Chapter 3
Theories and Lagrangians I: Matter Fields

Up to this point we have used a scalar field to illustrate our discussion of the quantization procedure. However, Nature is richer than that and it is necessary to consider other fields with more complicated behavior under Lorentz transformations. Before considering these other fields we pause and study the properties of the Lorentz group.

3.1 Representations of the Lorentz Group

The Lorentz group is the group of linear coordinate transformations that leave invariant the Minkowskian line element. It has a very rich mathematical structure that we review in Appendix B. Here our interest is focused on its representations.

In four dimensions the Lorentz group has six generators. Three of them are the generators J_i of the group of rotations in three dimensions SO(3). A finite rotation of angle φ with respect to the axis determined by a unitary vector $\mathbf{e}$ can be written as

$$R(\mathbf{e}, \varphi) = e^{-i\varphi\,\mathbf{e}\cdot\mathbf{J}}, \quad \mathbf{J} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}. \tag{3.1}$$

The other three generators of the Lorentz group are associated with boosts M_i along the three spatial directions. A boost with rapidity λ along a direction $\mathbf{u}$ is given by

$$B(\mathbf{u}, \lambda) = e^{-i\lambda\mathbf{u}\cdot\mathbf{M}}, \quad \mathbf{M} = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}. \tag{3.2}$$

The six generators $J_i,\ M_i$ satisfy the algebra

$$\begin{aligned} \left[J_i, J_j\right] &= i\varepsilon_{ijk}J_k, \\ [J_i, M_k] &= i\varepsilon_{ijk}M_k, \\ \left[M_i, M_j\right] &= -i\varepsilon_{ijk}J_k, \end{aligned} \tag{3.3}$$

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The first line are the commutation relations of SO(3), while the second one implies that the generators of the boosts transform like a vector under rotations. The six generators of the Lorentz group can be collected into the six independent components of an antisymmetric rank-two tensor $\mathscr{J}_{\mu\nu}$ according to

$$\mathscr{J}_{0i} = M_i, \quad \mathscr{J}_{ij} = \varepsilon_{ijk} J_k. \tag{3.4}$$

They satisfy

$$[\mathscr{J}_{\mu\nu}, \mathscr{J}_{\sigma\lambda}] = i\eta_{\mu\sigma}\mathscr{J}_{\nu\lambda} - i\eta_{\mu\lambda}\mathscr{J}_{\nu\sigma} + i\eta_{\nu\lambda}\mathscr{J}_{\mu\sigma} - i\eta_{\nu\sigma}\mathscr{J}_{\mu\lambda}. \tag{3.5}$$

The Lorentz algebra in terms of $\mathscr{J}_{\mu\nu}$ has the same form in any space-time dimension.

The task of finding representations of the algebra (3.3) [or (3.5)] might seem difficult at first sight. In four dimensions the problem is greatly simplified by combining the generators in the following way

$$J_k^{\pm} = \frac{1}{2}(J_k \pm i M_k). \tag{3.6}$$

Using (3.3), the new generators $J_k^{\pm}$ are found to satisfy

$$\begin{aligned} \left[J_i^{\pm}, J_j^{\pm}\right] &= i\varepsilon_{ijk} J_k^{\pm}, \\ \left[J_i^{+}, J_j^{-}\right] &= 0. \end{aligned} \tag{3.7}$$

Thus, the four-dimensional Lorentz algebra is equivalent to two copies of the algebra of SU(2) $\approx$ SO(3). Their irreducible representations are identified by $(\mathbf{s}_+, \mathbf{s}_-)$, where $\mathbf{s}_\pm = k_\pm$ or $k_\pm + \frac{1}{2}$ (with $k_\pm \in \mathbb{N}$) are the spins of the representations of the two copies of SU(2).

To get familiar with this way of labeling the representations of the Lorentz group we study some particular examples. Let us start with the simplest one $(\mathbf{s}_+, \mathbf{s}_-) = (\mathbf{0}, \mathbf{0})$. This state is a singlet under $J_i^{\pm}$ and therefore also under rotations and boosts. Therefore we have a scalar.

