Chapter 7
Symmetries I: Continuous Symmetries

The concept of symmetry is paramount in modern Physics. In this chapter we are going to deal with the implementation of symmetries in quantum field theory. After reviewing the relation between continuous symmetries and conservations laws, we study how symmetries are realized quantum mechanically and in which way different realizations reflect in the spectrum of the theory. Our aim is to describe the concept of spontaneous symmetry breaking, which is crucial to our current understanding of how particle masses emerge in the standard model. A number of subtleties in how and when spontaneous symmetry breaking can occur are described towards the end of the chapter. The focus of the present chapter centers on continuous symmetries. The physics of discrete symmetries will be taken up in Chap. 11.

7.1 Noether's Theorem

In classical mechanics and classical field theory there is a basic result relating symmetries and conserved charges. This is called Noether's theorem and states that for each continuous symmetry of the system there is a conserved current. In its simplest version in classical mechanics it is easy to prove. Let us consider a system whose action $S[q_i]$ is invariant under a transformation $q_i(t) \rightarrow q_i'(t,\varepsilon)$ labelled by a continuous parameter ε. This means that, without using the equations of motion, the Lagrangian changes at most by a total time derivative

$$L(q',\dot{q}') = L(q,\dot{q}) + \frac{d}{dt} f(q,\varepsilon), \tag{7.1}$$

where $f(q,\varepsilon)$ is a function of the coordinates. If $\varepsilon \ll 1$ we can consider an infinitesimal variation of the coordinates $\delta_\varepsilon q_i(t)$ and the transformation (7.1) of the Lagrangian implies

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$$
\begin{aligned}
\frac{d}{dt} f(q, \delta\varepsilon) = \delta_\varepsilon L(q_i, \dot{q}_i) &= \frac{\partial L}{\partial q_i}\delta_\varepsilon q_i + \frac{\partial L}{\partial \dot{q}_i}\delta_\varepsilon \dot{q}_i \\
&= \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right]\delta_\varepsilon q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta_\varepsilon q_i\right),
\end{aligned} \tag{7.2}
$$

When $\delta_\varepsilon q_i$ is applied on a solution to the equations of motion, the term inside the square brackets vanishes and we conclude that there is a conserved quantity

$$
\dot{Q} = 0 \quad \text{with} \quad Q \equiv \frac{\partial L}{\partial \dot{q}_i}\delta_\varepsilon q_i - f(q, \delta\varepsilon). \tag{7.3}
$$

Notice that in this derivation it is crucial that the symmetry depends on a continuous parameter since otherwise the infinitesimal variation of the Lagrangian in Eq. (7.2) does not make sense.

In classical field theory a similar result holds. Let us consider for simplicity a theory of a single field $\phi(x)$. We say that the variation $\delta_\varepsilon\phi$ depending on a continuous parameter ε is a symmetry of the theory if, again without using the equations of motion, the Lagrangian density changes by

$$
\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu. \tag{7.4}
$$

If this happens, the action remains invariant and so do the equations of motion. Working out now the variation of $\mathscr{L}$ under $\delta_\varepsilon\phi$ we find

$$
\begin{aligned}
\delta_\varepsilon \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\partial_\mu \delta_\varepsilon \phi + \frac{\partial \mathscr{L}}{\partial \phi}\delta_\varepsilon \phi \\
&= \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi\right) + \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\right]\delta_\varepsilon \phi \\
&= \partial_\mu K^\mu.
\end{aligned} \tag{7.5}
$$

If $\phi(x)$ is a solution to the equations of motion, the last term in the second line disappears, and we find a conserved current

$$
\partial_\mu J^\mu = 0 \quad \text{with} \quad J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi - K^\mu. \tag{7.6}
$$

A conserved current implies the existence of a charge

$$
Q \equiv \int d^3x\, J^0(t, \mathbf{x}) \tag{7.7}
$$

which is conserved

$$
\frac{dQ}{dt} = \int d^3x\, \partial_0 J^0(t, \mathbf{x}) = -\int d^3x\, \partial_i J^i(t, \mathbf{x}) = 0, \tag{7.8}
$$

provided the fields vanish at infinity fast enough. Moreover, the conserved charge Q is a Lorentz scalar. After canonical quantization Q is promoted to an operator generating the symmetry on the fields

$$\delta\phi = i[\phi, Q]. \tag{7.9}$$

As an example of how Noether's theorem works, we consider a scalar field $\phi(x)$ with Lagrangian density $\mathscr{L}$. Being $\phi(x)$ a scalar, its transformation under the Poincaré group $x \to x'$ is $\phi'(x') = \phi(x)$. Performing in particular a space-time translation $x^{\mu'} = x^\mu + a^\mu$ we have

$$\phi'(x) - \phi(x) = -a^\mu \partial_\mu \phi + \mathscr{O}(a^2) \Longrightarrow \delta\phi = -a^\mu \partial_\mu \phi. \tag{7.10}$$

That the theory is invariant under the Poincaré group means that the Lagrangian density is also a scalar quantity. Thus, it should also transform under translations as

$$\delta\mathscr{L} = -a^\mu \partial_\mu \mathscr{L}. \tag{7.11}$$

Noether's theorem implies then the existence of a conserved current. Applying the previous results we conclude that this is given by

$$J^\mu = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} a^\nu \partial_\nu \phi + a^\mu \mathscr{L} \equiv -a_\nu T^{\mu\nu}, \tag{7.12}$$

where we introduced the energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathscr{L}. \tag{7.13}$$

We have found that associated with the invariance of the theory with respect to space-time translations there are four conserved currents defined by $T^{\mu\nu}$ with $\nu = 0, \ldots, 3$, each one associated with the translation along a space-time direction. These four currents form a rank-two tensor under Lorentz transformations satisfying

$$\partial_\mu T^{\mu\nu} = 0. \tag{7.14}$$

The associated conserved charges are given by

$$P^\nu = \int d^3x T^{0\nu} \tag{7.15}$$

and correspond to the total energy-momentum content of the field configuration. Therefore the energy density of the field is given by T^{00} while T^{0i} is the momentum density. In the quantum theory P^μ are the generators of space-time translations.

