Appendix A
Notation, Conventions and Units

For the benefit of the reader we summarize in this Appendix the main conventions used throughout the book.

A.1 Covariant Notation

We have used the "mostly minus" metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

Derivatives with respect to the four-vector $x^\mu = (ct, \mathbf{x})$ are denoted by the shorthand

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right).$$

Sporadically we have used the notation

$$f(x)\overleftrightarrow{\partial}_\mu g(x) = f(x)\partial_\mu g(x) - \partial_\mu f(x) g(x). \tag{A.2}$$

As usual space-time indices will be labelled by Greek letters ($\mu, \nu, \ldots = 0, 1, 2, 3$) while Latin indices will be used for spatial directions ($i, j, \ldots = 1, 2, 3$). We reserved α, β for Dirac and $a, b, c, \ldots$ for Weyl spinor indices.

The electromagnetic four-vector potential A^μ is defined in terms of the scalar φ and vector potential $\mathbf{A}$ by

$$A^\mu = (\varphi,\ \mathbf{A}). \tag{A.3}$$

L. Ávarez-Gaumé and M.Á. Vázquez-Mozo, *An Invitation to Quantum Field Theory*, Lecture Notes in Physics 839, DOI: 10.1007/978-3-642-23728-7,

The components of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and its dual $\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}F^{\sigma\lambda}$ are given respectively by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad \widetilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}, \tag{A.4}$$

with $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ the electric and magnetic fields. Similar expressions are valid in the nonabelian case.

A.2 Pauli and Dirac Matrices

We have used the notation $\sigma^\mu_\pm = (\mathbf{1}, \pm\sigma_i)$ where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.5}$$

They satisfy the identity

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + \varepsilon_{ijk}\sigma_k, \tag{A.6}$$

from where their commutator and anticommutator can be easily obtained.

Dirac matrices have always been used in the chiral representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_- \\ \sigma^\mu_+ & 0 \end{pmatrix}. \tag{A.7}$$

The chirality matrix is normalized as $\gamma_5^2 = \mathbf{1}$ and defined by $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. In many places we have used the Feynman's slash notation $\not{a} = \gamma^\mu a_\mu$.

A.3 Units

Unless stated otherwise, we work in natural units $\hbar = c = 1$. Electromagnetic Heaviside-Lorentz units have been used, where the Coulomb and Ampère laws take the form

$$\mathbf{F} = \frac{1}{4\pi}\frac{qq'}{r^3}\mathbf{r}, \quad \frac{dF}{d\ell} = \frac{1}{2\pi c^2}\frac{II'}{d}. \tag{A.8}$$

In these units the fine structure constant is

$$\alpha = \frac{e^2}{4\pi\hbar c}. \tag{A.9}$$

The electron charge in natural units is dimensionless and equal to $e \approx 0.303$.

Appendix B
A Crash Course in Group Theory

Group theory is one of the most useful mathematical tools in Physics in general and in quantum field theory in particular. To make the presentation self-contained we summarize in this Appendix some basic facts about group theory. Here we limit ourselves to the statement of basic results. Proofs and more detailed discussions can be found in the many books on the subject, such as the ones listed in Ref. [1, 2, 3, 4].

B.1 Generalities

Physical transformations have a number of interesting properties. To have an intuitive example in mind let us think of rotations in three-dimensional space. These transformations have interesting properties: if two rotations are performed in sequence the result is another one, and any rotation can be "undone".

Group theory is a way to translate these elementary properties of rotations or any other physical transformations into mathematical terms. A group G is a set of elements among which an operation $G \times G \to G$ is defined that associates to every ordered pair of elements (g_1, g_2) of the group another element, their product $g_1 g_2$. In order to be a group, the set G and the product operation have to satisfy a number of properties:

- The group product should be associative. This means that given three elements $g_1, g_2, g_3 \in G$ they satisfy $g_1(g_2 g_3) = (g_1 g_2) g_3$.
- G has a unit element $\mathbf{1}$ such that $g\mathbf{1} = \mathbf{1}g = g$ for every element g of the group.
- The group G contains together with every element $g \in G$ of the group its inverse, $g^{-1} \in G$, that satisfies the property $g^{-1}g = gg^{-1} = \mathbf{1}$.

