Chapter 8
Renormalization

The computation of quantum corrections to observables in quantum field theory requires making sense of expressions that are formally divergent. In this chapter we are going to show how this is done systematically. The renormalization program has nevertheless a much more profound meaning than just taming infinities. Using concepts borrowed from statistical mechanics we are going to see how the notion of renormalization is linked to the way physics looks like at different scales. These ideas will be further developed in Chap. 12.

8.1 Removing Infinities

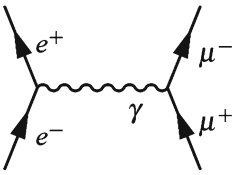
From its very early stages, quantum field theory was faced with infinities. They emerged in the calculation of important physical quantities, such as the corrections to the charge of the electron due to the interactions with the radiation field. The way these divergences where handled in the 1940s, starting with Kramers, was physically very much in the spirit of the quantum theory emphasis in observable quantities: the measured magnitude of a physical quantity, such as the electron mass, results from adding the quantum corrections to its unobservable "bare" value. The fact that both of these quantities are divergent is not a problem physically, since only their finite sum is observable. To make things mathematically consistent, the handling of infinities requires the introduction of some regularization procedure cutting off the divergent integrals at some momentum scale Λ. The physical value of an observable $\mathcal{O}_{\text{physical}}$ is then given by

$$\mathcal{O}_{\text{physical}} = \lim_{\Lambda\to\infty} \left[\mathcal{O}(\Lambda)_{\text{bare}} + \Delta\mathcal{O}(\Lambda)_{\hbar}\right], \tag{8.1}$$

where $\Delta\mathcal{O}(\Lambda)_{\hbar}$ represents the regularized quantum corrections.

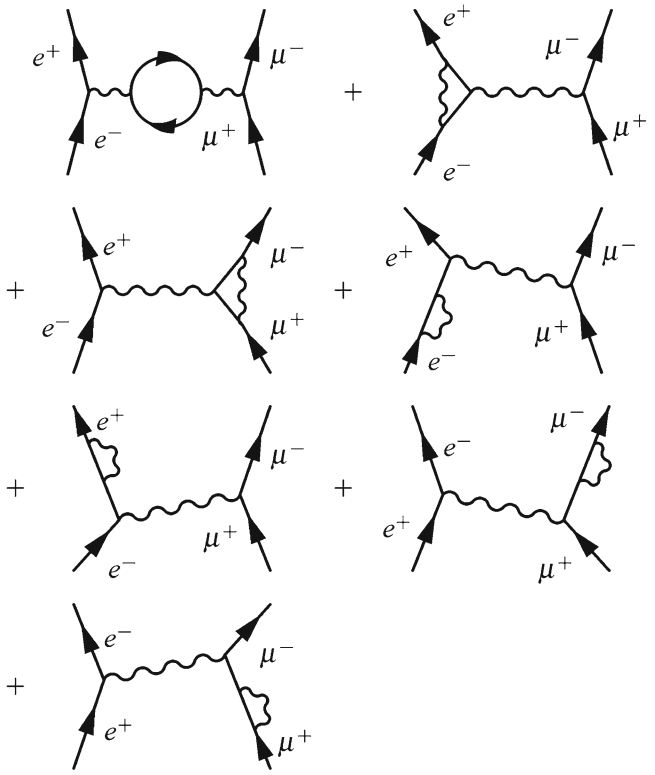
To make this qualitative discussion more precise we compute the corrections to the electric charge in QED. We consider the process of annihilation of an electron-positron pair to create a muon–antimuon pair $e^-e^+ \to \mu^+\mu^-$. To lowest order in the electric charge e the only diagram contributing is

L. Álvarez-Gaumé and M. Á. Vázquez-Mozo, *An Invitation to Quantum Field Theory*, Lecture Notes in Physics 839, DOI: 10.1007/978-3-642-23728-7_8,



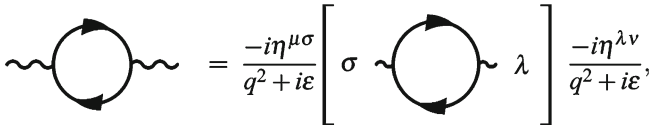
(8.2)

The corrections to order e^4 require the calculation of seven more diagrams



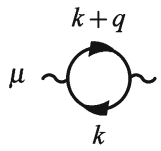
(8.3)

In order to compute the renormalization of the charge we consider the first diagram. We begin by factoring out the propagators associated with the external photon legs



(8.4)

where in between brackets we have the *amputated* diagram whose contribution defines the photon polarization tensor



$$\equiv \Pi^{\mu\nu}(q)$$

$$= i^2(-ie)^2(-1)\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[(\not{k}+m_e)\gamma^\mu(\not{k}+\not{q}+m_e)\gamma^\nu\right]}{\left[k^2-m_e^2+i\varepsilon\right]\left[(k+q)^2-m_e^2+i\varepsilon\right]}. \tag{8.5}$$

Physically, this diagram includes the correction to the propagator due to the polarization of the vacuum, i.e. the creation of virtual electron-positron pairs by the propagating photon. The momentum q is the total momentum of the electron-positron pair in the intermediate channel. Notice that the one-loop diagram (8.5) is the Fourier transform of the vacuum expectation value of the time-ordered product of two U(1) gauge currents, namely

$$\Pi^{\mu\nu}(q) = \int d^4x e^{iq\cdot x}\langle 0|T[j^\mu(x)j^\nu(0)]|0\rangle. \tag{8.6}$$

It is instructive to look at the one loop correction to the photon propagator from the point of view of perturbation theory in nonrelativistic quantum mechanics. In each vertex the interaction consists of the annihilation (resp. creation) of a photon and the creation (resp. annihilation) of an electron-positron pair. This can be implemented by the interaction Hamiltonian

$$H_{\mathrm{int}} = e\int d^3x\overline{\psi}\gamma^\mu\psi A_\mu. \tag{8.7}$$

All fields inside the integral can be expressed in terms of the corresponding creation-annihilation operators for photons, electrons and positrons. In quantum mechanics, the change in the wave function to first order in the perturbation H_{int} is given by

$$|\gamma,\mathrm{in}\rangle = |\gamma,\mathrm{in}\,\rangle_0 + \sum_n \frac{\langle n|H_{\mathrm{int}}|\gamma,\ \mathrm{in}\,\rangle_0}{E_{\mathrm{in}}-E_n}|n\rangle \tag{8.8}$$

and similarly for $|\gamma,\mathrm{out}\rangle$, where we have denoted symbolically by $|n\rangle$ all the possible states of the electron-positron pair. Since these states are orthogonal to $|\gamma,\ \mathrm{in}\,\rangle_0, |\gamma,\ \mathrm{out}\,\rangle_0$, we find to order e^2

$$\langle\gamma,\mathrm{in}\,|\gamma',\mathrm{out}\rangle = {}_0\langle\gamma,\mathrm{in}\,|\gamma',\mathrm{out}\,\rangle_0 + \sum_n \frac{{}_0\langle\gamma,\mathrm{in}\,|H_{\mathrm{int}}|n\rangle\langle n|H_{\mathrm{int}}|\gamma',\mathrm{out}\rangle_0}{(E_{\mathrm{in}}-E_n)(E_{\mathrm{out}}-E_n)} + \mathscr{O}(e^4). \tag{8.9}$$

Hence, we see that the diagram of Eq. (8.4) really corresponds to the order-e^2 correction to the photon propagator $\langle \gamma,\ \text{in}\,|\gamma',\ \text{out}\,\rangle$

$$\longrightarrow\ {}_0\langle \gamma, \text{in}|\gamma', \text{out}\rangle_0$$

$$\longrightarrow\quad \sum_n \frac{{}_0\langle \gamma, \text{in}|H_{\text{int}}|n\rangle\langle n|H_{\text{int}}|\gamma', \text{out}\rangle_0}{(E_{\text{in}} - E_n)(E_{\text{out}} - E_n)}. \tag{8.10}$$

Once we understand the physical meaning of the Feynman diagram to be computed we proceed to its evaluation. In principle there is no problem in computing the integral in Eq. (8.5) for nonzero values of the electron mass. However since here we are going to be mostly interested in how the divergence of the integral results in an energy scale dependent renormalization of the electric charge, we will set $m_e = 0$. This is something safe to do, since in the case of this diagram we are not inducing new infrared divergences in taking the electron as massless.

