Chapter 10
The Origin of Mass

The time has come to finally address a central problem left pending in the discussion of the standard model carried out in Chap. 5: how particle masses can be generated preserving gauge invariance. We apply the Brout–Englert–Higgs mechanism introduced in Chap. 7 to solve the problem of mass in the electroweak theory.

We will see, however, that this is not the end of the story. The masses generated by spontaneous symmetry breaking in the standard model cannot account for the mass of protons and neutrons, and therefore for most of the mass we see around us, including our own. We will see that its origin is a purely quantum mechanical effect in QCD.

10.1 The Masses in the Standard Model

We are finally ready to give a solution to the double problem that we left unsolved in Chap. 5. First, the chiral nature of the electroweak interaction forbade writing mass terms for the quark and lepton fields, while we know for sure that electrons, muons and other particles are massive. Secondly, the phenomenology of weak decays indicated that this interaction should be mediated by massive gauge bosons, something that at face value is impossible to reconcile with gauge invariance.

Chapter 7 has provided the crucial hint on how this problem can be cured: by breaking the $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ gauge symmetry spontaneously to the electromagnetic $\mathrm{U}(1)$ one could give mass to three of the gauge bosons mediating the electroweak interaction leaving a massless photon behind. To do so we have to introduce a new field, the Higgs field, transforming under the electroweak gauge group and whose vacuum expectation value breaks it properly. Since we are not interested in breaking Lorentz invariance, the field has to be a scalar.

To find the transformation of the Higgs field under the gauge group we take into account that, in acquiring its vacuum expectation value, it should also give mass to the matter fields. To see how this can be done we go back for a moment to the Abelian Higgs model discussed in Chap. 7 [see Eq. (7.78)]. We add a massless fermion ψ

L. Álvarez-Gaumé and M. Á. Vázquez-Mozo, *An Invitation to Quantum Field Theory*, 193
Lecture Notes in Physics 839, DOI: 10.1007/978-3-642-23728-7_10,

and couple it to the complex scalar field $\varphi(x)$ introducing the Yukawa coupling term

$$\mathscr{L}_{\text{Yukawa}} = -c\varphi\overline{\psi}\psi, \tag{10.1}$$

where c is a real constant. Upon symmetry breaking, this term in the Lagrangian takes the form

$$\mathscr{L}_{\text{Yukawa}} = -\frac{cv}{\sqrt{2}}\overline{\psi}\psi - \frac{c}{\sqrt{2}}\sigma\overline{\psi}\psi. \tag{10.2}$$

The first term gives a Dirac mass $m_\psi = \frac{1}{\sqrt{2}}cv$ to the fermion $\psi(x)$, while the second one couples it to the scalar field $\sigma(x)$.

This shows the way to solve the problem of giving mass to fermions coupling to gauge fields in a chiral way without breaking gauge invariance. In Chap. 5 we learned that in the standard model the left-handed fermions transform as doublets under the SU(2) factor of the gauge group, whereas the right-handed components are singlets. Then, the gauge invariance of the Yukawa couplings indicates that the Higgs field has to be a SU(2) doublet

$$\mathbf{H} = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}, \tag{10.3}$$

where H^+ and H^0 are complex scalar fields. Taking this into account we add to the standard model Lagrangian the piece

$$\mathscr{L}^{(\ell)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(\ell)}_{ij}\overline{\mathbf{L}}^i\mathbf{H}\ell^j_R + C^{(\ell)*}_{ji}\overline{\ell}^i_R\mathbf{H}^\dagger\mathbf{L}^j\right), \tag{10.4}$$

invariant under SU(2) gauge transformations. Here $C^{(\ell)}_{ij}$ are dimensionless coupling constants and we have used the notation introduced in Table 5.1. The Yukawa couplings have been constructed in such a way that neutrinos do not get Dirac masses.

