_{\rm tri}}{\partial u} = 4\lambda^{\eta} u^3 + 2\Big[\mu^2_{\eta} + \Big(\lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \lambda_4^{\eta\eta'}\Big) u'^{\,2} + \lambda_1^{\phi'\eta} v'^{\,2} + \lambda_1^{\phi\eta} v^2 + \lambda_1^{\eta\rho} v_\rho^2 + \omega^2 \lambda_1^{\eta\chi}\Big] u + \Big(\lambda_1^1 + \lambda_1^2\Big) u' v v' - \mu_1 \omega v , \tag{75}$$

$$\frac{\partial V_{\rm tri}}{\partial u'} = 4\lambda^{\eta'} u'^3 + 2\Big[\mu^2_{\eta'} + \Big(\lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \lambda_4^{\eta\eta'}\Big) u^2 + \lambda_1^{\phi\eta'} v^2 + \lambda_1^{\phi'\eta'} v'^{\,2} + \lambda_1^{\eta'\rho} v_\rho^2 + \omega^2 \lambda_1^{\eta'\chi}\Big] u' + \Big(\lambda_1^1 + \lambda_1^2\Big) u v v' - \mu_1' \omega v' . \tag{76}$$

It is easy to see that the derivatives of $V_{\rm tri}$ with respect to the variable ω and v_ρ shown in (71) and (72) are symmetric to each other. Similarly, the two pairs (v, v') and (u, u') behave the same as shown in Eqs. (73)–(76). The parameters $\lambda_2^{\chi\phi}$, $\lambda_2^{\chi\phi'}$ in Eq. (71) vanish because of the interaction $(\phi^+\chi)(\chi^+\phi)$ in (64), and the parameters $\lambda^{\chi\phi}_{2,3,4}$, $\lambda^{\chi\phi'}_{2,3,4}$ in Eqs. (71) and (72) vanish due to the symmetries of the model (such as L or X or $\mathcal{L}$ or D_4 or one of their combinations).

The system of equations (71)–(76) always has the solution (u, v, u', v') as expected, even though the complication. It is also noted that the above alignment is only one of the solutions to be imposed to have the desirable results. We have evaluated that the Eqs. (73)–(76) have the same structure solution. The solution is as follows

$$u = u' = v' = v = \pm\sqrt{\alpha}\,, \tag{77}$$

with

$$\begin{aligned}\alpha = {} & \Big\{-\omega^2\Big(\lambda_1^{\chi\eta} + \lambda_1^{\chi\eta'} + \lambda_1^{\chi\phi} + \lambda_1^{\chi\phi'}\Big) + \omega(\mu_1 + \mu_1') - \mu_\eta^2 - \mu_{\eta'}^2 - \mu_\phi^2 - \mu_{\phi'}^2 \\ & + \Big(\lambda_1^{\eta\rho} + \lambda_1^{\eta'\rho} + \lambda_1^{\phi\rho} + \lambda_1^{\phi'\rho} + \lambda_2^{\phi\rho} + \lambda_2^{\phi'\rho}\Big) v_\rho^2\Big\} \Big/ \\ & \Big\{2\Big[\lambda_1^1 + \lambda_1^2 + \lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \lambda_4^{\eta\eta'} + \lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} \\ & + \lambda_4^{\phi\phi'} + \lambda_1^{\phi\eta} + \lambda_1^{\phi\eta'} + \lambda_1^{\phi'\eta} + \lambda_1^{\phi'\eta'} + \lambda^{\eta} + \lambda^{\eta'} + \lambda^{\phi} + \lambda^{\phi'}\Big]\Big\} \\ \simeq {} & \frac{-2\omega^2\lambda_1^{\chi\phi} + \omega\mu_1 - 2\mu_\phi^2 + 3\lambda_1^{\phi\rho} v_\rho^2}{2\lambda_1^1 + 12\lambda_1^{\phi\phi'} + 4\lambda^{\phi}}\,.\end{aligned} \tag{78}$$

Substituting (78) into (71) and (72) we obtain

$$\frac{\partial V_{\rm tri}}{\partial \omega} = 4\lambda^{\chi}\omega^3 + 2\Big(\mu_\chi^2 + 4\lambda_1^{\chi\phi} v^2 + \lambda_1^{\chi\rho} v_\rho^2\Big)\omega - 2\mu_1 v^2 , \tag{79}$$

$$\frac{\partial V_{\rm tri}}{\partial v_\rho} = 4\lambda^{\rho} v_\rho^3 + 2\Big(\mu_\rho^2 + 6\lambda_1^{\rho\phi} v^2 + \lambda_1^{\chi\rho}\omega^2\Big) v_\rho . \tag{80}$$

Noting that the solution (77) leads to special relations among coupling constants: $\lambda^{\eta} = \lambda^{\eta'} = \lambda^{\phi} = \lambda^{\phi'}$ and mass parameters $\mu^2_{\eta} = \mu^2_{\eta'} = \mu^2_{\phi} = \mu^2_{\phi'}$ and so on. In general, these couplings and mass parameters are independent, however, the neutrino data and the discrete D_4 symmetry force them being related. This is the common property of the discrete flavor symmetries.

Considering the potential V_{sex} and $V_{\text{tri-sex}}$, we urge that the contribution of $V_{\chi\phi\phi'\eta\eta'\rho}$ in (65) is very small in comparison with the other terms in V_{tri}, so it can be neglected. From (62) to (65) and with the help of (4), (10), (29), and imposing that

$$\lambda_1^* = \lambda_1\,, \quad \lambda_2^* = \lambda_2\,, \quad v_1^* = v_1\,, \quad v_2^* = v_2\,, \quad \Lambda_1^* = \Lambda_1\,, \quad \Lambda_2^* = \Lambda_2\,,$$
$$\lambda_\sigma^* = \lambda_\sigma\,, \quad v_\sigma^* = v_\sigma\,, \quad \Lambda_\sigma^* = \Lambda_\sigma\,,$$
$$v^* = v\,, \quad v'^* = v'\,, \quad u^* = u\,, \quad u'^* = u'\,, \quad v_\chi^* = v_\chi\,, \quad v_\rho^* = v_\rho\,,$$

we obtain a system of equations of the potential minimization for antisextets:

$$\begin{aligned}\frac{\partial V_1}{\partial \lambda_1^*} &= v_\chi^2 \lambda_1^{\chi s}\lambda_1 + 2\big(\lambda_1^s + \lambda_2^s + \lambda_5^s + \lambda_6^s\big)\lambda_1^3 + 4\lambda_7^s\lambda_2\Lambda_1\Lambda_2 \\
&\quad + \lambda_4^{s\sigma}\Lambda_1\Lambda_\sigma\lambda_\sigma + 2(\lambda_1^s + \lambda_2^s)\Lambda_1 v_1^2 + 2\big(\lambda_1^s\lambda_2 + \lambda_1^s\Lambda_2 - \lambda_2^s\lambda_2 - \lambda_2^s\Lambda_2 \\
&\quad + 3\lambda_3^s\lambda_2 + \lambda_3^s\Lambda_2 + \lambda_4^s\lambda_2 - \lambda_4^s\Lambda_2 + 4\lambda_7^s\lambda_2\big)v_1 v_2 \\
&\quad + 2\big(\lambda_3^s + \lambda_4^s\big)\Lambda_1 v_2^2 + \big[\big(\lambda_1^{s\sigma} + \lambda_3^{s\sigma} + 2\lambda_4^{s\sigma}\big)\lambda_\sigma + \big(\Lambda_1^{s\sigma} + \lambda_3^{s\sigma}\big)\Lambda_\sigma\big]v_1 v_\sigma \\
&\quad + \lambda_1\Big\{2\big(\lambda_5^s + \lambda_6^s\big)\Lambda_1^2 + \big(\lambda_1^{s\sigma} + \lambda_2^{s\sigma} + \lambda_3^{s\sigma} + \lambda_4^{s\sigma}\big)\lambda_\sigma^2 + \lambda_2^{s\sigma}\Lambda_\sigma^2 \\
&\quad + \mu_s^2 + \big(\lambda_1^{\eta s} + \lambda_2^{\eta s}\big)u^2 + \big(\lambda_1^{\eta' s} + \lambda_2^{\eta' s}\big)u'^2 + \lambda_1^{\phi s}v^2 + \lambda_1^{\phi' s}v'^2 \\
&\quad + 2\Big[\big(\lambda_5^s - \lambda_6^s\big)\Lambda_2^2 + \lambda_2^2\big(\lambda_1^s - \lambda_2^s + 2\lambda_3^s + \lambda_5^s - \lambda_6^s + 2\lambda_7^s\big) \\
&\quad + 2\big(\lambda_1^s + \lambda_2^s + \lambda_5^s + \lambda_6^s\big)v_1^2 + \big(\lambda_1^s - \lambda_2^s + \lambda_3^s - \lambda_4^s + 2\lambda_5^s - 2\lambda_6^s\big)v_2^2\Big] \\
&\quad + \lambda_1^{\rho s}v_\rho^2 + \big(\lambda_1^{\sigma s} + 2\lambda_2^{\sigma s} + \lambda_3^{\sigma s}\big)v_\sigma^2\Big\}\,,\end{aligned} \tag{81}$$

$$\begin{aligned}\frac{\partial V_1}{\partial \lambda_2^*} &= v_\chi^2 \lambda_1^{\chi s}\lambda_2 + 2\big(\lambda_1^s + \lambda_2^s + \lambda_5^s + \lambda_6^s\big)\lambda_2^3 + 4\lambda_7^s\lambda_1\Lambda_1\Lambda_2 \\
&\quad + \lambda_4^{s\sigma}\Lambda_2\Lambda_\sigma\lambda_\sigma + 2\big(\lambda_1^s + \lambda_2^s\big)\Lambda_2 v_2^2 + 2\big(\lambda_1^s\lambda_1 + \lambda_1^s\Lambda_1 - \lambda_2^s\lambda_1 \\
&\quad - \lambda_2^s\Lambda_1 + 3\lambda_3^s\lambda_1 + \lambda_3^s\Lambda_1 + \lambda_4^s\lambda_1 - \lambda_4^s\Lambda_1 + 4\lambda_7^s\lambda_1\big)v_1 v_2 \\
&\quad + 2\big(\lambda_3^s + \lambda_4^s\big)\Lambda_2 v_1^2 + \Big[\big(\lambda_1^{s\sigma} + \lambda_3^{s\sigma} + 2\lambda_4^{s\sigma}\big)\lambda_\sigma + \big(\Lambda_1^{s\sigma} + \lambda_3^{s\sigma}\big)\Lambda_\sigma\Big]v_2 v_\sigma \\
&\quad + \lambda_2\Big\{2\big(\lambda_5^s + \lambda_6^s\big)\Lambda_2^2 + \big(\lambda_1^{s\sigma} + \lambda_2^{s\sigma} + \lambda_3^{s\sigma} + \lambda_4^{s\sigma}\big)\lambda_\sigma^2 + \lambda_2^{s\sigma}\Lambda_\sigma^2\end{aligned}$$

