Chapter 13
Special Topics

In this closing chapter we have decided to present a few special topics in quantum field theory that have applications in cosmology and particle phenomenology. Given that we have not covered so many things in this book, the number of subjects to choose from is vast. Our choice is to study the creation of particles by classical external sources, including the Schwinger effect: the creation of electron–positron pair in a strong electric field; and then to explore the general properties of supersymmetric theories. Currently a large fraction of theories beyond the standard model are based on various supersymmetric completions. The reader will find the most rudimentary properties of such theories and the basic representation of this new symmetry.

13.1 Creation of Particles by Classical Fields

Particle Creation by a Classical Source

In a free quantum field theory the total number of particles is a conserved quantity. For example, in the case of the quantum scalar field studied in Chap. 2 we have that the number operator commutes with the Hamiltonian

$$\hat{n} \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}), \quad [\hat{H},\hat{n}] = 0. \tag{13.1}$$

This means that any states with a well-defined number of particle excitations will preserve this number at all times. The situation, however, changes as soon as interactions are introduced, since in this case particles can be created and/or destroyed as a result of the dynamics.

Another case in which the number of particles might change is if the quantum theory is coupled to a classical source. The archetypical example of such a situation is the Schwinger effect, in which a classical strong electric field produces the creation of electron–positron pairs out of the vacuum. However, before plunging into this more

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involved situation we can illustrate the relevant physics involved in the creation of particles by classical sources with the help of the simplest example: a free scalar field theory coupled to a classical external source $J(x)$. The action for such a theory can be written as

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2 + J(x)\phi(x) \right], \tag{13.2}$$

where $J(x)$ is a real function of the coordinates. Its identification with a classical source is obvious once we calculate the equations of motion

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi(x) = J(x). \tag{13.3}$$

Our plan is to quantize this theory but, unlike the case analyzed in Chap. 2, now the presence of the source $J(x)$ makes the situation a bit more involved. The general solution to the equation of motion can be written in terms of the retarded Green function for the Klein–Gordon equation as

$$\phi(x) = \phi_0(x) + i\int d^4x' G_R(x-x')J(x'), \tag{13.4}$$

where $\phi_0(x)$ is a general solution to the homogeneous equation and

$$\begin{aligned} G_R(t,\mathbf{x}) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon\,\mathrm{sign}(k^0)}e^{-ik\cdot x} \\ &= i\theta(t)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\left(e^{-iE_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x}} - e^{iE_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}\right), \end{aligned} \tag{13.5}$$

with $\theta(x)$ the Heaviside step function. The denominator in the first integral is reminding us that the integration contour over k^0 surrounds the poles at $k^0 = \pm E_{\mathbf{k}}$ from above. Since $G_R(t,\mathbf{x}) = 0$ for $t < 0$, the function $\phi_0(x)$ corresponds to the solution of the field equation at $t \to -\infty$, before the interaction with the external source.[1]

To make the argument simpler we assume that $J(x)$ is switched on at $t = 0$, and only lasts for a time τ, that is

$$J(t,\mathbf{x}) = 0 \quad \text{if } t < 0 \text{ or } t > \tau. \tag{13.6}$$

We are interested in a solution of (13.3) for times after the external source has been switched off, $t > \tau$. In this case the expression (13.5) can be written in terms of the Fourier modes $\tilde{J}(E,\vec{k})$ of the source as

$$\begin{aligned} \phi(t,\mathbf{x}) = \phi_0(t,\mathbf{x}) + i\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\Big[&\tilde{J}(E_{\mathbf{k}},\mathbf{k})e^{-iE_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} \\ &- \tilde{J}(E_{\mathbf{k}},\mathbf{k})^* e^{iE_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}\Big]. \end{aligned} \tag{13.7}$$

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[1] We could have taken instead the advanced propagator $G_A(x)$ in which case $\phi_0(x)$ would correspond to the solution to the equation at large times, after the interaction with $J(x)$.

The general solution $\phi_0(t, \mathbf{x})$ has been already computed in Eq. (2.55). Combining this result with Eq. (13.7) we find the following expression for the late time general solution to the Klein–Gordon equation in the presence of the source

$$\phi(t,x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\left\{\left[\alpha(\mathbf{k}) + i\tilde{J}(E_{\mathbf{k}},\mathbf{k})\right]e^{-iE_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} + \left[\alpha^*(\mathbf{k}) - i\tilde{J}(E_{\mathbf{k}},\mathbf{k})^*\right]e^{iE_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}\right\}. \tag{13.8}$$

On the other hand, for $t < 0$ we find from Eqs. (13.4) and (13.5) that the general solution is given by Eq. (2.55).