The masses of the quarks are generated by Yukawa couplings similar to the ones already written for the leptons. One important difference, however, lies in the fact that now we want to give mass to the two components of the left-handed SU(2) doublets. To achieve this we need to couple the fermions not only to the Higgs doublet $\mathbf{H}$ but also to its "charge conjugate"

$$\tilde{\mathbf{H}} \equiv i\sigma_2\mathbf{H}^* = \begin{pmatrix} H^{0*} \\ -H^{+*} \end{pmatrix}. \tag{10.5}$$

From the identity

$$(i\sigma_2)e^{-i\mathbf{a}\cdot\frac{\boldsymbol{\sigma}^*}{2}} = e^{i\mathbf{a}\cdot\frac{\boldsymbol{\sigma}}{2}}(i\sigma_2), \tag{10.6}$$

it follows that the conjugated Higgs field $\tilde{\mathbf{H}}$ also transforms as a SU(2) doublet. Then, the Dirac masses of the quark fields can be obtained from the following Yukawa couplings

$$\mathscr{L}^{(q)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(q)}_{ij}\overline{\mathbf{Q}}^{i}\mathbf{H}D^{j}_{R} + C^{(q)*}_{ji}\overline{D}^{i}_{R}\mathbf{H}^{\dagger}\mathbf{Q}^{j}\right)$$
$$-\sum_{i,j=1}^{3}\left(\tilde{C}^{(q)}_{ij}\overline{\mathbf{Q}}^{i}\tilde{\mathbf{H}}U^{j}_{R} + \tilde{C}^{(q)*}_{ji}\overline{U}^{i}_{R}\tilde{\mathbf{H}}^{\dagger}\mathbf{Q}^{j}\right). \tag{10.7}$$

The notation also follows Table 5.2.

We have constructed an interaction term between the Higgs field and the fermions demanding invariance under SU(2) gauge transformations. It is easy to see that the Yukawa couplings (10.4) and (10.7) are invariant also under the $\mathrm{U}(1)_Y$ gauge symmetry factor provided the Higgs field is assigned the weak hypercharge $Y(\mathbf{H}) = \frac{1}{2}$. The Gell–Mann–Nishijima formula then implies that

$$Q(H^{+}) = 1, \quad Q(H^{0}) = 0, \tag{10.8}$$

thus justifying our notation.

To implement symmetry breaking we have to add the following term to the standard model Lagrangian

$$\mathscr{L}_{\text{Higgs}} = (D_{\mu}\mathbf{H})^{\dagger}D^{\mu}\mathbf{H} - V(\mathbf{H},\mathbf{H}^{\dagger}), \tag{10.9}$$

where D_{μ} is the corresponding $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ covariant derivative (see Sect. 5.4). The potential has to be wisely chosen in such a way that spontaneous symmetry breaking takes place and solves our problems with the particle masses in a satisfactory way. In fact, gauge invariance and the condition that the theory is renormalizable (see Chap. 8) imply that the Higgs potential should be of the form

$$V(\mathbf{H},\mathbf{H}^{\dagger}) = \frac{\lambda}{4}\left(\mathbf{H}^{\dagger}\mathbf{H} - \frac{v^{2}}{2}\right)^{2}. \tag{10.10}$$

The system exhibits spontaneous symmetry breaking if $v^2 > 0$. Then, the theory has a degenerate family of vacua defined by $\mathbf{H}^{\dagger}\mathbf{H} = \frac{1}{2}v^2$.

The only surviving gauge symmetry in the electroweak sector at low energies is the U(1) invariance of QED. This means that this symmetry is realized à la Wigner–Weyl and therefore the vacuum has zero electric charge. Taking into account Eq. (10.8) this means that we are forced to take[1]

$$\langle\mathbf{H}\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}v \end{pmatrix}. \tag{10.11}$$

Since $Y(\mathbf{H}) = \frac{1}{2}$ this vacuum expectation value breaks not only SU(2) but also $\mathrm{U}(1)_Y$. It however preserves the electromagnetic U(1) and therefore implements correctly the symmetry breaking pattern, $\mathrm{SU}(2)\times\mathrm{U}(1)_Y \to \mathrm{U}(1)$.

[1] It can be shown that, by appropriate gauge transformations, any other vacuum expectation value can always be brought to this form.