$$
\begin{aligned}
&+\mu_s^2+\left(\lambda_1^{\eta s}+\lambda_2^{\eta s}\right)u^2+\left(\lambda_1^{\eta' s}+\lambda_2^{\eta' s}\right)u'^2+\lambda_1^{\phi s}v^2+\lambda_1^{\phi' s}v'^2\\
&+2\Big[\left(\lambda_5^s-\lambda_6^s\right)\Lambda_1^2+\lambda_1^2\left(\lambda_1^s-\lambda_2^s+2\lambda_3^s+\lambda_5^s-\lambda_6^s+2\lambda_7^s\right)\\
&+2\left(\lambda_1^s+\lambda_2^s+\lambda_5^s+\lambda_6^s\right)v_2^2+\left(\lambda_1^s-\lambda_2^s+\lambda_3^s-\lambda_4^s+2\lambda_5^s-2\lambda_6^s\right)v_1^2\Big]\\
&+\lambda_1^{\rho s}v_\rho^2+\left(\lambda_1^{\sigma s}+2\lambda_2^{\sigma s}+\lambda_3^{\sigma s}\right)v_\sigma^2\Big\}\,, \qquad (82)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_1}{\partial v_1^*}=&\Big[v_\chi^2\left(2\lambda_1^{\chi s}+\lambda_2^{\chi s}\right)+2\mu_s^2+\left(2\lambda_1^{\eta s}+\lambda_2^{\eta s}\right)u^2\\
&+\left(2\lambda_1^{\eta' s}+\lambda_2^{\eta' s}\right)u'^2+2\lambda_1^{\phi s}v^2+2\lambda_1^{\phi' s}v'^2\Big]v_1\\
&-2\lambda_4^s(\lambda_2-\Lambda_2)\left[(\lambda_2-\Lambda_2)v_1-(\lambda_1-\Lambda_1)v_2\right]\\
&+8\lambda_7^s v_2(\lambda_1\lambda_2+\Lambda_1\Lambda_2+2v_1v_2)\\
&+4\lambda_6^s v_1\left[\left(\lambda_1^2-\lambda_2^2\right)+\left(\Lambda_1^2-\Lambda_2^2\right)+2\left(v_1^2-v_2^2\right)\right]\\
&+2\Big\{v_1\left[\left(\lambda_2^2+\Lambda_2^2\right)\left(\lambda_1^s-\lambda_2^s\right)+2\left(\lambda_1^2+\lambda_1\Lambda_1+\Lambda_1^2+v_1^2\right)\left(\lambda_1^s+\lambda_2^s\right)\right]\\
&+(\lambda_1+\Lambda_1)(\lambda_2+\Lambda_2)\left(\lambda_1^s-\lambda_2^s\right)v_2+2\left(\lambda_1^s-\lambda_2^s\right)v_1v_2^2\Big\}\\
&+2\lambda_3\left\{(\lambda_2+\Lambda_2)^2v_1+\left[\lambda_1(3\lambda_2+\Lambda_2)+\Lambda_1(\lambda_2+3\Lambda_2)\right]v_2+4v_1v_2^2\right\}\\
&+4\lambda_5^s v_1\left[\lambda_1^2+\lambda_2^2+\Lambda_1^2+\Lambda_2^2+2\left(v_1^2+v_2^2\right)\right]\\
&+2\lambda_1^{\rho s}v_1v_\rho^2+2\lambda_4^{s\sigma}v_\sigma(\lambda_1\lambda_\sigma+\Lambda_1\Lambda_\sigma+2v_1v_\sigma)+2\lambda_2^{s\sigma}v_1\left(\lambda_\sigma^2+\Lambda_\sigma^2+2v_\sigma^2\right)\\
&+\left(\lambda_1^{s\sigma}+\lambda_3^{s\sigma}\right)\left[\left(\lambda_\sigma^2+\Lambda_\sigma^2\right)v_1+(\lambda_1+\Lambda_1)(\lambda_\sigma+\Lambda_\sigma)v_\sigma+2v_1v_\sigma^2\right]\,, \qquad (83)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_1}{\partial v_2^*}=&\Big[v_\chi^2\left(2\lambda_1^{\chi s}+\lambda_2^{\chi s}\right)+2\mu_s^2+\left(2\lambda_1^{\eta s}+\lambda_2^{\eta s}\right)u^2\\
&+\left(2\lambda_1^{\eta' s}+\lambda_2^{\eta' s}\right)u'^2+2\lambda_1^{\phi s}v^2+2\lambda_1^{\phi' s}v'^2\Big]v_2\\
&-2\lambda_4^s(\lambda_1-\Lambda_1)[(\lambda_1-\Lambda_1)v_2-(\lambda_2-\Lambda_2)v_1]\\
&+8\lambda_7^s v_1(\lambda_1\lambda_2+\Lambda_1\Lambda_2+2v_1v_2)\\
&+4\lambda_6^s v_2\left[\left(\lambda_2^2-\lambda_1^2\right)+\left(\Lambda_2^2-\Lambda_1^2\right)+2\left(v_2^2-v_1^2\right)\right]\\
&+2\Big\{v_2\left[\left(\lambda_1^2+\Lambda_1^2\right)\left(\lambda_1^s-\lambda_2^s\right)+2\left(\lambda_2^2+\lambda_2\Lambda_2+\Lambda_2^2+v_2^2\right)\left(\lambda_1^s+\lambda_2^s\right)\right]\\
&+(\lambda_1+\Lambda_1)(\lambda_2+\Lambda_2)\left(\lambda_1^s-\lambda_2^s\right)v_1+2\left(\lambda_1^s-\lambda_2^s\right)v_2v_1^2\Big\}\\
&+2\lambda_3\left\{(\lambda_1+\Lambda_1)^2v_2+\left[\lambda_1(3\lambda_2+\Lambda_2)+\Lambda_1(\lambda_2+3\Lambda_2)\right]v_1+4v_2v_1^2\right\}
\end{aligned}
$$

$$+ 4\lambda_5^s v_2\big[\lambda_1^2 + \lambda_2^2 + \Lambda_1^2 + \Lambda_2^2 + 2\big(v_1^2 + v_2^2\big)\big] + 2\lambda_1^{\rho s} v_2 v_\rho^2$$
$$+ 2\lambda_4^{s\sigma} v_\sigma(\lambda_2\lambda_\sigma + \Lambda_2\Lambda_\sigma + 2v_2 v_\sigma) + 2\lambda_2^{s\sigma} v_2\big(\lambda_\sigma^2 + \Lambda_\sigma^2 + 2v_\sigma^2\big)$$
$$+ \big(\lambda_1^{s\sigma} + \lambda_3^{s\sigma}\big)\big[\big(\lambda_\sigma^2 + \Lambda_\sigma^2\big)v_2 + (\lambda_2 + \Lambda_2)(\lambda_\sigma + \Lambda_\sigma)v_\sigma + 2v_2 v_\sigma^2\big]\,, \qquad (84)$$

$$\frac{\partial V_1}{\partial \Lambda_1^*} = v_\chi^2\big(\lambda_1^{\chi s} + \lambda_2^{\chi s}\big)\Lambda_1 + 2\big(\lambda_1^s + \lambda_2^s + \lambda_5^s + \lambda_6^s\big)\Lambda_1^3$$
$$+ 4\lambda_7^s\lambda_2\lambda_1\Lambda_2 + \lambda_4^{s\sigma}\lambda_1\Lambda_\sigma\lambda_\sigma + 2\big(\lambda_1^s + \lambda_2^s\big)\lambda_1 v_1^2$$
$$+ 2\big(\lambda_1^s\lambda_2 + \lambda_1^s\Lambda_2 - \lambda_2^s\lambda_2 - \lambda_2^s\Lambda_2 + 3\lambda_3^s\Lambda_2 + \lambda_3^s\lambda_2$$
$$+ \lambda_4^s\Lambda_2 - \lambda_4^s\lambda_2 + 4\lambda_7^s\Lambda_2)v_1 v_2 + 2\big(\lambda_3^s + \lambda_4^s\big)\lambda_1 v_2^2$$
$$+ \Big[\big(\lambda_1^{s\sigma} + \lambda_3^{s\sigma} + 2\lambda_4^{s\sigma}\big)\Lambda_\sigma + \big(\Lambda_1^{s\sigma} + \lambda_3^{s\sigma}\big)\lambda_\sigma\Big]v_1 v_\sigma$$
$$+ \Lambda_1\Big\{2\big(\lambda_5^s - \lambda_6^s\big)\lambda_2^2 + \big(\lambda_1^{s\sigma} + \lambda_3^{s\sigma} + \lambda_4^{s\sigma}\big)\Lambda_\sigma^2$$
$$+ \mu_s^2 + \lambda_1^{\eta s}u^2 + \lambda_1^{\eta' s}u'^2 + \lambda_1^{\phi s}v^2 + \lambda_1^{\phi' s}v'^2$$
$$+ 2\Big[\Lambda_2^2\big(\lambda_1^s - \lambda_2^s + 2\lambda_3^s + \lambda_5^s - \lambda_6^s + 2\lambda_7^s\big) + 2\big(\lambda_1^s + \lambda_2^s + \lambda_5^s + \lambda_6^s\big)v_1^2$$
$$+ \big(\lambda_1^s - \lambda_2^s + \lambda_3^s - \lambda_4^s + 2\lambda_5^s - 2\lambda_6^s\big)v_2^2\Big]$$
$$+ \lambda_1^{\rho s}v_\rho^2 + \big(\lambda_1^{\sigma s} + \lambda_3^{\sigma s}\big)v_\sigma^2 + \lambda_2^{\sigma s}\big(\lambda_\sigma^2 + \Lambda_\sigma^2 + 2v_\sigma^2\big)\Big\}\,, \qquad (85)$$

