Dynamics of the universe

How is the evolution of the universe determined?

- **Friedmann equation**

Einstein equation : $G_{\mu\nu} = \frac{1}{M_P^2} T_{\mu\nu}$

- Geometry of our universe

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega \right]$$

Scale Factor ($a(t)$)

- Matter distribution

$$T^\mu{}_\nu = \begin{pmatrix} -\rho(t) & 0 \\ 0 & p(t)\delta_{ij} \end{pmatrix}$$

Energy density ($\rho(t)$) pressure ($p(t)$)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3M_P^2} \sum_i \rho_i$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2} \sum_i (\rho_i + 3p_i)$$

$H(t) \equiv \frac{\dot{a}}{a}$ **Hubble parameter** Expansion rate

Pressure also gravitates. ($3p_i$)

- **Evolution of the scale factor is determined by the matter content.**

Einstein Eq. : Geometry ↔ Matter distribution

Friedmann eq. : scale factor change ↔ matter species, amount

- **Species** – **equation of state** (relation of energy density(ρ) and pressure(p)
  - Simple form of eq. of state : $p = p(\rho) = w\,\rho$ ⇨ $\rho = \rho_0\,(a/a_0)^{-3(1+w)}$

Name		Radiation	Matter	Vacuum E
Eq. of state	$p = w\rho$	$1/3$	0	-1
Energy density	$\rho \propto a^{-3(1+w)}$	a^{-4}	a^{-3}	$\mathrm{constant}$
Scale Factor (K=0)	$a \propto t^{2/3(1+w)}$	$t^{1/2}$	$t^{2/3}$	e^{Ht}

- **Amount** – **Density parameter**, ratio to the critical density

$$\Omega_i = \frac{\rho_i}{\rho_c}$$

- The critical density is determined by the Hubble constant.
- Roughly, 6 protons per $1\mathrm{m}^3$

$$\rho_c = 3M_P^2 H_0^2 = 1.9h^2 \times 10^{-26}\,\mathrm{kg/m^3}$$

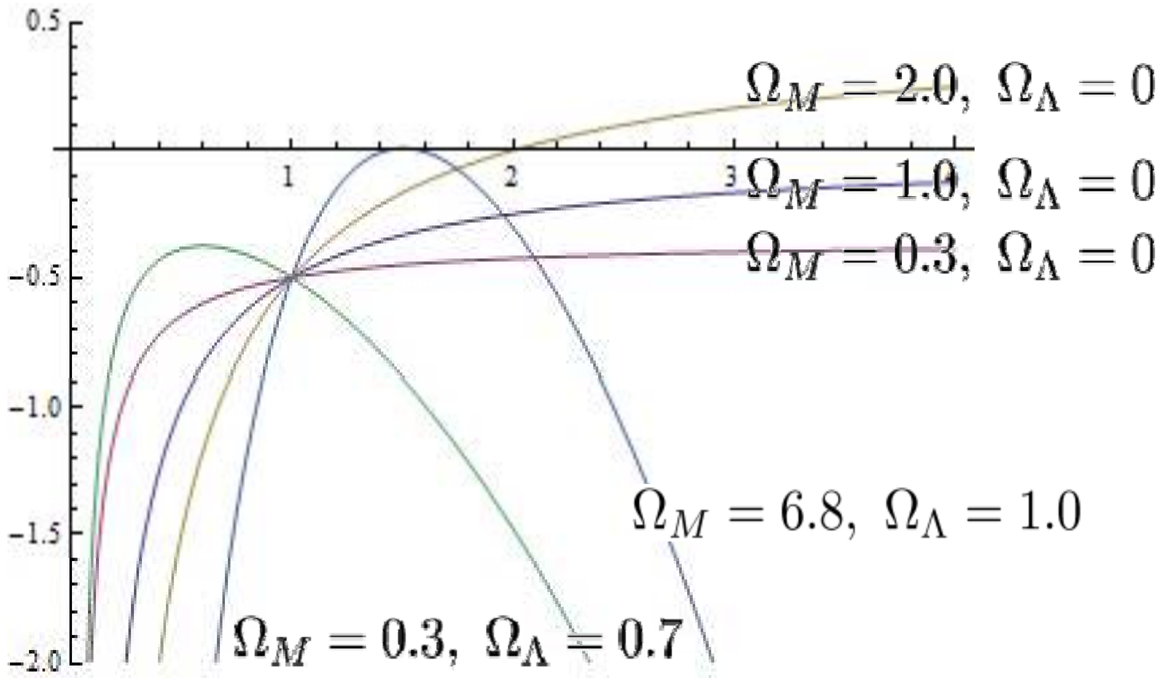
- **Solving Friedmann eq.**

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3M_P^2}\sum_i \rho_i = \frac{1}{3M_P^2}\sum_i \rho_{i0} a^{-3(1+w_i)}$$