Following the example of the Abelian Higgs model, the fluctuations around this vacuum can be parametrized as [cf. Eq. (7.81)]

$$\mathbf{H}(x) = \frac{1}{\sqrt{2}} e^{i\mathbf{a}(x)\cdot\frac{\boldsymbol{\sigma}}{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{10.12}$$

There are four different fields associated with these fluctuations, here denoted by $\mathbf{a}(x)$ and $h(x)$. The factor $e^{i\mathbf{a}(x)\cdot\frac{\boldsymbol{\sigma}}{2}}$ represents the action of the three broken generators,[2] and can be eliminated by a SU(2) gauge transformation. This removes the three would-be Nambu–Goldstone bosons $\mathbf{a}(x)$ that are transmuted into the longitudinal components of the massive gauge bosons W^+, W^- and Z^0. The remaining propagating degree of freedom $h(x)$ is the neutral scalar whose elementary excitation is known as the Higgs boson. Inserting

$$\mathbf{H}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{10.13}$$

in $V(\mathbf{H}, \mathbf{H}^\dagger)$ and expanding the result in powers of the field $h(x)$, the mass of the Higgs particle is found to be

$$m_{\text{H}} = v\sqrt{\frac{\lambda}{2}}. \tag{10.14}$$

The dimensionless coupling λ governs the self-interaction of the Higgs bosons.

Substituting Eq. (10.13) in the Yukawa couplings (10.4), we find, at low energies, the following lepton mass terms

$$\mathscr{L}^{(\ell)}_{\text{mass}} = -(\overline{e}_L, \overline{\mu}_L, \overline{\tau}_L) M^{(\ell)} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} + \text{h.c.} \tag{10.15}$$

We notice that no mass term for the neutrinos is generated through the Brout–Englert–Higgs mechanism, so neutrino masses have to be explained in some other way. On the other hand, for the quarks we find

$$\mathscr{L}^{(q)}_{\text{mass}} = -(\overline{d}_L, \overline{s}_L, \overline{b}_L) M^{(q)} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - (\overline{u}_L, \overline{c}_L, \overline{t}_L) \tilde{M}^{(q)} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + \text{h.c.} \tag{10.16}$$

and the mass matrices are given by

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[2] It might seem strange that, apparently, we have included only the action of the SU(2) generators on the vacuum. As a matter of fact, this is not the case. What happens is that the electromagnetic U(1) remains unbroken and therefore $Q_{\text{vac}} = 0$. Then, using the Gell–Mann–Nishijima relation, the action of the weak hypercharge generator Y on the vacuum can be written in terms of the generators of SU(2) as $Y = -2T_3 = -\sigma_3$.

$$M_{ij}^{(\ell,q)} = \frac{1}{\sqrt{2}} v C_{ij}^{(\ell,q)}, \quad \tilde{M}_{ij}^{(q)} = \frac{1}{\sqrt{2}} v \tilde{C}_{ij}^{(q)}, \tag{10.17}$$

with $C_{ij}^{(\ell,q)}$ and $\tilde{C}_{ij}^{(q)}$ the strength of the Yukawa couplings defining general complex 3×3 matrices. We notice as well that the mass scale of all charged fermion is set by the Higgs vacuum expectation value v.

So far we have written the standard model Lagrangian in terms of fields with well defined transformations under the gauge group (this we call flavor eigenstates). Now, however, there is no a priori reason for the mass matrices in (10.15) and (10.16) to be diagonal. This means that the corresponding propagators are not diagonal and therefore the different flavor eigenstates mix with each other as they propagate. In order to quantize the theory, however, it is more convenient to work with fields whose propagators, at low energies, are diagonal and therefore have well-defined masses. These fields are constructed by noticing that a general complex matrix can always be diagonalized by a biunitary transformation. More precisely, this means that there are unitary matrices $V_{L,R}^{(\ell,q)}$, $\tilde{V}_{L,R}^{(q)}$ such that

$$V_L^{(\ell)\dagger} M^{(\ell)} V_R^{(\ell)} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \tag{10.18}$$

for the leptons, whereas for the quarks we have

$$V_L^{(q)\dagger} M^{(q)} V_R^{(q)} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad \tilde{V}_L^{(q)\dagger} \tilde{M}^{(q)} \tilde{V}_R^{(q)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}. \tag{10.19}$$