In Physics one usually deals with group representations. These are realizations of abstract groups in terms of finite or infinite dimensional matrices. In more

L. Ávarez-Gaumé and M.Á. Vázquez-Mozo, *An Invitation to Quantum Field Theory*, Lecture Notes in Physics 839, DOI: 10.1007/978-3-642-23728-7,

technical terms, it can be said that a representation of a group G is a correspondence between its elements and the set of linear operators acting on a vector space V. This correspondence

$$D(g) : V \longrightarrow V \tag{B.1}$$

has to "mimic" the group product: given $g_1, g_2 \in G$

$$D(g_1)D(g_2) = D(g_1 g_2), \quad D(g_1^{-1}) = D(g_1)^{-1}. \tag{B.2}$$

A representation of a group is a set of operators acting of a certain vector space V. It might well happen that all these operators leave a proper subspace $U \subset V$ (i.e. $U \neq V$ and $U \neq \emptyset$) invariant, $D(g)U \subset U$ for any element $D(g)$ of the representation. When this happens it is said that the representation is reducible. A reducible representation can be decomposed into irreducible ones. These latter are the ones that satisfy that if $D(g)U \subset U$ for any element of the representations then either $U = \emptyset$ or $U = V$.

A very important result concerning irreducible representations is Schur's lemma: if $D(g)$ is a irreducible representation of a group G acting on a complex vector space V, and if there is an operator $A : V \to V$ that commutes with all the elements of this representation, then A must be proportional to the identity, $A = \lambda \mathbf{1}$. Here λ is some complex number.

Schur's lemma can be a useful tool in deciding whether a representation is reducible. If given a group representation we manage to find an operator that, commuting with all elements of such representations, is not proportional to the identity this automatically implies that the representation is reducible. This criterium was used in Chap. 3 to show that Dirac spinors transform in a reducible representation of the Lorentz group.

B.2 Lie Groups and Lie Algebras

Specially interesting for their applications in quantum field theory are the Lie groups whose elements are labelled by a number of continuous parameters. In mathematical terms this means that a Lie group G can be seen as a manifold where the parameters provide a set of (local) coordinates. The simplest example of a Lie group is SO(2), the group of rotations in the plane. Each element $R(\theta)$ is labelled by the rotation angle θ, with the multiplication acting as $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$. The angle θ is defined modulo 2π, therefore the manifold of SO(2) is a circumference S^1.

One of the interesting properties of Lie groups is that in a neighborhood of the identity any element can be expressed in terms of a set of generators T^A ($A = 1, \ldots, \dim G$) as

$$D(g) = \exp(-i\alpha^A T^A) \equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \alpha^{A_1} \ldots \alpha^{A_n} T^{A_1} \ldots T^{A_n}, \tag{B.3}$$

where $\alpha^A \in \mathbb{C}$ are a set of coordinates of G in a neighborhood of **1**. Using the general Baker-Campbell-Haussdorf formula (see for example [5], p. 81–82), the multiplication of two group elements is encoded in the value of the commutator of two generators, that in general has the form

$$[T^A, T^B] = if^{ABC}T^C, \tag{B.4}$$

where $f^{ABC} \in \mathbb{C}$ are called the structure constants. The generators can be normalized in such a way that f^{ABC} is completely antisymmetric in all its indices.

The set of generators T^A with the commutator operation (B.4) define the Lie algebra $\mathfrak{g}$ associated with the Lie group G. Hence, given a representation of the Lie algebra of generators we can construct a representation of the group by exponentiation (at least locally near the identity).

Besides their dimension, i.e. the number of generators, Lie algebras are characterized by their rank. This is defined as the maximal number of generators that commute among themselves. It is easy to see that those commuting generators form a subalgebra, called the Cartan subalgebra of the Lie algebra. The rank of a Lie algebra is therefore equal to the dimension of its Cartan subalgebra.

We illustrate these concepts with three particular examples of physical relevance.

U(1)

This is about the simplest Lie group one can imagine. Its Lie algebra consists of a single generator, T. Group elements can then be written as

$$U(\alpha) = e^{-i\alpha T}. \tag{B.5}$$

with α a real number. This group is abelian and all its irreducible representations are one-dimensional. This last result can be easily proved using Schur's lemma. This means that irreducible representations are of the form

$$D_q(\alpha) = e^{-iq\alpha}, \tag{B.6}$$

where q is a real number labeling the representation. This number is the analog of the electric charge for the U(1) gauge group of QED.