To compute the vacuum polarization tensor we are going to exploit what we can expect from gauge symmetry or current conservation. If we contract the external legs of the diagram (8.5) with the polarization tensors of the incoming and outgoing photon $\varepsilon_\mu(q)$ and $\varepsilon'_\nu(q)$, the result must be gauge invariant. That is, the amplitude cannot change under the replacement $\varepsilon_\mu(q) \to \varepsilon_\mu(q) + \lambda q_\mu, \varepsilon'_\mu(q) \to \varepsilon'_\mu(q) + \lambda' q_\mu$, for arbitrary λ and λ'. The consequence is that

$$q_\mu \Pi^{\mu\nu}(q) = 0 = q_\nu \Pi^{\mu\nu}(q). \tag{8.11}$$

This implies the following tensor structure for the polarization tensor

$$\Pi_{\mu\nu}(q) = \left(q^2 \eta_{\mu\nu} - q_\mu q_\nu\right) \Pi(q^2). \tag{8.12}$$

Manipulating (8.5) with techniques to be learned in Chap. 12 [using (12.37) and shifting the integration variable], we obtain

$$\Pi(q) = 8e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{x(1-x)}{[k^2 - m^2 + x(1-x)q^2 + i\varepsilon]^2}. \tag{8.13}$$

From the representation (8.6) of the polarization tensor we see that the gauge invariance conditions (8.11) implement current conservation.

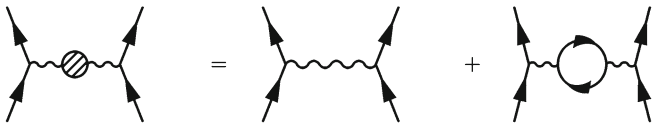
A more intuitive way to obtain this same result is to think of the diagram in (8.5) as the Fourier transform of the time-ordered correlation function of two gauge currents (8.6). Naively, the conservation of each current implies condition (8.11), and thus the form (8.12) of the polarization tensor. Notice that here we said "naively" because for this to be true we should have a way to compute the correlation function that either preserves gauge invariance (i.e., current conservation) or, if it breaks it, the damage can be fixed without much difficulty.

By looking at the powers of k in the numerator and denominator of the integrand of (8.5) we would conclude that the integral is quadratically divergent. It can be seen, however, that the quadratic divergence does cancel leaving behind only a logarithmic one.[1] In order to handle this divergent integral we have to figure out some procedure to render it finite. This can be done in several ways, but here we choose to cut the integrals off at a high energy scale Λ, where new physics might be at work, $|p| < \Lambda$. This gives the result

$$\Pi(q^2) \simeq \frac{e^2}{12\pi^2} \log\left(\frac{q^2}{\Lambda^2}\right) + \text{ finite terms} . \tag{8.14}$$

As a matter of fact, we have cheated a little bit in this analysis. Regularizing the integral (8.5) using a momentum cutoff does not lead to an expression of the form (8.12). In addition to this piece there is another one proportional to $\Lambda^2 \eta_{\mu\nu}$ that spoils gauge invariance. Here we are not very concerned about this term because it can be regarded as an artifact of the chosen regularization. Indeed, in the case of QED there are other regularization methods that preserve gauge invariance, such as dimensional regularization that we will introduce in Chap. 12. In any case the term proportional to Λ^2 could be dealt with by adding an appropriate local counterterm (see Sect. 8.3). Therefore in the following we will pretend that the offending term is absent.

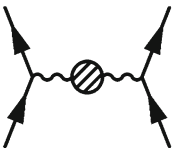
If we want to make sense out of Eq. (8.14), we have to go back to the physical question that led us to compute Eq. (8.4). Our primary motivation was to find the corrections to the annihilation of two electrons into two muons. Including the virtual photon propagation correction, we obtain



$$\begin{aligned} &= \eta_{\alpha\beta} \left(\bar{v}_e \gamma^\alpha u_e\right) \frac{e^2}{4\pi q^2} \left(\bar{v}_\mu \gamma^\beta u_\mu\right) + \eta_{\alpha\beta} \left(\bar{v}_e \gamma^\alpha u_e\right) \frac{e^2}{4\pi q^2} \Pi(q^2) \left(\bar{v}_\mu \gamma^\beta u_\mu\right) \\ &= \eta_{\alpha\beta} \left(\bar{v}_e \gamma^\alpha u_e\right) \left\{ \frac{e^2}{4\pi q^2} \left[1 + \frac{e^2}{12\pi^2} \log\left(\frac{q^2}{\Lambda^2}\right) \right] \right\} \left(\bar{v}_\mu \gamma^\beta u_\mu\right) . \end{aligned} \tag{8.15}$$

The reader is invited to check that the contribution of the terms proportional to $q_\mu q_\nu$ in (8.12) cancel after using the wave equation for the spinor wave functions. Now let us imagine that in the scattering $e^- e^+ \to \mu^- \mu^+$ we have a center of mass energy μ. From the previous result we can identify the effective charge of the particles at this energy scale $e(\mu)$ as

[1] The change from a quadratically to a logarithmically divergent integral is a consequence of the tensor structure (8.12) of the polarization tensor, and therefore a consequence of gauge invariance.



$$= \eta_{\alpha\beta}\,(\bar{v}_e\gamma^\alpha u_e)\left[\frac{e(\mu)^2}{4\pi q^2}\right]\left(\bar{v}_\mu\gamma^\beta u_\mu\right). \tag{8.16}$$

This charge, $e(\mu)$, is the physically measurable quantity in our experiment. Now we can make sense of the formally divergent result (8.15) by assuming that the charge appearing in the classical Lagrangian of QED is just a "bare" value that depends on the scale Λ at which we cut off the theory, $e \equiv e_0(\Lambda)$. In order to reconcile (8.15) with the physical results (8.16), we must assume that the dependence of the bare (unobservable) charge $e_0(\Lambda)$ on the cutoff Λ is determined by the identity

$$e(\mu)^2 = e_0(\Lambda)^2\left[1 + \frac{e_0(\Lambda)^2}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right]. \tag{8.17}$$

If we still insist in removing the cutoff, $\Lambda \to \infty$, we have to send the bare charge to zero, $e_0(\Lambda) \to 0$, in such a way that the effective coupling has the finite value given by the experiment at the energy scale μ. All observable quantities should be expressed in perturbation theory as a power series in the physical coupling $e(\mu)^2$ and not in terms of the unphysical bare coupling $e_0(\Lambda)$.

8.2 The Beta-Function and Asymptotic Freedom

We can look at the previous discussion, and in particular Eq. (8.17), from a different point of view. In order to remove the ambiguities associated with infinities we have introduced a dependence of the coupling constant on the energy scale at which a process takes place. From the expression of the physical coupling in terms of the bare charge (8.17) we can eliminate the cutoff Λ, whose value after all should not affect the value of physical quantities. Taking into account that we are working in perturbation theory in $e(\mu)^2$, we can express the bare charge $e_0(\Lambda)^2$ in terms of $e(\mu)^2$ as

$$e_0(\Lambda)^2 = e(\mu)^2\left[1 + \frac{e(\mu)^2}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right] + \mathcal{O}[e(\mu)^6]. \tag{8.18}$$

This expression allows us to eliminate all dependence in the cutoff in the expression of the effective charge at a scale μ by replacing $e_0(\Lambda)$ in Eq. (8.17) by the one computed using (8.18) at a given reference energy scale μ_0

$$e(\mu)^2 = e(\mu_0)^2\left[1 + \frac{e(\mu_0)^2}{12\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]. \tag{8.19}$$

From this equation we can compute, at this order in perturbation theory, the effective value of the coupling constant at an energy μ, once we know its value at some reference energy scale μ_0. In the case of the electron charge we can use as a reference Thompson's scattering at energies of the order of the electron mass $m_e \simeq 0.5\,\text{MeV}$, where the value of the electron charge is given by the well known value

$$\alpha(m_e) = \frac{e(m_e)^2}{4\pi} \simeq \frac{1}{137}. \tag{8.20}$$

With this, we can compute $e(\mu)^2$ at any other energy scale by applying Eq. (8.19). In computing the electromagnetic coupling constant at any other scale we must take into account the fact that other charged particles can run in the loop in Eq. (8.15). Suppose, for example, that we want to calculate the fine structure constant at the mass of the Z^0-boson $\mu = m_Z \simeq 92$ GeV. Then, we should include in Eq. (8.19) the effect of other standard model fermions with masses below m_Z. Thus

$$e(m_Z)^2 = e(m_e)^2 \left[1 + \frac{e(m_e)^2}{12\pi^2}\left(\sum_i q_i^2\right)\log\left(\frac{m_Z^2}{m_e^2}\right)\right], \tag{8.21}$$

where q_i is the charge in units of the electron charge of the ith fermionic species running in the loop, and we sum over all fermions with masses below the mass of the Z^0 boson. This expression shows how the electromagnetic coupling grows with energy. To compare with the experimental value of $e(m_Z)^2$ it is not enough to include the effect of fermionic fields, since also the $W^\pm$ bosons can run in the loop ($m_W < m_Z$). Taking this into account, as well as threshold effects, the value of the electron charge at the scale m_Z is found to be [1]

$$\alpha(m_Z) = \frac{e(m_Z)^2}{4\pi} \simeq \frac{1}{128.9}. \tag{8.22}$$

This growth of the effective fine structure constant with energy can be understood heuristically by remembering that the effect of the polarization of the vacuum shown in the diagram of Eq. (8.4) amounts to the creation of virtual electron-positron pairs around the location of the charge. These virtual pairs behave as dipoles that, as in a dielectric medium, tend to screen this charge and to decrease its value at long distances (i.e. lower energies).