Now we can proceed to quantize the theory. The conjugate momentum $\pi(x) = \partial_0\phi(x)$ can be computed from Eqs. (2.55) and (13.8). Imposing the canonical equal time commutation relations (2.52) we find that $\alpha(\mathbf{k})$, $\alpha^\dagger(\mathbf{k})$ satisfy the creation-annihilation algebra (2.29). From our previous calculation we find that for $t > \tau$ the expansion of the operator $\phi(x)$ in terms of the creation-annihilation operators $\alpha(\mathbf{k})$, $\alpha^\dagger(\mathbf{k})$ can be obtained from the one for $t < 0$ by the replacement

$$\begin{aligned}\alpha(\mathbf{k}) &\longrightarrow \beta(\mathbf{k}) \equiv \alpha(\mathbf{k}) + i\tilde{J}(E_{\mathbf{k}},\mathbf{k}),\\ \alpha^\dagger(\mathbf{k}) &\longrightarrow \beta^\dagger(\mathbf{k}) \equiv \alpha^\dagger(\mathbf{k}) - i\tilde{J}(E_{\mathbf{k}},\mathbf{k})^*.\end{aligned} \tag{13.9}$$

Since $\tilde{J}(E_{\mathbf{k}},\mathbf{k})$ is a c-number, the operators $\beta(\mathbf{k})$, $\beta^\dagger(\mathbf{k})$ satisfy the same algebra as $\alpha(\mathbf{k})$, $\alpha^\dagger(\mathbf{k})$ and therefore can be interpreted as well as a set of creation-annihilation operators. This means that we can define two vacuum states, $|0_-\rangle$, $|0_+\rangle$ associated with both sets of operators

$$\left.\begin{aligned}\alpha(\mathbf{k})|0_-\rangle &= 0\\ \beta(\mathbf{k})|0_+\rangle &= 0\end{aligned}\right\} \quad \text{for all } \mathbf{k}. \tag{13.10}$$

For an observer at $t < 0$, $\alpha(\mathbf{k})$ and $\alpha^\dagger(\mathbf{k})$ are the natural set of creation-annihilation operators in terms of which to expand the field operator $\phi(x)$. After the usual zero-point energy subtraction the Hamiltonian is given by [cf. 2.59]

$$\hat{H}^{(-)} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}) \tag{13.11}$$

and the ground state of the spectrum for this observer is the vacuum $|0_-\rangle$. At the same time, a second observer at $t > \tau$ will also see a free scalar quantum field (the source has been switched off at $t = \tau$), and consequently will expand ϕ in terms of the second set of creation-annihilation operators $\beta(\mathbf{k})$, $\beta^\dagger(\mathbf{k})$. In terms of these operators the Hamiltonian is written as

$$\hat{H}^{(+)} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\beta^\dagger(\mathbf{k})\beta(\mathbf{k}). \tag{13.12}$$

Then for this late-time observer the ground state of the Hamiltonian is the second vacuum state $|0_+\rangle$.

In our analysis we have been working in the Heisenberg picture, where states are time-independent and the time dependence is in the operators. This means that the states of the theory and defined globally in time. Suppose now that the system is in the "in" ground state $|0_-\rangle$. An observer at $t < 0$ will find that there are no particles

$$\hat{n}^{(-)}|0_-\rangle = 0. \tag{13.13}$$

However the late-time observer will find that the state $|0_-\rangle$ contains an average number of particles given by

$$\langle 0_-|\hat{n}^{(+)}|0_-\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\left|\tilde{J}(E_{\mathbf{k}},\mathbf{k})\right|^2. \tag{13.14}$$

Moreover, $|0_-\rangle$ is no longer the ground state for the "out" observer. On the contrary, this state has a vacuum expectation value for $\hat{H}^{(+)}$

$$\langle 0_-|\hat{H}^{(+)}|0_-\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left|\tilde{J}(E_{\mathbf{k}},\mathbf{k})\right|^2. \tag{13.15}$$

The key to understand what is going on here lies in the fact that the external source breaks the invariance of the theory under space-time translations. In the particular case we have studied here where $J(x)$ has support on a finite time interval $0 < t < \tau$, this implies that the vacuum is not invariant under time translations, so observers at different times will make different choices of vacua that will not necessarily agree with each other. This is clear in our example. An observer in $t < 0$ will choose the vacuum to be the lowest energy state of her Hamiltonian, $|0_-\rangle$. On the other hand, the second observer at late times $t > \tau$ will naturally choose $|0_+\rangle$ as the vacuum. For this second observer, the state $|0_-\rangle$ is not the vacuum of his Hamiltonian, but an excited state that is a superposition of states with well-defined number of particles. In this sense it can be said that the external source has the effect of creating particles out of the "in" vacuum. Besides, this breaking of time translation invariance produces a violation in the energy conservation as we see from Eq. (13.15). Particles are created from the energy pumped into the system by the external source.