The next interesting cases are $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$ and $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$. States transforming in these representations are respectively right and left-handed Weyl spinors. Their properties will be studied in more detail below. Next we deal with $(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$. Equation (3.6) shows that $J_i = J_i^+ + J_i^-$. Applying the rules of addition of angular momenta we find that the states transforming in this representations decompose into a vector and a scalar under three-dimensional rotations. A more detailed analysis shows that the singlet state is identified with the time component of a four-vector, combining with the triplet to form a vector under the Lorentz group.

We can consider more "exotic" representations. For example the $(\mathbf{1},\mathbf{0})$ and $(\mathbf{0},\mathbf{1})$ representations correspond respectively to selfdual and anti-selfdual rank-two antisymmetric tensors $T^{\mu\nu} = -T^{\nu\mu}$,

$$T_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}T^{\sigma\lambda} \quad (+\,\text{selfdual}, -\,\text{anti-selfdual}), \tag{3.8}$$

Table 3.1 Representations of the Lorentz group in terms of the representations of SU(2)× SU(2)

Representation	Type of field
$(\mathbf{0},\mathbf{0})$	Scalar
$(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0})$	Right-handed spinor
$(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}})$	Left-handed spinor
$(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$	Vector
$(\mathbf{1},\mathbf{0})$	Selfdual antisymmetric 2-tensor
$(\mathbf{0},\mathbf{1})$	Anti-selfdual antisymmetric 2-tensor

where $\varepsilon_{\mu\nu\sigma\lambda}$ is the Levi-Civita symbol with four indices. Table 3.1 summarizes the previous discussion.

To conclude our analysis of the representations of the Lorentz group we notice that under parity the generators of SO(1,3) transform as[1]

$$P\colon J_i \longrightarrow J_i, \quad P : M_i \longrightarrow -M_i. \tag{3.9}$$

This implies that $P\colon J_i^{\pm} \longrightarrow J_i^{\mp}$ and therefore a representation $(\mathbf{s}_1,\mathbf{s}_2)$ is transformed into $(\mathbf{s}_2,\mathbf{s}_1)$. As a consequence a vector $(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$ is invariant under parity, whereas a left-handed Weyl spinor $(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0})$ transforms into a right-handed one $(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}})$ and vice versa.

It is instructive to see how the representations of the Lorentz group differ from those of SO(4), the isometry group of four-dimensional Euclidean space. Like the Lorentz group, it is generated by a set of six generators $\mathscr{J}_{\mu\nu}$ whose algebra can be obtained from Eq. (3.5) by replacing $\eta_{\mu\nu} \to -\delta_{\mu\nu}$. The Lie algebra of SO(4) is isomorphic to that of SU(2)× SU(2). This can be seen by introducing the generators

$$N^a = \eta^a_{\mu\nu} J^{\mu\nu}, \quad \overline{N}^a = \overline{\eta}^a_{\mu\nu} J^{\mu\nu}. \tag{3.10}$$

The numerical coefficients $\eta^a_{\mu\nu}$ and $\overline{\eta}^a_{\mu\nu}$ (with $a = 1, 2, 3$ and $\mu, \nu = 0, \ldots, 3$) are called *'t Hooft symbols* and are given by

$$\begin{aligned} \eta^a_{\mu\nu} &= \varepsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 0} - \delta_{a\nu}\delta_{\mu 0}, \\ \overline{\eta}^a_{\mu\nu} &= \varepsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 0} + \delta_{a\nu}\delta_{\mu 0}. \end{aligned} \tag{3.11}$$

Here $\varepsilon_{a\mu\nu}$ represents the Levi-Civita antisymmetric symbol with three indices and it is taken to be zero whenever μ or ν are equal to zero. Now it is not difficult to check that the generators (3.10) satisfy the Lie algebra of SU(2)× SU(2)

$$\left[N^a, N^b\right] = i\varepsilon^{abc} N^c, \quad \left[\overline{N}^a, \overline{N}^b\right] = i\varepsilon^{abc}\overline{N}^c, \quad \left[N^a, \overline{N}^b\right] = 0. \tag{3.12}$$

This shows that the representations of SO(4) can also be labelled in terms of the irreducible representations of SU(2).

[1] Parity and other discrete symmetries are studied in detail in Chap. 11.