The variation of the coupling constant with energy is usually given in quantum field theory in terms of the *beta function* defined by

$$\beta(g) = \mu \frac{dg}{d\mu}. \tag{8.23}$$

In the case of QED the beta function can be computed from Eq. (8.19) with the result

$$\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2}. \tag{8.24}$$

The fact that the coefficient of the leading term in the beta-function is positive gives us the overall behavior of the coupling as we change the scale. Equation (8.24) means that, if we start at an energy where the electric coupling is small enough for our perturbative treatment to be valid, the effective charge grows with the energy scale. This growth of the effective coupling constant with energy means that QED is infrared safe, since the perturbative approximation gives better and better results as we go to lower energies. In fact, since the electron is the lightest electrically charged particle and has a finite nonvanishing mass, the running of the fine structure constant stops at the scale m_e in the well-known value $\frac{1}{137}$. Would other charged fermions with masses below m_e be present in Nature, the effective value of the fine structure constant would run further to lower values at energies below the electron mass.

When we increase the energy scale, $e(\mu)^2$ grows until at some scale the coupling is of order one and the perturbative approximation breaks down. In QED this is known as the problem of the Landau pole but in fact it does not pose any serious threat to the reliability of QED perturbation theory: a calculation including the effect of all standard model fermions shows that the energy scale at which the theory would become strongly coupled is $\Lambda_{\text{Landau}} \simeq 10^{34}$ GeV [2]. However, we expect QED to be unified with other interactions below that scale, and even if this is not the case we will enter the uncharted territory of quantum gravity at energies of the order of 10^{19} GeV.

So much for QED. The next question that one may ask at this stage is whether it is possible to find quantum field theories with a behavior opposite to that of QED, i.e. such that they become weakly coupled at high energies. This is not a purely academic question. In the late 1960s a series of deep inelastic scattering experiments carried out at SLAC showed that the quarks behave essentially as free particles inside hadrons. The apparent problem was that no theory was known at the time that would become free at very short distances: the QED behavior was encountered in all the theories that were analyzed. This posed a very serious problem for quantum field theory as a way to describe subnuclear physics, since it seemed that its predictive power was restricted to electrodynamics but failed when applied to the strong interactions.

This critical time for quantum field theory turned out to be its finest hour. In 1973 David Gross and Frank Wilczek [3] and David Politzer [4] showed that nonabelian gauge theories display the required behavior. For the QCD Lagrangian in Eq. (9.38) the beta function is given by [2]

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right). \tag{8.25}$$

In particular, for real QCD ($N_c = 3$, and N_f equal the number of active flavors) we have that $\beta(g) < 0$. This means that for a weakly coupled theory at an energy scale μ_0 the coupling constant decreases as energy increases $\mu \to \infty$. This explain the

[2] This result has an interesting history. See, for example, [5].

apparent freedom of quarks inside hadrons: when the quarks are very close together their effective color charge tends to zero. This phenomenon is called *asymptotic freedom*.

Asymptotically free theories display a behavior opposite to QED. At high energies their coupling constant approaches zero, whereas at low energies they become strongly coupled (infrared slavery). This features are at the heart of the success of QCD as a theory of the strong interactions, since this is exactly the type of behavior found in quarks: they are quasi-free particles inside the hadrons but the interaction potential between them increases at large distances.

Although asymptotically free theories can be handled in the ultraviolet, they have remarkable properties in the infrared. In the case of QCD they are responsible for color confinement and chiral symmetry breaking (9.52).

In general, the ultraviolet and infrared properties of a theory are controlled by the fixed points of the beta function, i.e. those values of the coupling constant g for which it vanishes

$$\beta(g^*) = 0. \tag{8.26}$$

Using perturbation theory we have seen that for both QED and QCD a fixed point occurs at zero coupling, $g^* = 0$. However, our analysis also showed that the two theories present radically different behavior at high and low energies. From the point of view of the beta function, the difference lies in the energy regime at which the coupling constant approaches its critical value. This is in fact governed by the sign of the beta function around the critical coupling.

If the beta function is negative close to the fixed point (the case of QCD) the coupling tends to its critical value, $g^* = 0$, as the energy is increased. This means that the critical point is *ultraviolet stable*, i.e. it is an attractor as we evolve towards higher energies. If, on the contrary, the beta function is positive (as it happens in QED) the coupling constant approaches the critical value as the energy decreases. This is the case of an *infrared stable* fixed point.

This analysis that we have motivated with the examples of QED and QCD is completely general and can be carried out for any quantum field theory. In Fig. 8.1 we have represented the beta function for a hypothetical theory with three fixed points located at couplings g_1^*, g_2^* and g_3^*. The arrows in the line below the plot represent the evolution of the coupling constant as the energy increases. We learn that $g_1^* = 0$ and g_3^* are ultraviolet stable fixed points, while g_2^* is infrared stable.

In order to understand the high and low energy behavior of a quantum field theory it is crucial to know the structure of the beta functions associated with its couplings. This can be a very difficult task, since perturbation theory only allows the study of the theory around “trivial" fixed points, i.e. those that occur at zero coupling like the case of g_1^* in Fig. 8.1. Any “nontrivial” fixed point occurring in a theory (like g_2^* and g_3^*) cannot be captured in perturbation theory and requires a full nonperturbative analysis.

The lesson to be learned from this discussion is that dealing with the ultraviolet divergences in a quantum field theory has as a consequence the introduction of an energy dependence in the measured value of the coupling constants of the

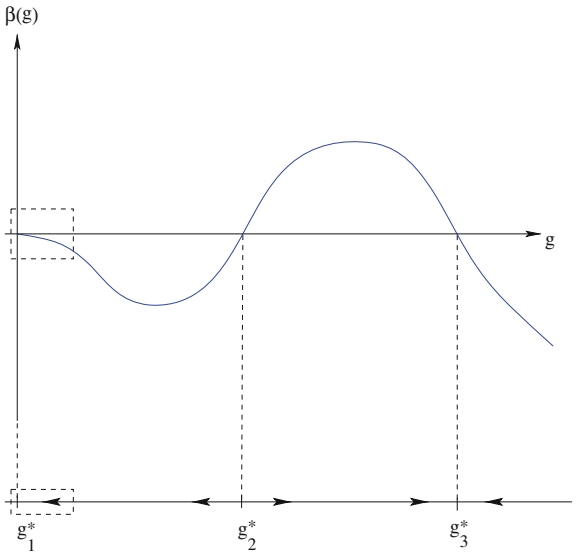


Fig. 8.1 Beta function for a hypothetical theory with three fixed points g_1^*, g_2^* and g_3^*. A perturbative analysis would capture only the regions shown in the *boxes*

theory. This happens even in the case of theories without dimensionful couplings. These theories are scale invariant at the classical level because the action does not contain dimensionful parameters. In this case the running of the coupling constants can be seen as resulting from a quantum breaking of classical scale invariance: different energy scales in the theory are distinguished by different values of the coupling constants. We say that classical scale invariance is an *anomalous* symmetry (see Chap. 9). A heuristic way to understand how the conformal anomaly comes about is to notice that the regularization of an otherwise scale invariant field theory requires the introduction of an energy scale (e.g. a cutoff). In general, the classical invariance cannot be restored after renormalization.

Scale invariance is not completely lost in quantum field theory, however. It is recovered at the fixed points of the beta function where, by definition, the coupling does not run. We consider a scale invariant classical field theory whose field $\phi(x)$ transform under coordinate rescalings as

$$x^\mu \longrightarrow x'^\mu = \lambda x^\mu, \quad \phi(x) \longrightarrow \phi'(x) = \lambda^{-\Delta}\phi(\lambda^{-1}x), \tag{8.27}$$

where Δ is called the canonical scaling dimension of the field. An example of such a theory is a massless ϕ^4 theory in four dimensions

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{4!}\phi^4, \tag{8.28}$$

where the scalar field has canonical scaling dimension $\Delta = 1$. The Lagrangian density transforms as

$$\mathscr{L} \longrightarrow \lambda^{-4}\mathscr{L}[\phi] \tag{8.29}$$

and the classical action remains invariant [3] under (8.27).

If scale invariance is preserved by quantization, the Green's functions transform as

$$\langle\Omega|T[\phi'(x_1)\dots\phi'(x_n)]|\Omega\rangle = \lambda^{-n\Delta}\langle\Omega|T[\phi(\lambda^{-1}x_1)\dots\phi(\lambda^{-1}x_n)]|\Omega\rangle. \quad (8.30)$$

This is what happens in a free theory, whereas in an interacting theory the running of the coupling constant destroys classical scale invariance at the quantum level. In spite of this, at the fixed points of the beta function the Green's functions transform again according to Eq. (8.30) where Δ is replaced by

$$\Delta_{\text{anom}} = \Delta + \gamma^*. \quad (8.31)$$

Thus, the canonical scaling dimension of the fields are corrected by γ^*, called the *anomalous dimension*. A more detailed discussion of this issue is postponed to Chap. 12.

The previous discussion exhibits some of the high-energy properties of asymptotically free theories like QCD. In the critical theory, the fields have anomalous dimensions different from those in the free theory. These carry the dynamical information about the high-energy behavior.