Another example of a symmetry related with a physically relevant conserved charge is the global phase invariance of the Dirac Lagrangian (3.36), $\psi \to e^{i\theta}\psi$.

For small θ this corresponds to the variations $\delta_\theta\psi = i\theta\psi$, $\delta_\theta\overline{\psi} = -i\theta\overline{\psi}$ and using Noether's theorem we obtain the conserved current

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \quad \partial_\mu j^\mu = 0. \tag{7.16}$$

The associated charge is

$$Q = \int d^3x\overline{\psi}\gamma^0\psi = \int d^3x\psi^\dagger\psi. \tag{7.17}$$

In physics there are several instances of global U(1) symmetries acting as phase shifts on spinors. This is the case, for example, of the baryon and lepton number symmetries in the standard model. A more familiar case is the U(1) local symmetry associated with electromagnetism. Although this is a local symmetry, $\theta \to q\varepsilon(x)$, the Lagrangian is invariant also under global transformations with $\varepsilon(x)$ constant and there is a conserved current $j^\mu = q\overline{\psi}\gamma^\mu\psi$. In Eq. (4.40) we learned how spinors in QED are coupled to the photon field precisely through this current. Its time component is the electric charge density ρ, while the spatial components make the current density vector $\mathbf{j}$.

The previous analysis can be extended also to nonabelian unitary global symmetries acting on N species of fermions as

$$\psi_i \longrightarrow U_{ij}\psi_j, \tag{7.18}$$

where U_{ij} is a $N \times N$ unitary matrix, $U^\dagger U = UU^\dagger = \mathbf{1}$. This transformation leaves invariant the Lagrangian

$$\mathscr{L} = i\overline{\psi}_j \, \partial\!\!\!/ \psi_j - m\overline{\psi}_j\psi_j, \tag{7.19}$$

where we sum over repeated indices. If we write the matrix U in terms of the N^2 hermitian group generators T^A of U(N) as

$$U = \exp\left(i\alpha^A T^A\right), \quad (T^A)^\dagger = T^A, \tag{7.20}$$

the conserved currents are found to be

$$j^{\mu A} = \overline{\psi}_i T^A_{ij}\gamma^\mu\psi_j, \quad \partial_\mu j^\mu = 0. \tag{7.21}$$

with N^2 conserved charges

$$Q^A = \int d^3x\psi_i^\dagger T^A_{ij}\psi_j \tag{7.22}$$

The group U(N) of $N \times N$ unitary matrices admits the decomposition U(N) $= U(1) \times \mathrm{SU}(N)$. The U(1) factor corresponds to the element $U = e^{i\alpha^0}\mathbf{1}$ multiplying all spinor fields by the same phase. The corresponding charge

$$Q^0 = \int d^3x \psi_i^\dagger \psi_i \tag{7.23}$$

measures, in the quantum theory, the number of fermions minus the number of antifermions. It commutes with the other N^2-1 charges associated with the nontrivial SU(N) part of the global symmetry group.

As an example of these internal unitary symmetries, we mention the approximate flavor symmetries in hadron physics. Ignoring charge and mass differences, the QCD Lagrangian is invariant under the following unitary symmetry acting on the quarks u and d

$$\begin{pmatrix} u \\ d \end{pmatrix} \longrightarrow M \begin{pmatrix} u \\ d \end{pmatrix}, \tag{7.24}$$

where $M \in \mathrm{U}(2) = \mathrm{U}(1)_B \times \mathrm{SU}(2)$. The $\mathrm{U}(1)_B$ factor corresponds to the baryon number, whose conserved charge assigns $\pm\frac{1}{3}$ to quarks and antiquarks respectively. On the other hand, the SU(2) part mixes the u and d quarks. Since the proton is a bound state of two quarks u and one quark d, while the neutron is made out of one quark u and two quarks d, this symmetry reduces at low energies to the well-known isospin transformations of nuclear physics mixing protons and neutrons.

7.2 Quantum Mechanical Realizations of Symmetries

In a quantum theory physical symmetries are maps in the Hilbert space of the theory preserving the probability amplitudes. In more precise terms, a symmetry is a one-to-one transformation that, acting on two arbitrary states $|\alpha\rangle, |\beta\rangle \in \mathscr{H}$

$$|\alpha\rangle \longrightarrow |\alpha'\rangle, \quad |\beta\rangle \longrightarrow |\beta'\rangle, \tag{7.25}$$

satisfies

$$|\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|. \tag{7.26}$$

Wigner's theorem states that these transformations are implemented by operators that are either unitary or antiunitary. Unitary operators are well-known objects from any quantum mechanics course. They are *linear* operators $\mathscr{U}$ satisfying[1]

$$\langle\mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle, \tag{7.27}$$

for any two states in the Hilbert space. In addition, the transformation of an operator $\mathscr{O}$ under $\mathscr{U}$ is

$$\mathscr{O} \longrightarrow \mathscr{O}' = \mathscr{U}\mathscr{O}\mathscr{U}^{-1}, \tag{7.28}$$

from where it follows that $\langle\alpha|\mathscr{O}|\beta\rangle = \langle\alpha'|\mathscr{O}'|\beta'\rangle$.

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1 Here we use the notation $|\mathscr{U}\alpha\rangle \equiv \mathscr{U}|\alpha\rangle$ and $\langle\mathscr{U}\alpha| \equiv \langle\alpha|\mathscr{U}^\dagger$.