3.2 Weyl Spinors

A Weyl spinor $u_{\pm}$ is a complex two-component object that transforms in the representations $(\mathbf{\frac{1}{2}}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{\frac{1}{2}})$ respectively. The generators $J_i^{\pm}$ can be explicitly constructed using the Pauli matrices as

$$\begin{aligned} J_i^{+} &= \frac{1}{2}\sigma_i, \quad J_i^{-} = 0 \quad \text{for} \quad \left(\mathbf{\tfrac{1}{2}}, \mathbf{0}\right), \\ J_i^{+} &= 0, \qquad J_i^{-} = \frac{1}{2}\sigma_i \quad \text{for} \quad \left(\mathbf{0}, \mathbf{\tfrac{1}{2}}\right). \end{aligned} \tag{3.13}$$

Going back to J^i and K^i, we find that under a rotation of angle θ and axis $\mathbf{n}$ and a boost of rapidity $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ the spinors $u_{\pm}$ transform as

$$u_{\pm} \longrightarrow e^{-\frac{i}{2}(\theta\mathbf{n} \mp i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}} u_{\pm}. \tag{3.14}$$

To construct a free Lagrangian for the fields $u_{\pm}$ we have to look for quadratic combinations of the fields that are Lorentz scalars. Defining $\sigma_{\pm}^{\mu} = (\mathbf{1}, \pm\sigma_i)$, we can construct the following quantities

$$u_{+}^{\dagger}\sigma_{+}^{\mu}u_{+}, \qquad u_{-}^{\dagger}\sigma_{-}^{\mu}u_{-}. \tag{3.15}$$

The first thing to point out is that, since $(J_i^{\pm})^{\dagger} = J_i^{\mp}$, the hermitian conjugate fields $u_{\pm}^{\dagger}$ are in the $(\mathbf{0}, \mathbf{\frac{1}{2}})$ and $(\mathbf{\frac{1}{2}}, \mathbf{0})$ representation respectively. The combinations (3.15) transform as a four-vector under (3.14), due to the property

$$e^{\frac{i}{2}(\theta\mathbf{n} \pm i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}}\sigma_{\pm}^{\mu}e^{-\frac{i}{2}(\theta\mathbf{n} \mp i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}} = \Lambda^{\mu}_{\ \nu}(\theta\mathbf{n}, \boldsymbol{\beta})\sigma_{\pm}^{\nu}, \tag{3.16}$$

where $\Lambda^{\mu}_{\ \nu}(\theta\mathbf{n}, \boldsymbol{\beta})$ gives the transformation of the coordinates x^{μ}.

Once the transformation properties of (3.15) are known we can start building invariants. If, in addition, we also demand that the Lagrangian be invariant under global phase rotations

$$u_{\pm} \longrightarrow e^{i\theta}u_{\pm} \tag{3.17}$$

we are left with just one possibility up to a sign, namely

$$\mathscr{L}_{\text{Weyl}}^{\pm} = iu_{\pm}^{\dagger}\left(\partial_t \pm \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)u_{\pm} = iu_{\pm}^{\dagger}\sigma_{\pm}^{\mu}\partial_{\mu}u_{\pm}. \tag{3.18}$$

This is the Weyl Lagrangian. In order to get a more clear idea of the physical meaning of the spinors $u_{\pm}$ we write the equations of motion

$$\left(\partial_0 \pm \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)u_{\pm} = 0. \tag{3.19}$$

Multiplying this equation on the left by $(\partial_0 \mp \boldsymbol{\sigma}\cdot\boldsymbol{\nabla})$ and applying the algebraic properties of the Pauli matrices, we conclude that $u_{\pm}$ satisfy the massless Klein-Gordon equation

$$\partial_\mu \partial^\mu u_\pm = 0, \tag{3.20}$$

whose solutions are

$$u_\pm(x) = u_\pm(k) e^{-ik\cdot x}, \quad \text{with} \quad k^0 = |\mathbf{k}|. \tag{3.21}$$

Plugging them back into the equations of motion (3.19) we find

$$(|\mathbf{k}| \mp \mathbf{k}\cdot\boldsymbol{\sigma})\, u_\pm = 0, \tag{3.22}$$

implying the following conditions

$$\begin{aligned} u_+ : \quad & \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{|\mathbf{k}|} = 1, \\ u_- : \quad & \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{|\mathbf{k}|} = -1. \end{aligned} \tag{3.23}$$