$\Omega_0 \equiv \sum_i \Omega_i$

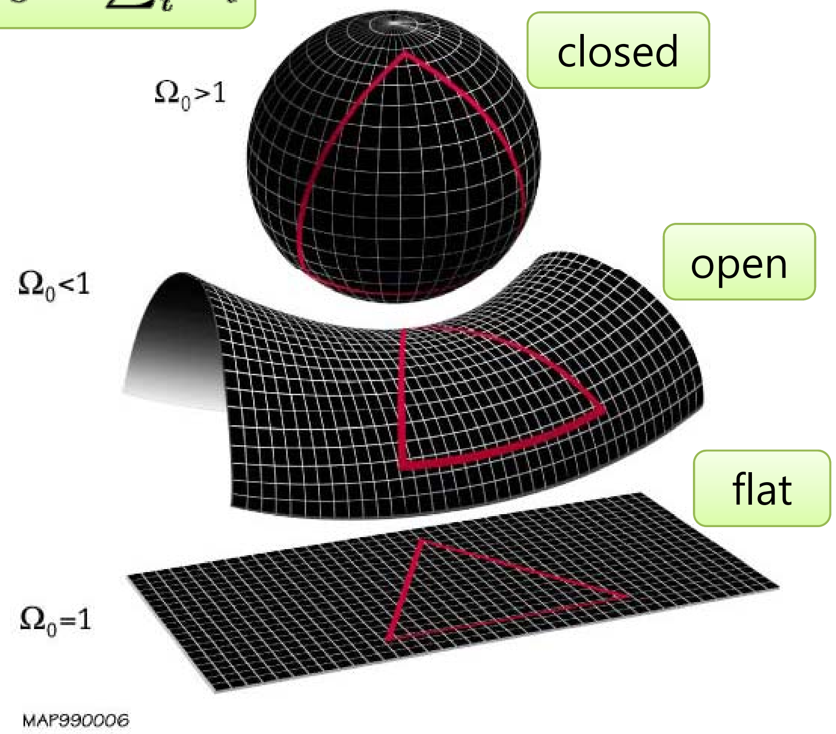
$$\frac{1}{2}\dot{a}^2 + V(a) = 0, \quad V(a) = \frac{1}{2}H_0^2\left[(\Omega_0 - 1) - \sum_i \Omega_i a^{-1-3w_i}\right]$$

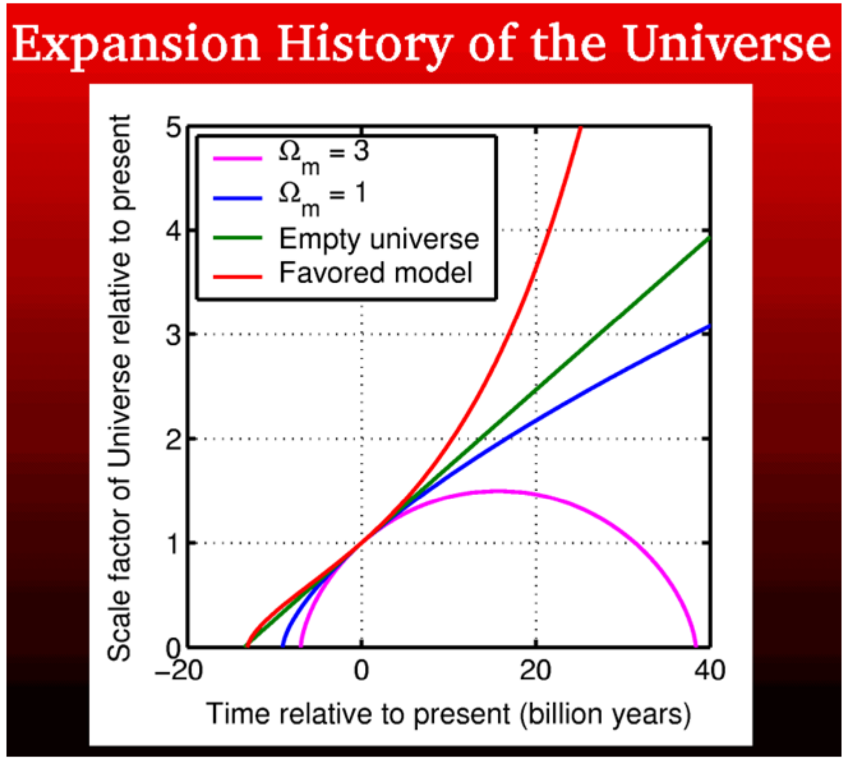
⇒ the motion of a particle with E=0 under the potential V(a)