Antiunitary operators, on the other hand, have the property

$$\langle \mathscr{U}\alpha | \mathscr{U}\beta \rangle = \langle \alpha | \beta \rangle^* \tag{7.29}$$

and are *antilinear*, i.e.

$$\mathscr{U}\,(a|\alpha\rangle + b|\beta\rangle) = a^*|\mathscr{U}\alpha\rangle + b^*|\mathscr{U}\beta\rangle, \quad a, b \in \mathbb{C}. \tag{7.30}$$

To find the transformation of operator matrix elements under an antiunitary transformation we compute

$$\langle \alpha|\mathscr{O}|\beta\rangle = \langle \mathscr{O}^\dagger \alpha|\beta\rangle = \langle \mathscr{U}\beta|\mathscr{U}\mathscr{O}^\dagger\alpha\rangle. \tag{7.31}$$

Writing now $|\mathscr{U}\mathscr{O}^\dagger\alpha\rangle = \mathscr{U}\mathscr{O}^\dagger|\alpha\rangle$ and inserting the identity we arrive at the final result

$$\langle \alpha|\mathscr{O}|\beta\rangle = \langle \beta'|\mathscr{U}\mathscr{O}^\dagger\mathscr{U}^{-1}|\alpha'\rangle. \tag{7.32}$$

Continuous symmetries are implemented only by unitary operators. This is because they are continuously connected with the identity, which is a unitary operator. Discrete symmetries, on the other hand, can be implemented by either unitary or antiunitary operators. An example of the latter is time reversal, that we will study in detail in Chap. 11.

In the previous section we have seen that in canonical quantization the conserved charges Q^a associated with a continuous symmetry by Noether's theorem are operators generating the infinitesimal transformations of the quantum fields. The conservation of the classical charges $\{Q^a, H\}_{\rm PB} = 0$ implies that the operators Q^a commute with the Hamiltonian

$$[Q^a, H] = 0. \tag{7.33}$$

The symmetry group generated by the operators Q^a is implemented in the Hilbert space of the theory by a set of unitary operators $\mathscr{U}(\alpha)$, where α^a (with $a = 1, \ldots, \dim \mathfrak{g}$) labels the transformation.[2] That the group is generated by the conserved charges means that in a neighborhood of the identity, the operators $\mathscr{U}(\alpha)$ can be written as

$$\mathscr{U}(\alpha) = e^{i\alpha^a Q^a}. \tag{7.34}$$

A symmetry group can be realized in the quantum theory in two different ways, depending on how its elements act on the ground state of the theory. Implementing it in one way or the other has important consequences for the spectrum of the theory, as we now learn.

[2] A quick survey of group theory can be found in Appendix B.

Wigner–Weyl Realization

In this case the ground state of the theory $|0\rangle$ is invariant under all the elements of the symmetry group $\mathscr{U}(\alpha)|0\rangle = |0\rangle$. Equation (7.34) implies that the vacuum is annihilated by them

$$Q^a|0\rangle = 0. \tag{7.35}$$

The field operators of the quantum theory have to transform according to some irreducible representation of the symmetry group. It is easy to see that the finite form of the infinitesimal transformation (7.9) is given by

$$\mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1} = U_{ij}(\alpha)\phi_j, \tag{7.36}$$

where the matrices $U_{ij}(\alpha)$ form the group representation in which the field ϕ_i transforms. If we consider now the quantum state associated with the operator ϕ_i

$$|i\rangle = \phi_i|0\rangle \tag{7.37}$$

we find that, due to the invariance of the vacuum (7.35), the states $|i\rangle$ have to transform in the same representation as ϕ_i

$$\mathscr{U}(\alpha)|i\rangle = \mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1}\mathscr{U}(\alpha)|0\rangle = U_{ij}(\alpha)\phi_j|0\rangle = U_{ij}(\alpha)|j\rangle. \tag{7.38}$$

Therefore the spectrum of the theory is classified in multiplets of the symmetry group.

Any two states within a multiplet can be "rotated" into one another by a symmetry transformation. Now, since $[H, \mathscr{U}(\alpha)] = 0$ the conclusion is that all states in the same multiplet have the same energy. If we consider one-particle states, then going to the rest frame we see how all states in the same multiplet have exactly the same mass.

Nambu–Goldstone Realization

In our previous discussion we have seen how the invariance of the ground state of a theory under a symmetry group has as a consequence that the spectrum splits into multiplets transforming under irreducible representations of the symmetry group. This shows in degeneracies in the mass spectrum.

The condition (7.35) is not mandatory and can be relaxed by considering theories where the vacuum state is not preserved by the symmetry

$$e^{i\alpha^a Q^a}|0\rangle \neq |0\rangle \Longrightarrow Q^a|0\rangle \neq 0. \tag{7.39}$$

The symmetry is said to be spontaneously broken by the vacuum.

To illustrate the consequences of (7.39) we consider the example of a number of scalar fields φ^i $(i = 1, \ldots, N)$ whose dynamics is governed by the Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\varphi^i\partial^\mu\varphi^i - V(\varphi^i), \tag{7.40}$$

where we assume that $V(x)$ is bounded from below and depends on the fields through the combination $\varphi^i\varphi^i$. The theory is invariant under the transformations

$$\delta\varphi^i = \varepsilon^a (T^a)^i_{\ j}\varphi^j, \tag{7.41}$$

with T^a, $a = 1, \ldots, \frac{1}{2}N(N-1)$ the generators of the group SO(N).