Since the spin operator is $\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}$, the previous expressions give the helicity of the states with wave function $u_\pm$, i.e. the projection of the spin along the momentum of the particle

$$\lambda = \mathbf{s}\cdot\frac{\mathbf{k}}{|\mathbf{k}|}. \tag{3.24}$$

We conclude that u_+ is a Weyl spinor of positive helicity $\lambda = \frac{1}{2}$, while u_- has negative helicity $\lambda = -\frac{1}{2}$. This agrees with our assertion in the previous section that the representation $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$ corresponds to a right-handed Weyl fermion (positive helicity) whereas $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$ is a left-handed Weyl fermion (negative helicity). For example, the standard model neutrinos are left-handed Weyl spinors and therefore transform in the representation $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$ of the Lorentz group.

Nevertheless, it is possible that we were too restrictive in constructing the Weyl Lagrangian (3.18). There we constructed the invariants from the vector bilinears (3.15) corresponding to the product representations

$$\left(\tfrac{\mathbf{1}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) = \left(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \otimes \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) \quad \text{and} \quad \left(\tfrac{\mathbf{1}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) = \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) \otimes \left(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right). \tag{3.25}$$

In particular our insistence in demanding the Lagrangian to be invariant under the global symmetry $u_\pm \to e^{i\theta} u_\pm$ rules out the scalar term that appears in the product representations

$$\begin{aligned} \left(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \otimes \left(\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) &= (\mathbf{1}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{0}), \\ \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) \otimes \left(\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}\right) &= (\mathbf{0}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{0}). \end{aligned} \tag{3.26}$$

The singlet representations corresponds to the antisymmetric combinations

$$\varepsilon_{ab} u^a_{\pm} u^b_{\pm}, \tag{3.27}$$

where ε_{ab} is the antisymmetric symbol $\varepsilon_{12} = -\varepsilon_{21} = 1$.

At first sight it might seem that the term (3.27) vanishes identically due to the antisymmetry of the ε-symbol. However we should keep in mind that the spin-statistics theorem (more on this later) demands that fields with half-integer spin have to satisfy the Fermi-Dirac statistics and therefore satisfy anticommutation relations, whereas fields of integer spin follow the statistic of Bose-Einstein and, as a consequence, quantization replaces Poisson brackets by commutators. This implies that the components of the Weyl fermions $u_{\pm}$ are anticommuting Grassmann fields

$$u^a_{\pm} u^b_{\pm} + u^b_{\pm} u^a_{\pm} = 0. \tag{3.28}$$

It is important to realize that, strictly speaking, fermions (i.e., objects that satisfy the Fermi-Dirac statistics) do not exist classically. The reason is that they satisfy the Pauli exclusion principle and therefore each quantum state can be occupied, at most, by one fermion. Therefore the naive definition of the classical limit as a limit of large occupation numbers cannot be applied. Fermion fields do not really make sense classically.

Since the combination (3.27) does not vanish, we can construct a new Lagrangian

$$\mathscr{L}^{\pm}_{\text{Weyl}} = i u^{\dagger}_{\pm} \sigma^{\mu}_{\pm} \partial_{\mu} u_{\pm} - \frac{m}{2} \left(\varepsilon_{ab} u^a_{\pm} u^b_{\pm} + \text{h.c.} \right) \tag{3.29}$$

This mass term, called of Majorana type, is allowed if we do not worry about breaking the global U(1) symmetry $u_{\pm} \to e^{i\theta} u_{\pm}$. This is not the case, for example, of charged chiral fermions, since the Majorana mass violates the conservation of electric charge or any other gauge U(1) charge. In the standard model, however, there is no such a problem if we introduce Majorana masses for right-handed neutrinos, since they are singlets under all standard model gauge groups. Such a term will break, however, the global U(1) lepton number charge, the operator $\varepsilon_{ab} \nu^a_R \nu^b_R$ changes the lepton number by two units. We will have more to say about this later.