$$\frac{\partial V_1}{\partial \Lambda_2^*} = v_\chi^2\big(\lambda_1^{\chi s} + \lambda_2^{\chi s}\big)\Lambda_2 + 2\big(\lambda_1^s + \lambda_2^s + \lambda_5^s + \lambda_6^s\big)\Lambda_2^3$$
$$+ 4\lambda_7^s\lambda_2\lambda_1\Lambda_1 + \lambda_4^{s\sigma}\lambda_2\Lambda_\sigma\lambda_\sigma + 2\big(\lambda_3^s + \lambda_4^s\big)\lambda_2 v_1^2$$
$$+ 2\big(\lambda_1^s\lambda_1 + \lambda_1^s\Lambda_1 - \lambda_2^s\lambda_1 - \lambda_2^s\Lambda_1 + 3\lambda_3^s\Lambda_1 + \lambda_3^s\lambda_1$$
$$+ \lambda_4^s\Lambda_1 - \lambda_4^s\lambda_1 + 4\lambda_7^s\Lambda_1\big)v_1 v_2 + 2\big(\lambda_1^s + \lambda_2^s\big)\lambda_2 v_2^2$$
$$+ \big[\big(\lambda_1^{s\sigma} + \lambda_3^{s\sigma} + 2\lambda_4^{s\sigma}\big)\Lambda_\sigma + \big(\Lambda_1^{s\sigma} + \lambda_3^{s\sigma}\big)\lambda_\sigma\big]v_2 v_\sigma$$
$$+ \Lambda_2\Big\{2\big(\lambda_5^s - \lambda_6^s\big)\lambda_1^2 + \big(\lambda_1^{s\sigma} + \lambda_3^{s\sigma} + \lambda_4^{s\sigma}\big)\Lambda_\sigma^2$$
$$+ \mu_s^2 + \lambda_1^{\eta s}u^2 + \lambda_1^{\eta' s}u'^2 + \lambda_1^{\phi s}v^2 + \lambda_1^{\phi' s}v'^2$$
$$+ 2\big[\Lambda_1^2\big(\lambda_1^s - \lambda_2^s + 2\lambda_3^s + \lambda_5^s - \lambda_6^s + 2\lambda_7^s\big) + 2\big(\lambda_1^s + \lambda_2^s + \lambda_5^s + \lambda_6^s\big)v_2^2$$
$$+ \big(\lambda_1^s - \lambda_2^s + \lambda_3^s - \lambda_4^s + 2\lambda_5^s - 2\lambda_6^s\big)v_1^2\big]$$
$$+ \lambda_1^{\rho s}v_\rho^2 + \big(\lambda_1^{\sigma s} + \lambda_3^{\sigma s}\big)v_\sigma^2 + \lambda_2^{\sigma s}\big(\lambda_\sigma^2 + \Lambda_\sigma^2 + 2v_\sigma^2\big)\Big\}\,, \qquad (86)$$

where V_1 is a sum of V_{sext} and $V_{\text{tri-sext}}$:

$$V_1 = V_{\text{sext}} + V_{\text{tri-sext}} \,. \tag{87}$$

It is easy to see that Eqs. (81)–(86) take the same form in couples. This system of equations yields the following solutions

$$\lambda_1 = \beta\lambda_2 \,, \quad v_1 = \beta v_2 \,, \quad \Lambda_1 = \beta\Lambda_2 \,, \tag{88}$$

where β is a constant. It means that there are several alignments for VEVs. In this work, to have the desirable results, we have imposed the two directions for breaking $D_4 \to Z_2 \otimes Z_2$ and $D_4 \to Z_2$ as mentioned, in which $\beta = 1$ and $\beta \neq 1$ but is approximate to the unit. In the case that $\beta = 1$ or $\langle s_1\rangle = \langle s_2\rangle$, we have

$$\frac{\partial V_1}{\partial \lambda_1} = \frac{\partial V_1}{\partial \lambda_2} \equiv \frac{\partial V_1}{\partial \lambda_s} \,, \quad \frac{\partial V_1}{\partial v_1} = \frac{\partial V_1}{\partial v_2} \equiv \frac{\partial V_1}{\partial v_s} \,, \quad \frac{\partial V_1}{\partial \Lambda_1} = \frac{\partial V_1}{\partial \Lambda_2} \equiv \frac{\partial V_1}{\partial \Lambda_s} \,, \tag{89}$$

and this system reduces to

$$\begin{aligned}
\frac{\partial V_1}{\partial \lambda_s} &= 4\big(\lambda^s_1 + \lambda^s_3 + \lambda^s_5 + \lambda^s_7\big)\lambda^3_s \\
&\quad + \lambda_s\big[\mu^2_s + 4\big(\lambda^s_5 + \lambda^s_7\big)\Lambda^2_s + \big(\lambda^{s\sigma}_1 + \lambda^{s\sigma}_2 + \lambda^{s\sigma}_3 + \lambda^{s\sigma}_4\big)\lambda^2_\sigma \\
&\quad + \lambda^{s\sigma}_2\Lambda^2_\sigma + 8\big(\lambda^s_1 + \lambda^s_3 + \lambda^s_5 + \lambda^s_7\big)v^2_s + \big(\lambda^{\sigma s}_1 + 2\lambda^{\sigma s}_2 + \lambda^{\sigma s}_3\big)v^2_\sigma\big] \\
&\quad + 4\big(\lambda^s_1 + \lambda^s_3\big)\Lambda_s v^2_s + \big(\lambda^{s\sigma}_1 + \lambda^{s\sigma}_3\big)\big(\lambda_\sigma + \Lambda_\sigma\big)v_s v_\sigma \\
&\quad + \lambda^{s\sigma}_4\big(\Lambda_s\Lambda_\sigma + 2v_s v_\sigma\big)\lambda_\sigma \,,
\end{aligned} \tag{90}$$

$$\begin{aligned}
\frac{\partial V_1}{\partial v_s} &= 4\big(\lambda^s_1 + \lambda^s_3 + 2\lambda^s_5 + 2\lambda^s_7\big)v^3_s \\
&\quad + v_s\big[2\mu^2_s + 8\big(\lambda^s_1 + \lambda^s_3 + \lambda^s_5 + \lambda^s_7\big)\big(\lambda^2_s + \Lambda^2_s\big) \\
&\quad + \big(\lambda^{s\sigma}_1 + 2\lambda^{s\sigma}_2 + \lambda^{s\sigma}_3\big)\big(\lambda^2_\sigma + \Lambda^2_\sigma\big) \\
&\quad + 8\big(\lambda^s_1 + \lambda^s_3\big)\lambda_s\Lambda_s + 2\big(\lambda^{\sigma s}_1 + 2\lambda^{\sigma s}_2 + \lambda^{\sigma s}_3 + 2\lambda^{\sigma s}_4\big)v^2_\sigma\big] \\
&\quad + \big[\big(\lambda^{s\sigma}_1 + \lambda^{s\sigma}_3\big)\big(\lambda_\sigma + \Lambda_\sigma\big)(\lambda_s + \Lambda_s) + 2\lambda^{s\sigma}_4(\lambda_s\Lambda_s + \lambda_\sigma\Lambda_\sigma)\big]v_\sigma \,,
\end{aligned} \tag{91}$$

$$\begin{aligned}
\frac{\partial V_1}{\partial \Lambda_s} &= 4\big(\lambda^s_1 + \lambda^s_3 + \lambda^s_5 + \lambda^s_7\big)\Lambda^3_s \\
&\quad + \Lambda_s\big[\mu^2_s + 4\big(\lambda^s_5 + \lambda^s_7\big)\lambda^2_s + \big(\lambda^{s\sigma}_1 + \lambda^{s\sigma}_2 + \lambda^{s\sigma}_3 + \lambda^{s\sigma}_4\big)\Lambda^2_\sigma + \lambda^{s\sigma}_2\lambda^2_\sigma \\
&\quad + 8\big(\lambda^s_1 + \lambda^s_3 + \lambda^s_5 + \lambda^s_7\big)v^2_s + \big(\lambda^{\sigma s}_1 + 2\lambda^{\sigma s}_2 + \lambda^{\sigma s}_3\big)v^2_\sigma\big] + 4\big(\lambda^s_1 + \lambda^s_3\big)\lambda_s v^2_s \\
&\quad + \big(\lambda^{s\sigma}_1 + \lambda^{s\sigma}_3\big)(\lambda_\sigma + \Lambda_\sigma)v_s v_\sigma + \lambda^{s\sigma}_4(\lambda_s\lambda_\sigma + 2v_s v_\sigma)\Lambda_\sigma \,.
\end{aligned} \tag{92}$$

The derivatives of V_1 with respect to the variable λ_s and Λ_s as shown in (90), (92) are symmetric to each other.

7. Gauge Bosons

The covariant derivative of a triplet is given by

$$D_\mu = \partial_\mu - ig\frac{\lambda_a}{2}W_{\mu a} - ig_X X\frac{\lambda_9}{2}B_\mu = \partial_\mu - iP_\mu\,, \tag{93}$$

where λ_a $(a = 1, 2, \ldots, 8)$ are Gell-Mann matrices, $\lambda_9 = \sqrt{\frac{2}{3}}\mathrm{diag}(1, 1, 1)$, $\mathrm{Tr}\,\lambda_a\lambda_b = 2\delta_{ab}$, $\mathrm{Tr}\,\lambda_9\lambda_9 = 2$, and X is X-charged of Higgs triplets. Let us denote the following combinations:

$$\begin{aligned} W'^{+}_\mu &= \frac{W_{\mu 1} - iW_{\mu 2}}{\sqrt{2}}\,, \quad X'^{0}_\mu = \frac{W_{\mu 4} - iW_{\mu 5}}{\sqrt{2}}\,, \\ Y'^{-}_\mu &= \frac{W_{\mu 6} - iW_{\mu 7}}{\sqrt{2}}\,, \quad W'^{-}_\mu = (W'^{+}_\mu)^*\,, \quad Y'^{+}_\mu = (Y'^{-}_\mu)^*\,, \end{aligned} \tag{94}$$

then P_μ is rewritten in a convenient form as follows:

$$\frac{g}{2}\begin{pmatrix} W_{\mu 3} + \frac{W_{\mu 8}}{\sqrt{3}} + t\sqrt{\frac{2}{3}}XB_\mu & \sqrt{2}W'^{+}_\mu & \sqrt{2}X'^{0}_\mu \\ \sqrt{2}W'^{-}_\mu & -W_{\mu 3} + \frac{W_{\mu 8}}{\sqrt{3}} + t\sqrt{\frac{2}{3}}XB_\mu & \sqrt{2}Y'^{-}_\mu \\ \sqrt{2}X'^{0*}_\mu & \sqrt{2}Y'^{+}_\mu & -\frac{2}{\sqrt{3}}W_{\mu 8} + t\sqrt{\frac{2}{3}}XB_\mu \end{pmatrix}, \tag{95}$$

with

$$t = \frac{g_X}{g}\,.$$

We note that W_4 and W_5 are pure real and imaginary parts of X^0 and X^{0*}, respectively.