To analyze the structure of vacua in this theory we construct its Hamiltonian

$$H[\pi^i, \varphi^i] = \int d^3x \left[\frac{1}{2}\pi^i\pi^i + \frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right] \tag{7.42}$$

and look for the minimum of the potential energy functional, given by

$$\mathscr{V}[\varphi^i] = \int d^3x \left[\frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right]. \tag{7.43}$$

We want the vacuum to preserve translational invariance, so we will be looking for field configurations satisfying $\nabla\varphi = \mathbf{0}$. This means that the vacua of the potential $\mathscr{V}[\varphi^i]$ coincides with those of $V(\varphi^i)$. The corresponding values of the scalar fields φ^i we denote by[3]

$$\langle\varphi^i\rangle : V(\langle\varphi^i\rangle) = 0, \quad \left.\frac{\partial V}{\partial\varphi^i}\right|_{\varphi^i = \langle\varphi^i\rangle} = 0. \tag{7.44}$$

Let us divide now the generators T^a of SO(N) into two groups: the first set consists of H^α ($\alpha = 1, \ldots, h$) satisfying

$$(H^\alpha)^i_{\ j}\langle\varphi^j\rangle = 0. \tag{7.45}$$

Thus, the vacuum configuration $\langle\varphi^i\rangle$ is left invariant by the group transformations generated by H^α. For this reason they are called *unbroken generators*. Notice that the commutator of two unbroken generators also annihilates the vacuum expectation value, $[H^\alpha, H^\beta]_{ij}\langle\varphi^j\rangle = 0$. They form a subalgebra of the algebra of the generators of SO(N). The subgroup they generate preserves the vacuum and hence it is realized à la Wigner–Weyl. This means in particular that the spectrum is classified in multiplets with respect to this unbroken subgroup.

The remaining generators we denote by K^A, with $A = 1, \ldots, \frac{1}{2}N(N-1) - h$, and by definition they satisfy

$$(K^A)^i_{\ j}\langle\varphi^j\rangle \neq 0. \tag{7.46}$$

These are called the *broken generators*. They generate group transformations that do not preserve the vacuum expectation value of the field. Next we prove a very

[3] For simplicity we consider that the minima of $V(x)$ occur at $V = 0$.

important result concerning these broken generators known as Goldstone's theorem: for each generator broken by the vacuum there is a massless excitation in the theory.

The mass matrix of the field excitations around the vacuum $\langle\varphi^i\rangle$ is determined by the quadratic part of the potential. Since we have assumed that $V(\langle\varphi^i\rangle) = 0$ and we are expanding around a minimum, the leading term in the expansion of the potential around the vacuum expectation values is given by

$$V(\varphi^i\varphi^i) = \left.\frac{\partial^2 V}{\partial\varphi^i\partial\varphi^j}\right|_{\varphi=\langle\varphi\rangle}(\varphi^i - \langle\varphi^i\rangle)(\varphi^j - \langle\varphi^j\rangle) + \mathscr{O}\left[(\varphi - \langle\varphi\rangle)^3\right] \qquad (7.47)$$

and the mass matrix is

$$M^2_{ij} \equiv \left.\frac{\partial^2 V}{\partial\varphi^i\partial\varphi^j}\right|_{\varphi=\langle\varphi\rangle}. \qquad (7.48)$$

In order to avoid a cumbersome notation, we do not indicate explicitly the dependence of the mass matrix on $\langle\varphi^i\rangle$.

To extract information about the possible zero modes of M^2_{ij}, we write down the conditions that follow from the invariance of the potential $V(\varphi^i)$ under the field transformations $\delta\varphi^i = \varepsilon^a(T^a)^i_{\ j}\varphi^j$. At first order in ε^a

$$\delta V(\varphi) = \varepsilon^a\frac{\partial V}{\partial\varphi^i}(T^a)^i_{\ j}\varphi^j = 0. \qquad (7.49)$$

Differentiating this expression with respect to φ^k we arrive at

$$\frac{\partial^2 V}{\partial\varphi^i\partial\varphi^k}(T^a)^i_{\ j}\varphi^j + \frac{\partial V}{\partial\varphi^i}(T^a)^i_{\ k} = 0. \qquad (7.50)$$

We evaluate this expression in the vacuum $\varphi^i = \langle\varphi^i\rangle$. The derivative in the second term cancels while the second derivative in the first one gives the mass matrix. Hence we have found

$$M^2_{ik}(T^a)^i_{\ j}\langle\varphi^j\rangle = 0. \qquad (7.51)$$

Now we can write this expression for both broken and unbroken generators. For the unbroken ones, since $(H^\alpha)^i_{\ j}\langle\varphi^j\rangle = 0$, we find a trivial identity 0=0. Things are more interesting for the broken generators, for which we have

$$M^2_{ik}(K^A)^i_{\ j}\langle\varphi^j\rangle = 0. \qquad (7.52)$$

Since $(K^A)^i_{\ j}\langle\varphi^j\rangle \neq 0$ this equation implies that the mass matrix has as many zero modes as broken generators. Therefore we have proven Goldstone's theorem: associated with each broken symmetry there is a massless mode in the theory. These modes are known in the literature as Nambu–Goldstone modes. Here we have presented a classical proof of the theorem. In the quantum theory the proof follows the same lines as the one presented here but one has to consider the effective action containing the effects of the quantum corrections to the classical Lagrangian.