In view of this, we define the *mass eigenstate* quark fields as[3]

$$\begin{pmatrix} u'_{L,R} \\ c'_{L,R} \\ t'_{L,R} \end{pmatrix} = \tilde{V}_{L,R}^{(q)\,\dagger} \begin{pmatrix} u_{L,R} \\ c_{L,R} \\ t_{L,R} \end{pmatrix}, \quad \begin{pmatrix} d'_{L,R} \\ s'_{L,R} \\ b'_{L,R} \end{pmatrix} = V_{L,R}^{(q)\,\dagger} \begin{pmatrix} d_{L,R} \\ s_{L,R} \\ b_{L,R} \end{pmatrix}, \tag{10.20}$$

and similarly for the charged lepton fields, this time using the matrices $V_{L,R}^{(\ell)}$. By construction, the propagators are diagonal when expressed in terms of the new fields. The couplings with the gauge fields, on the other hand, can get a dependence on the unitary matrices involved in the diagonalization of the mass matrices. To see how this dependence comes about we look, for example, at the quark charged current coupling to the W^+ bosons

$$j_+^\mu = (\bar{u}_L, \bar{c}_L, \bar{t}_L)\gamma^\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = (\bar{u}'_L, \bar{c}'_L, \bar{t}'_L)\gamma^\mu \tilde{V}_L^{(q)\dagger} V_L^{(q)} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} \tag{10.21}$$

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[3] Our notation at this point differs from the usual one in the literature in that we use primed fields to indicate the mass eigenstates. The reason to use this notation is to avoid cluttering the equations with primes both in this chapter and in Chap. 5.

A similar calculation for the neutral quark current shows that it does not depend on the unitary matrices relating flavor to mass eigenstates. This means that at tree level there are no flavor changing neutral currents (FCNC), as a consequence of the quantum numbers of the three families. This is the tree-level version of the Glashow–Iliopoulos–Maiani (GIM) mechanism that works for complete families.

We have shown that the couplings of the quarks to the $W^{\pm}$ bosons mix the different mass eigenstates. This mixing is given by the 3×3 matrix

$$V \equiv \tilde{V}_L^{(q)\dagger} V_L^{(q)}, \tag{10.22}$$

called the Cabibbo–Kobayashi–Maskawa (CKM) quark mixing matrix. It is immediate to check that this matrix is unitary and therefore in general complex. In Chap. 11 we will see that this has important physical consequences.

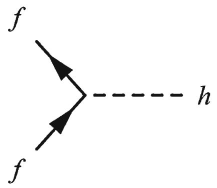
We analyze next the leptonic sector. The charged lepton-neutrino current is

$$j_+^{\mu} = (\overline{\nu}_{eL}, \overline{\nu}_{\mu L}, \overline{\nu}_{\tau L})\gamma^{\mu} \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} = (\overline{\nu}_{e,L}, \overline{\nu}_{\mu,L}, \overline{\nu}_{\tau,L})\gamma^{\mu} V_L^{(\ell)} \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix}. \tag{10.23}$$

Were the neutrino massless, the matrix $V_L^{(\ell)}$ could be reabsorbed in a redefinition of the neutrino fields without making the propagator nondiagonal. We know, however, that the neutrinos are massive and the only question is whether their mass terms are of Dirac, Majorana or a mixture of both. In either case one has to redefine the neutrino fields to diagonalize their mass matrix and this results in the introduction of a second CKM matrix in the leptonic sector.

Higgs Couplings

Having learned how fermion masses are generated, we would like to know how these states couple to the Higgs field itself. This is important because these couplings determine both how the Higgs particle can be produced in a scattering experiment and also what its decay signatures are. Looking at the terms linear in $h(x)$ in (10.4), we find that the Higgs boson couples to the *charged* mass eigenstates $f = (e', \mu', \tau', u', d', c', s', t', b')$ according to the vertices



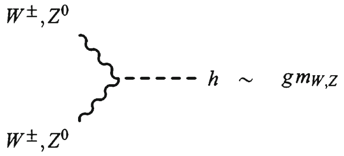
$$\sim \frac{m_f}{v},$$

where m_f is the mass of the charged fermion. Thus, the Higgs-fermion couplings are suppressed by the ratio between the fermion masses and the vacuum expectation value of the Higgs field.