It is useful to make a distinction between noncompact and compact U(1) groups. The difference lies in the fact that in the first case α takes its values over the whole real line. For a compact U(1), on the other hand, the parameter α varies in a compact range. This latter case is realized when all irreducible representations of the group U(1) are characterized by values of q that are integer multiples of some real number q_0, i.e. $q = nq_0$ with $n \in \mathbb{Z}$. If this is the case one has

$$D_q\left(\alpha + \frac{2\pi}{q_0}\right) = D_q(\alpha) \tag{B.7}$$

for every q. This periodicity is not satisfied when the U(1) is noncompact, in which case q can take any real value.

SU(2)

The group SU(2) is well-known from the theory of angular momentum in quantum mechanics. Its Lie algebra has three generators $\{T^1, T^2, T^3\}$ that satisfy

$$[T^k, T^\ell] = i\varepsilon^{k\ell m} T^m. \tag{B.8}$$

The generators

$$T^{\pm} = \frac{1}{\sqrt{2}}(T^1 \pm iT^2), \quad T^3 \tag{B.9}$$

can alternatively be used to write the SU(2) Lie algebra as

$$[T^3, T^{\pm}] = \pm T^{\pm}, \quad [T^+, T^-] = T^3. \tag{B.10}$$

Either form of the algebra shows that no subset of generators is mutually commuting. Therefore the Cartan subalgebra of SU(2) can be taken to be made of a single generator that, by convention, we can take to be T^3.

Using (B.10), the irreducible representations of the Lie algebra of SU(2) can be constructed following the standard techniques familiar from quantum mechanics. They are characterized by their spin s, a nonnegative integer or half-integer, and have dimension $2s+1$. Here we focus on two basic representations. One is the fundamental two-dimensional representation with spin $s = \frac{1}{2}$. The generators can be written in terms of the Pauli matrices as

$$T^k = \frac{1}{2}\sigma_k, \quad k = 1, 2, 3, \tag{B.11}$$

whereas finite transformations in the connected component of the identity are

$$D_{\frac{1}{2}}(\alpha^k) = e^{-\frac{i}{2}\alpha^k \sigma_k}. \tag{B.12}$$

The second representation of SU(2) that we mention here is the three-dimensional adjoint (or spin 1) representation which can be written as

$$D_{\mathbf{1}}(\alpha^k) = e^{-i\alpha^k J^k}, \tag{B.13}$$

with the generators given by

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.14}$$

The J^k $(k = 1, 2, 3)$ generate rotations around the x, y and z axis respectively.

SU(3)

This group has eight generators and two basic three-dimensional irreducible representations, the fundamental and antifundamental denoted respectively by **3** and $\overline{\mathbf{3}}$. In QCD these representations are associated with the transformation of quarks and antiquarks under the color gauge symmetry SU(3). The elements of these representations can be written as

$$D_{\mathbf{3}}(\alpha^k) = e^{\frac{i}{2}\alpha^k\lambda_k}, \quad D_{\overline{\mathbf{3}}}(\alpha^a) = e^{-\frac{i}{2}\alpha^k\lambda_k^T} \quad (k = 1, \ldots, 8), \tag{B.15}$$

where λ_n are the eight hermitian Gell-Mann matrices

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.
\end{aligned} \tag{B.16}$$

Hence the generators of the representations $\mathbf{3}$ and $\overline{\mathbf{3}}$ are given by

$$T^k(\mathbf{3}) = \frac{1}{2}\lambda_k, \quad T^k(\overline{\mathbf{3}}) = -\frac{1}{2}\lambda_k^T. \tag{B.17}$$

The rank of SU(3) is 2, its Cartan subalgebra being generated by T^3 and T^8.

Given a representation *D(g)* of a group *G*, it is easy to see that the set of operators obtained by complex conjugation $D(g)^*$ are also a representation of the same group. In the case of a Lie group this is reflected in the fact that the generators $-(T^A)^*$ satisfy the Lie algebra relations (B.4) with the same structure constants. In fact, irreducible representations of a Lie algebra can be classified in three types, real, complex and pseudoreal, depending on whether $-(T^A)^*$ is or is not related to the original generators T^A by a similarity transformation:

- Real representations: a representation is said to be real if there is a *symmetric matrix S* which acts as intertwiner between the generators and their complex conjugates, namely

$$(T^A)^* = -ST^AS^{-1}, \quad S^T = S. \tag{B.18}$$

  This is the case of the adjoint representation of SU(2) generated by the matrices (B.14). In this example all the generators are real matrices and the intertwiner is just the identity.
- Pseudoreal representations: are the ones for which an *antisymmetric matrix S* exists with the property

$$(T^A)^* = -ST^AS^{-1}, \qquad S^T = -S. \tag{B.19}$$

As an example we can mention the spin-$\frac{1}{2}$ representation of SU(2) generated by $\frac{1}{2}\sigma_i$. The intertwiner is $S = -i\sigma_2$.