The covariant derivative for an antisextet with the VEV part is[150,158]

$$D_\mu\langle s_i\rangle = \frac{ig}{2}\left\{W^a_\mu\lambda^*_a\langle s_i\rangle + \langle s_i\rangle W^a_\mu\lambda^{*T}_a\right\} + ig_X T_9 X B_\mu\langle s_i\rangle\,. \tag{96}$$

The covariant derivative (96) acting on the antisextet VEVs are given by

$$[D_\mu\langle s_i\rangle]_{11} = ig\left(\lambda_i W_{\mu 3} + \frac{\lambda_i}{\sqrt{3}}W_{\mu 8} + \frac{1}{3}\sqrt{\frac{2}{3}}t\lambda_i B_\mu + \sqrt{2}v_i X'^{0*}\right),$$

$$[D_\mu\langle s_i\rangle]_{12} = \frac{ig}{\sqrt{2}}\left(\lambda_i W'^{+}_\mu + v_i Y'^{+}_\mu\right),$$

$$[D_\mu\langle s_i\rangle]_{13} = \frac{ig}{2}\left(v_i W_{\mu 3} - \frac{v_i}{\sqrt{3}}W_{\mu 8} + \frac{2}{3}\sqrt{\frac{2}{3}}tv_i B_\mu + \sqrt{2}\lambda_i X'^{0}_\mu + \sqrt{2}\Lambda_i X'^{0*}_\mu\right),$$

$$[D_\mu\langle s_i\rangle]_{22} = 0\,, \quad [D_\mu\langle s_i\rangle]_{23} = \frac{ig}{\sqrt{2}}\left(v_i W'^{+}_\mu + \Lambda_i Y'^{+}_\mu\right),$$

$$[D_\mu\langle s_i\rangle]_{33} = ig\left(-\frac{2}{\sqrt{3}}\Lambda_i W_{\mu 8} + \frac{1}{3}\sqrt{\frac{2}{3}}t\Lambda_i B_\mu + \sqrt{2}v_i X'^0_\mu\right),$$

$$[D_\mu\langle s_i\rangle]_{21} = [D_\mu\langle s_i\rangle]_{12}\,, \quad [D_\mu\langle s_i\rangle]_{31} = [D_\mu\langle s_i\rangle]_{13}\,, \quad [D_\mu\langle s_i\rangle]_{32} = [D_\mu\langle s_i\rangle]_{23}\,.$$

The masses of gauge bosons in this model are defined

$$\begin{aligned}\mathcal{L}^{\rm GB}_{\rm mass} &= (D_\mu\langle\phi\rangle)^+(D^\mu\langle\phi\rangle) + (D_\mu\langle\phi'\rangle)^+(D^\mu\langle\phi'\rangle) + (D_\mu\langle\chi\rangle)^+(D^\mu\langle\chi\rangle)\\ &\quad + (D_\mu\langle\eta\rangle)^+(D^\mu\langle\eta\rangle) + (D_\mu\langle\eta'\rangle)^+(D^\mu\langle\eta'\rangle) + (D_\mu\langle\rho\rangle)^+(D^\mu\langle\rho\rangle)\\ &\quad + {\rm Tr}[(D_\mu\langle s_1\rangle)^+(D^\mu\langle s_1\rangle)] + {\rm Tr}[(D_\mu\langle s_2\rangle)^+(D^\mu\langle s_2\rangle)]\\ &\quad + {\rm Tr}[(D_\mu\langle\sigma\rangle)^+(D^\mu\langle\sigma\rangle)]\,,\end{aligned} \tag{97}$$

where $\mathcal{L}^{\rm GB}_{\rm mass}$ in (97) is different from one in Ref. 150, by the contribution from the ρ and the term relating to the antisextet σ. In Ref. 150 the ρ and s' contributions were skipped at the first-order. In the following, we note that $\langle s_1\rangle = \langle s_1\rangle$, namely $\lambda_1 = \lambda_2 = \lambda_s$, $v_1 = v_2 = v_s$, $\Lambda_1 = \Lambda_2 = \Lambda_s$ are taken into account.

Substitute the Higgs VEVs of the model from (4), (10), (27), (28) and (30) into (97) we obtain

$$\begin{aligned}\mathcal{L}^{\rm GB}_{\rm mass} &= \frac{v^2}{324}\big[81g^2(W^2_{\mu1} + W^2_{\mu2}) + 81g^2(W^2_{\mu6} + W^2_{\mu7})\\ &\quad + (-9gW_{\mu3} + 3\sqrt{3}gW_{\mu8} + 2\sqrt{6}g_X B_\mu)^2\big]\\ &\quad + \frac{v'^2}{324}\big[81g^2(W^2_{\mu1} + W^2_{\mu2}) + 81g^2(W^2_{\mu6} + W^2_{\mu7})\\ &\quad + (-9gW_{\mu3} + 3\sqrt{3}gW_{\mu8} + 2\sqrt{6}g_X B_\mu)^2\big]\\ &\quad + \frac{\omega^2}{108}\big[27g^2(W^2_{\mu4} + W^2_{\mu5}) + 27g^2(W^2_{\mu6} + W^2_{\mu7})\\ &\quad + 36g^2W^2_{\mu8} + 12\sqrt{2}gg_x W_{\mu8}B_\mu + 2g^2_X B^2_\mu\big]\\ &\quad + \frac{u^2}{324}\big[81g^2(W^2_{\mu1} + W^2_{\mu2}) + 81g^2(W^2_{\mu4} + W^2_{\mu5})\\ &\quad + (-9gW_{\mu3} - 3\sqrt{3}gW_{\mu8} + \sqrt{6}g_X B_\mu)^2\big]\\ &\quad + \frac{u'^2}{324}\big[81g^2(W^2_{\mu1} + W^2_{\mu2}) + 81g^2(W^2_{\mu4} + W^2_{\mu5})\\ &\quad + (-9gW_{\mu3} - 3\sqrt{3}gW_{\mu8} + \sqrt{6}g_X B_\mu)^2\big]\\ &\quad + \frac{v^2_\rho}{324}\big[81g^2(W^2_{\mu1} + W^2_{\mu2}) + 81g^2(W^2_{\mu6} + W^2_{\mu7})\\ &\quad + (-9gW_{\mu3} + 3\sqrt{3}gW_{\mu8} + 2\sqrt{6}g_X B_\mu)^2\big]\end{aligned}$$

$$+\frac{g^2}{6}\Big[2(2\Lambda_s v_s+\Lambda_\sigma v_\sigma)\big(3W_{\mu3}W_{\mu4}+3W_{\mu1}W_{\mu6}-3W_{\mu2}W_{\mu7}-5\sqrt{3}W_{\mu4}W_{\mu8}\big)$$
$$+3(2v_s^2+v_\sigma^2+2\lambda_s^2+\lambda_\sigma^2)(W_{\mu1}^2+W_{\mu2}^2)+3(2v_s^2+v_\sigma^2+4\lambda_s^2+2\lambda_\sigma^2)W_{\mu3}^2$$
$$+3(8v_s^2+4v_\sigma^2+2\lambda_s^2+\lambda_\sigma^2+2\Lambda_s^2+\Lambda_\sigma^2+4\Lambda_s\lambda_s+2\Lambda_\sigma\lambda_\sigma)W_{\mu4}^2$$
$$+3\big(8v_s^2+4v_\sigma^2+2\lambda_s^2+\lambda_\sigma^2+2\Lambda_s^2+\Lambda_\sigma^2-4\Lambda_s\lambda_s-2\Lambda_\sigma\lambda_\sigma\big)W_{\mu5}^2$$
$$+3\big(2v_s^2+v_\sigma^2+2\Lambda_s^2+\Lambda_\sigma^2\big)W_{\mu6}^2+3\big(2v_s^2+v_\sigma^2+2\Lambda_s^2+\Lambda_\sigma^2\big)W_{\mu7}^2$$
$$+2\sqrt{3}\big(-2v_s^2-v_\sigma^2+4\lambda_s^2+2\lambda_\sigma^2\big)W_{\mu3}W_{\mu8}$$
$$+\big(2v_s^2+v_\sigma^2+4\lambda_s^2+2\lambda_\sigma^2+18\Lambda_s^2+8\Lambda_\sigma^2\big)W_{\mu8}^2$$
$$+18\big(2\lambda_s v_s+\lambda_\sigma v_\sigma\big)W_{\mu3}W_{\mu4}+6(2\lambda_s v_s+\lambda_\sigma v_\sigma)W_{\mu1}W_{\mu6}$$
$$-6(2\lambda_s v_s+\lambda_\sigma v_\sigma)W_{\mu2}W_{\mu7}+2\sqrt{3}(2\lambda_s v_s+\lambda_\sigma v_\sigma)W_{\mu4}W_{\mu8}\Big]$$
$$+\frac{2}{27}t^2g^2\big(2\lambda_s^2+\lambda_\sigma^2+2\Lambda_s^2+\Lambda_\sigma^2+4v_s^2+2v_\sigma^2\big)B_\mu^2$$
$$-\frac{2\sqrt{6}}{9}tg^2\big(2\lambda_s^2+\lambda_\sigma^2+2v_s^2+v_\sigma^2\big)W_{\mu3}B_\mu$$
$$-\frac{4\sqrt{6}}{9}tg^2\left[(2\lambda_s+2\Lambda_s)v_s+(\lambda_\sigma+\Lambda_\sigma)v_\sigma\right]W_{\mu4}B_\mu$$
$$+\frac{2\sqrt{2}}{9}tg^2\big(2v_s^2+v_\sigma^2+4\Lambda_s^2+2\Lambda_\sigma^2-2\lambda_s^2-\lambda_\sigma^2\big)W_{\mu8}B_\mu\,. \tag{98}$$

We can separate $\mathcal{L}^{\rm GB}_{\rm mass}$ in (98) into

$$\mathcal{L}^{\rm GB}_{\rm mass}=\mathcal{L}^{W_5}_{\rm mass}+\mathcal{L}^{\rm CGB}_{\rm mix}+\mathcal{L}^{\rm NGB}_{\rm mix}\,, \tag{99}$$

where $\mathcal{L}^{W_5}_{\rm mass}$ is the Lagrangian of the imaginary part W_5. This boson is decoupled with mass given by

$$M^2_{W_5}=\frac{g^2}{2}\big(\omega^2+u^2+u'^2+16v_s^2+8v_\sigma^2+4\lambda_s^2$$
$$+2\lambda_\sigma^2+4\Lambda_s^2+2\Lambda_\sigma^2-8\Lambda_s\lambda_s-4\Lambda_\sigma\lambda_\sigma\big)\,. \tag{100}$$

In the limit λ_s, λ_σ, v_s, $v_\sigma\to 0$, $M^2_{W_5}$ reduces to

$$M^2_{W_5}=\frac{g^2}{2}\big(\omega^2+u^2+u'^2+4\Lambda_s^2+2\Lambda_\sigma^2\big)\,. \tag{101}$$