8.3 A Look at the Systematics of Renormalization

The renormalization program presented in Sect. 8.1 proceeds in two steps. First, the divergences appearing in the calculation of loop diagrams are tamed by introducing a regulator Λ setting an energy scale above which the theory is modified.[4] The second step consists of absorbing the divergences appearing as $\Lambda \to \infty$ in the perturbative calculation of S-matrix amplitudes (or Green's functions) in the parameters of the Lagrangian.

In the particular case of QED this implies a dependence on the regulator of the bare electron charge and mass $e_0(\Lambda)$ and $m_0(\Lambda)$ and also of the global normalization factor of the fields. That is, the "bare" fields $\psi_0(x)$ and $A_{0\mu}(x)$ appearing in the Lagrangian get a dependence on Λ of the form

$$\psi_0(x,\Lambda) = \sqrt{Z_\psi(\Lambda)}\psi(x), \quad A_{0\mu}(x,\Lambda) = \sqrt{Z_A(\Lambda)}A_\mu(x), \quad (8.32)$$

where $\psi(x)$ and $A_\mu(x)$ are called the renormalized fields and are regulator independent. The dependence on Λ of all bare quantities has to be chosen in such a way that

[3] In a d-dimensional theory the canonical scaling dimensions of the fields coincide with its engineering dimension: $\Delta = \frac{d-2}{2}$ for bosonic fields and $\Delta = \frac{d-1}{2}$ for fermionic ones. For a Lagrangian with no dimensionful parameters classical scale invariance follows then from dimensional analysis.

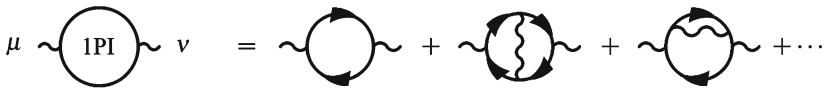
[4] In the following we denote by Λ any regulator, not necessarily the momentum cutoff used in Sect. 8.1 By convention we consider that the removal of the regulator corresponds to the limit $\Lambda \to \infty$.

the time-ordered Green's functions for the renormalized fields, as computed from the renormalized Lagrangian density

$$\begin{aligned}\mathscr{L}_{\text{ren}} &= Z_\psi(\Lambda)\overline{\psi}\left[i\gamma^\mu\partial_\mu - m_0(\Lambda)\right]\psi - \frac{1}{4}Z_A(\Lambda)F_{\mu\nu}F^{\mu\nu} \\ &\quad - e_0(\Lambda)Z_\psi(\Lambda)\sqrt{Z_A(\Lambda)}A_\mu\overline{\psi}\psi,\end{aligned} \tag{8.33}$$

remain finite when $\Lambda \to \infty$. The dependence of Z_ψ, Z_A, e_0 and m_0 on the regulator has to be corrected at each order in perturbation theory.

Since the bare parameters are unphysical we need to identify their physical counterparts that would be measurable in experiments. This we did in Sect. 8.1 by a proper physical interpretation of the sum of two diagrams in Eq. (8.15). We can be more general and instead of considering the one-loop diagram to compute the photon self-energy we can add all *one-particle-irreducible* (1PI) diagrams with two amputated photon external legs. This class of diagrams is defined as those that cannot be split into two disconnected pieces by slicing a single internal line. For example, all diagrams in (8.2) and (8.3) are reducible whereas the one on the left-hand side of Eq. (8.4) is 1PI. We consider the sum



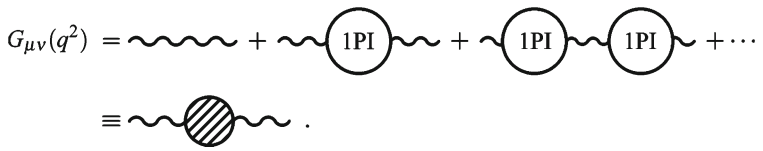
$$= (q^2\eta_\mu - q_\mu q_\nu)\Pi(q^2), \tag{8.34}$$

where the function $\Pi(q^2)$ receives contributions to all orders in perturbation theory. The tensor structure of the previous sum of diagrams is imposed by gauge invariance, as explained in page 148.

What makes the quantity $\Pi(q^2)$ interesting is that the full photon propagator

$$G_{\mu\nu}(q^2) = \int d^4x e^{iq\cdot x}\langle\Omega|T\left[A_\mu(x)A_\nu(0)\right]|\Omega\rangle \tag{8.35}$$

can be written in terms of it. The diagrammatic expansion of $G_{\mu\nu}(q^2)$ has the form



(8.36)

Using the simple identity

$$\left(\delta^\mu_\alpha - \frac{q^\mu q_\alpha}{q^2}\right)\left(\delta^\alpha_\nu - \frac{q^\alpha q_\nu}{q^2}\right) = \delta^\mu_\nu - \frac{q^\mu q_\nu}{q^2} \tag{8.37}$$

it is not difficult to show that (8.36) reduces to a geometric series whose sum is

$$G_{\mu\nu}(q^2) = \frac{-i}{q^2[1-\Pi(q^2)]}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - i\frac{q_\mu q_\nu}{q^4}. \tag{8.38}$$

Using the Lagrangian (8.33) we compute the full propagator $G_{\mu\nu}(q^2, \Lambda)_0$ of the bare gauge field

$$G_{\mu\nu}(q^2, \Lambda)_0 = \frac{-i\eta_{\mu\nu}}{q^2[1-\Pi_0(q^2, \Lambda)]} \tag{8.39}$$

Here we have removed the terms in the full propagator proportional to $q_\mu q_\nu$ since they vanish once contracted with the fermion lines. The divergence of $\Pi_0(q^2, \Lambda)$ can be absorbed in the normalization factor of the bare field. To see this we notice that

$$G_{\mu\nu}(q^2, \Lambda)_0 = Z_A(\Lambda) G_{\mu\nu}(q^2), \tag{8.40}$$

where the Green's function on the right-hand side is that of the renormalized photon field and therefore remains finite as $\Lambda \to \infty$. We choose $Z_A(\Lambda)$ satisfying the condition

$$\lim_{\Lambda\to\infty} Z_A(\Lambda)[1-\Pi_0(q^2, \Lambda)] < \infty. \tag{8.41}$$

This does not determine uniquely $Z_A(\Lambda)$. To fix the ambiguity we impose a *renormalization condition*. For example, we demand that the renormalized Green's function $G_{\mu\nu}(q^2)$ behaves close to the pole in the same way as the free photon propagator

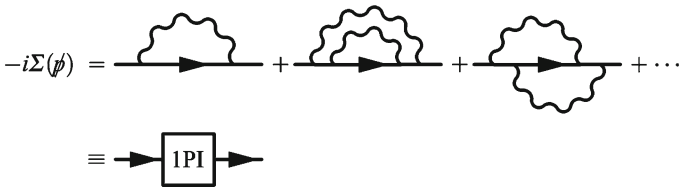
$$G_{\mu\nu}(q^2) \sim \frac{-i\eta_{\mu\nu}}{q^2} \quad \text{as} \quad q^2 \to 0. \tag{8.42}$$

This fixes the wave function renormalization to be

$$Z_A(\Lambda) = \frac{1}{1-\Pi_0(0, \Lambda)}. \tag{8.43}$$

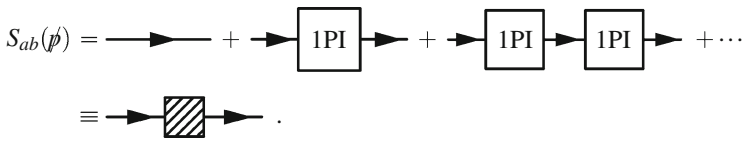
We should bear in mind that the denominator on the right-hand side of this expression does not vanish. This follows from the condition that the photon remains massless to all orders in perturbation theory. Hence, (8.39) should have a single pole at $q^2 = 0$, and we get the condition $\Pi_0(0, \Lambda) \neq 1$.

The calculation of other physical parameters can be done along similar lines. In the case of the electron mass we start by summing the contributions of all 1PI corrections that defines the fermion self-energy $\Sigma_{ab}(\not{p})$



(8.44)

Similarly to the photon case, the whole perturbative expansion of the full fermion propagator $S_{ab}(p)$ can be formally obtained by iterating the insertion of self-energy blobs in the fermion line



(8.45)

The resulting geometric series yields

$$S(\not{p}) = \frac{i}{\not{p} - m} \sum_{n=0}^{\infty} \left[\frac{1}{\not{p} - m} \Sigma(\not{p}) \right]^n = \frac{i}{\not{p} - m - \Sigma(\not{p})}. \tag{8.46}$$

In a free fermion theory the mass of the particle can be identified as the pole in the propagator

$$S(\not{p}) = \frac{i}{\not{p} - m}. \tag{8.47}$$

We extend this definition of the physical fermion mass to QED. Working with the Lagrangian (8.33) the complete propagator for the bare fermion field reads

$$S_0(\not{p}, \Lambda) = \frac{i}{\not{p} - m_0(\Lambda) - \Sigma_0(\not{p}, \Lambda)}. \tag{8.48}$$

The physical mass is identified with the value $\not{p} = m$ at which the denominator of the full propagator vanishes

$$m = m_0(\Lambda) + \Sigma_0(\not{p}, \Lambda)\Big|_{\not{p}=m}. \tag{8.49}$$

This gives the dependence of the bare mass on the regulator Λ.