- Complex representations: finally, a representation is complex if the generators and their complex conjugates are not related by a similarity transformation. This is for instance the case of the two three-dimensional representations **3** and $\overline{\mathbf{3}}$ of SU(3).

B.3 Invariants

There are a number of invariants that can be constructed associated with an irreducible representation **R** of a Lie group G and that can be used to label such a representation. Let $T^A_{\mathbf{R}}$ be the generators in a certain representation **R** of the Lie algebra $\mathfrak{g}$. Using the antisymmetry of f^{ABC} it can be proved that the matrix $\sum_{A=1}^{\dim G} T^A_{\mathbf{R}} T^A_{\mathbf{R}}$ commutes with every generator $T^A_{\mathbf{R}}$. Therefore, according to Schur's lemma, it has to be proportional to the identity.[1] This defines the Casimir invariant $C_2(\mathbf{R})$ as

$$\sum_{A=1}^{\dim G} T^A_{\mathbf{R}} T^A_{\mathbf{R}} = C_2(\mathbf{R})\mathbf{1}. \tag{B.20}$$

A second invariant $T_2(\mathbf{R})$ associated with a representation **R** can also be defined by the identity

$$\mathrm{Tr}\, T^A_{\mathbf{R}} T^B_{\mathbf{R}} = T_2(\mathbf{R})\delta^{AB}. \tag{B.21}$$

Taking the trace in Eq. (B.20) and combining the result with (B.21) we find that both invariants are related by

$$C_2(\mathbf{R})\dim \mathbf{R} = T_2(\mathbf{R})\dim G, \tag{B.22}$$

with dim **R** the dimension of the representation **R**.

These two invariants appear frequently in quantum field theory calculations with nonabelian gauge fields. For example $T_2(\mathbf{R})$ comes about as the coefficient of the one-loop calculation of the beta-function for a Yang-Mills theory with gauge group G. In the case of SU(N), for the fundamental representation, we find the values

$$C_2(\mathbf{fund}) = \frac{N^2-1}{2N}, \quad T_2(\mathbf{fund}) = \frac{1}{2}, \tag{B.23}$$

---

[1] Schur's lemma also applies to the representations of a Lie algebra: if a representation is irreducible and there is a matrix of the same dimension as the representation that commutes with all the generators then this element has to be proportional to the identity.

whereas for the adjoint representation the results are

$$C_2(\mathbf{adj}) = N, \qquad T_2(\mathbf{adj}) = N. \tag{B.24}$$

A third invariant $A(\mathbf{R})$ is specially important in the calculation of anomalies. As discussed in Chap. 9, the chiral anomaly in gauge theories is proportional to the group-theoretical factor $\mathrm{Tr}\big[T^A_{\mathbf{R}}\{T^B_{\mathbf{R}}, T^C_{\mathbf{R}}\}\big]$. This leads us to define $A(\mathbf{R})$ as

$$\mathrm{Tr}\big[T^A_{\mathbf{R}}\{T^B_{\mathbf{R}}, T^C_{\mathbf{R}}\}\big] = A(\mathbf{R})d^{ABC}, \tag{B.25}$$

where d^{ABC} is symmetric in its three indices and does not depend on the representation. The cancellation of anomalies in a gauge theory with fermions transformed in the representation $\mathbf{R}$ of the gauge group is guaranteed if the corresponding invariant $A(\mathbf{R})$ vanishes.

It is not difficult to prove that $A(\mathbf{R}) = 0$ if the representation $\mathbf{R}$ is either real or pseudoreal. Indeed, if this is the case, then there is a matrix S (symmetric or antisymmetric) that intertwins the generators $T^A_{\mathbf{R}}$ and their complex conjugates $(T^A_{\mathbf{R}})^* = -ST^A_{\mathbf{R}}S^{-1}$. Then, using the hermiticity of the generators we can write

$$\mathrm{Tr}\Big[T^A_{\mathbf{R}}\Big\{T^B_{\mathbf{R}}, T^C_{\mathbf{R}}\Big\}\Big] = \mathrm{Tr}\Big[T^A_{\mathbf{R}}\Big\{T^B_{\mathbf{R}}, T^C_{\mathbf{R}}\Big\}\Big]^T = \mathrm{Tr}\Big[(T^A_{\mathbf{R}})^*\Big\{(T^B_{\mathbf{R}})^*, (T^C_{\mathbf{R}})^*\Big\}\Big]. \tag{B.26}$$