$\mathcal{L}_{\rm mix}^{\rm CGB}$ is the Lagrangian part of the charged gauge bosons W and Y,

$$\begin{aligned}\mathcal{L}_{\rm mix}^{\rm CGB} = {} & \frac{g^2}{4}\left(v^2 + v'^2 + \omega^2 + u^2 + u'^2 + v_\rho^2\right)\left(W_{\mu 1}^2 + W_{\mu 2}^2 + W_{\mu 6}^2 + W_{\mu 7}^2\right) \\ & + \frac{g^2}{6}\Big[2(2\Lambda_s v_s + \Lambda_\sigma v_\sigma)(3W_{\mu 1}W_{\mu 6} - 3W_{\mu 2}W_{\mu 7}) \\ & + 3\left(2v_s^2 + v_\sigma^2 + 2\lambda_s^2 + \lambda_\sigma^2\right)W_{\mu 1}^2 + 3\left(2v_s^2 + v_\sigma^2 + 2\lambda_s^2 + \lambda_\sigma^2\right)W_{\mu 2}^2 \\ & + 3\left(2v_s^2 + v_\sigma^2 + 2\Lambda_s^2 + \Lambda_\sigma^2\right)W_{\mu 6}^2 + 3\left(2v_s^2 + v_\sigma^2 + 2\Lambda_s^2 + \Lambda_\sigma^2\right)W_{\mu 7}^2 \\ & + 6(2\lambda_s v_s + \lambda_\sigma v_\sigma)W_{\mu 1}W_{\mu 6} - 6(2\lambda_s v_s + \lambda_\sigma v_\sigma)W_{\mu 2}W_{\mu 7}\Big]. \end{aligned} \tag{102}$$

We can rewrite $\mathcal{L}_{\rm mix}^{\rm CGB}$ in matrix form

$$\mathcal{L}_{\rm mix}^{\rm CGB} = \frac{g^2}{4}\left(W_\mu'^- \quad Y_\mu'^-\right) M_{WY}^2 \left(W'^{+\mu} \quad Y'^{+\mu}\right)^T,$$

where

$$M_{WY}^2 = \begin{pmatrix} m_{11}^2 & m_{12}^2 \\ m_{21}^2 & m_{22}^2 \end{pmatrix}, \tag{103}$$

with

$$\begin{aligned} m_{11}^2 &= 2(v^2 + v'^2 + u^2 + u'^2 + v_\rho^2 + 4v_s^2 + 2v_\sigma^2 + 4\lambda_s^2 + 2\lambda_\sigma^2), \\ m_{12}^2 &= m_{21}^2 = 4(2\Lambda_s v_s + 2\lambda_s v_s + \Lambda_\sigma v_\sigma + \lambda_\sigma v_\sigma), \\ m_{22}^2 &= 2(v^2 + v'^2 + \omega^2 + v_\rho^2 + 4v_s^2 + 2v_\sigma^2 + 4\Lambda_s^2 + 2\Lambda_\sigma^2). \end{aligned} \tag{104}$$

The matrix M_{WY}^2 in (103) with the elements in (104) can be diagonalized as follows

$$U_2^T M_{WY}^2 U_2 = \mathrm{diag}\left(M_W^2, M_Y^2\right),$$

where

$$\begin{aligned} M_W^2 &= \frac{g^2}{4}\Big\{2v^2 + 2v'^2 + u^2 + u'^2 + \omega^2 + 2v_\rho^2 + 4\lambda_s^2 \\ &\quad + 4\Lambda_s^2 + 8v_s^2 + 2\lambda_\sigma^2 + 2\Lambda_\sigma^2 + 4v_\sigma^2 - \sqrt{\Gamma}\Big\}, \\ M_Y^2 &= \frac{g^2}{4}\Big\{2v^2 + 2v'^2 + u^2 + u'^2 + \omega^2 + 2v_\rho^2 + 4\lambda_s^2 \\ &\quad + 4\Lambda_s^2 + 8v_s^2 + 2\lambda_\sigma^2 + 2\Lambda_\sigma^2 + 4v_\sigma^2 + \sqrt{\Gamma}\Big\}, \end{aligned} \tag{105}$$

with

$$\begin{aligned} \Gamma = {} & 16\lambda_s^4 + 16\Lambda_s^4 + (2\lambda_\sigma^2 - 2\Lambda_\sigma^2 - \omega^2 + u^2 + u'^2)^2 \\ & - 8\lambda_s^2(4\Lambda_s^2 - 2\lambda_\sigma^2 + 2\Lambda_\sigma^2 + \omega^2 - u^2 - u'^2 - 8v_s^2) \\ & - 8\Lambda_s^2(2\lambda_\sigma^2 - 2\Lambda_\sigma^2 - \omega^2 + u^2 + u'^2 - 8v_s^2) + 64\Lambda_s(\lambda_\sigma + \Lambda_\sigma)v_s v_\sigma \\ & + 16(\lambda_\sigma + \Lambda_\sigma)^2 v_\sigma^2 + 64\lambda_s v_s(2\Lambda_s v_s + \lambda_\sigma v_\sigma + \Lambda_\sigma v_\sigma). \end{aligned} \tag{106}$$

With corresponding eigenstates, the charged gauge boson mixing matrix takes the form:

$$U_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where, the mixing angle θ is given by

$$\tan\theta = \frac{8(\lambda_s + \Lambda_s)v_s + 4(\lambda_\sigma + \Lambda_\sigma)v_\sigma}{4\lambda_s^2 - 4\Lambda_s^2 + 2\lambda_\sigma^2 - 2\Lambda_\sigma^2 - \omega^2 + u^2 + u'^2 - \sqrt{\Gamma}}. \tag{107}$$

The physical charged gauge bosons is defined

$$W_\mu^- = \cos\theta W_\mu'^- + \sin\theta Y_\mu'^-,$$
$$Y_\mu^- = -\sin\theta W_\mu'^- + \cos\theta Y_\mu'^-.$$

In our model, the following limit is often taken into account:

$$\lambda_s^2,\ \lambda_\sigma^2,\ v_s^2,\ v_\sigma^2 \ll u^2, \quad u'^2,\ v^2,\ v'^2 \ll \omega^2 \sim \Lambda_s^2 \sim \Lambda_\sigma^2. \tag{108}$$

With the help of (108), the Γ in (106) becomes

$$\sqrt{\Gamma} \simeq (4\Lambda_s^2 + 2\Lambda_\sigma^2 + \omega^2 - u^2 - u'^2) + \frac{32\Lambda_s\Lambda_\sigma v_s v_\sigma + 8\Lambda_\sigma^2 v_\sigma^2}{4\Lambda_s^2 + 2\Lambda_\sigma^2 + \omega^2 - u^2 - u'^2}. \tag{109}$$

It is then

$$M_W^2 \simeq \frac{g^2}{2}\left(u^2 + u'^2 + v^2 + v'^2 + v_\rho^2\right) - \frac{g^2}{2}\Delta_{M_w^2}, \tag{110}$$

with

$$\Delta_{M_w^2} = \frac{16\Lambda_s\Lambda_\sigma v_s v_\sigma + 4\Lambda_\sigma^2 v_\sigma^2}{4\Lambda_s^2 + 2\Lambda_\sigma^2 + \omega^2 - u^2 - u'^2}. \tag{111}$$

Notice that in the limit $\lambda_s, \lambda_\sigma, v_s, v_\sigma \to 0$ then $\Gamma \simeq 4\Lambda_s^2 + 2\Lambda_\sigma^2 + \omega^2 - u^2 - u'^2$, the mixing angle θ tends to zero, and M_W^2, M_Y^2 in (105) reduces to

$$\begin{aligned} M_W^2 &= \frac{g^2}{2}(u^2 + u'^2 + v^2 + v'^2 + v_\rho^2), \\ M_Y^2 &= \frac{g^2}{2}\left(4\Lambda_s^2 + 2\Lambda_\sigma^2 + \omega^2 + v^2 + v'^2 + v_\rho^2\right), \end{aligned} \tag{112}$$

and one can evaluate

$$\tan\theta \simeq -\frac{8\Lambda_s v_s + 4\Lambda_\sigma v_\sigma}{8\Lambda_s^2 + 4\Lambda_\sigma^2 + 2\omega^2} \sim \frac{v_s}{\Lambda_s} \sim \frac{v_\sigma}{\Lambda_\sigma}. \tag{113}$$

In addition, from (112), it follows that M_W^2 is much smaller than M_Y^2.

$\mathcal{L}_{\rm mix}^{\rm NGB}$ is the Lagrangian part of the neutral gauge bosons W_3, W_8, B, W_4. The mass Lagrangian in this case has the form

$$\begin{aligned} \mathcal{L}_{\rm mix}^{\rm NGB} &= \frac{(v^2 + v'^2 + v_\rho^2)}{324}\left(-9gW_{\mu 3} + 3\sqrt{3}gW_{\mu 8} + 2\sqrt{6}g_X B_\mu\right)^2 \\ &\quad + \frac{\omega^2}{108}\left(27g^2 W_{\mu 4}^2 + 36g^2 W_{\mu 8}^2 + 12\sqrt{2}gg_X W_{\mu 8}B_\mu + 2g_X^2 B_\mu^2\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{(u^2+u'^2)}{324}\Big[81g^2W_{\mu4}^2+\big(-9gW_{\mu3}-3\sqrt{3}gW_{\mu8}+\sqrt{6}g_XB_\mu\big)^2\Big]\\
&+\frac{g^2}{6}\Big[2(2\Lambda_sv_s+\Lambda_\sigma v_\sigma)\big(3W_{\mu3}W_{\mu4}-5\sqrt{3}W_{\mu4}W_{\mu8}\big)\\
&+3\big(2v_s^2+v_\sigma^2+4\lambda_s^2+2\lambda_\sigma^2\big)W_{\mu3}^2\\
&+3\big(8v_s^2+4v_\sigma^2+2\lambda_s^2+\lambda_\sigma^2+2\Lambda_s^2+\Lambda_\sigma^2+4\Lambda_s\lambda_s+2\Lambda_\sigma\lambda_\sigma\big)W_{\mu4}^2\\
&+2\sqrt{3}\big(-2v_s^2-v_\sigma^2+4\lambda_s^2+2\lambda_\sigma^2\big)W_{\mu3}W_{\mu8}\\
&+\big(2v_s^2+v_\sigma^2+4\lambda_s^2+2\lambda_\sigma^2+16\Lambda_s^2+8\Lambda_\sigma^2\big)W_{\mu8}^2\\
&+18(2\lambda_sv_s+\lambda_\sigma v_\sigma)W_{\mu3}W_{\mu4}+2\sqrt{3}(2\lambda_sv_s+\lambda_\sigma v_\sigma)W_{\mu4}W_{\mu8}\Big]\\
&+\frac{2t^2g^2}{27}\big(2\lambda_s^2+\lambda_\sigma^2+2\Lambda_s^2+\Lambda_\sigma^2+4v_s^2+2v_\sigma^2\big)B_\mu^2\\
&-\frac{2}{3}\sqrt{\frac{2}{3}}tg^2\big(2\lambda_s^2+\lambda_\sigma^2+2v_s^2+v_\sigma^2\big)W_{\mu3}B_\mu\\
&-\frac{4}{3}\sqrt{\frac{2}{3}}tg^2[(2\lambda_s+2\Lambda_s)v_s+(\lambda_\sigma+\Lambda_\sigma)v_\sigma]W_{\mu4}B_\mu\\
&+\frac{2\sqrt{2}}{9}tg^2\big(2v_s^2+v_\sigma^2+4\Lambda_s^2+2\Lambda_\sigma^2-2\lambda_s^2-\lambda_\sigma^2\big)W_{\mu8}B_\mu\,. \qquad (114)
\end{aligned}
$$