7.3 Some Applications of Goldstone's Theorem

To illustrate Goldstone's theorem we consider a three-component real scalar field $\Phi = (\varphi^1, \varphi^2, \varphi^3)$ with the SO(3)-invariant "mexican hat" potential

$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^2 - a^2\right)^2. \tag{7.53}$$

The vacua of the theory are the field configurations satisfying the condition $\langle\Phi\rangle^2 = a^2$. In field space this equation describes a two-dimensional sphere and each vacuum is represented by a point in that sphere. It is easy to visualize geometrically how choosing one of these vacua results in symmetry breaking: while the whole sphere is invariant under the global SO(3) symmetry, each vacuum (i.e. each point) is preserved only by the SO(2) rotations around the axis of the sphere that passes through that point. Hence, the vacuum expectation value of the scalar field breaks the symmetry according to

$$\langle\Phi\rangle : \mathrm{SO(3)} \longrightarrow \mathrm{SO(2)}. \tag{7.54}$$

The symmetry group SO(3) has three generators while the symmetry of the vacuum SO(2) has only one. This means that the vacuum breaks two generators and using the Goldstone theorem we conclude that the system should have two massless Nambu–Goldstone bosons. These are easy to identify heuristically because they correspond to excitations along the surface of the sphere $\langle\Phi\rangle^2 = a^2$. That they are indeed massless follows from the fact that the potential (7.53) is flat along the directions of these excitations.

Once a minimum of the potential has been chosen, we can proceed to quantize the excitations around it. Since the vacuum only leaves invariant a SO(2) subgroup of the original SO(3) global symmetry group, it seems that in expanding around a particular vacuum expectation value of the scalar field we have lost part of the symmetry of the Lagrangian. This is however not the case. The full quantum theory is indeed symmetric under the whole SO(3). This is reflected in the fact that the physical properties of the theory do not depend on the particular point of the sphere $\langle\Phi\rangle^2 = a^2$ that we have chosen for our vacuum. In fact, different vacua are related by the full SO(3) symmetry and therefore should give the same physics.

A very important point to keep in mind is that once the system described by the theory chooses a vacuum determined by a value of $\langle\Phi\rangle$, all other possible vacua of the theory are inaccessible in the infinite volume limit. This means that any two vacuum states $|0_1\rangle$, $|0_2\rangle$ corresponding to different vacuum expectation values of the scalar field are orthogonal $\langle 0_1|0_2\rangle = 0$ and, moreover, cannot be connected by any local observable $\mathscr{O}(x)$, $\langle 0_1|\mathscr{O}(x)|0_2\rangle = 0$. Heuristically, this can be understood by thinking that in the infinite volume limit switching from one vacuum into another requires changing the vacuum expectation value of the field everywhere in space at the same time, something that cannot be done by any local operator of the theory. Notice that this is radically different from our expectations based on the quantum mechanics of a system with a finite number of degrees of freedom where symmetries do not break spontaneously, i.e. the ground state is always symmetrical.

Let us make these arguments a bit more explicit since they are very important in understanding how symmetry breaking works. Consider a relatively simple system: a set of spin-$\frac{1}{2}$ magnets, the Heisenberg ferromagnet model, with nearest neighbors interactions. Space is replaced by a lattice with spacing a and lattice vectors $\mathbf{x} = (n_1 a, n_2 a, n_3 a)$. At each lattice site $\mathbf{x}$ there is a spin-$\frac{1}{2}$ degree of freedom

$$\mathbf{s} = \left(\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{2}\sigma_3\right), \tag{7.55}$$

with σ_i the Pauli matrices. The Heisenberg Hamiltonian is defined by

$$H = -J \sum_{\langle \mathbf{x},\mathbf{x}' \rangle} \mathbf{s}(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}') \quad \text{with} \quad J > 0, \tag{7.56}$$

where the symbol $\langle \mathbf{x}, \mathbf{x}' \rangle$ indicates that we are summing over nearest neighbors on the lattice.

At each lattice site we have a two-dimensional Hilbert space whose basis we can take to be the two $s_3(\mathbf{x})$ eigenstates $\{|\mathbf{x};\uparrow\rangle, |\mathbf{x};\downarrow\rangle\}$. The state corresponding to the spin at the site $\mathbf{x}$ being aligned along the direction $\hat{\mathbf{r}}$

$$\hat{\mathbf{r}} \cdot \mathbf{s}(\mathbf{x})|\mathbf{x}; \hat{\mathbf{r}}\rangle = \frac{1}{2}|\mathbf{x}; \hat{\mathbf{r}}\rangle, \tag{7.57}$$

can be written in this basis as

$$|\mathbf{x}; \hat{\mathbf{r}}\rangle = \cos\left(\frac{\theta}{2}\right)|\mathbf{x};\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\mathbf{x};\downarrow\rangle, \tag{7.58}$$

where θ and ϕ are the polar an azimuthal angle associated with the unit vector $\hat{\mathbf{r}}$. Using rotational invariance it is an easy exercise to show that

$$\langle \mathbf{x}; \hat{\mathbf{r}}|\mathbf{x}; \hat{\mathbf{r}}'\rangle = \cos\left(\frac{\alpha}{2}\right), \tag{7.59}$$

where α is the angle between the unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$, i.e. $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos\alpha$.

We can construct now the ground states of the Hamiltonian (7.56). They correspond to states where all spins in the ferromagnet are aligned along the same direction, that we indicate by the unit vector $\hat{\mathbf{r}}$. Thus we write

$$|\hat{\mathbf{r}}\rangle = \bigotimes_{\mathbf{x}} |\mathbf{x}; \hat{\mathbf{r}}\rangle, \tag{7.60}$$

From this result we conclude that the overlap between two ground states characterized by unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ is given by

$$\langle \hat{\mathbf{r}}|\hat{\mathbf{r}}\rangle = \left[\cos\left(\frac{\alpha}{2}\right)\right]^N \tag{7.61}$$

where $N = Va^{-3}$ is the number of lattice sites, with V the spatial volume.

It is clear that unless the directions $\hat{\mathbf{r}}$, $\hat{\mathbf{r}}'$ are parallel, the scalar product vanishes in the infinite volume limit. Hence we can build disconnected Hilbert spaces for each different direction $\hat{\mathbf{r}}$. If the spatial volume is finite, the scalar product is non-vanishing and the ground states associated to different directions will mix, so that the lowest ground state will preserve the symmetry. It is only in the limit $V \to \infty$, when the states are orthogonal, that we obtain spontaneous symmetry breaking. It is clear that if the volume is finite but large, the mixing of the different ground states is very highly suppressed, so that for many practical purposes we can approximate this finite volume theory by the theory with Goldstone bosons.