The Schwinger Effect

A typical example of creation of particles by external fields is the Schwinger effect [1] consisting in the creation of electron–positron pairs by a strong electric field. To illustrate the main physical features of this effect we use a heuristic argument based on the Dirac sea picture and the WKB approximation.

In the absence of an electric field the vacuum state of a spin-$\frac{1}{2}$ field is constructed by filling all the negative energy states as depicted in Fig. 1.2. We switch on a constant electric field $\mathscr{E}\mathbf{u}_x$ in the range $0 < x < L$. The associated electrostatic potential is taken to be

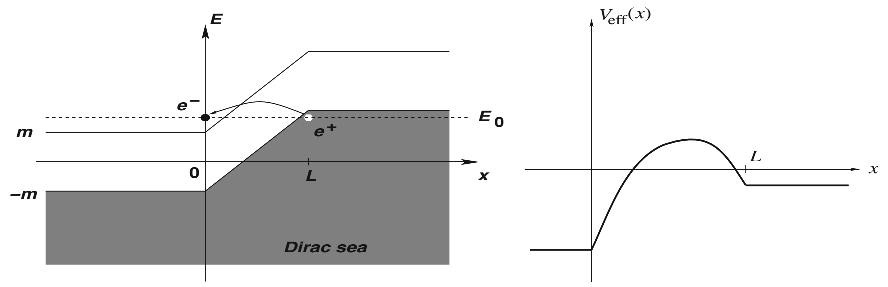

Fig. 13.1 Left: pair creation by a electric field in the Dirac sea picture. Right: effective potential felt by the electron in the x direction. The pair creation corresponds to the tunneling of the particle from the right to the left of the potential bump

$$V(\mathbf{r}) = \begin{cases} 0 & x < 0 \\ -\mathscr{E}x & 0 < x < L \\ -\mathscr{E}L & x > L \end{cases}. \tag{13.16}$$

The Dirac sea is deformed into the shape shown in the left panel of Fig. 13.1 (in drawing this figure we have to bear in mind that electrons have negative electric charge $q = -e$). When the electric field satisfies $e\mathscr{E}L > 2m$ there are states in the Dirac sea with $x > L$ having the same energy as some positive energy states in the region $x < 0$. It is therefore possible for a Dirac sea electron with energy $m \lesssim E_0 \lesssim e\mathscr{E}L - m$ to tunnel through the classically forbidden region leaving a hole behind. The physical interpretation of such process is the production of an electron–positron pair out of the vacuum by the effect of the electric field.

We can make this heuristic picture more precise with a simplified model where electrons are described by a single component wave function $\Psi(x)$ satisfying the equation[2]

$$\left\{\left[i\frac{\partial}{\partial t} + eV(x)\right]^2 + \frac{\partial^2}{\partial x^2} + \nabla_T^2 - m^2\right\}\Psi(t, x, \mathbf{x}_T) = 0. \tag{13.17}$$

This is obtained from the dispersion relation

$$(E + eV)^2 - \mathbf{p}^2 - m^2 = 0 \tag{13.18}$$

using the correspondence principle (1.2). Since the potential only depends on the coordinate x, we have separated it from the transverse coordinates denoted by $\mathbf{x}_T$. This also suggests the following ansatz for the single-particle wave function

$$\Psi(t, x, \mathbf{x}_T) = f(x)e^{-iE_0 t + i\mathbf{p}_T\cdot\mathbf{x}_T}. \tag{13.19}$$

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[2] Our analysis essentially ignores the effect of the spin of the electron, the two helicities being treated as scalar fields. A more careful treatment of the problem using the Dirac equation can be found in [2].