With this we do not get rid of all infinities since the fermion self-energy can have a divergent piece linear in the momentum p. To deal with this problem we write the following expansion around the physical fermion mass m

$$\Sigma_0(\not{p}, \Lambda) = \Sigma_0(m, \Lambda) + (\not{p} - m)\Sigma_0'(m, \Lambda) + (\not{p} - m)^2 \tilde{\Sigma}_0(\not{p}, \Lambda), \tag{8.50}$$

where

$$\Sigma_0'(m, \Lambda) = \frac{d}{d\not{p}} \Sigma_0(\not{p}, \Lambda)\Big|_{\not{p}=m}. \tag{8.51}$$

Plugging this in the full propagator (8.48) and taking into account the definition (8.49) we have

$$S_0(\not{p}, \Lambda) = \frac{i}{(\not{p} - m)[1 - \Sigma_0'(m, \Lambda) - (\not{p} - m)\tilde{\Sigma}(\not{p}, \Lambda)]}. \tag{8.52}$$

The divergence of $\Sigma_0'(m, \Lambda)$ as $\Lambda \to \infty$ can now be absorbed in the normalization of the bare field. Indeed, the bare propagator is written in terms of the renormalized one as

$$S_0(\not{p}, \Lambda) = Z_\psi(\Lambda) S(\not{p}), \tag{8.53}$$

where $Z_\psi(\Lambda)$ is chosen to satisfy

$$\lim_{\Lambda \to \infty} Z_\psi(\Lambda) \Big[1 - \Sigma_0'(m, \Lambda) - (\not{p} - m)\tilde{\Sigma}(\not{p}, \Lambda) \Big] < \infty. \tag{8.54}$$

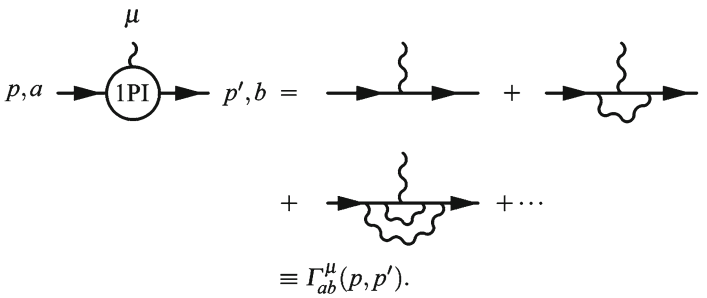
To fix the freedom in choosing the field normalization we use a renormalization condition similar to the one for the photon propagator and demand that the renormalized fermion propagator satisfies

$$S(\not{p}) \sim \frac{i}{\not{p} - m} \quad \text{when} \quad \not{p} \to m. \tag{8.55}$$

With this we find

$$Z_\psi(\Lambda) = \frac{1}{1 - \Sigma_0'(m, \Lambda)}. \tag{8.56}$$

We still have to express the bare charge $e_0(\Lambda)$ in terms of renormalized parameters. This requires to know how the fermion-photon interaction is corrected by quantum effects. These corrections are contained in the 1PI diagrams with one photon and two fermion lines



$$\equiv \Gamma^\mu_{ab}(p, p'). \tag{8.57}$$

We compute these diagrams in the regularized theory with the Lagrangian (8.33)

$$\Gamma^\mu(p, p'; \Lambda)_0 = -i e_0(\Lambda) \Big[\gamma^\mu + \Lambda_0^\mu(\not{p}, \not{p}'; \Lambda) \Big], \tag{8.58}$$

where the function $-ie(\Lambda)\Lambda_0^\mu(\not{p},\not{p}';\Lambda)$ contains the contributions of all 1PI *loop* diagrams in Eq. (8.57)

To define the renormalized charge e we point out that at tree level the electric charge is read from the vertex $-ie\gamma^\mu$. We use this as a guiding principle and define the renormalized coupling in terms of the *renormalized* 1PI vertex function using the renormalization condition

$$\lim_{\not{p},\not{p}'\to m}\Gamma^\mu(p,p') = -ie\gamma^\mu. \tag{8.59}$$

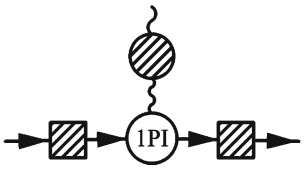
We only need to express $\Gamma^\mu(p,p')$ in terms of the bare function $\Gamma^\mu(p,p';\Lambda)_0$ computed in (8.58). Were we dealing with the complete Green's function

$$\begin{aligned} &G^\mu_{ab}(\not{p},\not{p}';\Lambda)_0(2\pi)^4\delta^{(4)}(p+p'+q)\\ &\quad=\int d^4x_1d^4x_2d^4x_3e^{ip\cdot x_1+ip'\cdot x_2+iq\cdot x_3}\langle\Omega|T[\psi_{0a}(x_1)\overline{\psi}_{0b}(x_2)A^\mu_0(x_3)]|\Omega\rangle, \end{aligned} \tag{8.60}$$

we would only need to multiply by the corresponding field renormalizations

$$G^\mu(\not{p},\not{p}';\Lambda)_0 = \sqrt{Z_A(\Lambda)}Z_\psi(\Lambda)G^\mu(\not{p},\not{p}'), \tag{8.61}$$

as dictated by (8.32). This Green's function can be expressed in terms of the exact propagators and the 1PI vertex (8.57) as shown in the following "skeleton" diagram

$G^\mu(p,p';\Lambda)_0$ = 

(8.62)

It is now clear that in order to write the 1PI bare vertex function in terms of the renormalized one we have to divide Eq. (8.61) by the factor $[Z_A(\Lambda)Z_\psi(\Lambda)^2]^{-1}$ associated with the external propagators. The final result is then

$$\Gamma^\mu(p,p';\Lambda)_0 = Z_\psi(\Lambda)^{-1}Z_A(\Lambda)^{-\frac{1}{2}}\Gamma^\mu(p,p'). \tag{8.63}$$

Using the Lorentz transformation properties of the function $\Lambda^\mu(\not{p},\not{p}';\Lambda)_0$, it is possible to show that when evaluated at the point $\not{p}=\not{p}'=m$ we have $\Lambda^\mu_0(m,m;\Lambda) = \gamma^\mu\Lambda_0(m,m;\Lambda)$. Thus, defining

$$Z_1(\Lambda) = \frac{1}{1+\Lambda_0(m,m;\Lambda)} \tag{8.64}$$

we find the renormalized coupling to be

$$e = e_0(\Lambda)\frac{Z_\psi(\Lambda)\sqrt{Z_A(\Lambda)}}{Z_1(\Lambda)}. \tag{8.65}$$

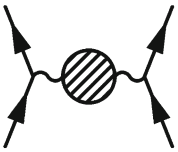
The form of the renormalized coupling we have found can in fact be simplified due to the following identity

$$\Sigma_0'(m,\Lambda) = -\Lambda_0(m,m;\Lambda) \quad \Longrightarrow \quad Z_1(\Lambda) = Z_\psi(\Lambda), \tag{8.66}$$

valid to all orders in QED whenever the theory is regularized in a way preserving gauge invariance (see Ref. [1–15] in Chap. 1). Taking this into account, Eq. (8.65) gives

$$e^2 = e_0(\Lambda)^2 Z_A(\Lambda) = \frac{e_0(\Lambda)^2}{1-\Pi_0(0,\Lambda)}. \tag{8.67}$$

Physically, the renormalized charge e is identified with the physical coupling at low transferred momentum, as can be seen from the diagram describing the interaction of an electron and a muon by the interchange of a full photon propagator



$$= \frac{e_0(\Lambda)^2}{4\pi}(\bar{v}_e\gamma^\mu u_e)\frac{\eta_{\mu\nu}}{q^2[1-\Pi_0(q^2,\Lambda)]}(\bar{v}_\mu\gamma^\nu u_\mu). \tag{8.68}$$

In fact, the identity (8.66) guarantees that the charge renormalization is universal and independent of the fermion species.

All divergences in QED can be handled order by order in the bare coupling using the renormalization procedure that we just overviewed. Using the relation between the bare and renormalized quantities derived previously, it is possible to express every physical quantity, such as S-matrix amplitudes or the effective charge at different energy scales, solely in terms of the renormalized parameters e and m.

A very practical way of implementing the renormalization program systematically to all orders is to use renormalized perturbation theory. This means that instead of using the bare couplings as expansion parameters we use the renormalized ones that are cutoff independent. In the case of QED the starting point is the action written in terms of the renormalized fields, mass and charge

$$\mathscr{L} = \overline{\psi}\Big(i\gamma^\mu\partial_\mu - m\Big)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\overline{\psi}\gamma^\mu\psi. \tag{8.69}$$

The divergences appearing in the computation of loop diagrams from this action are dealt with in the following way: for each divergent 1PI diagram we add a *counterterm* to the action such that the new vertex induced by this counterterm cancels the divergence.