3.3 Dirac Spinors

We have seen that parity interchanges the representations $(\mathbf{\frac{1}{2}}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{\frac{1}{2}})$, i.e. it changes right-handed with left-handed fermions

$$P : u_{\pm} \longrightarrow u_{\mp}. \tag{3.30}$$

An obvious way to build a parity invariant theory is to combine a pair or Weyl fermions u_+ and u_- of opposite helicity in a single four-component spinor

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{3.31}$$

transforming in the reducible representation $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \oplus (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$.

Since now we have both u_+ and u_- simultaneously at our disposal, the equations of motion for $u_\pm$, $i\sigma^\mu_\pm \partial_\mu u_\pm = 0$ can be modified, while keeping them linear, to introduce a mass term

$$\left.\begin{aligned} i\sigma^\mu_+ \partial_\mu u_+ &= m u_- \\ i\sigma^\mu_- \partial_\mu u_- &= m u_+ \end{aligned}\right\} \quad \Longrightarrow \quad i \begin{pmatrix} \sigma^\mu_+ & 0 \\ 0 & \sigma^\mu_- \end{pmatrix} \partial_\mu \psi = m \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \psi. \tag{3.32}$$

These equations of motion can be derived from the Lagrangian density

$$\mathscr{L}_{\text{Dirac}} = i\psi^\dagger \begin{pmatrix} \sigma^\mu_+ & 0 \\ 0 & \sigma^\mu_- \end{pmatrix} \partial_\mu \psi - m\psi^\dagger \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \psi. \tag{3.33}$$

To simplify the notation it is useful to define the Dirac γ-matrices as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_- \\ \sigma^\mu_+ & 0 \end{pmatrix}. \tag{3.34}$$

and the Dirac conjugate spinor $\overline{\psi}$

$$\overline{\psi} \equiv \psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{3.35}$$

The Lagrangian (3.33) can be written in the more compact form

$$\mathscr{L}_{\text{Dirac}} = \overline{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi, \tag{3.36}$$

whose equations of motion give the Dirac equation (1.10) with the identifications

$$\gamma^0 = \beta, \qquad \gamma^i = i\alpha^i. \tag{3.37}$$

The γ-matrices defined in (3.34) satisfy the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{3.38}$$

In d dimensions this algebra admits representations of dimension $2^{\left[\frac{d}{2}\right]}$. Equation (3.34) gives the chiral representation of the algebra (3.38). Other equivalent representations can be constructed exploiting the invariance of (3.38) under unitary transformations $\gamma^\mu \to U\gamma^\mu U^\dagger$.

A representation of the Lorentz algebra SO$(1, d-1)$ can be constructed using the γ-matrices as

$$\mathscr{J}^{\mu\nu} = -\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \equiv \sigma^{\mu\nu}. \tag{3.39}$$

By definition, Dirac fermions ψ in d dimensions transform under the Lorentz group in this representation.

When d is even the representation (3.39) is reducible. In the case of interest $d = 4$ this result is easy to prove by defining the chirality matrix

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{3.40}$$

The matrix γ_5 anticommutes with all other γ-matrices and as a consequence

$$\left[\gamma_5, \sigma^{\mu\nu}\right] = 0. \tag{3.41}$$

Using Schur's lemma (see Appendix B) this implies that the representation of the Lorentz group provided by $\sigma^{\mu\nu}$ is reducible into subspaces spanned by the eigenvectors of γ_5 with the same eigenvalue. Introducing the projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ these subspaces correspond to

$$P_+\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix}, \quad P_-\psi = \begin{pmatrix} 0 \\ u_- \end{pmatrix}, \tag{3.42}$$

which are precisely the Weyl spinors introduced above.

Our next task is to quantize the Dirac Lagrangian. This will be done along the lines followed for the free real scalar field, starting with a general solution to the Dirac equation and introducing the corresponding set of creation–annihilation operators. Therefore we start by looking for a complete basis of solutions to the Dirac equation. In the case of the scalar field the elements of the basis were labelled by their four-momentum k^μ. Now, however, the field has several components so we have to add an extra label. Equation (3.23) suggest the following definition of the helicity operator of a Dirac spinor

$$\lambda = \begin{pmatrix} \frac{1}{2}\boldsymbol{\sigma}\cdot\frac{\mathbf{k}}{|\mathbf{k}|} & 0 \\ 0 & \frac{1}{2}\boldsymbol{\sigma}\cdot\frac{\mathbf{k}}{|\mathbf{k}|} \end{pmatrix}. \tag{3.43}$$

Each element of the basis of functions is labelled by its four-momentum k^μ and the corresponding eigenvalue s of the helicity operator.