In the basis of $(W_{\mu3}, W_{\mu8}, B_\mu, W_{\mu4})$, the $\mathcal{L}_{\rm mix}^{\rm NGB}$ in (114) can be rewritten:

$$
\mathcal{L}_{\rm mix}^{\rm NGB}\equiv\frac{1}{2}V^TM^2V\,, \qquad (115)
$$

$$
V^T=\big(W_{\mu3},W_{\mu8},B_\mu,W_{\mu4}\big)\,,
$$

$$
M^2=\frac{g^2}{4}\begin{pmatrix} M_{11}^2 & M_{12}^2 & M_{13}^2 & M_{14}^2\\ M_{12}^2 & M_{22}^2 & M_{23}^2 & M_{24}^2\\ M_{13}^2 & M_{23}^2 & M_{33}^2 & M_{34}^2\\ M_{14}^2 & M_{24}^2 & M_{34}^2 & M_{44}^2\end{pmatrix}, \qquad (116)
$$

where

$$
M_{11}^2=2\big(v^2+v'^2+u^2+u'^2+v_\rho^2+4v_s^2+2v_\sigma^2+8\lambda_s^2+4\lambda_\sigma^2\big)\,,
$$

$$
M_{12}^2=-\frac{2\sqrt{3}}{3}\big(v^2+v'^2-u^2-u'^2+v_\rho^2+4v_s^2+2v_\sigma^2-8\lambda_s^2-4\lambda_\sigma^2\big)\,,
$$

$$
M_{13}^2=-\frac{2}{3}\sqrt{\frac{2}{3}}t\big(2v^2+2v'^2+u^2+u'^2+2v_\rho^2+8\lambda_s^2+4\lambda_\sigma^2+8v_s^2+4v_\sigma^2\big)\,,
$$

$$
M_{14}^2=8(3\lambda_sv_s+\Lambda_sv_s)+4(3\lambda_\sigma v_\sigma+\Lambda_\sigma v_\sigma)\,,
$$

$$
\begin{aligned}
M_{22}^2 &= \frac{2}{3}\big(v^2 + v'^2 + 4\omega^2 + u^2 + u'^2 + v_\rho^2 + 4v_s^2 \\
&\quad + 2v_\sigma^2 + 8\lambda_s^2 + 4\lambda_\sigma^2 + 32\Lambda_s^2 + 16\Lambda_\sigma^2\big)\,, \\
M_{23}^2 &= \frac{2\sqrt{2}t}{9}\big(2v^2 + 2v'^2 + 2\omega^2 - u^2 - u'^2 \\
&\quad + 2v_\rho^2 + 8v_s^2 + 4v_\sigma^2 + 16\Lambda_s^2 + 8\Lambda_\sigma^2 - 8\lambda_s^2 - 4\lambda_\sigma^2\big)\,, \\
M_{24}^2 &= \frac{8}{\sqrt{3}}\big(\lambda_s v_s - 5\Lambda_s v_s\big) + \frac{4}{\sqrt{3}}\big(\lambda_\sigma v_\sigma - 5\Lambda_\sigma v_\sigma\big)\,, \\
M_{33}^2 &= \frac{4t^2}{27}\big(4v^2 + 4v'^2 + \omega^2 + u^2 + u'^2 + 4v_\rho^2 \\
&\quad + 8\lambda_s^2 + 4\lambda_\sigma^2 + 8\Lambda_s^2 + 4\Lambda_\sigma^2 + 16v_s^2 + 8v_\sigma^2\big)\,, \\
M_{34}^2 &= -\frac{32}{3}\sqrt{\frac{2}{3}}t\big(\lambda_s v_s + \Lambda_s v_s\big) - \frac{16}{3}\sqrt{\frac{2}{3}}t\big(\lambda_\sigma v_\sigma + \Lambda_\sigma v_\sigma\big)\,, \\
M_{44}^2 &= 2\big(\omega^2 + u^2 + u'^2 + 16v_s^2 + 8v_\sigma^2 + 4\lambda_s^2 + 2\lambda_\sigma^2 + 4\Lambda_s^2 \\
&\quad + 2\Lambda_\sigma^2 + 8\Lambda_s\lambda_s + 4\Lambda_\sigma\lambda_\sigma\big)\,.
\end{aligned} \tag{117}
$$

The matrix M^2 in (116) with the elements in (117) has one exact eigenvalue, which is identified with the photon mass,

$$M_\gamma^2 = 0\,. \tag{118}$$

The corresponding eigenvector of M_γ^2 is

$$A_\mu = \begin{pmatrix} \frac{\sqrt{3}t}{\sqrt{4t^2+18}} \\ -\frac{t}{\sqrt{4t^2+18}} \\ \frac{3\sqrt{2}}{\sqrt{4t^2+18}} \\ 0 \end{pmatrix}. \tag{119}$$

Note that in the limit λ_s, λ_σ, v_s, $v_\sigma \to 0$, $M_{14}^2 = M_{24}^2 = M_{34}^2 = 0$ and W_4 does not mix with $W_{3\mu}$, $W_{8\mu}$, B_μ. In the general case λ_s, λ_σ, v_s, $v_\sigma \neq 0$, the mass matrix in (116) contains one exact eigenvalues as in (118) with the corresponding eigenstate being given in (119).

The diagonalization of the mass matrix M^2 in (116) is done via two steps. In the first step, the basic $(W_{\mu 3}, W_{\mu 8}, B'_\mu, W_{4\mu})$ is transformed into the basic

$(A_\mu, Z_\mu, Z'_\mu, W_{4\mu})$ by the matrix:

$$U_{\mathrm{NGB}} = \begin{pmatrix} s_W & -c_W & 0 & 0 \\ -\frac{c_W t_W}{\sqrt{3}} & -\frac{s_W t_W}{\sqrt{3}} & \sqrt{1-\frac{t_W^2}{3}} & 0 \\ c_W\sqrt{1-\frac{t_W^2}{3}} & s_W\sqrt{1-\frac{t_W^2}{3}} & \frac{t_W}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{120}$$

where $s_W = \sin\theta_W$, $c_W = \cos\theta_W$, $t_W = \tan\theta_W$, and we have used the continuation of the gauge coupling constant g of the $SU(3)_L$ at the spontaneous symmetry breaking point,[150,158]

$$t = \frac{3\sqrt{2}s_W}{\sqrt{3-4s_W^2}}\,. \tag{121}$$

The corresponding eigenstates are rewritten as follows

$$\begin{aligned} A_\mu &= s_W W_{3\mu} + c_W\left(-\frac{t_W}{\sqrt{3}}W_{8\mu} + \sqrt{1-\frac{t_W^2}{3}}B_\mu\right), \\ Z_\mu &= -c_W W_{3\mu} + s_W\left(-\frac{t_W}{\sqrt{3}}W_{8\mu} + \sqrt{1-\frac{t_W^2}{3}}B_\mu\right), \\ Z'_\mu &= \sqrt{1-\frac{t_W^2}{3}}W_{8\mu} + \frac{t_W}{\sqrt{3}}B_\mu\,. \end{aligned} \tag{122}$$

In this basis, the mass matrix M^2 becomes

$$M'^2 = U^+_{\mathrm{NGB}} M^2 U_{\mathrm{NGB}} = \frac{g^2}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & M'^2_{22} & M'^2_{23} & M'^2_{24} \\ 0 & M'^2_{23} & M'^2_{33} & M'^2_{34} \\ 0 & M'^2_{24} & M'^2_{34} & M'^2_{44} \end{pmatrix}, \tag{123}$$

where

$$\begin{aligned} M'^2_{22} &= \frac{4(2t^2+9)}{t^2+18}\left(8\lambda_s^2 + 4\lambda_\sigma^2 + u^2 + u'^2 + v^2 + v'^2 + v_\rho^2 + 4v_s^2 + 2v_\sigma^2\right), \\ M'^2_{23} &= \frac{4}{3\sqrt{3}}\frac{\sqrt{2t^2+9}}{(t^2+18)}\big[(t^2-9)(8\lambda_s^2 + 4\lambda_\sigma^2 + u^2 + u'^2) \\ &\quad + (2t^2+9)\left(v^2 + v'^2 + v_\rho^2 + 4v_s^2 + 2v_\sigma^2\right)\big], \\ M'^2_{24} &= -4\sqrt{2}\sqrt{\frac{2t^2+9}{t^2+18}}[2(\Lambda_s + 3\lambda_s)v_s + (\Lambda_\sigma + 3\lambda_\sigma)v_\sigma], \end{aligned}$$

$$
\begin{aligned}
M_{33}'^{\,2} &= \frac{4}{27(t^2+18)}\Big\{8\lambda_s^2(t^2-9)^2 + 8\Lambda_s^2(t^2+18)^2 \\
&\quad + 81\big(4\lambda_\sigma^2 + 16\Lambda_\sigma^2 + 4\omega^2 + u^2 + u'^{\,2} + v^2 + v'^{\,2} + v_\rho^2 + 4v_2^2 + 2v_\sigma^2\big) \\
&\quad + 4\lambda_\sigma^2(t^2-18)t^2 + t^2\big(144\Lambda_\sigma^2 + 36\omega^2 - 18u^2 \\
&\quad - 18u'^{\,2} + 36v^2 + 36v'^{\,2} + 36v_\rho^2 + 72v_s^2 + 36v_\sigma^2\big) \\
&\quad + t^4\big(4\Lambda_\sigma^2 + \omega^2 + u^2 + u'^{\,2} + 4v^2 + 4v'^{\,2} + 4v_\rho^2 + 16v_s^2 + 8v_\sigma^2\big)\Big\}\,, \\
M_{34}'^{\,2} &= -\frac{4\sqrt{2}}{3\sqrt{3}}\frac{1}{\sqrt{t^2+18}}\left[(2\lambda_s v_s + \lambda_\sigma v_\sigma)(4t^2-9) + (2\Lambda_s v_s + \Lambda_\sigma v_\sigma)(4t^2+45)\right], \\
M_{44}'^{\,2} &= 2\big(4\lambda_s^2 + 8\lambda_s\Lambda_s + 4\Lambda_s^2 + 2\lambda_\sigma^2 + 4\lambda_\sigma\Lambda_\sigma \\
&\quad + 2\Lambda_\sigma^2 + \omega^2 + u^2 + u'^{\,2} + 16v_s^2 + 8v_\sigma^2\big)\,.
\end{aligned}
\tag{124}
$$