Now, using (B.18) or (B.19) we have

$$\begin{aligned}\mathrm{Tr}\Big[(T^A_{\mathbf{R}})^*\Big\{(T^B_{\mathbf{R}})^*, (T^C_{\mathbf{R}})^*\Big\}\Big] &= -\,\mathrm{Tr}\Big[ST^A_{\mathbf{R}}S^{-1}\Big\{ST^B_{\mathbf{R}}S^{-1}, ST^C_{\mathbf{R}}S^{-1}\Big\}\Big] \\ &= -\,\mathrm{Tr}\Big[T^A_{\mathbf{R}}\Big\{T^B_{\mathbf{R}}, T^C_{\mathbf{R}}\Big\}\Big],\end{aligned} \tag{B.27}$$

which proves that $\mathrm{Tr}\big[T^A_{\mathbf{R}}\{T^B_{\mathbf{R}}, T^C_{\mathbf{R}}\}\big] = 0$ and therefore $A(\mathbf{R}) = 0$ whenever the representation is real or pseudoreal. Since the gauge anomaly in four dimensions is proportional to $A(\mathbf{R})$, anomalies appear only when the fermions transform in a complex representation of the gauge group.

B.4 A Look at the Lorentz and Poincaré Groups

Finally, we close this Appendix with the review of some features of the Lorentz group used at several places in this book. We avoid getting into detailed proofs. They can be found in a number of textbooks (for example [6, 7]), as well as in reference [6] of Chap. 11.

The Lorentz Group

The Lorentz group SO(1,3) is defined as the group of space-time transformations that preserve the Minkowski metric, that is

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \quad \text{such that} \quad \eta_{\mu\nu}\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\lambda} = \eta_{\sigma\lambda}. \tag{B.28}$$

From its very definition we find that $\Lambda^{\mu}{}_{\nu}$ satisfies $\det\Lambda = \pm 1$ and

$$(\Lambda^{0}{}_{0})^{2} - \sum_{i=1}^{3}(\Lambda^{i}{}_{0})^{2} = 1, \tag{B.29}$$

which follows from the 00 component of the second equation in (B.28). From Eq. (B.29) we find $(\Lambda^{0}{}_{0})^{2} \geq 1$ and the Lorentz group can be split into the following four disconnected components

- $\mathfrak{L}_{+}^{\uparrow}$: proper, orthochronous transformations with $\det\Lambda = 1$, $\Lambda^{0}{}_{0} \geq 1$.
- $\mathfrak{L}_{-}^{\uparrow}$: improper, orthochronous transformations with $\det\Lambda = -1$, $\Lambda^{0}{}_{0} \geq 1$.
- $\mathfrak{L}_{-}^{\downarrow}$: improper, non-orthochronous transformations with $\det\Lambda = -1$, $\Lambda^{0}{}_{0} \leq -1$.
- $\mathfrak{L}_{+}^{\downarrow}$: proper, non-orthochronous transformations with $\det\Lambda = 1$, $\Lambda^{0}{}_{0} \leq -1$.

The term (non)-orthochronous refers to whether the Lorentz transformation preserves or not the direction of time. Notice that the identity is included in $\mathfrak{L}_{+}^{\uparrow}$ and therefore this is the only branch of the Lorentz group that forms a subgroup. The other three branches are connected to the orthochronous, proper Lorentz subgroup by parity and time reversal in the following way (see Chap. 11)

$$\mathfrak{L}_{+}^{\uparrow} \xrightarrow{\mathscr{P}} \mathfrak{L}_{-}^{\uparrow}, \quad \mathfrak{L}_{+}^{\uparrow} \xrightarrow{\mathscr{T}} \mathfrak{L}_{-}^{\downarrow}, \quad \mathfrak{L}_{+}^{\uparrow} \xrightarrow{\mathscr{P}\mathscr{T}} \mathfrak{L}_{+}^{\downarrow}. \tag{B.30}$$

We focus then on $\mathfrak{L}_{+}^{\uparrow}$. We are going to see that transformations in this subgroup can be written in terms of complex 2×2 matrices of unit determinant. We consider a four-vector V^{μ} and construct the Hermitian matrix

$$V = V^{0}\mathbf{1} + \sum_{i=1}^{3} V^{i}\sigma_{i} = \begin{pmatrix} V^{0} + V^{3} & V^{1} - iV^{2} \\ V^{1} + iV^{2} & V^{0} - V^{3} \end{pmatrix}. \tag{B.31}$$