- **Expansion history of the universe depends on the species and amounts of matter in the universe.**

$$\Omega_0 \equiv \sum_i \Omega_i$$



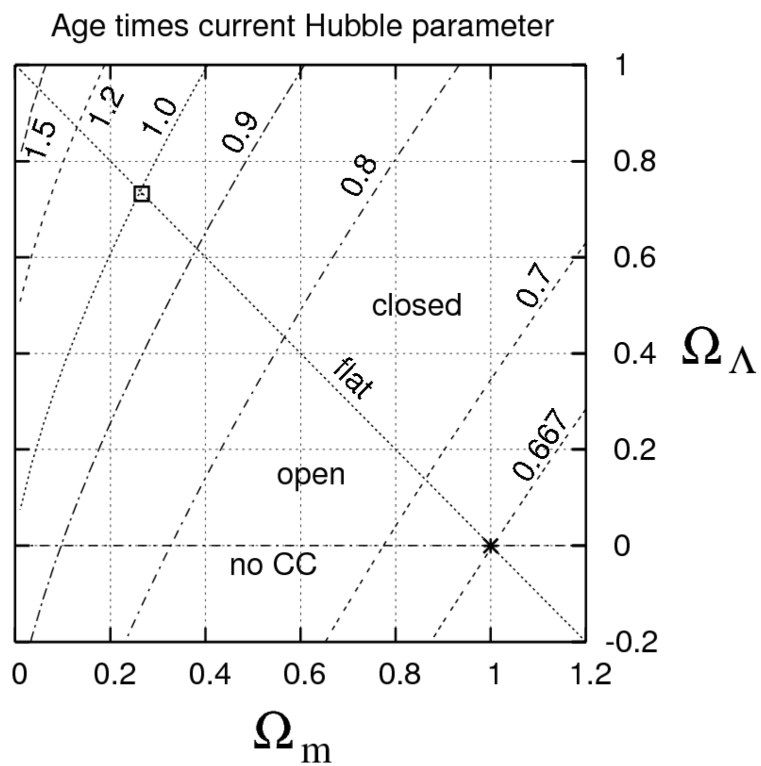


▪ Age of the universe

$$t_0 = H_0^{-1} f(\Omega_i), \quad f(\Omega_i) = \int_0^1 \left[-\Omega_K + \sum_i \Omega_i x^{-1-3w_i} \right]^{-1/2} dx$$

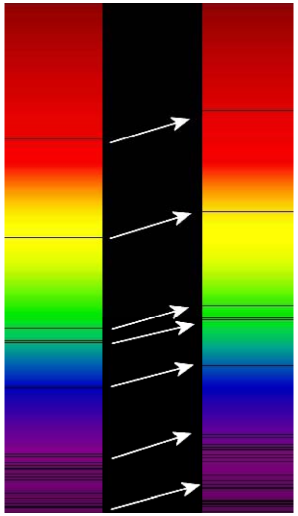
Hubble time

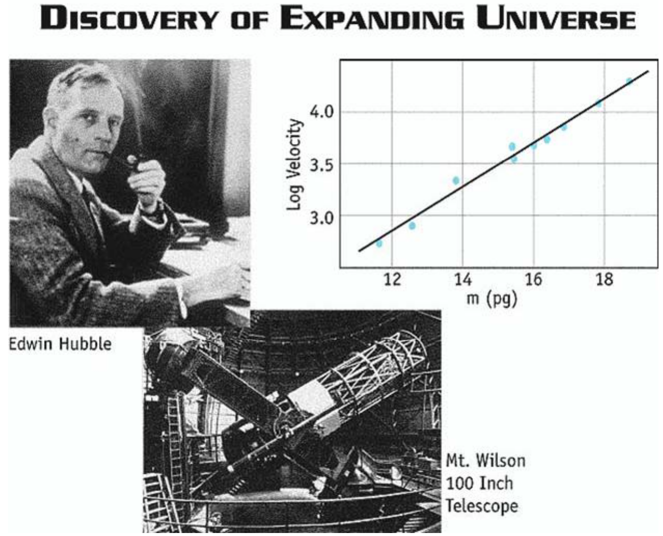
$$H_0^{-1} = \left(\frac{0.71}{h} \right) \times 13.8\,\mathrm{Gyr}$$



Discovery of Expansion

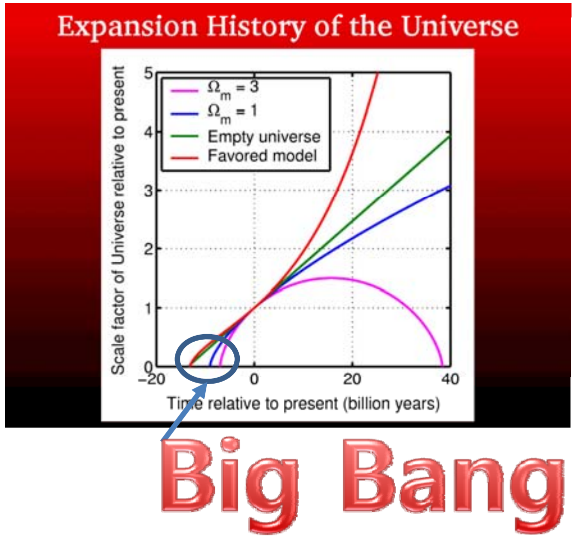
- **Redshift**
  - Vesto Slipher discovered the redshift of nebula (1912)
  - Absorption spectra from distant galaxies are red shifted.
  - Interpretation – Distant galaxies are receding from us.
- **Evidence for the expansion**
  - **Red shift proportional to distance (Hubble's law, 1929)**





Edwin Hubble discovered that redshift is proportional to distance by observing Cepheid variables in distant galaxies.