Substituting this expression in (13.17) results in the Schrödinger-like equation

$$-\frac{1}{2}f''(x) + V_{\text{eff}}(x)f(x) = 0, \tag{13.20}$$

where the effective potential $V_{\text{eff}}(x)$ is given by

$$V_{\text{eff}}(x) = \frac{1}{2}\Big\{\mathbf{p}_T^2 + m^2 - [E_0 + eV(x)]^2\Big\}. \tag{13.21}$$

This effective potential has two flat regions ($x < 0$ and $x > L$) joined by an inverted parabola, as shown in the right panel of Fig. 13.1. In the language of the Schrödinger equation (13.20) the production of particle pairs corresponds to the tunneling of an "analogue particle" of unit mass and zero energy through this potential bump. To solve the problem semiclassically we compute the classical turning points

$$x_\pm = \frac{1}{e\mathscr{E}}\Big(E_0 \pm \sqrt{\mathbf{p}_T^2 + m^2}\Big), \tag{13.22}$$

in terms of which the WKB transmission coefficient is given by

$$\begin{aligned} T_{\text{WKB}} &= \exp\left(-2\int_{x_-}^{x_+} dx\sqrt{2|V_{\text{eff}}(x)|}\right) \\ &= \exp\left[-2\int_{x_-}^{x_+} dx\sqrt{m^2 + \mathbf{p}_T^2 - (E_0 - e\mathscr{E}x)^2}\right]. \end{aligned} \tag{13.23}$$

The calculation of the integral yields the result

$$T_{\text{WKB}} = e^{-\frac{\pi}{e\mathscr{E}}(\mathbf{p}_T^2 + m^2)}. \tag{13.24}$$

Integrating the transmission coefficient over transverse momenta gives the number of pairs produced per unit time and unit transverse volume with energies between E_0 and $E_0 + dE$

$$\begin{aligned} \frac{dN}{dt\, d^2x_T} &= 2e^{-\frac{\pi m^2}{e\mathscr{E}}}\left(\frac{dE}{2\pi}\right)\int \frac{d^2p_T}{(2\pi)^2}e^{-\frac{\pi}{e\mathscr{E}}\mathbf{p}_T^2} \\ &= \frac{e\mathscr{E}}{2\pi^2}e^{-\frac{\pi m^2}{e\mathscr{E}}}\left(\frac{dE}{2\pi}\right), \end{aligned} \tag{13.25}$$

where the factor of 2 takes into account the two polarizations of the electron. To find the production rate per unit volume we notice that in the tunneling picture the turning points $x_\pm$ are the coordinates at which the two particles of the pair are produced. Shifting the energy by dE results in a change in the positions of the particles by

$dx = \frac{dE}{e\mathscr{E}}$. Using this relation in Eq. (13.25) we find the pair production rate per unit volume to be

$$W = \frac{e^2 \mathscr{E}^2}{4\pi^3 c \hbar^2} e^{-\frac{\pi m^2 c^3}{\hbar e \mathscr{E}}}, \tag{13.26}$$

where we have restored the powers of $\hbar$ and c.

The production of electron–positron pairs is exponentially suppressed for "weak" electric fields. This suppression ceases when the exponent becomes of order one, i.e., when the electric field reaches the critical value

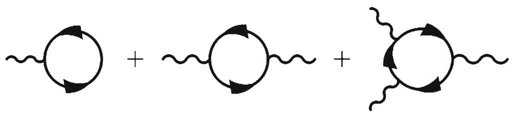
$$\mathscr{E}_{\text{crit}} = \frac{m^2 c^3}{\hbar e} \simeq 1.3 \times 10^{16}\,\text{V cm}^{-1}. \tag{13.27}$$

This is indeed a very strong electric field which is extremely difficult to generate in a laboratory. The Schwinger effect can also be produced by time-varying electric fields [3]. It is expected that pair production could be observed in the strong alternating fields produced by lasers.

In QED the decay of the vacuum into electron–positron pairs induced by an external field can be computed from the imaginary part of the effective action $\Gamma[A_\mu]$ in the presence of a classical gauge potential A_μ. In terms of path integrals this quantity is defined by

$$e^{i\Gamma[A_\mu]} \equiv \int \mathscr{D}\overline{\Psi}\mathscr{D}\Psi\, e^{i\int d^4x \overline{\Psi}(i\partial\!\!\!/ - m - eA\!\!\!/)\Psi}. \tag{13.28}$$

Expanding the integrand in powers of the electric charge e gives the diagrammatic expansion

$i\Gamma[A_\mu] \equiv$  $+ \ldots$ (13.29)

$$= \log\det\left(1 - eA\!\!\!/ \frac{1}{i\partial\!\!\!/ - m}\right).$$

The determinant can be computed using standard heat kernel techniques [1, 3]. The probability of pair production is proportional to the imaginary part of $i\Gamma[A_\mu]$ and gives Schwinger's result

$$W = \frac{e^2\mathscr{E}^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\frac{\pi m^2}{e\mathscr{E}}}. \tag{13.30}$$

Comparing this with (13.26) we see that our semiclassical analysis only captured the leading term in (13.30). The subleading contributions can also be obtained semiclassically by taking into account the probability of production of several particle pairs, i.e. the tunneling of more than one electron through the barrier.