This can be done systematically to each order of the perturbative expansion in powers of the renormalized couplings. By construction, after adding the counterterms to (8.69) the Green's functions calculated using the renormalized Lagrangian

$$\mathscr{L}_{\text{ren}} = \mathscr{L} + \mathscr{L}_{\text{ct}} \tag{8.70}$$

are finite in the limit where the cutoff is removed, $\Lambda \to \infty$. For QED, the counterterm Lagrangian has the form

$$\mathscr{L}_{\rm ct} = iA(\Lambda)\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - mB(\Lambda)\overline{\psi}\psi - \frac{1}{4}C(\Lambda)F_{\mu\nu}F^{\mu\nu} - eD(\Lambda)A_{\mu}\overline{\psi}\psi. \quad (8.71)$$

Adding this to (8.69) and comparing with the form of the renormalized Lagrangian given in (8.33) we find the bare mass, charge and field renormalizations in terms of the counterterm couplings

$$\begin{aligned} m_0(\Lambda) &= m\frac{1+B(\Lambda)}{1+A(\Lambda)}, \\ e_0(\Lambda) &= e\frac{1+D(\Lambda)}{[1+A(\Lambda)]\sqrt{1+C(\Lambda)}} \\ Z_{\psi}(\Lambda) &= 1+A(\Lambda), \\ Z_A(\Lambda) &= 1+C(\Lambda). \end{aligned} \quad (8.72)$$

In general, the renormalized parameters m and e do not have to correspond to physical values of the mass and the electric charge. They are finite parameters determined by the renormalization conditions in terms of which all physical (i.e., observable) quantities are expressed. All these issues will become clear in Chap. 12, where we will study the one-loop renormalization of an interacting scalar field theory in some detail using the techniques of renormalized perturbation theory.

QED, and in general Yang-Mills theories, belong to a class of quantum field theories called renormalizable. This means that the operators appearing in the renormalized Lagrangian are exactly the same ones as those of the classical action. In other words, the counterterms needed to cancel the divergences in the Green's functions have the same structure as the operators already present in the original Lagrangian [cf. Eqs. (8.69) and (8.71)]. This is not necessarily the case for other theories where the elimination of the divergences at higher orders in perturbation theory requires the introduction of new operators to absorb them in their couplings. When the number of new couplings grows with the order of perturbation theory we say that the quantum field theory is nonrenormalizable. Until the 1970s it was believed that nonrenormalizability would render a theory inconsistent. Nowadays, however, we know that nonrenormalizable theories are perfectly consistent and can be used to compute observables at energies below the natural scale of the theory. We will have more to say about effective field theories in Sect. 8.5 and in Chap. 12.

8.4 Renormalization in Statistical Mechanics

In spite of its successes, the renormalization procedure presented above could still be seen as some kind of prescription or recipe to get rid of the divergences in an ordered way. This discomfort about renormalization was expressed in occasions by

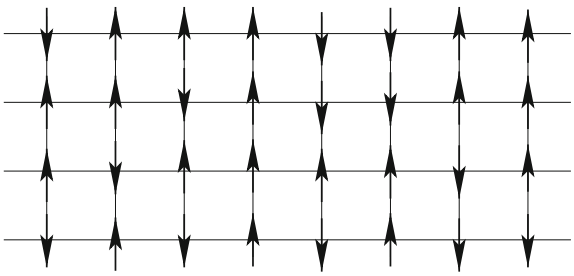


Fig. 8.2 Systems of spins in a two-dimensional square lattice

comparing it with "sweeping the infinities under the rug". After the work of Ken Wilson [6–8], the process of renormalization is now understood in a very profound way as a procedure to incorporate the effects of physics at high energies by modifying the value of the parameters that appear in the Lagrangian.

Wilson's ideas are both simple and profound and consist of thinking about quantum field theory as the analog of a thermodynamical description of a statistical system. To be more precise, let us consider an Ising spin system in a two-dimensional square lattice as the one depicted in Fig. 8.2. In terms of the spin variables $s_i = \pm\frac{1}{2}$, where i labels the lattice site, the Hamiltonian of the system is given by

$$H = -J \sum_{\langle i,\, j\rangle} s_i s_j, \tag{8.73}$$

where $\langle i,\ j\rangle$ indicates that the sum extends over nearest neighbors and J is the coupling constant between neighboring spins (no interaction with an external magnetic field is considered). The starting point to study the statistical mechanics of this system is the partition function defined as

$$\mathscr{Z} = \sum_{\{s_i\}} e^{-\beta H}, \tag{8.74}$$

where the sum is over all possible configurations of the spins and $\beta = \frac{1}{T}$ is the inverse temperature. For $J > 0$ the Ising model presents spontaneous magnetization below a critical temperature T_c, in any dimension higher than one. Away from this temperature correlations between spins decay exponentially at large distances

$$\langle s_i s_j\rangle \sim e^{-\frac{|x_{ij}|}{\xi}}, \tag{8.75}$$

with $|x_{ij}|$ the distance between the spins located in the ith and jth sites of the lattice. This expression serves as a definition of the correlation length ξ setting the characteristic length scale at which spins can influence each other by their interaction through their nearest neighbors.

Suppose now that we are interested in a macroscopic description of this spin system. We can capture the relevant physics by integrating out the physics at short

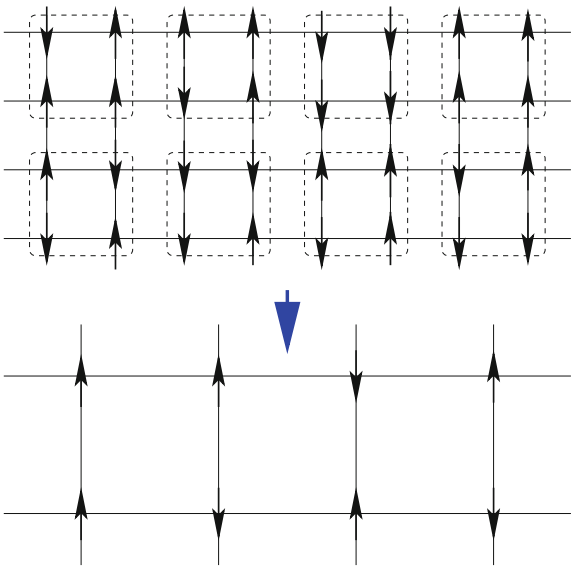


Fig. 8.3 Decimation of the spin lattice. Each block in the *upper lattice* is replaced by an effective spin computed according to the rule (8.77). Notice also that the size of the lattice spacing is doubled in the process

scales. A way in which this can be done was proposed by Leo Kadanoff [9] and consists of dividing our spin system in spin-blocks like the ones showed in Fig. 8.3. Now, we can construct another spin system where each spin-block of the original lattice is replaced by an effective spin calculated according to some rule from the spins contained in each block B_a

$$\{s_i : i \in B_a\} \longrightarrow s_a^{(1)}. \tag{8.76}$$

For example we can define the effective spin associated with the block B_a by taking the majority rule with an additional prescription in case of a draw

$$s_a^{(1)} = \frac{1}{2}\,\text{sign}\left(\sum_{i\in B_a} s_i\right), \tag{8.77}$$

where we have used the sign function, $\text{sign}\,(x) \equiv \frac{x}{|x|}$, with the additional definition $\text{sign}\,(0) = 1$. This procedure is called decimation and leads to a new spin system with a double lattice space.

The idea now is to rewrite the partition function (8.74) only in terms of the new effective spins $s_a^{(1)}$. We start by splitting the sum over spin configurations into two nested sums, one over the spin blocks and the other over the spins within each block

$$\mathscr{Z} = \sum_{\{\mathbf{s}\}} e^{-\beta H[s_i]} = \sum_{\{\mathbf{s}^{(1)}\}} \sum_{\{\mathbf{s}\in B_a\}} \delta\left[s_a^{(1)} - \text{sign}\left(\sum_{i\in B_a} s_i\right)\right] e^{-\beta H[s_i]}. \tag{8.78}$$

The interesting point is that the sum over spins inside each block can be written as the exponential of a new effective Hamiltonian depending only on the effective spins, $H^{(1)}[s_a^{(1)}]$

$$\sum_{\{s\in B_a\}} \delta\left[s_a^{(1)} - \text{sign}\left(\sum_{i\in B_a} s_i\right)\right] e^{-\beta H[s_i]} = e^{-\beta H^{(1)}[s_a^{(1)}]}. \tag{8.79}$$

The new Hamiltonian is of course more complicated

$$H^{(1)} = -J^{(1)} \sum_{\langle i,j\rangle} s_i^{(1)} s_j^{(1)} + \cdots \tag{8.80}$$

where the dots stand for other interaction terms between the effective block spins. The new terms appear because in the process of integrating out short distance physics we induce interactions between the new effective degrees of freedom. For example, the interaction between the spin block variables $s_i^{(1)}$ will not in general be restricted to nearest neighbors in the new lattice. The important point is that we have managed to rewrite the partition function solely in terms of this new (renormalized) spin variables $s^{(1)}$ interacting through a new Hamiltonian $H^{(1)}$

$$\mathscr{Z} = \sum_{\{s^{(1)}\}} e^{-\beta H^{(1)}[s_a^{(1)}]}. \tag{8.81}$$

We can think about the space of all possible Hamiltonians for our statistical system including all kinds of possible couplings between the individual spins compatible with the symmetries of the system. If we denote by $\mathscr{R}$ the decimation operation, it defines a map in the space of Hamiltonians

$$\mathscr{R} : H \to H^{(1)}. \tag{8.82}$$

At the same time the operation $\mathscr{R}$ replaces a lattice with spacing a by another one with double spacing $2a$. As a consequence, the correlation length in the new lattice measured in units of the lattice spacing is divided by two, $\mathscr{R} : \xi \to \frac{\xi}{2}$.