- Meaning of expansion – Dynamic universe
  - Static universe : eternal, no expansion.
  - **Dynamic universe : beginning and end, changes.**
  - Steady state universe : expanding but no changes.
- **Our universe has the beginning.**
  - If we trace back the expansion history, we meet a singular (infinite energy density) point of a=0 in a finite time.

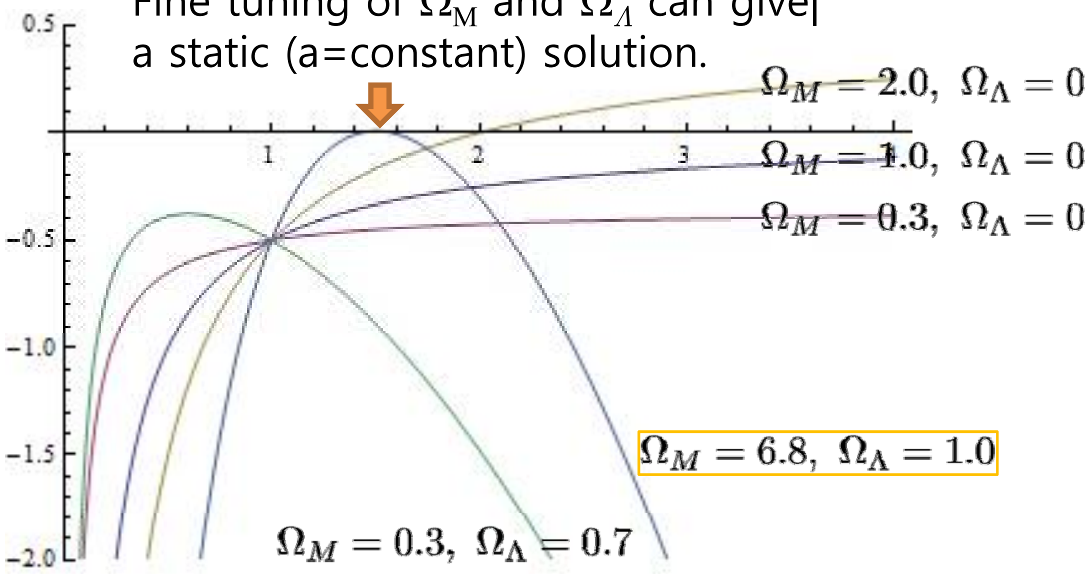


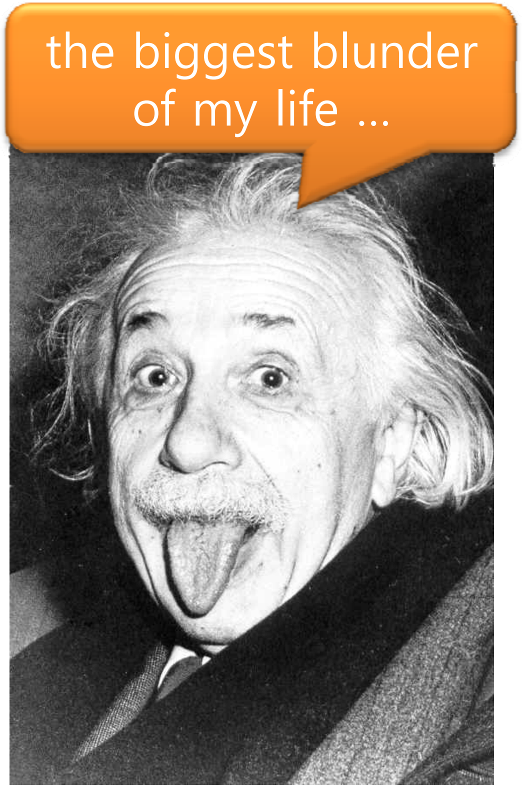
Einstein's Biggest Blunder

- **Introduced the cosmological constant to obtain the static universe** (1917)

$$\frac{1}{2}\dot{a}^2 + V(a) = 0, \quad V(a) = \frac{1}{2}H_0^2\left[(\Omega_0 - 1) - \sum_i \Omega_i a^{-1-3w_i}\right]$$

Fine tuning of Ω_M and Ω_Λ can give a static (a=constant) solution.