Here we have illustrated the creation of particles by semiclassical sources in quantum field theory using simple examples. Our results can be summarized as follows: in Minkowski spacetime quantum fields have a vacuum state invariant under the Poincaré group. This, together with the covariance of the theory under Lorentz transformations, implies that all inertial observers agree on the number of particles contained in a quantum state. Coupling the theory to a space- or time-varying external source results in the vacuum not being invariant under space(time) translations. The consequence is that it is no longer possible to define a state which would be recognized as the vacuum by all observers.

This is also the case when fields are quantized on curved backgrounds. If the background is time-dependent (as it happens in a cosmological setup or for a collapsing star) different observers will identify different vacuum states: what one observer calls the vacuum will contain particles for a different one. This is what is behind the phenomenon of Hawking radiation [4]. The emission of particles by a physical black hole formed by gravitational collapse follows from the fact that what an observer in the asymptotic past would identify as the vacuum is full of particles for an observer in the asymptotic future. Thus, a particle detector located far away from the black hole detects a stream of thermal radiation with temperature

$$T_{\text{Hawking}} = \frac{\hbar c^3}{8\pi G_N k M}, \tag{13.31}$$

where M is the mass of the black hole, G_N is Newton's constant and k is Boltzmann's constant. As in the case of the Schwinger effect, particle creation by black holes can be heuristically understood as resulting from quantum tunneling of particles through the barrier created by the black hole gravitational potential [5].

13.2 Supersymmetry

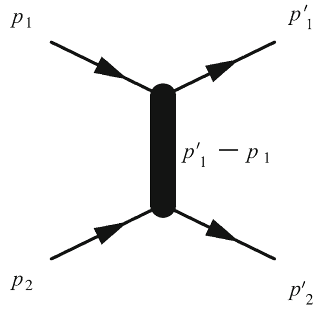
One of the things that we have learned in our journey around the landscape of quantum field theory is that our knowledge of the fundamental interactions in Nature is based on the idea of symmetry, and in particular gauge symmetry. The Lagrangian of the standard model can be written just including all possible renormalizable terms (i.e. with canonical dimension smaller o equal to 4) compatible with the gauge symmetry $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y$ and Poincaré invariance. All attempts to go beyond start with the question of how to extend the symmetries of the standard model.

As explained in Sect. 6.1, in a quantum field theoretical description of the interaction of elementary particles the basic observable quantity to compute is the scattering or S-matrix giving the probability amplitude for the scattering of a number of incoming particles with a certain momentum into some final products

$$S(\text{in} \to \text{out}) = \langle p_1', \ldots ; \text{out} | p_1, \ldots ; \text{in} \rangle. \tag{13.32}$$

An explicit symmetry of the theory has to be necessarily a symmetry of the S-matrix. Hence it is fair to ask what is the largest symmetry of the S-matrix.

Let us ask this question in the simple case of the scattering of two particles with incoming four-momenta p_1 and p_2 described by the graph



We will make the usual assumptions regarding positivity of the energy and analyticity of the S-matrix. Invariance of the theory under the Poincaré group implies that the amplitude can only depend on the scattering angle ϑ through the square of the transferred momentum $p_1' - p_1$

$$\begin{aligned} t = (p_1' - p_1)^2 &= 2\left(m_1^2 - p_1 \cdot p_1'\right) \\ &= 2\left(m_1^2 - E_1 E_1' + |\mathbf{p}_1||\mathbf{p}_1'|\cos\vartheta\right). \end{aligned} \tag{13.33}$$

We assume now the existence of an extra symmetry with a bosonic conserved charge transforming as a tensor under the Poincaré group. Due to its tensor properties, the charge of the asymptotic states would depend nontrivially on their momentum eigenvalues. Therefore, charge conservation would restrict the scattering angle to a set of discrete values. In this case the S-matrix cannot be analytic, since it would vanish everywhere except for the discrete values selected by the extra symmetry.[3] Thus, the condition of having nontrivial scattering implies that the conserved charges associated with internal symmetries cannot transform as tensors under the Poincaré group (this result is the Coleman–Mandula theorem).