The masses of the gauge fields $W^{\pm}$ and Z^0 and their couplings to the Higgs are obtained by expanding $\mathbf{H}(x)$ around the vacuum in the covariant derivative terms in Eq. (10.9). For the masses one finds

$$m_W = \frac{1}{2} g v, \quad m_Z = \frac{g v}{2 \cos \theta_w}, \tag{10.24}$$

with g the electroweak coupling constant and θ_w the weak mixing angle (see Chap. 5). As for the coupling of the vector bosons to the Higgs field, the terms linear in $h(x)$ give rise to the following interaction vertices



In addition, the theory contains also vertices that couple two vector bosons to two Higgs fields, as well as self-interaction vertices with three and four Higgs bosons. They can be found, for example, in Ref. [4–8] of Chap. 5.

The implementation of symmetry breaking has resulted in the introduction of a new energy scale, the Higgs vacuum expectation value v, and a number of dimensionless couplings: the Higgs self-interaction λ, and the Yukawa couplings for leptons and quarks, $C_{ij}^{(\ell)}$, $C_{ij}^{(q)}$ and $\tilde{C}_{ij}^{(q)}$. In fact, the Higgs vacuum expectation value v is related to the Fermi coupling constant G_F introduced in Chap. 5. Using the relation between v and the mass of the W boson (10.24) we find

$$G_F = \frac{1}{\sqrt{2} v^2}. \tag{10.25}$$

Since G_F can be measured, for example, from muon decay we learn that the Higgs vacuum expectation value is

$$v \approx 246\,\text{GeV}. \tag{10.26}$$

Once the value of the only energy scale v is determined, one can use the relations (10.17) to fix the Yukawa couplings for quarks and leptons from measurement of the mass matrices for the different matter fields. With this, however, we still get no information about the value of the Higgs self-coupling constant λ, or equivalently, the Higgs boson mass m_H. This is the last standard model parameter that remains to be measured and the Higgs boson the last particle of the model to be detected.

What makes the Higgs field so elusive? First, our ignorance of the value of the Higgs mass makes its detection difficult because it is not possible to know a priori "where" to look for it. Depending on the value of m_H different channels have to be considered for the production of this particle. A second aspect is that the Higgs boson couples to other standard model particles with a strength proportional to their masses. Thus, its coupling to light fermions is very small and to produce Higgs particles one must begin by producing heavy fermions, $W^{\pm}$, or Z^0 vector bosons in large quantities. The situation is complicated by the fact that many decay channels of the Higgs boson produce signals that are quite common also in other standard model processes not involving Higgs particles (or, in technical jargon, they have "large backgrounds").

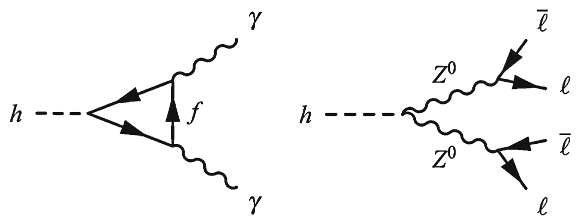
This however does not mean that we do not know anything about the Higgs mass. The Higgs particle enters in the calculation of higher order corrections to standard model processes and bounds to m_H can be found by comparing these calculations with the precision measurements carried out at the Large Electron Positron (LEP) collider, running at CERN between the years 1989 and 2000. Additional bounds for m_H can also be found from consistency requirements. For example, if the Higgs boson is too light the quantum corrections to the Higgs coupling constant λ could make it negative, thus rendering the theory unstable. On the other side, a too-heavy Higgs boson would have unpleasant effects on the good behavior of the theory at high energies. Combining these with other pieces of information a likely range for the Higgs mass can be obtained depending on the energy scale Λ up to which we consider the standard models to describe the physics correctly [1]. Taking, for example, $\Lambda \sim 1$ TeV one finds

$$50\,\text{GeV} \lesssim m_H \lesssim 800\,\text{GeV}, \tag{10.27}$$

while if $\Lambda \sim 10^{16}$ GeV the range narrows to

$$130\,\text{GeV} \lesssim m_H \lesssim 180\,\text{GeV}. \tag{10.28}$$

Searches for Higgs boson are currently underway at both the Tevatron at Fermilab and the LHC at CERN. Particularly promising channels are the decay of the Higgs into two photons or into two Z^0, that in turn decay into a couple of lepton–antilepton pairs:



The first process would give a distinctive signature for a Higgs with mass $m_H \lesssim 150\,\text{GeV}$, whereas the second would be important in the regime $m_H \gtrsim 2m_Z$.