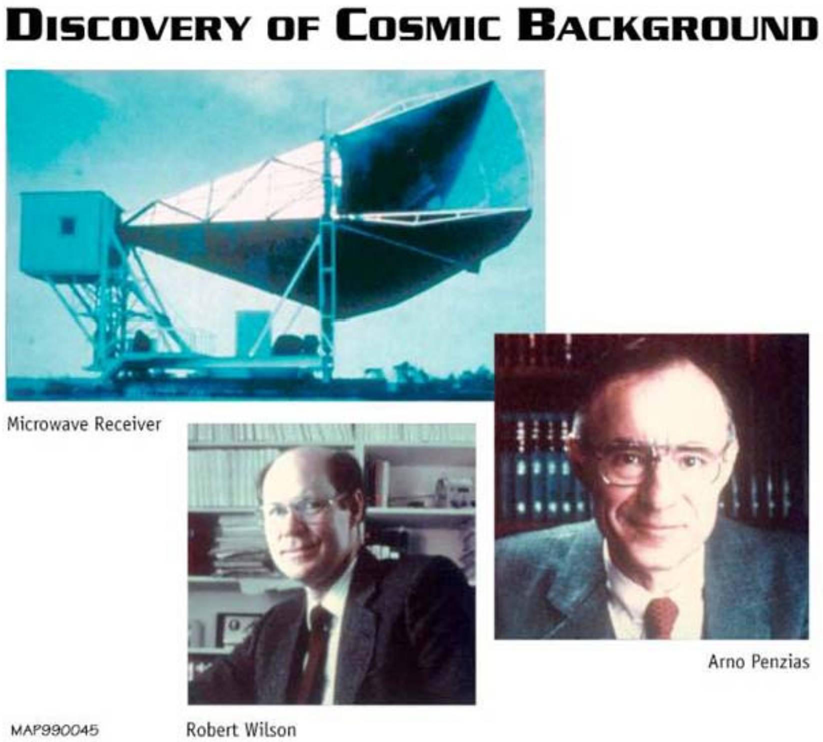




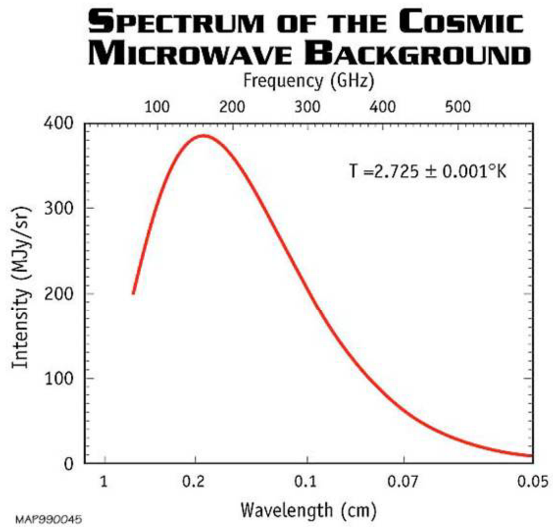
- Gave up the static universe after Hubble's discovery of expansion (1929)
- Resurrection of CC to explain the accelerating expansion (1998)

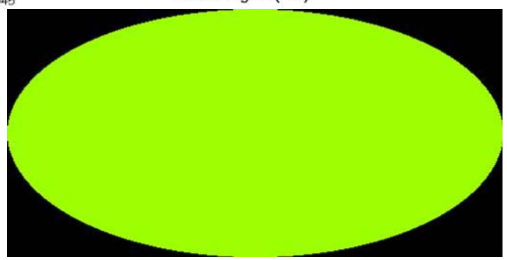
Discovery of CMB

- Matter content of the universe - Radiation
  - Eq. of state $p = \frac{1}{3}\rho$ (Ideal gas of photons)
  - Cosmic Microwave Background Radiation (CMB)
    - George Gamow and Ralph Alpher's prediction (1948)
    - Arno Penzias and Robert Wilson's discovery (1965)
    - Very isotropic, perfect black body spectrum with T=**2.73 K**



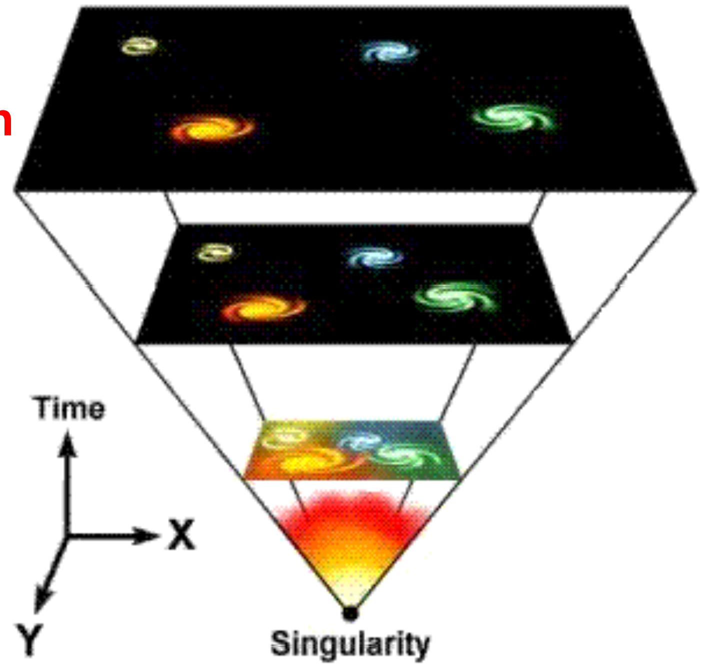
Penzias and Wilson, and the antenna they used in the discovery of CMB





CMB is very isotropic. ($\delta T/T \sim 10^{-5}$)

Consequences of Expansion



- Meaning of the existence of CMB
  - Black body spectrum of 2.73K
    $\Rightarrow$ Our universe was in **thermal equilibrium** in the past.