A similar argument can be carried out in field theory. The simplest theory with a Nambu–Goldstone boson is a free real massless scalar with Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi. \tag{7.62}$$

It is invariant under shifts of the field by a real constant, $\varphi(x) \to \varphi(x) + \alpha$, with $\alpha \in \mathbb{R}$. This symmetry has an associated Noether current given by $j_\mu = \partial_\mu\varphi$, whose conservation can be checked by applying the equations of motion. The vacuum of the theory is not invariant under the symmetry. Indeed, when we quantize the theory by expanding about some particular constant value of the field $\varphi(x) = \varphi_0$ the symmetry is broken, and the massless particle created by $\varphi(x)$ is the associated Goldstone boson.

In fact, at low energies all Nambu–Goldstone bosons are well represented by this approximation. Consider for instance a scalar doublet $\Phi = (\varphi_1, \varphi_2)$ with a "Mexican hat" potential of the type (7.53). The vacuum breaks the global SO(2) symmetry completely. We parametrize the fields $\varphi_1(x)$ and $\varphi_2(x)$ in terms of a single complex scalar field

$$\zeta(x) = \frac{1}{\sqrt{2}}\left[\varphi_1(x) + i\varphi_2(x)\right] \equiv \frac{1}{\sqrt{2}}\left[a + h(x)\right]e^{i\theta(x)}. \tag{7.63}$$

The field $\theta(x)$ is the Nambu–Goldstone boson. Using this parametrization, the action can be written as

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}\partial_\mu\Phi\cdot\partial^\mu\Phi - \frac{\lambda}{4}\left(\Phi^2 - a^2\right)^2 \\ &= \partial_\mu\zeta^*\partial^\mu\zeta - \lambda\left(|\zeta|^2 - \frac{a^2}{2}\right)^2 = \frac{a^2}{2}\partial_\mu\theta\partial^\mu\theta + \cdots,\end{aligned} \tag{7.64}$$

where the dots stand for the other terms involving the field $h(x)$ and its couplings to the Nambu–Goldstone boson $\theta(x)$. If we do not excite the $h(x)$ field, the Lagrangian for $\theta(x)$ is at leading order of the form (7.62). This shows that, although we consider a simplified example, the analysis can be extended to more general situations.

We wish to study the quantization of (7.62) in a spatial box of side L and periodic boundary conditions. A complete set of properly normalized plane waves solutions to the field equation $\partial_\mu\partial^\mu\varphi(x) = 0$ is provided by

$$\varphi_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{\sqrt{V}} e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n} \tag{7.65}$$

where $\mathbf{n} = (n_1, n_2, n_3)$, with $n_i \in \mathbb{Z}$, and $V = L^3$. Their completeness relation reads

$$\delta(\mathbf{x}_1 - \mathbf{x}_2) = \sum_{\mathbf{k}} \varphi_{\mathbf{k}}(t,\mathbf{x}_1)\varphi_{\mathbf{k}}(t,\mathbf{x}_2)^* = \frac{1}{V} + \frac{1}{V}\sum_{\mathbf{k}\neq\mathbf{0}} e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}, \tag{7.66}$$

where we have extracted explicitly the zero mode $\mathbf{k} = \mathbf{0}$.

There is a very important difference between the quantization of this massless scalar field in a spatial box and the one we carried out in Chap. 2 for a free scalar field in $\mathbb{R}^3$. This difference lies in the treatment of the zero mode that in finite volume is a normalizable state that has to be quantized independently. In our case the most general position-independent solution to the equation of motion compatible with the periodic boundary conditions is linear in time. Taking this into account, we write the following mode expansion

$$\varphi(t,\mathbf{x}) = \varphi_0 + \pi_0 t + \sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{\sqrt{2V|\mathbf{k}|}}\Big[\alpha(\mathbf{k})e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + \alpha^\dagger(\mathbf{k})e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\Big]. \tag{7.67}$$

Imposing the canonical equal-time commutation relations

$$[\varphi(t,\mathbf{x}_1), \dot{\varphi}(t,\mathbf{x}_2)] = i\delta(\mathbf{x}_1 - \mathbf{x}_2) = \frac{i}{V} + \frac{i}{V}\sum_{\mathbf{k}\neq\mathbf{0}} e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \tag{7.68}$$

yields the standard canonical commutation relations for creation-annihilation operators $\alpha(\mathbf{k})$, $\alpha^\dagger(\mathbf{k})$, as well as the commutation relations for φ_0 and π_0

$$[\varphi_0, \pi_0] = \frac{i}{V}. \tag{7.69}$$

After a little work, we find a simple form for the normal-ordered Hamiltonian

$$:H:= \frac{V}{2}\pi_0^2 + \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}). \tag{7.70}$$

Since we are interested in states where $\varphi(x)$ acquires an expectation value, we follow by analogy the treatment of coherent states in elementary quantum mechanics. Let us introduce the operators a and $a^\dagger$ associated to the zero modes

$$a = \frac{1}{\sqrt{2}}\Big(\varphi_0 + iV^{\frac{1}{3}}\pi_0\Big), \quad a^\dagger = \frac{1}{\sqrt{2}}\Big(\varphi_0 - iV^{\frac{1}{3}}\pi_0\Big), \tag{7.71}$$

that satisfy

$$[a, a^\dagger] = V^{-\frac{2}{3}}. \tag{7.72}$$

The conserved charge associated with the Noether current $j_\mu = \partial_\mu \varphi$ has the simple form

$$Q = \int d^3x \partial_0 \varphi = V\pi_0 = \frac{V^{\frac{2}{3}}}{i\sqrt{2}} \left(a - a^\dagger\right). \tag{7.73}$$

This charge generates constants shifts in the value of the field, namely

$$e^{-i\xi Q}\varphi(x)e^{i\xi Q} = \varphi(x) + \xi, \tag{7.74}$$

for any real ξ.