Now we can iterate the operation $\mathscr{R}$ an indefinite number of times. Eventually we might reach a Hamiltonian $H_\star$ that is not further modified by the operation $\mathscr{R}$

$$H \xrightarrow{\mathscr{R}} H^{(1)} \xrightarrow{\mathscr{R}} H^{(2)} \xrightarrow{\mathscr{R}} \ldots \xrightarrow{\mathscr{R}} H_\star. \tag{8.83}$$

The fixed point Hamiltonian $H_\star$ is *scale invariant* because it does not change as $\mathscr{R}$ is performed. As a consequence of this invariance, the correlation length of the system at the fixed point does not change under $\mathscr{R}$. This fact is compatible with the transformation $\xi \to \frac{\xi}{2}$ only if $\xi = 0$ or $\xi = \infty$. Here we will focus in the case of nontrivial fixed points with infinite correlation length.

The space of Hamiltonians can be parametrized by specifying the values of the coupling constants associated with all possible interaction terms between individual

spins of the lattice. If we denote by $\mathscr{O}_a[s_i]$ these (possibly infinite) interaction terms, the most general Hamiltonian for the spin system under study can be written as

$$H[s_i] = \sum_{a=1}^{\infty} \lambda_a \mathscr{O}_a[s_i], \tag{8.84}$$

where $\lambda_a \in \mathbb{R}$ are the coupling constants for the corresponding operators. These constants can be thought of as coordinates in the space of all Hamiltonians. Therefore the operation $\mathscr{R}$ defines a transformation in the set of coupling constants

$$\mathscr{R} : \lambda_a \longrightarrow \lambda_a^{(1)}. \tag{8.85}$$

For example, in our case we started with a Hamiltonian in which only one of the coupling constants is different from zero (say $\lambda_1 = -J$). As a result of the decimation $\lambda_1 \equiv -J \to -J^{(1)}$ while some of the originally vanishing coupling constants will take nonzero values. Of course, for the fixed point Hamiltonian the coupling constants do not change under the scale transformation $\mathscr{R}$.

Physically, the transformation $\mathscr{R}$ integrates out short distance physics. The consequence for physics at long distances is that we have to replace our Hamiltonian by a new one with different values for the coupling constants. That is, our ignorance of the details of the physics going on at short distances result in a *renormalization* of the coupling constants of the Hamiltonian describing the long range physical processes. It is important to stress that although $\mathscr{R}$ is sometimes called a renormalization group transformation in fact this is a misnomer. Transformations between Hamiltonians defined by $\mathscr{R}$ do not form a group: since these transformations proceed by integrating out degrees of freedom at short scales they cannot be inverted.

In statistical mechanics fixed points under renormalization group transformations with $\xi = \infty$ are associated with phase transitions. From our previous discussion we can conclude that the space of Hamiltonians is divided into regions corresponding to the basins of attraction of the different fixed points. We can ask ourselves now about the stability of those fixed points. Suppose we have a statistical system described by a fixed-point Hamiltonian $H_\star$ and we perturb it by changing the coupling constant associated with an interaction term $\mathscr{O}$. This is equivalent to replace $H_\star$ by the perturbed Hamiltonian

$$H = H_\star + \delta\lambda \mathscr{O}, \tag{8.86}$$

where $\delta\lambda$ is a perturbation of the coupling constant corresponding to $\mathscr{O}$ (we can also consider perturbations in more than one coupling constant). Thinking of the λ_a's as coordinates in the space of all Hamiltonians this corresponds to moving slightly away from the position of the fixed point.

The question to decide now is in which direction the renormalization group flow will take the perturbed system. Working to first order in $\delta\lambda$ there are three possibilities:

- The renormalization group flow takes the system back to the fixed point. In this case the corresponding interaction $\mathscr{O}$ is called *irrelevant*.

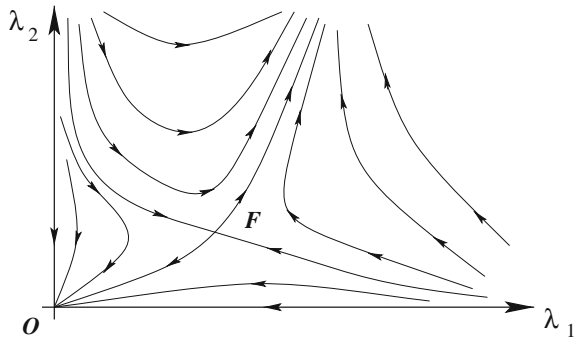


Fig. 8.4 Example of a renormalization group flow

- $\mathscr{R}$ takes the system away from the fixed point. If this is what happens the interaction is called *relevant*.
- It is possible that the perturbation does not take the system away from the fixed point at first order in $\delta\lambda$. In this case the interaction is said to be *marginal* and it is necessary to go to higher orders in $\delta\lambda$ in order to decide whether the system moves to or away the fixed point, or whether we have a family of fixed points.

We can picture the action of the renormalization group transformation as a flow in the space of coupling constants. In Fig. 8.4 we have depicted an example of such a flow in the case of a system with two coupling constants λ_1 and λ_2. In this example we find two fixed points, one at the origin O and another at F for a finite value of the couplings. The arrows indicate the direction in which the renormalization group flow acts. The free theory at $\lambda_1 = \lambda_2 = 0$ is a stable fix point since any perturbation $\delta\lambda_1, \delta\lambda_2 > 0$ makes the theory flow back to the free theory at long distances. On the other hand, the fixed point F is stable with respect to certain perturbations (along the line with incoming arrows) whereas for any other perturbations the system flows either to the free theory at the origin or to a theory with infinite values for the couplings.

8.5 The Renormalization Group in Quantum Field Theory

In the renormalization program in quantum field theory a key role is played by the renormalization conditions. It is through them that the renormalized parameters are related to the bare ones. In the case of QED that we have analyzed in Sect. 8.3, we have used what is called on-shell renormalization in which the renormalized parameters are defined by evaluating the corresponding Green's functions for on-shell values of the external momenta.

In carrying out the renormalization of QED, we could have defined the renormalized charge and mass at any other value of the momentum (physical or unphysical) $p^2 = \mu^2$. This is called a change in the renormalization scheme. It would have lead to different values of the renormalized parameters and Green's functions, although all physical quantities would be independent of the chosen renormalization scheme.

The dependence of the renormalized Green's functions on the renormalization scheme μ is given by the renormalization group equation, or Callan–Symanzik equation, that we study next. To make things as simple as possible, we consider a theory of a single field $\phi(x)$ and ignore all possible indices (they are mostly irrelevant for the problem). Doing perturbative quantization with a regulator Λ we work with the renormalized Lagrangian written in terms of the bare mass and coupling constants $m_0(\Lambda)$ and $g_0(\Lambda)$, as well as the bare fields $\phi_0(x)$ appearing in the classical Lagrangian

$$\phi_0(x) = \sqrt{Z_\phi(\Lambda)}\phi(x), \tag{8.87}$$

where $\phi(x)$ is the renormalized field.

Using Feynman diagrams we can compute the bare Green's functions

$$\begin{aligned} &G_n(p_1, \ldots, p_n; \Lambda)_0 (2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_n) \\ &\quad = \int d^4x_1 \ldots d^4x_n e^{ip_1\cdot x_1 + \cdots + ip_n \cdot x_n} \langle \Omega | T[\phi_0(x_1) \ldots \phi_0(x_n)] | \Omega \rangle \end{aligned} \tag{8.88}$$

order by order in perturbation theory in the bare coupling constant $g_0(\Lambda)$. The renormalized Green's function $G(p_1, \ldots, p_n)$, on the other hand, is regulator independent. It only depends on the renormalized quantities and the energy scale μ at which the renormalization conditions are implemented. From the relation (8.87) between the bare and renormalized fields we find the following identity

$$G(p_1, \ldots, p_n; \Lambda)_0 = Z_\phi(\Lambda)^{\frac{n}{2}} G(p_1, \ldots, p_n; \mu). \tag{8.89}$$

In the following we focus on the 1PI Green's functions $\Gamma_n(p_1, \ldots, p_n)$, obtained by summing the all 1PI diagrams contributing to the corresponding amplitude. Any other Green's function can be computed in terms of them (see the example of QED studied in Sect. 8.3). The relation between the bare and renormalized 1PI functions can be obtained from (8.89) taking into account that in passing from the Green's functions to the 1PI functions we have to remove the contribution from the external propagators, each one contributing a factor of $Z_\phi(\Lambda)$. This leads to

$$\Gamma_n(p_1, \ldots, p_n; \Lambda)_0 = Z_\phi(\Lambda)^{-\frac{n}{2}} \Gamma_n(p_1, \ldots, p_n; \mu). \tag{8.90}$$

The function on the left-hand side of this equation depends on the regulator Λ both explicitly and through the bare parameters $m_0(\Lambda)$ and $g_0(\Lambda)$. The renormalized function, on the other hand, does not depend on Λ but only on the renormalized parameters m and g as well as on the energy scale μ.