- Relation between the scale factor and the temperature in thermal equilibrium

$$a(t)T(t) = \text{constant}.$$

- Temperature and expansion
  - small a in the past → high T in the past.
  - **Hot Big Bang : Our universe started in thermal equilibrium with high temperature.**

- High Temperature (T) ⇔ High Energy (E)
  ⇔ Short Distance (quantum principle)
  - **To understand the high temperature state of the early universe, we need the knowledge at short distance (high energy, that is particle physics).**

Particles in equilibrium

- Direct evidence for thermal equilibrium in the early universe

  CMB : isotropic, accurate black body spectrum

  The early universe is filled with **hot ideal gases in thermal equilibrium**.

- Energy density and pressure at T

$$\rho_i(T) = g_i \int \frac{d^3\vec{p}}{(2\pi)^3} f_i(\vec{p}) E(\vec{p}) \qquad f(\vec{p}) = \frac{1}{e^{(E-\mu)/T} \pm 1}$$

$$p_i(T) = g_i \int \frac{d^3\vec{p}}{(2\pi)^3} f_i(\vec{p}) \frac{\vec{p}^2}{3E(\vec{p})} \qquad E = \sqrt{p^2 + m^2}$$

- Relativistic, non-degenerate : $T \gg m, \mu$

$$n = \left[\frac{3}{4}\right] \frac{\zeta(3)}{\pi^2} g T^3, \quad \rho = \left[\frac{7}{8}\right] \frac{\pi^2}{30} g T^4, \quad p = \frac{1}{3}\rho$$

- Non-relativistic : $T \ll m$

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}, \quad \rho = mn + \frac{3}{2}p, \quad p = nT \ll \rho$$

◆ Total energy density and pressure

- In equilibrium, the energy density of non-relativistic species is exponentially smaller than that of relativistic species.

$$\rho_R = \frac{\pi^2}{30} g_*(T) T^4, \quad p_R = \frac{1}{3}\rho_R, \quad g_*(T) = \sum_{\text{bosons}} g_b + \frac{7}{8}\sum_{\text{fermions}} g_f$$

◆ Entropy

- The entropy in a comoving volume is conserved in thermal equilibrium.
- Entropy in the early universe is dominated by relativistic species

  Entropy density $\quad s = \sum \frac{\rho_i + p_i}{T_i} = \frac{2\pi^2}{45} g_* T^3$

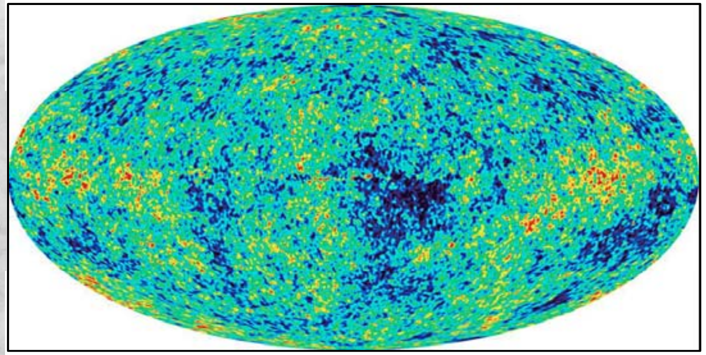
- Evolution of T $\quad \boldsymbol{T \propto g_*^{-1/3} a^{-1}}$

- Since $n \propto a^{-3}$ and $s \propto a^{-3}$, $\quad Y_i \equiv n_i / s$ is a convenient quantity to represent the abundance of decoupled particle.

Remnants of expansion

- Thermal equilibrium and its break-down
  - To keep thermal equilibrium, the reaction rate must be larger than the expansion rate.
  - As temperature goes down, the reaction rate decreases faster than the expansion rate and thermal equilibrium is broken.
  - If thermal equilibrium is kept on, no remnant from the past can be found.
- **Out-of-equilibrium make the history of the universe.**
  - **Baryogenesis, Big ban nucleosynthesis**
  - **Decoupling of dark matter, neutrinos, cosmic background radiation**

The expansion of the universe makes the history.



Cosmology is similar to archeology in the sense that it deduces the past from the remnants.

Thermal history of the universe

- The universe has been very nearly in thermal equilibrium for much of its history.
- Departure from thermal equilibrium might make fossil record of the early universe.
- Rule of thumb for thermal equilibrium

$$\text{Interaction rate } \boldsymbol{\Gamma_{\text{int}}} > \text{Expansion rate } \boldsymbol{H}$$

$$\Gamma_{\text{int}}(T) = n(T)\langle\sigma|v|\rangle^T \qquad H(T) \approx \frac{T^2}{M_P}$$

- Rough understanding of decoupling of species
  - Interaction mediated by a massive gauge boson

$$\sigma \sim \frac{\alpha^2 s}{m_X^2} \quad \Rightarrow \quad \Gamma_{\text{int}} \sim T^3 \cdot \frac{\alpha^2 T^2}{m_X^4} = \frac{\alpha^2 T^5}{m_X^4}$$

$$T \lesssim \left(\frac{m_X^4}{\alpha^2 M_P}\right)^{1/3} \sim \left(\frac{m_X}{100\,\text{GeV}}\right)^{4/3} \text{MeV} \quad \Rightarrow \quad \text{freeze out}$$