For positive energy solutions of the Dirac equation we take

$$u(k, s)e^{-ik\cdot x}, \qquad s = \pm\frac{1}{2}, \tag{3.44}$$

where $u_\alpha(k, s)$ $(\alpha = 1, \ldots, 4)$ is a four-component spinor. Substituting in the Dirac equation we obtain[2]

$$(\not{k} - m)u(k, s) = 0. \tag{3.45}$$

In the same way, for negative energy solutions we have

2 From now on we will frequently use the Feynman slash notation, $\not{a} \equiv \gamma^\mu a_\mu$.

$$v(k,s)e^{ik\cdot x}, \qquad s=\pm\frac{1}{2}, \tag{3.46}$$

where $v_\alpha(k,s)$ has to satisfy

$$(\not{k}+m)v(k,s)=0. \tag{3.47}$$

Multiplying Eqs. (3.45) and (3.47) on the left respectively by $(\not{k}\mp m)$ we find that the momentum is on the mass shell, $k^2=m^2$. Hence, the wave function for both positive- and negative-energy solutions is labelled by the three-momentum $\mathbf{k}$ of the particle, $u(\mathbf{k},s)$, $v(\mathbf{k},s)$.

Before proceeding any further we consider the case of a massless Dirac fermion. Using the equation $\not{k}u(\mathbf{k},s)=0$ it is not difficult to show that the helicity operator (3.43) satisfies

$$\lambda u(\mathbf{k},s)=\frac{1}{2}\gamma_5 u(\mathbf{k},s), \tag{3.48}$$

and similarly for $v(\mathbf{k},s)$. This means that when $m=0$ helicity (i.e., the projection of the spin along the direction of motion) and chirality (the eigenvalue of the γ_5 matrix) are equivalent concepts. In this case the helicity of the spinor is a relativistic invariant. This is no longer true when $m\neq 0$ because when the particle moves with a speed smaller than the speed of light the sign of λ can be changed by a boost reversing the direction of $\mathbf{k}$. Hence, the helicity of a massive Dirac spinor has no invariant meaning and moreover it is not equivalent to its chirality.

The spinors $u(\mathbf{k},s)$, $v(\mathbf{k},s)$ can be normalized according to

$$\begin{aligned}\overline{u}(\mathbf{k},s)u(\mathbf{k},s)&=2m,\\ \overline{v}(\mathbf{k},s)v(\mathbf{k},s)&=-2m.\end{aligned} \tag{3.49}$$

Given this normalization, the following identities can be obtained

$$\begin{aligned}\overline{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s)&=2k^\mu,\\ \overline{v}(\mathbf{k},s)\gamma^\mu v(\mathbf{k},s)&=2k^\mu,\end{aligned} \tag{3.50}$$

as well as the completeness relations

$$\begin{aligned}\sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\overline{u}_\beta(\mathbf{k},s)&=(\not{k}+m)_{\alpha\beta},\\ \sum_{s=\pm\frac{1}{2}} v_\alpha(\mathbf{k},s)\overline{v}_\beta(\mathbf{k},s)&=(\not{k}-m)_{\alpha\beta},\end{aligned} \tag{3.51}$$

with $k^0=E_{\mathbf{k}}=\sqrt{\mathbf{k}^2+m^2}$. A general solution to the Dirac equation including creation and annihilation operators can be written as

$$\hat{\psi}_\alpha(t, \mathbf{x}) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \Big[u_\alpha(\mathbf{k}, s)\hat{b}(\mathbf{k}, s) e^{-iE_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}} + v_\alpha(\mathbf{k}, s)\hat{d}^\dagger(\mathbf{k}, s) e^{iE_{\mathbf{k}}t - i\mathbf{k}\cdot\mathbf{x}} \Big]. \tag{3.52}$$

Unlike the real scalar field studied in the previous chapter, the Dirac field is not hermitian. As a consequence, the operators $\hat{b}(\mathbf{k}, s)$ and $\hat{d}(\mathbf{k}, s)$ are independent and not related by Hermitian conjugation.