Remarks on Symmetry Breaking in the Standard Model

The Higgs sector of the standard model cannot be regarded as a mere attachment to it, as just a smart "trick" intended to circumvent the conflict between masses and gauge invariance. There are more fundamental reasons to think that the Higgs particle, or something very similar, should be there. It is an experimental fact that the $W^{\pm}$ and

Z^0 bosons are massive and therefore have longitudinal components that have been detected.

If we only worry about giving masses to the vector bosons and fermions, it is clear that freezing the field $h(x)$ in Eq. (10.12) suffices. For all practical purposes the theory we obtain has massive $W^{\pm}$ and Z^0 bosons and massive fermions, but no elementary Higgs scalar. So long as we work at low enough energies, this may be a reasonably good phenomenological description.

This naive Higgsless standard model has problems: scattering amplitudes involving the longitudinal components of the gauge bosons behaves badly as the energy approaches the scale $v \sim m_{W,Z}/g$. The amplitudes grow so fast with the energy as to be incompatible with something as basic as the conservation of probability. This problem is automatically solved by including a neutral scalar field in the theory that couples to the massive gauge bosons and fermions in precisely the same way as the Higgs particle does. But this is not the only possibility.

We illustrate this point in more detail using the example of a SU(2) massive gauge field coupled to a pair of chiral doublets Ψ_L, Ψ_R transforming as

$$\Psi_L(x) \longrightarrow g(x)\Psi_L(x), \quad \Psi_R(x) \longrightarrow \Psi_R(x), \tag{10.29}$$

where $g(x)$ belongs to the fundamental representation of SU(2). The Lagrangian

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + M^2\mathrm{Tr}\left(A_\mu A^\mu\right) + i\overline{\Psi}_L \not{D}\Psi_\mathrm{L} + i\overline{\Psi}_\mathrm{R}\not{D}\Psi_R \\ &- m\left(\overline{\Psi}_\mathrm{L}\Psi_R + \overline{\Psi}_\mathrm{R}\Psi_\mathrm{L}\right)\end{aligned} \tag{10.30}$$

is not gauge invariant due to the presence of mass terms for the gauge and fermion fields. Gauge invariance can be "restored" using a trick originally due to Stückelberg [3] (see [4] for a review). We introduce a scalar field $U(x)$, called the Stückelberg field, taking values in the gauge group and transforming under SU(2) as $U(x) \to g(x)U(x)$. The Lagrangian

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{M^2}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[(U^\dagger D_\mu U)(U^\dagger D^\mu U)\right] \\ &+ i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{D}\Psi_R - m\left(\overline{\Psi}_L U\Psi_R + \overline{\Psi}_R U^\dagger \Psi_L\right)\end{aligned} \tag{10.31}$$

is gauge invariant. Using this gauge freedom we can set $U(x) = \mathbf{1}$ and recover the original Lagrangian (10.30). In this picture the breaking of gauge invariance in the massive theory can be seen as resulting from gauge fixing. In the process, the field $U(x)$ becomes the longitudinal component of the massive vector field.

Replacing (10.30) by (10.31) does not solve our ultraviolet problems. The theory is still ill-defined at energies of order M/g_{YM} and should be completed by specifying the dynamics of $U(x)$ at high energies. Here we are faced with various alternatives. One of them is the Brout–Englert–Higgs mechanism presented: a gauge invariant potential implementing symmetry breaking is added

$$V(U^\dagger U) = \frac{\lambda}{4}\left(\frac{M}{g_{\rm YM}}\right)^4 \left[\frac{1}{2}\text{Tr}(U^\dagger U) - 1\right]^2, \quad (10.32)$$

and the field $U(x)$ is linearized around the vacuum

$$U(x) = U_0(x)\left[1 + \frac{g_{\rm YM}}{M}h(x)\right], \quad (10.33)$$

where $U_0(x) \in \text{SU}(2)$ and $h(x)$ is the Higgs field of mass $m_H^2 = 2\lambda M^2/g_{\rm YM}^2$. At energies below m_H the Higgs field is frozen, $U(x) \simeq U_0(x)$, and the Stückelberg Lagrangian (10.31) provides a reliable phenomenological description.