One possible way to extend the symmetry of the theory without renouncing to the analyticity of the scattering amplitudes is to introduce "fermionic" symmetries, i.e. symmetries whose generators are anticommuting objects [6–8]. This means that in addition to the generators of the Poincaré group P^μ, $\mathscr{J}^{\mu\nu}$ and the ones for the internal gauge symmetries G, we can introduce a number of fermionic generators $Q_a^I, \overline{Q}_{\dot{a}I} (I = 1, \ldots, \mathscr{N})$, where $\overline{Q}_{\dot{a}I} = (Q_a^I)^\dagger$. The most general algebra that these generators satisfy is the $\mathscr{N}$-extended supersymmetry algebra [9]

$$\begin{aligned} \{Q_a^I, \overline{Q}_{\dot{b}J}\} &= 2\sigma^\mu_{a\dot{b}} P_\mu \delta^I_J, \\ \{Q_a^I, Q_b^J\} &= 2\varepsilon_{ab}\mathscr{Z}^{IJ}, \\ \{\overline{Q}_{\dot{a}}^I, \overline{Q}_{\dot{b}}^J\} &= 2\varepsilon_{\dot{a}\dot{b}}\overline{\mathscr{Z}}^{IJ}, \end{aligned} \tag{13.34}$$

[3] Technically this is correct only if the additional symmetry is additive in the incoming and outgoing Hilbert spaces. If this additivity is violated, then the conclusion does not hold.

where $\mathscr{Z}^{IJ} \in \mathbb{C}$ commute with any other generator and satisfies $\mathscr{Z}^{IJ} = -\mathscr{Z}^{JI}$. We also have the commutators determining the Poincaré transformations of the fermionic generators $Q^I_a,\ Q_{\dot{a}J}$

$$\begin{aligned}
\left[Q^I_a, P^\mu\right] &= \left[\overline{Q}_{\dot{a}I}, P^\mu\right] = 0, \\
\left[[Q^I_a, \mathscr{J}^{\mu\nu}\right] &= \frac{1}{2}(\sigma^{\mu\nu})_a{}^b Q^I_b, \\
\left[\overline{Q}_{\dot{a}I}, \mathscr{J}^{\mu\nu}\right] &= -\frac{1}{2}(\overline{\sigma}^{\mu\nu})_{\dot{a}}{}^{\dot{b}}\overline{Q}_{\dot{b}I},
\end{aligned} \tag{13.35}$$

where $\sigma^{0i} = -i\sigma_i,\ \sigma^{ij} = \varepsilon^{ijk}\sigma_k$ and $\overline{\sigma}^{\mu\nu} = (\sigma^{\mu\nu})^\dagger$. These identities simply mean that $Q^I_a,\ \overline{Q}_{\dot{a}J}$ transform respectively in the $\left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ and $\left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$ representations of the Lorentz group.

We know that the presence of a global symmetry in a theory implies that the spectrum can be classified in multiplets with respect to that symmetry. In the case of supersymmetry we start with $\mathscr{N} = 1$ where there is a single pair of supercharges $Q_a,\ \overline{Q}_{\dot{a}}$ satisfying the algebra

$$\{Q_a, \overline{Q}_{\dot{b}}\} = 2\sigma^\mu_{a\dot{b}}P_\mu, \quad \{Q_a, Q_b\} = \{\overline{Q}_{\dot{a}}, \overline{Q}_{\dot{b}}\} = 0. \tag{13.36}$$

Notice that in the $\mathscr{N} = 1$ case there is no possibility of having central charges.

We study the representations of the supersymmetry algebra (13.36), starting with the massless case. Given a state $|k\rangle$ satisfying $k^2 = 0$, we can always find a reference frame where the four-vector k^μ takes the form $k^\mu = (E, 0, 0, E)$. Since the theory is Lorentz covariant we can obtain the representation of the supersymmetry algebra in this frame where the expressions are simpler. In particular, the right-hand side of the first anticommutator in Eq. (13.36) is given by

$$2\sigma^\mu_{a\dot{b}}P_\mu = 2(P^0 - \sigma^3 P^3) = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}. \tag{13.37}$$

Therefore the algebra of supercharges in the massless case reduces to

$$\begin{aligned}
&\{Q_1, Q_1^\dagger\} = \{Q_1, Q_2^\dagger\} = 0, \\
&\{Q_2, Q_2^\dagger\} = 4E.
\end{aligned} \tag{13.38}$$

The commutator $\{Q_1, Q_1^\dagger\} = 0$ implies that the action of Q_1 on any state gives a zero-norm state of the Hilbert space $\|Q_1|\Psi\rangle\| = 0$. If we want the theory to preserve unitarity we must eliminate these null states from the spectrum. This is equivalent to setting $Q_1 \equiv 0$. On the other hand, in terms of the second generator Q_2 we can define the operators

$$a = \frac{1}{2\sqrt{E}}Q_2, \quad a^\dagger = \frac{1}{2\sqrt{E}}Q_2^\dagger, \tag{13.39}$$

which satisfy the algebra of a pair of fermionic creation-annihilation operators, $\{a, a^\dagger\} = 1,\ a^2 = (a^\dagger)^2 = 0$. Starting with a vacuum state $a|\lambda\rangle = 0$ with helicity λ we can build the massless multiplet

$$|\lambda\rangle, \quad \left|\lambda + \frac{1}{2}\right\rangle \equiv a^\dagger|\lambda\rangle. \tag{13.40}$$