This defines a one-to-one correspondence between four-vectors and Hermitian matrices, whose determinant gives the norm of the vector

$$\det V = \eta_{\mu\nu}V^{\mu}V^{\nu}. \tag{B.32}$$

Now, the determinant is preserved by any SL(2, $\mathbb{C}$) transformation acting as

$$V \longrightarrow AVA^{\dagger}, \quad \det A = 1. \tag{B.33}$$

Since the transformed matrix is also Hermitian, it defines a transformed four-vector V'^{μ} with the same norm. This means that the linear map (B.33) has to act on the components V^{μ} as a Lorentz transformation

$$V^{\mu} \longrightarrow V'_{\mu} = \Lambda^{\mu}{}_{\nu}(A)V^{\nu}. \tag{B.34}$$

That this Lorentz transformation belongs to $\mathfrak{L}_{+}^{\uparrow}$ can be seen as follows: the group SL(2, $\mathbb{C}$) is simply connected and the relation between SL(2,$\mathbb{C}$) and Lorentz transformations continuous. Since it includes the identity, the Lorentz transformation $\Lambda^{\mu}{}_{\nu}(A)$ has to lie in the connected component of the identity, i.e. $\mathfrak{L}_{+}^{\uparrow}$.

The correspondence between $\mathfrak{L}_{+}^{\uparrow}$ and SL(2, $\mathbb{C}$) is in fact two-to-one. This is obvious if we take into account that A and $-A$ define the same Lorentz transformation. This is why SL(2, $\mathbb{C}$) is said to be the double covering of the proper, orthochronous Lorentz group.

The relation between the Lorentz group and SL(2, $\mathbb{C}$) is very important for the definition of spinors. An undotted spinor is a two-component complex object ξ_a (with $a = 1, 2$) that under the Lorentz group transforms as

$$x^{\mu} \longrightarrow \Lambda^{\mu}{}_{\nu}(A)x^{\nu}, \quad \xi_a \longrightarrow A_a{}^b \xi_b. \tag{B.35}$$

Since the spinor ξ_a is a complex objects, its conjugate does not transform with the matrix A but with its complex conjugate A^*. Such objects are called dotted spinor. More precisely, they are two-component complex quantities $\eta_{\dot{a}}$ (with $\dot{a} = \dot{1}, \dot{2}$) that under $\mathfrak{L}_{+}^{\uparrow}$ transforms with the complex conjugate representation, namely

$$x^{\mu} \longrightarrow \Lambda^{\mu}{}_{\nu}(A)x^{\nu}, \quad \eta_{\dot{a}} \longrightarrow (A^*)_{\dot{a}}{}^{\dot{b}} \eta_{\dot{b}}. \tag{B.36}$$

Spinors with upper undotted and dotted indices are defined as objects transforming in the representations $(A^T)^{-1}$ and $(A^{\dagger})^{-1}$ respectively. In fact, these representations are equivalent to A and A^*, as can be seen from the identity

$$(A^T)^{-1} = \varepsilon A \varepsilon^{-1} \quad \text{where} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{B.37}$$

valid for any $A \in SL(2, \mathbb{C})$. This means that indices can be raised and lowered by contraction with ε^{ab}, $\varepsilon^{\dot{a}\dot{b}}$, ε_{ab} and $\varepsilon_{\dot{a}\dot{b}}$.

Bearing in mind the previous discussion and comparing with (B.33), we see that the matrix V associated with a Lorentz four-vector has an undotted and a dotted index, $V_{a\dot{b}}$. To connect with the SU(2)×SU(2) label of the representations of the Lorentz group introduced in Chap. 3, we notice that undotted spinors correspond to Weyl spinors in the representation $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$. The element of SL(2, $\mathbb{C}$) associated with a Lorentz transformation characterized by a rotation $\theta\mathbf{n}$ and a boost $\boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3)$ can be read from (3.14) to be

$$A = e^{-\frac{i}{2}(\theta\mathbf{n} - i\boldsymbol{\beta})\cdot\sigma}. \tag{B.38}$$