This linear realization is the simplest, and historically the first one used. Many other scenarios have been proposed as alternative ultraviolet cures of the mass generation mechanism. Among them, technicolor, where $U(x)$ is a bound state (analogous to the pion) of a set of strongly coupled new fermions. There is a large collection of alternatives to the standard Higgs mechanism (for a clear exposition see [4]), however they all share the same mechanism of giving masses to the vector bosons by absorbing the relevant Nambu–Goldstone bosons. This is reasonable, the masses of the $W^\pm$ and Z^0 bosons are infrared properties of the theory and their origin is not necessarily related to the high energy fate of the "Higgs"-mode.

This discussion should help clarifying the statement contained in the closing paragraph of Sect. 5.5. The Lagrangian (10.30) can be used to describe the physics of a nonabelian massive gauge field chirally coupled to massive fermions, as long as we restrict our attention to energies below the mass scales of the problem. In this regime, the absence of gauge invariance is no big deal. As the reader has repeatedly been reminded along the book, gauge invariance is not a real symmetry but rather a redundancy. The point of Stückelberg's trick is to "fake" this redundancy, allowing to write a formally gauge invariant Lagrangian.

The situation is different if we aim at constructing a theory whose predictions can be trusted to arbitrary high energies, in the spirit of good old QED.[4] In this case gauge invariance is a crucial ingredient for consistency. The Brout–Englert–Higgs mechanism provides a renormalizable, gauge invariant ultraviolet completion of the massive low energy theory. Historically, this explains the enormous effect the proof of renormalizability of spontaneously broken gauge theories by 't Hooft and Veltman [5–8] had on the acceptance of the Glashow–Weinberg–Salam theory.

10.2 Quark Masses

The previous presentation might have led to the mistaken conclusion that the Brout-Englert-Higgs mechanism settles once and for all the problem of accounting for the masses of the subatomic particles. The only task left is the experimental measurement

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[4] Let us forget for the moment about the presence of the Landau pole.

of the quark and lepton masses that in turn determine the value of the Yukawa couplings $C_{ij}^{(\ell,q)}$.

This idea works indeed for the leptons. Since they exist as asymptotic states, their masses can be unambiguously determined, and with them the corresponding parameters in the Lagrangian. The complication comes with the quarks. As they cannot be pulled out of the hadrons their masses cannot be measured directly.

One definition of the quark masses is provided by the nonrelativistic quark model. Here the hadrons are considered to be the bound states of a quark–antiquark pair (mesons) or three quarks (baryons). The mass of the hadron can be written in terms of the masses of its constituents plus the corresponding binding energy

$$M_{\text{meson}} = m_q + m_{\overline{q}} + \Delta E_{q\overline{q}}$$
$$M_{\text{baryon}} = m_{q_1} + m_{q_2} + m_{q_3} + \Delta E_{qqq}. \tag{10.34}$$

As quarks are considered to be nonrelativistic in the bound state, the binding energy is subleading with respect to the quark masses, $\Delta E \ll m_q$. In fact, it can be modelled as

$$\Delta E_{q\overline{q}} = \frac{4a}{m_q m_{\overline{q}}} \mathbf{s}_q \cdot \mathbf{s}_{\overline{q}}, \quad \Delta E_{qqq} = 4a' \sum_{i<j}^{3} \frac{1}{m_{q_i} m_{q_j}} \mathbf{s}_{q_i} \cdot \mathbf{s}_{q_j}, \tag{10.35}$$

where a, a' are undetermined numerical constants and $\mathbf{s}_q$ is the quark spin operator. Their products are numbers that depend on the total spin S of the system. This is easy to see in the case of the quark–antiquark bound state, where

$$\mathbf{s}_q \cdot \mathbf{s}_{\overline{q}} = \frac{1}{2}\left[S(S+1) - \frac{3}{2}\right], \tag{10.36}$$

with $S = 0, 1$ the spin of the corresponding meson.

At first sight the ansatz (10.35) for the binding energy might look surprising. It has the form of the hyperfine splitting of the hydrogen atom that we know is a small perturbation to the energy levels determined by the Coulomb interaction. This, however, is not the case for the quark bound states. In the hydrogen atom the smallness of the hyperfine splitting is due to the fact that the corresponding term in the Hamitonian comes suppressed by a factor $\frac{m_e}{m_p} \simeq 0.0004$. In the case of the quark system the factor in front of this term is of order one and therefore its contribution is expected to be of the same order as the quark-quark potential. Due to this we can parametrize our ignorance about the latter in terms of the numerical parameters a and a'.