Here we consider two important cases:

- Scalar multiplet: we take the vacuum state to have zero helicity and positive parity $|0^+\rangle$ so the multiplet consists of a scalar and a helicity-$\frac{1}{2}$ state

$$|0^+\rangle, \quad \left|\frac{1}{2}\right\rangle \equiv a^\dagger|0^+\rangle. \tag{13.41}$$

  This multiplet is not invariant under the CPT transformation which reverses the sign of the helicity of the states. In order to have a CPT-invariant theory we have to add to this multiplet its CPT-conjugate which can be obtained from a vacuum state with helicity $\lambda = -\frac{1}{2}$

$$|0^-\rangle, \quad \left|-\frac{1}{2}\right\rangle. \tag{13.42}$$

  Putting them together we can combine the two zero helicity states with the two fermionic ones into the degrees of freedom of a complex scalar field and a Weyl (or Majorana) spinor.
- Vector multiplet: now we take the vacuum state to have helicity $\lambda = \frac{1}{2}$, so the multiplet contains also a massless state with helicity $\lambda = 1$

$$\left|\frac{1}{2}\right\rangle, \quad |1\rangle \equiv a^\dagger\left|\frac{1}{2}\right\rangle. \tag{13.43}$$

  As with the scalar multiplet, we add the CPT conjugated obtained from a vacuum state with helicity $\lambda = -1$

$$\left|-\frac{1}{2}\right\rangle, \quad |-1\rangle, \tag{13.44}$$

  which together with (13.43) give the propagating states of a gauge field and a spin-$\frac{1}{2}$ gaugino.

In both cases we see the trademark of supersymmetric theories: the number of bosonic and fermionic states within a multiplet is the same.

In the case of extended supersymmetry we have to repeat the previous analysis for each supersymmetry charge. At the end, we have $\mathscr{N}$ sets of fermionic creation-annihilation operators $\{a^I, a_J^\dagger\} = \delta^I_J,\ (a_I)^2 = (a_I^\dagger)^2 = 0$. Let us work out the case of $\mathscr{N} = 8$ supersymmetry. Since for several reasons we do not want to have states with helicity larger than 2, we start with a vacuum state $|-2\rangle$ of helicity $\lambda = -2$. The rest of the states of the supermultiplet are obtained by applying the eight different creation operators $a_I^\dagger$ to the vacuum:

$$\begin{aligned}
&\lambda = 2 : a_1^\dagger \dots a_8^\dagger |-2\rangle && \tbinom{8}{8} = 1 \text{ state}, \\
&\lambda = \tfrac{3}{2} : a_{I_1}^\dagger \dots a_{I_7}^\dagger |-2\rangle && \tbinom{8}{7} = 8 \text{ states}, \\
&\lambda = 1 : a_{I_1}^\dagger \dots a_{I_6}^\dagger |-2\rangle && \tbinom{8}{6} = 28 \text{ states}, \\
&\lambda = \tfrac{1}{2} : a_{I_1}^\dagger \dots a_{I_5}^\dagger |-2\rangle && \tbinom{8}{5} = 56 \text{ states}, \\
&\lambda = 0 : a_{I_1}^\dagger \dots a_{I_4}^\dagger |-2\rangle && \tbinom{8}{4} = 70 \text{ states}, \\
&\lambda = -\tfrac{1}{2} : a_{I_1}^\dagger a_{I_2}^\dagger a_{I_3}^\dagger |-2\rangle && \tbinom{8}{3} = 56 \text{ states}, \\
&\lambda = -1 : a_{I_1}^\dagger a_{I_2}^\dagger |-2\rangle && \tbinom{8}{2} = 28 \text{ states}, \\
&\lambda = -\tfrac{3}{2} : a_{I_1}^\dagger |-2\rangle && \tbinom{8}{1} = 8 \text{ states}, \\
&\lambda = -2 : |-2\rangle && \tbinom{8}{0} = 1 \text{ state}.
\end{aligned} \tag{13.45}$$

Putting together the states with opposite helicity we find that the theory contains:

- 1 spin-2 field $g_{\mu\nu}$ (a graviton),
- 8 spin-$\frac{3}{2}$ gravitino fields ψ_μ^I,
- 28 gauge fields $A_\mu^{[IJ]}$,
- 56 spin-$\frac{1}{2}$ fermions $\psi^{[IJK]}$,
- 70 scalars $\phi^{[IJKL]}$,

where by $[IJ\dots]$ we indicated that the indices are antisymmetrized. We see that, unlike the massless multiplets of $\mathscr{N} = 1$ supersymmetry studied above, this multiplet is CPT invariant by itself. As in the case of the massless $\mathscr{N} = 1$ multiplet, here we also find as many bosonic as fermionic states:

$$\begin{aligned}
\text{bosons: } 1 + 28 + 70 + 28 + 1 &= 128 \text{ states}, \\
\text{fermions: } 8 + 56 + 56 + 8 &= 128 \text{ states}.
\end{aligned}$$

Now we study briefly the case of massive representations $|k\rangle$, $k^2 = M^2$. Things become simpler if we work in the rest frame where $P^0 = M$ and the spatial components of the momentum vanish. Then, the supersymmetry algebra becomes:

$$\{Q_a^I, \overline{Q}_{\dot{b}J}\} = 2M\delta_{a\dot{b}}\delta_J^I. \tag{13.46}$$

We proceed in a similar way to the massless case by defining the operators

$$a_a^I \equiv \frac{1}{\sqrt{2M}} Q_a^I, \quad a_{\dot{a}I}^\dagger \equiv \frac{1}{\sqrt{2M}} \overline{Q}_{\dot{a}I}. \tag{13.47}$$

The multiplets are found by choosing a vacuum state with a definite spin. For example, for $\mathscr{N} = 1$ and taking a spin-0 vacuum $|0\rangle$ we find three states in the multiplet transforming irreducibly with respect to the Lorentz group:

$$|0\rangle, \quad a^{\dagger}_{\dot{a}}|0\rangle, \quad \varepsilon^{\dot{a}\dot{b}} a^{\dagger}_{\dot{a}} a^{\dagger}_{\dot{b}}|0\rangle, \tag{13.48}$$

which, once transformed back from the rest frame, correspond to the physical states of two spin-0 bosons and one spin-$\frac{1}{2}$ fermion. For $\mathscr{N}$-extended supersymmetry the corresponding multiplets can be worked out in a similar way.

The equality between bosonic and fermionic degrees of freedom is at the root of many of the interesting properties of supersymmetric theories. For example, in Sect. 3 we computed the divergent vacuum energy contributions for each real bosonic or fermionic propagating degree of freedom[4]

$$E_{\text{vac}} = \pm\frac{1}{2}\delta(\mathbf{0})\int d^3p\,\omega_{\mathbf{p}}, \tag{13.49}$$

where the sign $\pm$ corresponds respectively to bosons and fermions. Hence, for a supersymmetric theory the vacuum energy contribution exactly cancels between bosons and fermions. This boson-fermion degeneracy is also responsible for supersymmetric quantum field theories being less divergent than nonsupersymmetric ones.

References

1. Schwinger, J.: On gauge invariance and vacuum polarization. Phys. Rev. **82**, 664 (1951)
2. Wang, R.-C., Wong, C.Y.: Finite size effect in the schwinger particle production mechanism. Phys. Rev. D **38**, 348–359 (1988)
3. Brezin, E., Itzykson, C.: Pair production in vacuum by an alternating field. Phys. Rev. D **2**, 1191 (1970)
4. Hawking, S.W.: Particle creation by black holes. Commun. Math. Phys. **43**, 199 (1975)
5. Parikh, M.K., Wilczek, F.: Hawking radiation as tunneling. Phys. Rev. Lett. **85**, 5042 (2000) (hep-th/9907001)
6. Golfand, Yu.A., Likhtman, E.P.: Extension of the algebra of Poincaré group generators and violations of P-invariance. JETP Lett. **13**, 323 (1971)
7. Volkov, D.V., Akulov, V.P.: Is the neutrino a goldstone particle. Phys. Lett. B **46**, 109 (1973)
8. Wess, J., Zumino, B.: A Lagrangian model invariant under supergauge transformations. Phys. Lett. B **49**, 52 (1974)
9. Haag, R., Ł opuszański, J., Sohnius, M.: All possible generators of supersymmetries of the S-matrix. Nucl. Phys. B **88**, 257 (1975)

[4] For a boson, this can be read off Eq. (2.59). In the case of fermions, the result of Eq. (3.59) gives the vacuum energy contribution of the four real propagating degrees of freedom of a Dirac spinor.