Since we are dealing with half-integer spin fields, the spin-statistics theorem forces a modification of the canonical quantization prescription (2.57). In the case of the Dirac field the canonical Poisson brackets are replaced by *anticommutators*

$$i\{\cdot, \cdot\}_{\mathrm{PB}} \longrightarrow \{\cdot, \cdot\}. \tag{3.53}$$

Thus we arrive to the following canonical anticommutation relations for $\hat{\psi}(t, \mathbf{x})$

$$\{\hat{\psi}_\alpha(t, \mathbf{x}), \hat{\psi}^\dagger_\beta(t, \mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta}, \tag{3.54}$$

with the other anticommutators vanishing. From Eq. (3.52) we find that the operators $\hat{b}^\dagger(\mathbf{k}, s)$, $\hat{b}(\mathbf{k}, s)$ satisfy the algebra[3]

$$\begin{aligned} \{b(\mathbf{k}, s), b^\dagger(\mathbf{k}', s')\} &= (2\pi)^3(2E_{\mathbf{k}})\delta(\mathbf{k} - \mathbf{k}')\delta_{ss'}, \\ \{b(\mathbf{k}, s), b(\mathbf{k}', s')\} &= \{b^\dagger(\mathbf{k}, s), b^\dagger(\mathbf{k}', s')\} = 0. \end{aligned} \tag{3.55}$$

They respectively create and annihilate a spin-$\frac{1}{2}$ particle (for example, an electron) out of the vacuum with momentum $\mathbf{k}$ and helicity s.

In the case of $d(\mathbf{k}, s)$, $d^\dagger(\mathbf{k}, s)$, they satisfy the fermionic algebra

$$\begin{aligned} \{d(\mathbf{k}, s), d^\dagger(\mathbf{k}', s')\} &= (2\pi)^3(2E_{\mathbf{k}})\delta(\mathbf{k} - \mathbf{k}')\delta_{ss'}, \\ \{d(\mathbf{k}, s), d(\mathbf{k}', s')\} &= \{d^\dagger(\mathbf{k}, s), d^\dagger(\mathbf{k}', s')\} = 0. \end{aligned} \tag{3.56}$$

Hence we have a set of creation–annihilation operators for the corresponding antiparticles (for example positrons). This is clear if we notice that $d^\dagger(\mathbf{k}, s)$ can be seen as the annihilation operator of a negative energy state of the Dirac equation with wave function $v_a(\mathbf{k}, s)$. In the Dirac picture this corresponds to the creation of an antiparticle out of the vacuum (see Fig. 1.2). Finally, all other anticommutators between $b(\mathbf{k}, s)$, $b^\dagger(\mathbf{k}, s)$ and $d(\mathbf{k}, s)$, $d^\dagger(\mathbf{k}, s)$ vanish.

The Hamiltonian operator for the Dirac field is

$$\hat{H} = \frac{1}{2} \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3} \Big[b^\dagger(\mathbf{k}, s)b(\mathbf{k}, s) - d(\mathbf{k}, s)d^\dagger(\mathbf{k}, s) \Big]. \tag{3.57}$$

[3] To simplify notation, and since there is no risk of confusion, we drop from now on the hats to indicate operators.

At this point we realize again the necessity of quantizing the theory using anticommutators instead of commutators. Had we used canonical *commutation* relations, the second term inside the integral in (3.57) would give the number operator $d^\dagger(\mathbf{k}, s)d(\mathbf{k}, s)$ with a minus sign in front. As a consequence, the Hamiltonian would be unbounded from below and we would be facing again the instability of the theory already noticed in the context of relativistic quantum mechanics. However, using the *anticommutation* relations (3.56), the Hamiltonian (3.57) takes the form

$$\begin{aligned}\hat{H} = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\Big[E_{\mathbf{k}}b^\dagger(\mathbf{k}, s)b(\mathbf{k}, s) + E_{\mathbf{k}}d^\dagger(\mathbf{k}, s)d(\mathbf{k}, s)\Big] \\ - 2\int d^3k\, E_{\mathbf{k}}\delta(\mathbf{0}). \end{aligned} \tag{3.58}$$