We consider a ground state $|0\rangle$ defined by $a|0\rangle = 0$, $\alpha(\mathbf{k})|0\rangle = 0$, for all $\mathbf{k} \neq \mathbf{0}$. It can be immediately shown that the field (7.67) has zero expectation value in this vacuum, $\langle 0|\varphi(x)|0\rangle = 0$. For every real ξ we define the state

$$|\xi\rangle \sim e^{i\xi Q}|0\rangle = e^{-\frac{1}{\sqrt{2}}\xi V^{\frac{2}{3}}(a^\dagger - a)}|0\rangle. \tag{7.75}$$

Using the properties of the creation-annihilation operators $a, a^\dagger$ we can compute the overlap

$$\langle 0|\xi\rangle = e^{-\frac{1}{4}\xi^2 V^{\frac{2}{3}}}\langle 0|0\rangle. \tag{7.76}$$

This vanishes exponentially as $V \to \infty$, as we found for the Heisenberg Hamiltonian. This shows once again that, strictly speaking, Goldstone bosons only appear in the infinite volume limit.

To be fair we must say that we have been a bit sloppy with the argument. In the cases considered so far, the field $\varphi(x)$ is itself an angle. Hence we should also impose the condition that in field space $\varphi(x) \sim \varphi(x) + 2\pi$, that means that the π_0 is not actually a momentum but an angular momentum variable. This complicates the argument technically, but does not change the conclusion: the overlap $\langle 0|\xi\rangle$ still vanishes exponentially in the infinite volume limit.

We close this discussion with a further comment. The argument presented works in space-times of dimension higher than two. In two-dimensions, space is a line and a number of specific subtleties appear. The conclusion is that in two-dimensions there are no Goldstone bosons. The quantum fluctuations always restore the original symmetry. This theorem appeared first in statistical mechanics, where it is known as the Mermin–Wagner theorem [1, 2]. Its field-theoretical version (Coleman theorem) was proved in [3].

A typical example of a Goldstone boson in high energy physics are the pions, associated with the spontaneous breaking of the global chiral isospin $SU(2)_L \times SU(2)_R$ symmetry and that we will study in some detail in Chap. 9. This symmetry acts independently in the left- and right-handed u and d quark spinors as

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \longrightarrow M_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}, \quad M_{L,R} \in SU(2)_{L,R} \tag{7.77}$$

Since quarks are confined at low energies, this symmetry is expected to be spontaneously broken by a nonvanishing vacuum expectation value of quark bilinears of the type $\langle \overline{u}_R u_L \rangle \neq 0$.

This breaking of the global $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry to the diagonal SU(2) acting in the same way on the two chiralities has three Nambu–Goldstone modes which are identified with the pions (see Sect. 9.3). This identification, however, might seem a bit puzzling at first sight, because pions are massive contrary to what is expected of a Goldstone boson. The solution to this apparent riddle is that the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ would be an exact global symmetry of the QCD Lagrangian only in the limit when the masses of the quarks are zero $m_u, m_d \to 0$. As these quarks have nonzero masses, the chiral symmetry is only approximate and as a consequence the corresponding Goldstone bosons are not strictly massless. That is why pions have masses, although they are the lightest particles among the hadrons.

The phenomenon of spontaneous symmetry breaking is not confined to high energy physics, but appears also frequently in condensed matter physics [4]. For example, when a solid crystallizes from a liquid the translational invariance that is present in the liquid phase is broken to a discrete group of translations that represent the crystal lattice. This symmetry breaking has associated Goldstone bosons that are identified with acoustic phonons, which are the quantum excitation modes of the vibrational degrees of freedom of the lattice.

7.4 The Brout–Englert–Higgs Mechanism

Gauge symmetry seems to prevent a vector field from having a mass. This is obvious once we realize that a term in the Lagrangian like $m^2 A_\mu A^\mu$ is incompatible with gauge invariance.

Certain physical situations, however, seem to require massive vector fields. This became evident during the 1960s in the study of weak interactions. The Glashow model gave a common description of both the electromagnetic and weak forces based on a gauge theory with group $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$. However, in order to reproduce Fermi's four-fermion theory of the β-decay, it was necessary that three of the vector fields involved were massive. Also in condensed matter physics massive vector fields are required to describe certain systems, most notably in superconductivity.

The way out to this situation was found independently by Brout and Englert [5] and by Higgs [6, 7] using the concept of spontaneous symmetry breaking discussed above[4]: if the consistency of the quantum theory requires gauge invariance, this can also be realized à la Nambu–Goldstone. When this happens the full gauge symmetry is not explicitly present in the effective action constructed around the particular vacuum chosen for the theory. This makes possible the existence of mass terms for gauge fields without jeopardizing the consistency of the full theory, which is still invariant under the whole gauge group.

[4] In condensed matter the idea had been previously considered by Anderson [8].

To illustrate the Brout–Englert–Higgs mechanism we study the simplest example, the Abelian Higgs model: a U(1) gauge field coupled to a self-interacting charged complex scalar field φ with Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{\lambda}{4}\left(\varphi^\dagger\varphi - \frac{v^2}{2}\right)^2, \tag{7.78}$$

where the covariant derivative is given in Eq. (4.36). This theory is invariant under the gauge transformations

$$\varphi \longrightarrow e^{i\alpha(x)}\varphi, \quad A_\mu \longrightarrow A_\mu + \partial_\mu\alpha(x). \tag{7.79}$$

The minimum of the potential is defined by the equation $|\varphi| = \frac{v}{\sqrt{2}}$. Thus, there is a continuum of different vacua labelled by the phase of the scalar field. None of these vacua, however, is invariant under the gauge symmetry

$$\langle\varphi\rangle = \frac{v}{\sqrt{2}}e^{i\vartheta_0} \longrightarrow \frac{v}{\sqrt{2}}e^{i\vartheta_0 + i\alpha(x)} \tag{7.80}$$

and therefore the symmetry is spontaneously broken.