In the approximation $\lambda_s^2 \sim \lambda_\sigma^2$, v_s^2, $v_\sigma^2 \ll \Lambda_s^2 \sim \Lambda_\sigma^2 \sim \omega^2$, we have

$$
\begin{aligned}
M_{22}'^{\,2} &= \frac{2}{c_W^2}\big(u^2 + u'^{\,2} + v^2 + v'^{\,2} + v_\rho^2\big)\,, \\
M_{23}'^{\,2} &= \frac{2[(1-2c_W^2)(u^2+u'^{\,2}) + v^2 + v'^{\,2} + v_\rho^2]\sqrt{\alpha_0}}{c_W^2}\,, \\
M_{24}'^{\,2} &= -\frac{4}{c_W}(2\Lambda_s v_s + \Lambda_\sigma v_\sigma)\,, \\
M_{33}'^{\,2} &= 32(\Lambda_\sigma^2 + 2\Lambda_s^2)c_W^2\alpha_0 + 8\omega^2 c_W^2\alpha_0 \\
&\qquad + \frac{2}{c_W^2}(v^2 + v'^{\,2} + v_\rho^2)\alpha_0 + \frac{2}{c_W^2}(2c_W^2-1)^2(u^2+u'^{\,2})\alpha_0\,, \\
M_{34}'^{\,2} &= -\frac{8x_0\sqrt{\alpha}}{c_W}(\Lambda_s v_s + 4\Lambda_\sigma v_\sigma)\,, \\
M_{44}'^{\,2} &= 2(\omega^2 + u^2 + u'^{\,2} + 4\Lambda_s^2 + 2\Lambda_\sigma^2 + 8\lambda_s\Lambda_s + 4\lambda_\sigma\Lambda_\sigma)\,.
\end{aligned}
\tag{125}
$$

with

$$
x_0 = 4c_W^2 + 1\,, \quad \alpha_0 = (4c_W^2 - 1)^{-1}\,. \tag{126}
$$

It is noteworthy that in the limit $v_s = 0$ and $v_\sigma = 0$, the elements $M_{24}'^{\,2}$ and $M_{34}'^{\,2}$ vanish. In this case the mixing between there is no mixing between W_4 and Z_μ, Z'_μ.

In the second step, three remain bosons gain masses via seesaw mechanism

$$
M_Z^2 = \frac{g^2}{4}\left[M_{22}'^{\,2} - \big(M^{\rm off}\big)^T\big(M_{2\times2}'^{\,2}\big)^{-1}M^{\rm off}\right], \tag{127}
$$

where

$$M^{\rm off} = \begin{pmatrix} M'^{\,2}_{23} \\ M'^{\,2}_{24} \end{pmatrix}, \quad M'^{\,2}_{2\times 2} = \begin{pmatrix} M'^{\,2}_{33} & M'^{\,2}_{34} \\ M'^{\,2}_{34} & M'^{\,2}_{44} \end{pmatrix}. \tag{128}$$

Combining (127) and (128) yields:

$$M_Z^2 = \frac{g^2\big(u^2 + u'^{\,2} + v^2 + v'^{\,2} + v_\rho^2\big)}{2c_W^2} - \frac{g^2}{2c_W^2}\Delta_{M_z^2}\,,$$

where

$$\Delta_{M_z^2} = \frac{4\Delta_1^2}{x_2} + \frac{x_1(x_1x_2 - 16\Delta_1\Delta_2 x_0)}{x_2(4c_W^4 x_3 + x_4)}\,, \tag{129}$$

with

$$\begin{aligned}
x_1 &= (1-2c_W^2)(u^2+u'^{\,2}) + v^2 + v'^{\,2} + v_\rho^2\,,\\
x_2 &= 4\Lambda_s(2\lambda_s + \Lambda_s) + 2\Lambda_\sigma(2\lambda_\sigma + \Lambda_\sigma) + \omega^2 + u^2 + u'^{\,2}\,,\\
x_3 &= 8\Lambda_s^2 + 4\Lambda_\sigma^2 + \omega^2 + u^2 + u'^{\,2}\,,\\
x_4 &= (1-4c^2)(u^2+u'^{\,2}) + v^2 + v'^{\,2} + v_\rho^2\,,\\
\Delta_1 &= 2\Lambda_s v_s + \Lambda_\sigma v_\sigma\,, \quad \Delta_2 = \Lambda_s v_s + 4\Lambda_\sigma v_\sigma\,.
\end{aligned} \tag{130}$$

The ρ parameter in our model is given by

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} = 1 + \frac{\delta_{wz}}{M_z^2} \equiv 1 + \delta_{\rm tree}\,, \tag{131}$$

where

$$\delta_{wz} = \frac{g^2}{2c_W^2}\big(\Delta_{M_z^2} - \Delta_{M_w^2}\big)\,. \tag{132}$$

From the mass of W boson evaluated in (112) we can identify

$$2\big(u^2 + u'^{\,2} + v^2 + v'^{\,2} + 2v_\rho^2\big) = v_{\rm weak}^2 \simeq (246~{\rm GeV})^2$$

and then obtain

$$u \sim u' \sim v' \sim v \sim 100~{\rm GeV}\,, \tag{133}$$

provided that $v_\rho \sim 0$.

In addition, let us assume the relations (108) and put

$$\lambda_s = \frac{v_s^2}{\Lambda_s} = \frac{v_\sigma^2}{\Lambda_\sigma} = \lambda_\sigma\,, \quad \omega = \Lambda_s \equiv \Lambda_\sigma$$

then

$$\Delta_{M_z^2} - \Delta_{M_w^2} \simeq \frac{(2.28571\Lambda_s^2 - 45590.8)v_s^2 + 2.77644\times 10^6}{\Lambda_s^2}\,. \tag{134}$$

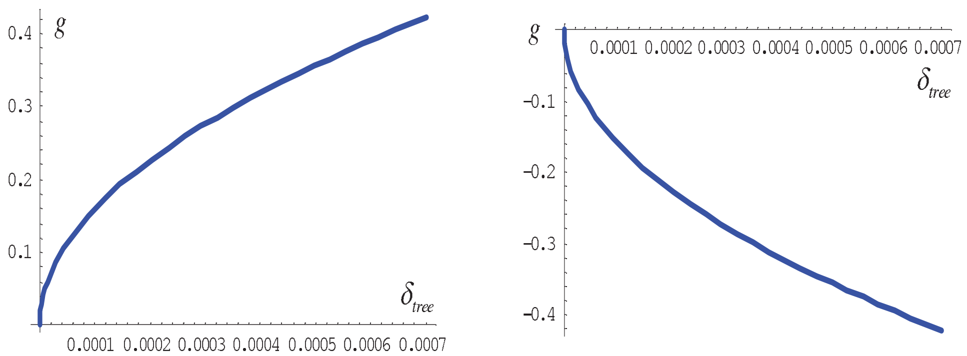

Fig. 12. The coupling g as a function of $\delta_{\rm tree}$ with $\delta_{\rm tree} \in (0, 0.0007)$ and $v_s = 10$ GeV.

From (131)–(134) we have:

$$\delta_{\rm tree} = \frac{g^2}{2c_W^2}\frac{(2.28571\Lambda_s^2 - 45590.8)v_s^2 + 2.77644\times 10^6}{M_z^2\Lambda_s^2}\,. \tag{135}$$

The experimental value of the ρ parameter and M_W are respectively given in Ref. 1,

$$\begin{aligned} \rho &= 1.0004^{+0.0003}_{-0.0004} \quad \left(\delta_{\rm tree} = 0.0004^{+0.0003}_{-0.0004}\right), \\ s_W^2 &= 0.23116 \pm 0.00012\,, \quad M_W = 80.358 \pm 0.015 \text{ GeV}\,. \end{aligned} \tag{136}$$

It means

$$0 \leq \delta_{\rm tree} \leq 0.0007\,. \tag{137}$$

The expression (135) gives the relations between g and $\delta_{\rm tree}$ as follows

$$g = \pm\frac{79.9648\sqrt{\delta_{\rm tree}}v_s^2}{\sqrt{1.14286v_s^6 - 22795.4v_s^2 - 45590.8\times 10^6}}\,.$$

In Fig. 12, we have plotted g as a function of $\delta_{\rm tree} \in (0, 0.0007)$ from which it provides that $v_s = 10$ GeV satisfying the condition (108). From Fig. 12, we can find out $|g| \in (0, 0.42)$.

Diagonalizing the mass matrix $M'^2_{2\times 2}$, we get two new physical gauge bosons

$$\begin{aligned} Z''_\mu &= \cos\phi Z'_\mu + \sin\phi W_{\mu 4}\,, \\ W'_{\mu 4} &= -\sin\phi Z'_\mu + \cos\phi W_{\mu 4}\,. \end{aligned} \tag{138}$$

The mixing angle ϕ is given by

$$\tan\phi = \frac{8\sqrt{\alpha_0}c_W(\Lambda_s v_s + 4\Lambda_\sigma v_\sigma)x_0}{-4c_W^4\alpha_0 x_3 + c_W^2 x_2 - \alpha_0 x_4 + \sqrt{F}}\,, \tag{139}$$

where

$$\begin{aligned} F = &\left(4c_W^4\alpha_0 x_3 + c_W^2 x_2 + \alpha_0 x_4\right)^2 \\ &- 4\alpha_0 c_W^2\big[(u^2+u'^2)x_5 - 4c_W^2(u^2+u'^2)x_2 + 4c_W^4 x_3 x_2 \\ &+ \left(v^2+v'^2+v_\rho^2\right)x_5 + 8\lambda_s\Lambda_s\left(u^2+u'^2+v^2+v'^2+v_\rho^2\right) \\ &- 16(\Lambda_s v_s + 4\Lambda_\sigma v_\sigma)^2 x_0^2\big]\,, \end{aligned} \tag{140}$$

and

$$x_5 = 4\Lambda_s^2 + 4\Lambda_\sigma\lambda_\sigma + 2\Lambda_\sigma^2 + \omega^2 + u^2 + u'^2\,.$$