As with the scalar field, we find a divergent vacuum energy contribution due to the zero-point energy of an infinite number of harmonic oscillators. Unlike the case of the scalar field, the vacuum energy here is negative. This is interesting because, as it will be explaned in Chap. 13, there is a certain type of theories called supersymmetric where the number of bosonic and fermionic degrees of freedom is the same. For this kind of theories the contribution of the vacuum energy of the bosonic field exactly cancels that of the fermions. The divergent contribution in the Hamiltonian (3.58) can be removed by the normal order prescription

$$:\hat{H}:= \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\Big[E_{\mathbf{k}}b^\dagger(\mathbf{k}, s)b(\mathbf{k}, s) + E_{\mathbf{k}}d^\dagger(\mathbf{k}, s)d(\mathbf{k}, s)\Big]. \tag{3.59}$$

Finally, let us mention that using the Dirac equation it is easy to prove the conservation of the four-current

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \quad \partial_\mu j^\mu = 0. \tag{3.60}$$

As we will explain further in Chap. 7, this current is associated to the invariance of the Dirac Lagrangian under the global phase shift $\psi \to e^{i\theta}\psi$. In electrodynamics the associated conserved charge

$$Q = q\int d^3x j^0 \tag{3.61}$$

is identified with the electric charge, with q the charge of the particle created by $b^\dagger(\mathbf{k}, s)$ acting on the vacuum.

Since we are dealing with a free theory, all correlation functions can be written in terms of those with two fields. The Feynman propagator is given by

$$\begin{aligned} S_{\alpha\beta}(x_1, x_2) &= \langle 0|T\Big[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\Big]|0\rangle \\ &= \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\alpha\beta} e^{-ip\cdot(x_1-x_2)}, \end{aligned} \tag{3.62}$$

while the other two-point correlation functions are zero

$$\langle 0|T\left[\psi_\alpha(x_1)\psi_\beta(x_2)\right]|0\rangle = \langle 0|T\left[\overline{\psi}_\alpha(x_1)\overline{\psi}_\beta(x_2)\right]|0\rangle = 0, \tag{3.63}$$

as can be seen by direct computation using the field expansion in terms of creation-annihilation operators. Due to the fermionic character of the Dirac field, the definition of the time-ordered product includes a number of minus signs associated with the permutation of the two fields. For the particular case of a Dirac spinor and its conjugate we have

$$T\Big[\psi_\alpha(x)\overline{\psi}_\beta(y)\Big] = \theta(x^0-y^0)\psi_\alpha(x)\overline{\psi}_\beta(y) - \theta(y^0-x^0)\overline{\psi}_\beta(y)\psi_\alpha(x). \tag{3.64}$$

The rule for higher order point functions is the same as in the bosonic case ("earlier" fields always to the right) apart from the fact that each term comes now multiplied by the sign needed to bring the original expression into the time order.

The computation of the vacuum expectation value of the time-ordered product of a number of ψ and $\overline{\psi}$ fields can be done using an extension of Wick's theorem introduced in Sect. 2.2 for a real scalar field. The main difference is that now the Wick contractions only occur between a Dirac field $\psi(x)$ and its conjugate $\overline{\psi}(x)$

$$\overbrace{\psi_\alpha(x_1)\overline{\psi}_\beta}(x_2) \longrightarrow S_{\alpha\beta}(x_1,x_2). \tag{3.65}$$

In addition, since the fields anticommute, there are extra signs associated with the permutations required to bring together in the correct order the fields that are Wick-contracted. The details can be found in the standard texts (see for example Ref. [1–15] in Chap. 1).

The Dirac field can also be quantized using the path integral formalism introduced in Chap. 2. The propagator (3.62) can be written as

$$S_{\alpha\beta}(x_1,x_2) = \frac{\displaystyle\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\,\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)e^{iS[\psi,\overline{\psi}]}}{\displaystyle\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, e^{iS[\psi,\overline{\psi}]}}. \tag{3.66}$$

This expression has, however, a very important difference with its bosonic counterpart shown in Eq. (2.89). Whereas in both cases all fields inside the path integral are functions and not operators, here ψ and $\overline{\psi}$ are *anticommuting* functions. This fact is crucial in performing the functional integration. Anticommuting objects have to be integrated using the so-called Berezin rules