Boltzmann eq. for annihilation

- Boltzmann Eq. : The rate of abundance change
  = the rate of production – the rate of elimination

- Consider the particle 1 in a process $1 + 2 \leftrightarrow 3 + 4$

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}$$

$$\times (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\,|\mathcal{M}(1+2\leftrightarrow 3+4)|^2$$

$$\times \{f_3 f_4(1\pm f_1)(1\pm f_2) - f_1 f_2(1\pm f_3)(1\pm f_4)\}$$

Change in comoving volume

Production of 1
$3 + 4 \rightarrow 1 + 2$

Elimination of 1
$1 + 2 \rightarrow 3 + 4$

Particle physics enters here !
Scattering amplitude
CP(or T) symmetry assumed

Distribution function and number density

$$n_1(t) = \int \frac{d^3\vec{p}_1}{(2\pi)^3} f_1(\vec{p}_1, t)$$

- Simplifying assumptions
  - Kinetic equilibrium

    Rapid elastic scattering ⇨ $f(\vec{p},t) = \dfrac{1}{e^{(E(\vec{p})-\mu(t))/T(t)} \pm 1}$

  - Annihilation in equilibrium : $\mu(t)$ → chemical potential

    $f(\vec{p},t)$: described by chemical potential (and temperature)

  - Low temperature approximation : $T \ll E - \mu$

    $f \approx e^{-(E-\mu(t))/T}, \quad 1 + f \approx 1$

  - Change of variables : chemical potential → number density

    $$\mu_i(t) \;\rightarrow\; n_i(t) = g_i e^{\mu_i(t)/T} \int \frac{d^3\vec{p}_i}{(2\pi)^3} e^{-E_i/T} = e^{\mu_i(t)/T} n_i^{(0)}$$

    ⇨ Ordinary differential equation for $n_i(t)$

$$f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)$$
$$\approx e^{-(E_1+E_2)/T} \left(e^{\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right)$$
$$= e^{-(E_1+E_2)/T} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} \right\}$$

- Define the thermally averaged cross section

$$\langle \sigma v \rangle \equiv \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3\vec{p}_4}{(2\pi)^3 2E_4} e^{-(E_1+E_2)/T}$$
$$\times (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) \left| \mathcal{M}(1 + 2 \leftrightarrow 3 + 4) \right|^2$$