From the same equation we see that a spinor u_- in the representation $\left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ transforms with $(A^{\dagger})^{-1}$ and therefore has an upper dotted spinor index. Thus, in the chiral representation of the γ-matrices, a Dirac spinor can be decomposed in dotted and undotted components as

$$\psi = \begin{pmatrix} \xi_a \\ \eta^{\dot{a}} \end{pmatrix}. \tag{B.39}$$

Since all other representations of the Lorentz group can be obtained by decomposing products of the two fundamental representations $\left(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0}\right)$ and $\left(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}}\right)$, any quantity transforming in a irreducible representation of the Lorentz group can be written as a mixed tensor

$$\Phi_{a_1\ldots a_n\dot{b}_1\ldots\dot{b}_m}, \tag{B.40}$$

where all undotted and dotted indices have to be symmetric among themselves.[2] They transform as

$$\Phi'_{a_1\ldots a_n\dot{b}_1\ldots\dot{b}_m} = A_{a_1}{}^{c_1}\ldots A_{a_n}{}^{c_n}(A^*)_{\dot{b}_1}{}^{\dot{d}_1}\ldots(A^*)_{\dot{b}_1}{}^{\dot{d}_1}\Phi_{c_1\ldots c_n\dot{d}_1\ldots\dot{d}_m}, \tag{B.41}$$

that in the language of SU(2) representations corresponds to $(\mathbf{s_1},\mathbf{s_2}) = \left(\frac{n}{2},\frac{m}{2}\right)$.

For some technical issues, such as the proof of the CPT theorem outlined in Sect. 11.6, it is necessary to study the complexification of the Lorentz group. This is defined again as in (B.28) but with $\Lambda^{\mu}{}_{\nu}$ complex. The only condition that follows from this equation now is that $\det\Lambda = \pm 1$. Therefore, unlike its real analog, the complexified Lorentz group has two connected components $\mathfrak{L}_{\pm}(\mathbb{C})$ labelled by the sign of the determinant.

Since now coordinates and four-vectors are complex as well, the matrix (B.31) associated to V^{μ} is not Hermitian. This means that

$$V \longrightarrow AVB^T, \qquad A,B \in SL(2,\mathbb{C}), \tag{B.42}$$

defines a complex Lorentz transformation

$$V^{\mu} \longrightarrow V'^{\mu} = \Lambda^{\mu}{}_{\nu}(A,B)V^{\nu}. \tag{B.43}$$

Undotted and dotted spinors transform under $\mathfrak{L}_{+}(\mathbb{C})$ with the matrices A and B belonging to the two factors of SL(2, $\mathbb{C}$)×SL(2, $\mathbb{C}$). For a general tensor (B.40) the transformation is

$$\Phi'_{a_1\ldots a_n\dot{b}_1\ldots\dot{b}_m} = A_{a_1}{}^{\dot{c}_1}\ldots A_{a_n}{}^{\dot{c}_n}B_{\dot{b}_1}{}^{\dot{d}_1}\ldots B_{\dot{b}_1}{}^{\dot{d}_1}\Phi_{c_1\ldots c_n\dot{d}_1\ldots\dot{d}_m}. \tag{B.44}$$

Using the same continuity arguments as for the real Lorentz group, we conclude that the correspondence $(A,B)\to\Lambda(A,B)$ defines a two-to-one isomorphism between SL(2, $\mathbb{C}$)×SL(2, $\mathbb{C}$) and the proper complex Lorentz group $\mathfrak{L}_{+}(\mathbb{C})$. The elements (A,B) and $(-A,-B)$ correspond to the same complex Lorentz transformation. An important thing achieved by the complexification of the Lorentz group is that now the space-time inversion $\mathscr{PT}: x^{\mu}\to -x^{\mu}$ lies in the connected component of the identity $\mathfrak{L}_{+}(\mathbb{C})$. This transformation acts on tensors by multiplying it by -1 for each dotted spinor,

[2] Notice that if the quantity were antisymmetric in a pair of dotted or undotted antisymmetric indices these can be eliminated using either ε_{ab} or $\varepsilon_{\dot{a}\dot{b}}$. For example, if $\Phi_{ab\cdots} = -\Phi_{ba\cdots}$ we can write $\Phi_{ab\cdots} = \varepsilon_{ab}\Phi_{\cdots}$ where $\Phi_{\cdots} = \frac{1}{2}\varepsilon^{cd}\Phi_{cd\cdots}$.

$$\mathscr{P}\mathscr{T} : \Phi_{a_1 \ldots a_n \dot{b}_1 \ldots \dot{b}_m} \longrightarrow (-1)^m \Phi_{a_1 \ldots a_n \dot{b}_1 \ldots \dot{b}_m}. \tag{B.45}$$