With the help of (108), we have

$$F \simeq c_W^2\left\{8\Lambda_s\lambda_s + 2\Lambda_\sigma(2\lambda_\sigma+\Lambda_\sigma) + \omega^2 - 4\left[(8\alpha_0 c_W^2 - 1)\Lambda_s^2 + \alpha_0 c_W^2(4\Lambda_\sigma^2+\omega^2)\right]\right\}$$

and one can evaluate

$$\begin{aligned} \tan\phi &\simeq \frac{4\sqrt{\alpha_0}(\Lambda_s v_s + 4\Lambda_\sigma v_\sigma)x_0}{c_W\left[(4-32\alpha_0 c_W^2)\Lambda_s^2 + (2-16\alpha_0 c_W^2)\Lambda_\sigma^2 + (1-4\alpha_0 c_W^2)\omega^2\right]} \\ &\sim \frac{v_s}{\Lambda_s} \sim \frac{v_\sigma}{\Lambda_\sigma}\,. \end{aligned} \tag{141}$$

The physical mass eigenvalues are defined by

$$M^2_{Z''_\mu, W'_{\mu4}} = \frac{g^2}{4c_W^2}\left\{4\alpha_0 c_W^4 x_3 + c_W^2 x_2 + a x_4 \pm \sqrt{F}\right\}. \tag{142}$$

In the limit λ_s, λ_σ, v_s, $v_\sigma \to 0$ the mixing angle ϕ tends to zero, and $M^2_{Z''_\mu, W'_{\mu4}}$ in (142) reduces to

$$\begin{aligned} M^2_{Z''_\mu} &= \frac{g^2}{2c_W^2}\left(x_4 + 4c_W^2 x_3\right), \\ M^2_{W'_{\mu4}} &= \frac{g^2}{2}\left(u^2+u'^2+\omega^2+4\Lambda_s^2+2\Lambda_\sigma^2\right). \end{aligned} \tag{143}$$

Thus, the $W'_{\mu4}$ and W_5 components have the same mass. With this result, we should identify the combination of $W'_{\mu4}$ and W_5

$$\sqrt{2}X_\mu^0 = W'_{\mu4} - iW_5\,, \tag{144}$$

as physical neutral non-Hermitian gauge boson. The superscript "0" denotes neutrality of gauge boson X. Notice that, the identification in (144) only can be acceptable with the limit $\lambda_{s,\sigma}$, $v_{s,\sigma} \to 0$. In general it is not true because of the difference in masses of $W'_{\mu4}$ and $W_{\mu5}$ as in (100) and (142).

The expressions (113) and (141) show that, with the limit (108), the mixings between the charged gauge bosons $W - Y$ and the neutral ones $Z' - W_4$ are in the same order since they are proportional to v_s/Λ_s (or v_σ/Λ_σ). In addition, from (112) $M^2_{Z''_\mu} \simeq 2g^2(8\Lambda_s^2 + 4\Lambda_\sigma^2 + \omega^2)$ is little bigger than $M^2_{W'_{\mu 4}} \simeq \frac{g^2}{2}\left(\omega^2 + 4\Lambda_s^2 + 2\Lambda_\sigma^2\right)$, (or $M^2_{X^0_\mu}$), and $\left|M^2_Y - M^2_{X^0_\mu}\right| = \frac{g^2}{2}(v^2 + v'^2 + v_\rho^2 + u^2 + u'^2)$ is little smaller than $M^2_W = \frac{g^2}{2}(u^2 + u'^2 + v^2 + v'^2 + v_\rho^2)$. In that limit, the masses of X^0_μ and Y is nearly degenerate.

8. Conclusions

In this paper, we have constructed the D_4 model based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ gauge symmetry responsible for fermion masses and mixing. Neutrinos get masses from antisextets which is in a singlet and a doublet under D_4. We argue how flavor mixing patterns and mass splitting are obtained with a perturbed D_4 symmetry. We have pointed out that this model is more simpler than those of S_3 and S_4 (Refs. 145 and 144) since the same number of Higgs multiplets are needed in order to allow the fermions to gain masses but with the simple scalar Higgs potential. The CKM matrix is the identity matrix at the tree-level, but it can be different from it by adding the soft violating terms. The realistic neutrino mixing, by old data with $\theta_{13} = 0$, can be obtained *only if the direction for breaking* $D_4 \to Z_2$. For the case with the nonvanishing θ_{13}, it is necessary to introduce one more Higgs triplet ρ which is in $1'''$ of the D_4 group *responsible for breaking the* $Z_2 \to \{\text{identity}\}$. As a result, the value of θ_{13} is a small perturbation by $\frac{v_\rho}{\Lambda}$.

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Appendix A. D_4 Group and Clebsch–Gordan Coefficients

D_4 is the symmetry group of a square.[159] It has eight elements divided into five conjugacy classes, with $\underline{1}$, $\underline{1}'$, $\underline{1}''$, $\underline{1}'''$ and $\underline{2}$ as its five irreducible representations. Any element of D_4 can be formed by multiplication of the generators a (the $\pi/2$ rotation) and b (the reflection) obeying the relations $a^4 = e$, $b^2 = e$ and $bab = a^{-1}$. D_4 has the following five conjugacy classes,

$$\begin{aligned} C_1 &: \quad \{a_1 \equiv e\}\,, \\ C_2 &: \quad \{a_2 \equiv a^2\}\,, \\ C_3 &: \quad \{a_3 \equiv a,\, a_4 \equiv a^3\}\,, \\ C_4 &: \quad \{a_5 \equiv b,\, a_6 \equiv a^2 b\}\,, \\ C_5 &: \quad \{a_7 \equiv ab,\, a_8 \equiv a^3 b\}\,. \end{aligned} \tag{A.1}$$

The character table of D_4 is given as follows:

Class	n	h	χ_{1}	$\chi_{1'}$	$\chi_{1''}$	$\chi_{1'''}$	χ_{2}
C_1	1	1	1	1	1	1	2
C_2	1	2	1	1	1	1	−2
C_3	2	4	1	−1	−1	1	0
C_4	2	2	1	1	−1	−1	0
C_5	2	2	1	−1	1	−1	0

where n is the order of class and h the order of elements within each class.

We have worked in real basis, in which the two-dimensional representation $\underline{2}$ of D_4 is real, $2^*(1^*, 2^*) = 2(1^*, 2^*)$. One possible choice of generators is given as follows

$$\begin{aligned}
\underline{1}:&\quad a = 1\,, && b = 1\,,\\
\underline{1}':&\quad a = 1\,, && b = -1\,,\\
\underline{1}'':&\quad a = -1\,, && b = 1\,,\\
\underline{1}''':&\quad a = -1\,, && b = -1\,,\\
\underline{2}:&\quad a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{A.2}$$

Using them we calculate the Clebsch–Gordan coefficients for all the tensor products as given below.

First, let us put $\underline{2}(1,2)$ which means some $\underline{2}$ doublet such as $x = (x_1, x_2) \sim \underline{2}$ or $y = (y_1, y_2) \sim \underline{2}$ and so on, and similarly for the other representations. Moreover, the numbered multiplets such as $(\ldots, ij, \ldots)$ mean $(\ldots, x_i y_j, \ldots)$ where x_i and y_j are the multiplet components of different representations x and y, respectively. In the following the components of representations on left-hand side will be omitted and should be understood, but they always exist in order in the components of decompositions on right-hand side:

$$\begin{aligned}
\underline{1}(1) \otimes \underline{1}(1) = \underline{1}(11)\,, &\qquad \underline{1}'(1) \otimes \underline{1}'(1) = \underline{1}(11)\,,\\
\underline{1}''(1) \otimes \underline{1}''(1) = \underline{1}(11)\,, &\qquad \underline{1}'''(1) \otimes \underline{1}'''(1) = \underline{1}(11)\,,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\underline{1}(1) \otimes \underline{1}'(1) = \underline{1}'(11)\,, &\qquad \underline{1}(1) \otimes \underline{1}''(1) = \underline{1}''(11)\,,\\
\underline{1}(1) \otimes \underline{1}'''(1) = \underline{1}'''(11)\,, &\qquad \underline{1}'(1) \otimes \underline{1}''(1) = \underline{1}'''(11)\,,\\
\underline{1}''(1) \otimes \underline{1}'''(1) = \underline{1}'(11)\,, &\qquad \underline{1}'''(1) \otimes \underline{1}'(1) = \underline{1}''(11)\,,
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\underline{1}(1) \otimes \underline{2}(1,2) = \underline{2}(11,12), &\qquad \underline{1}'(1) \otimes \underline{2}(1,2) = \underline{2}(11,-12)\,,\\
\underline{1}''(1) \otimes \underline{2}(1,2) = \underline{2}(12,11)\,, &\qquad \underline{1}'''(1) \otimes \underline{2}(1,2) = \underline{2}(-12,11)\,,
\end{aligned} \tag{A.5}$$

$$\underline{2}(1,2) \otimes \underline{2}(1,2) = \underline{1}(11+22) \oplus \underline{1}'(11-22) \oplus \underline{1}''(12+21) \oplus \underline{1}'''(12-21)\,. \tag{A.6}$$

In the text we usually use the following notations, for example, $(xy)_{\underline{1}} \equiv (x_1 y_1 + x_2 y_2)$ which is the Clebsch–Gordan coefficients of $\underline{1}$ in the decomposition of $\underline{2} \otimes \underline{2}$, where as mentioned $x = (x_1, x_2) \sim \underline{2}$ and $y = (y_1, y_2) \sim \underline{2}$.

The rules to conjugate the representations $\underline{1}$, $\underline{1}'$, $\underline{1}''$, $\underline{1}'''$ and $\underline{2}$ are given by

$$\underline{2}^*(1^*, 2^*) = \underline{2}\,(1^*, 2^*)\,, \tag{A.7}$$

$$\begin{aligned} \underline{1}^*(1^*) &= \underline{1}\,(1^*)\,, \qquad \underline{1}'^*(1^*) = \underline{1}'(1^*)\,, \\ \underline{1}''^*(1^*) &= \underline{1}''(1^*)\,, \quad \underline{1}'''^*(1^*) = \underline{1}'''(1^*)\,, \end{aligned} \tag{A.8}$$

where, for example, $\underline{2}^*(1^*, 2^*)$ denotes some $\underline{2}^*$ multiplet of the form $(x_1^*, x_2^*) \sim \underline{2}^*$.

Appendix B. The Numbers

In the following we will explicitly point out the lepton number (L) and lepton parity (P_l) of the model particles (notice that the family indices are suppressed):

Particles	L	P_l
N_R, u, d, ϕ_1^+, $\phi_1'^+$, ϕ_2^0, $\phi_2'^0$, η_1^0, $\eta_1'^0$, η_2^-, $\eta_2'^-$ χ_3^0, σ_{33}^0, s_{33}^0	0	1
ν_L, l, U, D^*, ϕ_3^+, $\phi_3'^+$, η_3^0, $\eta_3'^0$, χ_1^{0*}, χ_2^+, σ_{13}^0, σ_{23}^+, s_{13}^0, s_{23}^+	-1	-1
σ_{11}^0, σ_{12}^+, σ_{22}^{++}, s_{11}^0, s_{12}^+, s_{22}^{++}	-2	1

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