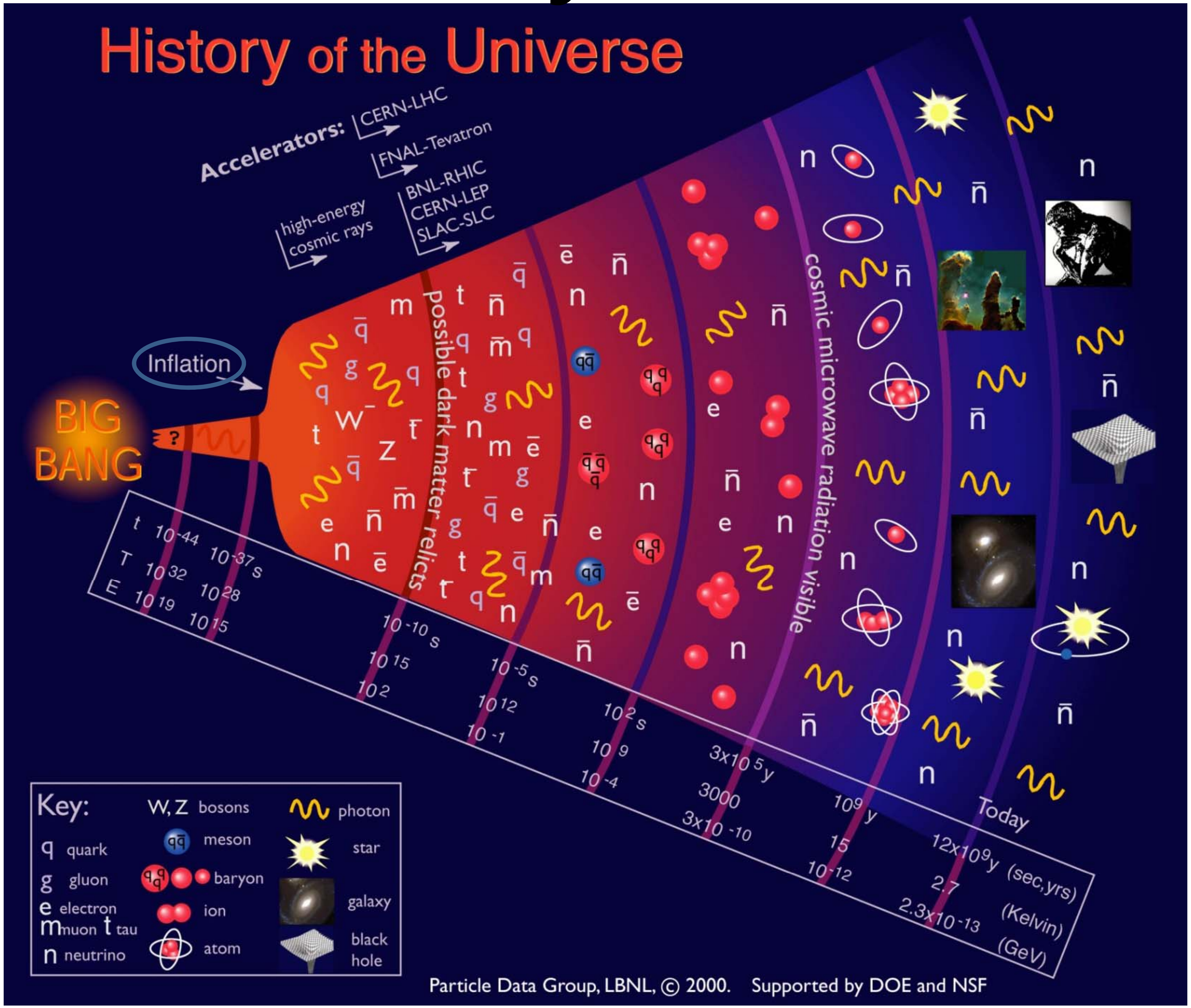
- Simplified Boltzmann equation

⇒ $$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)$$

$$\sim \frac{n_1}{t_H} \sim n_1 H \qquad n_1 n_2 \langle \sigma v \rangle \sim n_1 \Gamma_{\rm int}$$

$$H \ll \Gamma_{\rm int} \quad \Rightarrow \quad \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} = 0$$ chemical equilibrium

Thermal History of the Universe



Exercise 1

By integrating the Friedmann equation, derive the time-temperature relation during the radiation dominated era.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3M_P^2}\rho_R = \frac{1}{3M_P^2}\frac{\pi^2}{30}g_* T^4,$$

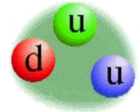
$$a\,T \propto g_*^{-1/3}$$

$$\Downarrow$$

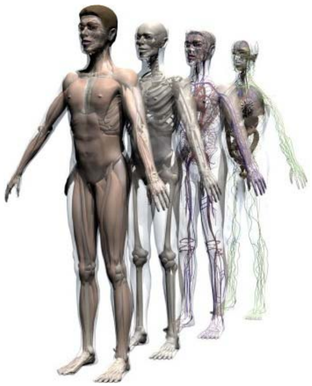
$$\frac{\dot{a}}{a} = -\frac{\dot{T}}{T}$$

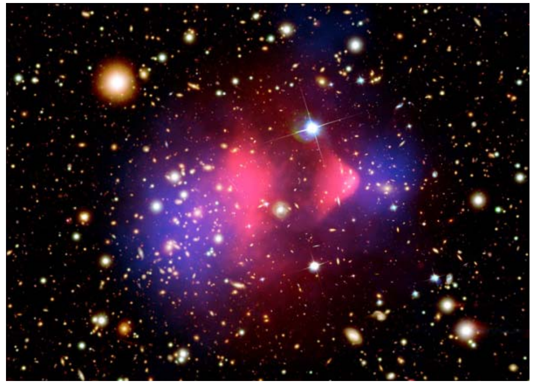
$$t = \left(\frac{90}{\pi^2 g_*}\right)^{1/2}\frac{M_P}{T^2} = 1.32\,\mathrm{s}\left(\frac{1\,\mathrm{MeV}}{T}\right)^2$$

Baryon Asymmetry



- Matter content of the universe – **Baryons**
  - Matter forming our body : Baryon(protons, neutron), lepton(electron)
  - Stars, planets, dust, gas, … (Most baryons are in intergalactic gases.)



- Baryon Asymmetry
  - SM of particle physics – very symmetric in baryon and anti-baryon
  - Our universe is dominated by baryons.

$$\frac{B-\bar{B}}{B+\bar{B}} \approx 1$$

- The amount of baryon in our universe
  - Good agreement in required amounts from BBN and CMBA

$$\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} \approx 4 \times 10^{-9}$$ (4 baryons to 1 billion photons)

Exercise 2

Suppose that there were no baryon asymmetry so that the number density of baryons exactly equaled that of anti-baryons. Determine the final relic density of (baryons+anti-baryons). At what temperature is this asymptotic value reached? (from Exercise 12 of Ch.3, Dodelson)

$$\Gamma_{\rm int}(T) = n(T)\langle\sigma|v|\rangle^T \approx (m_N T)^{3/2} e^{-m_N/T} \cdot \frac{\alpha^2}{T^2}$$

$$H(T) \approx \frac{T^2}{M_P}$$

$$(m_N/T)^{5/2} e^{-m_N/T} \approx \frac{m_N}{\alpha^2 M_P} \quad \Rightarrow \quad \frac{m_N}{T} \approx 41.6$$

$$\Rightarrow \quad \frac{n_b}{n_\gamma} \approx 10^{-16}$$