The Poincaré Group

The Poincaré group $\mathfrak{P}$ is the Lorentz group supplemented by space-time translations

$$\mathfrak{P} : x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{B.46}$$

The group has ten generators: six of the Lorentz group, $\mathscr{J}_{\mu\nu}$, plus the four of space-time translation, P^μ. In addition to (3.5) its Lie algebra contains the commutators

$$[P_\mu, P_\nu] = 0, \quad [\mathscr{J}_{\mu\nu}, P_\sigma] = i\eta_{\mu\sigma} P_\nu - i\eta_{\nu\sigma} P_\mu. \tag{B.47}$$

Each element of the Poincaré group is labelled by a Lorentz transformation and a four-vector. The restriction of the Lorentz transformations to the proper subgroup $\mathfrak{L}_+^\uparrow \approx SL(2, \mathbb{C})$ defines the proper Poincaré subgroup $\mathfrak{P}_+$.

The unitary irreducible representations of the Poincaré group are labelled by two Casimir operators. The first one is constructed from the generator of translations as

$$M^2 = P_\mu P^\mu. \tag{B.48}$$

The second one is defined by

$$W^2 = W_\mu W^\mu, \tag{B.49}$$

where W^μ is the Pauli-Lubański vector

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\sigma\lambda} J_{\nu\sigma} P_\lambda. \tag{B.50}$$

The representations are classified according to the sign of M in the following three classes:

- *Timelike or massive representations* ($M^2 > 0$). The representation acts on a linear space whose basis we take to be eigenstates of the translation operator P^μ with eigenvalue p^μ. Since $p_\mu p^\mu = M^2 > 0$, we can choose a reference frame where the eigenvalue takes the form $p^\mu = (M, \mathbf{0})$. Then, the Pauli-Lubański vector acting on these states has the form $W^\mu = (0, M\mathbf{J})$, with $\mathbf{J}$ the generator of spatial rotations. The rotation group generated by $\mathbf{J}$ defines the *little group*, i.e. the group preserving the form of the eigenvalue p^μ. The Casimir operator is easily computed to be

$$W^2 = -M^2 s(s+1), \tag{B.51}$$

  where s is the spin that takes positive integer or half-integer values. Notice that the second Casimir operator W^2 is a Lorentz scalar and therefore its value is independent of the particular system of coordinates used.

- *Light-like or massless representations* ($M^2 = 0$). We work again in a basis of eigenstates of P^μ. Since the eigenvalues satisfy $p^\mu p_\mu = 0$ the wise choice of reference frame is one where $p^\mu = (M, 0, 0, M)$. It takes a little bit of algebra to check that the transformations preserving this vector are generated by J_3, $K_1 + J_2$ and $K_2 - J_1$. Working out their commutation relations we find that they generate the two-dimensional euclidean group ISO(2) of rotations and translations in a plane. Its unitary finite dimensional representations are one-dimensional and labelled by the eigenvalue of J_3, the helicity, that takes values $\lambda = 0, \pm 1/2, \ldots$ If we want the representation to preserve CPT, we need to include together the positive and negative eigenvalues of J_3. Therefore, the representation associated with a massless particle contains the helicities λ and $-\lambda$. This is the reason why photons or other massless particles come only in two helicity states.
- *Space-like or tachyonic representations* ($M^2 < 0$). There are no known particles transforming under this class of representations. Therefore we will not elaborate on them.

Unitary irreducible representations of the Poincaré group are determined by the eigenvalue of P^2 and the irreducible representation of the corresponding little group [i.e., SO(3) $\approx$ SU(2) for massive and ISO(2) for massless representations]. What we usually call a particle is a state that transforms in one of these irreducible representations, which comes labelled by its mass and spin/helicity.

References

1. Sattinger, D.H., Weaver, O.L.: Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics. Springer (1986)
2. Georgi, H.: Lie Algebras in Particle Physics, 2nd edn. Perseus Books (1999)
3. Hall, B.C.: Lie Groups, Lie Algebras, and their representations. Springer (2003)
4. Ramond, P.: Group Theory. A Physicist's Survey. Cambridge (2010)
5. Galindo, A., Pascual, P.: Quantum Mechanics I. Springer (1990)
6. Cornwell, J.F.: Group Theory in Physics, vol. II. Academic (1984)
7. Tung, W.-K.: Group Theory in Physics. World Scientific (1985)

Index

L. Ávarez-Gaumé and M.Á. Vázquez-Mozo, *An Invitation to Quantum Field Theory*,
Lecture Notes in Physics 839, DOI: 10.1007/978-3-642-23728-7,