Let us study now the theory around one of these vacua, for example $\langle\varphi\rangle = \frac{v}{\sqrt{2}}$, by writing the field φ in terms of the excitations around this particular vacuum

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[v + \sigma(x)\right]e^{i\vartheta(x)}. \tag{7.81}$$

The whole Lagrangian is still gauge invariant under (7.79), independently of which vacuum we have chosen. This means that we are at liberty of performing a gauge transformation with parameter $\alpha(x) = -\vartheta(x)$ in order to get rid of the phase in Eq. (7.81). Substituting then $\varphi(x) = \frac{1}{\sqrt{2}}[v + \sigma(x)]$ in Eq. (7.78) we find

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2v^2A_\mu A^\mu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4}\lambda v^2\sigma^2 \\ &- \frac{1}{\sqrt{2}}\lambda v\sigma^3 - \frac{\lambda}{4}\sigma^4 + \frac{1}{\sqrt{2}}e^2vA_\mu A^\mu\sigma + e^2A_\mu A^\mu\sigma^2.\end{aligned} \tag{7.82}$$

We ask now about the excitation of the theory around the vacuum $\langle\varphi\rangle = v/\sqrt{2}$. There is a real scalar field $\sigma(x)$ with mass squared $\frac{1}{2}\lambda v^2$. What makes the construction interesting is that the gauge field A_μ has acquired a mass given by

$$m_\gamma^2 = e^2v^2. \tag{7.83}$$

What is really remarkable about this way of giving a mass to the photon is that at no point we have given up gauge invariance. The symmetry is only hidden. Therefore in quantizing the theory we can still enjoy all the advantages of having a gauge theory, while at the same time we have managed to generate a mass for the gauge field.

It might look surprising that in the Lagrangian (7.82) we did not find any massless mode. Since the vacuum chosen by the scalar field breaks the single generator of U(1), we would have expected from Goldstone's theorem to have one massless particle. To understand the fate of the missing Goldstone boson we have to revisit the calculation leading to the Lagrangian (7.82). Were we dealing with a global U(1) theory, the Goldstone boson would correspond to excitations of the scalar field along the valley of the potential associated with the phase $\vartheta(x)$. In writing the Lagrangian we managed to get rid of $\vartheta(x)$ using a gauge transformation. With this we shifted the Goldstone mode into the gauge field A_μ. In fact, by identifying the gauge parameter with the Goldstone excitation we have completely fixed the gauge and the Lagrangian (7.82) does not have any residual gauge symmetry.

A massive vector field has three polarizations: two transverse ones $\mathbf{k}\cdot\varepsilon(\mathbf{k}, \pm 1) = 0$ with helicities $\lambda = \pm 1$ plus a longitudinal one $\varepsilon_L(\mathbf{k}) \sim \mathbf{k}$. In gauging away the massless Goldstone boson $\vartheta(x)$ we have transformed it into the longitudinal polarization of the massive vector field. In the literature this is usually expressed by saying that the Goldstone mode is "eaten up" by the longitudinal component of the gauge field. One should not forget that, in spite of the fact that the Lagrangian (7.82) looks quite different from the one we started with, we have not lost any degrees of freedom. We started with two polarizations of the photon plus the two degrees of freedom associated with the real and imaginary components of the complex scalar field $\varphi(x)$. After symmetry breaking we ended up with the three polarizations for the massive vector field, plus the degree of freedom represented by the real scalar field $\sigma(x)$.

We can understand the Brout–Englert–Higgs mechanism in the light of our general discussion of gauge symmetry in Chap. 4 (see Sect. 4.7). Remember that there we had considered the set $\mathscr{G}$ of all gauge transformations $g(\mathbf{x}) \in G$ approaching the identity at infinity and the subset $\mathscr{G}_0 \subset \mathscr{G}$ formed by those contractible to the identity. These latter are the ones generated by Gauss' law, $[(\mathbf{D}\cdot\mathbf{E})_A - \rho_A]|\text{phys}\rangle = 0$ where ρ_a represents the matter contribution.

The set of *all* gauge transformations also contains elements $g(\mathbf{x})$ approaching any other element of G as $|\mathbf{x}| \to \infty$. This differs from $\mathscr{G}$ by a copy of the gauge group G at infinity. This is identified as the group of *global* transformations associated with the existence of conserved charges via Noether's theorem. When the gauge symmetry is spontaneously broken, the invariance of the theory under $\mathscr{G}$ is nevertheless preserved, while the invariance under global transformations (i.e. the copy of G at infinity) is broken. Notice that this in no way poses a threat to the consistency of the theory since properties like the decoupling of unphysical states are guaranteed by the fact that Gauss' law is satisfied quantum mechanically. This follows from the invariance under $\mathscr{G}_0$.

In Chap. 10 we will explain why the Abelian Higgs model discussed here can be regarded as a toy model of the Brout–Englert–Higgs mechanism responsible for giving masses to the $W^\pm$ and Z^0 gauge bosons in the standard model. In condensed matter physics the symmetry breaking described by the nonrelativistic version of the Abelian Higgs model can be used to characterize the onset of a superconducting phase in the Ginzburg–Landau and BCS (Bardeen–Cooper–Schrieffer) theories, where the complex scalar field Φ is associated with the Cooper pairs. In this case the parameter