We can find now how $\Gamma_n(p_1, \ldots, p_n; \mu)$ changes when we change the scale μ. Keeping fixed the bare mass $m_0(\Lambda)$ and coupling constant $g_0(\Lambda)$ while varying μ results in a change both in the renormalized parameters m and g and of the field renormalization $Z_\phi(\Lambda)$. Remembering that the left-hand side of (8.90) is independent of μ

$$\mu \frac{d}{d\mu} \left[Z_\phi(\Lambda)^{-\frac{n}{2}} \Gamma_n(p_1, \ldots, p_n; \mu) \right] = 0, \tag{8.91}$$

and using the μ-dependence of m, g and Z_ϕ we arrive at the Callan–Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\gamma_m(g)\frac{\partial}{\partial m}-n\gamma(g)\right]\Gamma_n(p_1,\ldots,p_n;\mu)=0, \tag{8.92}$$

where we have defined the functions

$$\begin{aligned}\beta(g)&=\mu\frac{\partial g}{\partial\mu},\\ \gamma_m(g)&=\mu\frac{\partial m}{\partial\mu},\\ \gamma(g)&=\frac{1}{2}\mu\frac{\partial}{\partial\mu}\log Z_\phi.\end{aligned} \tag{8.93}$$

Two of this functions already appeared in Sect. 8.2: the beta function $\beta(g)$, that governs the evolution of the coupling with the energy, and the anomalous dimension $\gamma(g)$.

The application to quantum field theory of the idea of the renormalization group in statistical mechanics introduced in Sect. 8.4 leads to a profound understanding of what renormalizing a quantum field theory means in physical terms. Consider a theory with a number of fields ϕ_a defined by a Lagrangian

$$\mathscr{L}[\phi_a]=\mathscr{L}_0[\phi_a]+\sum_i g_i\mathscr{O}_i[\phi_a], \tag{8.94}$$

where $\mathscr{L}_0[\phi_a]$ is the kinetic part and the g_i's are the coupling constants associated with the operators $\mathscr{O}_i[\phi_a]$. In order to make sense of the quantum theory we introduce a momentum cutoff Λ. In principle, we include all operators $\mathscr{O}_i$ compatible with the symmetries of the theory.

In Sect. 8.2 we learned how in the cases of QED and QCD the value of the coupling constants changed with the scale from their values at the scale Λ. We can understand this behavior along the lines of the analysis presented for the Ising model. If we would like to compute the effective dynamics of the theory at an energy scale $\mu<\Lambda$, we only have to integrate out all physical modes with energies between the cutoff Λ and the scale of interest μ. This is analogous to what we did in the Ising model by replacing the original spins by the block spins. In the case of field theory the effective action $S[\phi_a,\mu]$ at scale μ can be written in the language of functional integration as

$$e^{iS[\phi'_a,\mu]}=\int\limits_{\mu<p<\Lambda}\prod_a\mathscr{D}\phi_a e^{iS[\phi_a,\Lambda]}. \tag{8.95}$$

Here $S[\phi_a,\Lambda]$ is the action at the cutoff scale

$$S[\phi_a, \Lambda] = \int d^4x \left\{ \mathscr{L}_0[\phi_a] + \sum_i g_i(\Lambda) \mathscr{O}_i[\phi_a] \right\} \tag{8.96}$$

and the functional integral in Eq. (8.95) is carried out only over the field modes with momenta in the range $\mu < p < \Lambda$. The action resulting from integrating out the physics at the intermediate scales between Λ and μ depends not on the original field variable ϕ_a but on some renormalized field ϕ'_a. At the same time, the couplings $g_i(\mu)$ differ from their values at the cutoff scale $g_i(\Lambda)$. This is analogous to what we learned in the Ising model: by integrating out short distance physics we ended up with a new Hamiltonian depending on renormalized effective spin variables and with renormalized values for the coupling constants. Therefore the resulting effective action at scale μ can be written as

$$S[\phi'_a, \mu] = \int d^4x \left\{ \mathscr{L}_0[\phi'_a] + \sum_i g_i(\mu) \mathscr{O}_i[\phi'_a] \right\}. \tag{8.97}$$

This Wilsonian interpretation of renormalization sheds light to what in Sect. 8.1 might have looked just a smart way to get rid of the infinities. The running of the coupling constant with the energy scale can be understood instead as a way of incorporating into an effective action at scale μ the effects of field excitations at higher energies $E > \mu$.

As in statistical mechanics, there are also quantum field theories that are fixed points of the renormalization group flow, i.e. whose coupling constants do not change with the scale. We have encountered them already in Sect. 8.2 when studying the properties of the beta function. The most trivial example of such theories are massless free quantum field theories, but there are also examples of scale invariant, four-dimensional interacting quantum field theories. We can ask the question of what happens when a scale invariant theory is perturbed with some operator. In general, the perturbed theory is not scale invariant anymore but we may wonder whether the perturbed theory flows at low energies towards or away the fixed point theory.

In quantum field theory this can be decided by looking at the canonical dimension $D_{\mathscr{O}}$ of the operator $\mathscr{O}[\phi_a]$ used to perturb the theory at the fixed point. In four dimensions the three possibilities are:

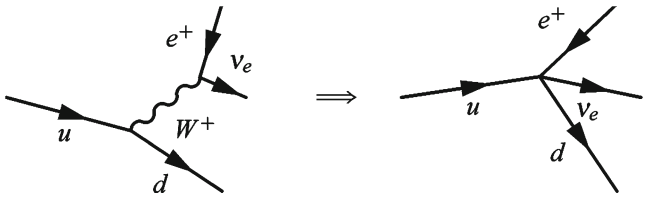
- $D_{\mathscr{O}} > 4$: irrelevant perturbation. The running of the coupling constants takes the theory back to the fixed point.
- $D_{\mathscr{O}} < 4$: relevant perturbation. At low energies the theory flows away from the scale-invariant theory.
- $D_{\mathscr{O}} = 4$: marginal deformation. The direction of the flow cannot be decided only on dimensional grounds.

As an example, let us consider first a massless fermion theory perturbed by a four-fermion interaction term

$$\mathscr{L} = i\overline{\psi}\partial\!\!\!/\psi - \frac{1}{M^2}(\overline{\psi}\psi)^2. \tag{8.98}$$

This is indeed a perturbation by an irrelevant operator, since in four-dimensions $D_\psi = \frac{3}{2}$. Interactions generated by the extra term are suppressed at low energies since typically their effects are weighted by the dimensionless factor E^2/M^2, where E is the energy scale of the process. This means that as we try to capture the relevant physics at lower and lower energies, the effect of the perturbation is weaker and weaker rendering in the infrared limit $E \to 0$ again a free theory. Hence, the irrelevant perturbation in (8.98) makes the theory flow back to the fixed point.

On the other hand, relevant operators dominate the physics at low energies. This is the case, for example, of a mass term. As we lower the energy the mass becomes more important, and once the energy goes below the mass of the field its dynamics is completely dominated by the mass term. This is, for example, how Fermi's theory of weak interactions emerges from the standard model at energies below the mass of the $W^\pm$ boson



At energies below $m_W = 80.4$ GeV the dynamics of the W^+ boson is dominated by its mass term and therefore becomes nonpropagating, giving rise to the effective four-fermion Fermi theory.

To summarize our discussion so far, we found that while relevant operators dominate the dynamics in the infrared, taking the theory away from the fixed point, irrelevant perturbations become suppressed in the same limit. Finally, we consider the effect of marginal operators. As an example we take the interaction term in massless QED, $\mathscr{O} = \overline{\psi}\gamma^\mu\psi A_\mu$. Taking into an account that in $d = 4$ the dimension of the electromagnetic potential is $[A_\mu] = 1$, the operator $\mathscr{O}$ is a marginal perturbation. In order to decide whether the fixed point theory

$$\mathscr{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\partial\!\!\!/\psi \tag{8.99}$$

is restored at low energies or not we need to study the perturbed theory in more detail. This we have done in Sect. 8.1 where we learned that the effective coupling in QED decreases at low energies. Then we conclude that the perturbed theory flows towards the fixed point in the infrared.