Using (10.35) the spectrum of hadrons can be fit to get m_q, a and a'. The masses thus obtained are the so-called *constituent quark masses*. They make up a large fraction of the mass of the hadron. For example, in the case of the u and d quarks their constituent masses have the values

$$m_u \simeq m_d \simeq 310\,\text{MeV}, \tag{10.37}$$

about $\frac{1}{3}$ of the proton mass.

Despite the success of the nonrelativistic quark model in accounting for certain properties of hadrons such as their masses and magnetic moments, the constituent masses of the quarks cannot be identified with the mass parameter appearing in the standard model Lagrangian in its broken phase. For historical reasons these parameters are called the *current-algebra quark masses*.

In fact, there is experimental evidence showing that there is much more stuff inside hadrons than the nonrelativistic quark model picture shows. The most compelling comes from the deep inelastic scattering of leptons off protons already described in Chap. 5. In these experiments it is possible to measure the distribution function of the proton momentum among the constituents of the hadron, collectively called partons. The remarkable thing is that about 50% of the total momentum is carried by constituents that do not participate in the electroweak interactions! These have to be identified with virtual gluons responsible for the interaction between the quarks. The remaining proton momentum is shared between the quarks responsible for the quantum numbers such as charge, spin and isospin of the hadron (called the *valence quarks*), and virtual quark–antiquark pairs (*sea quarks*).

With this picture of the hadron interior in mind, constituent quarks can be seen as effective "quasiparticles" resulting from the dressing of the valence quarks by the QCD interaction. This heuristic idea, that would explain the success of the nonrelativistic quark model, is unfortunately too hard to make quantitative due to computational difficulties.

10.3 Λ_{QCD} and the Hadron Masses

The techniques described in Chap. 8 can be used to calculate the beta function in perturbation theory. The running coupling constant can then be formally written in terms of a single dimensionful integration constant as

$$\Lambda = \mu \exp\left[-\int^{g(\mu)} \frac{dx}{\beta(x)}\right]. \tag{10.38}$$

We observe that, whenever the beta function is nonvanishing, quantum corrections generate a characteristic energy scale. This happens even when the classical Lagrangian contains no dimensionful parameters, a phenomenon called dimensional transmutation. It is important to keep in mind that the dynamically generated scale Λ is an integration constant and therefore has to be fixed experimentally. This is related to the fact that quantum field theory only determines the rate of change of the coupling constants with the energy through the renormalization group functions (8.93). Fixing the numerical values of the couplings requires measurements at a reference scale.

In the case of QCD, using the value of the one-loop beta function (8.25) particularized to the case of three colors ($N_c = 3$)

$$\beta(g) = -\frac{g^3}{48\pi^2}(33 - N_f), \tag{10.39}$$

we find the QCD energy scale to be

$$\Lambda_{\text{QCD}} = \mu e^{-\frac{24\pi}{(33-N_f)}\frac{1}{g(\mu)^2}}. \tag{10.40}$$

The strong coupling constant can be written in terms of it as

$$g(\mu)^2 = \frac{24\pi^2}{(33 - N_f)\log\left(\frac{\mu}{\Lambda_{\text{QCD}}}\right)}. \tag{10.41}$$

The physical meaning of Λ_{QCD} becomes clear: it sets the energy scale at which the theory becomes strongly coupled. Notice that the divergence of the coupling constant at $\mu = \Lambda_{\text{QCD}}$ following from the one loop computation cannot be taken literally. When the coupling constant grows the perturbative approximation used to compute the beta function (10.39) breaks down.

In the context of the physics of hadrons, Λ_{QCD} determines the characteristic size of a hadron. Indeed, the theory becomes strongly coupled when the hadron constituents are at distances larger than $\Lambda_{\text{QCD}}^{-1}$, setting thus the length scale inside which quarks are confined.

One of the big problems in QCD is to calculate the mass of particles such as the proton and the neutron in terms of the mass parameters of the quarks. The reason why this problem is difficult lies in the fact that the valence quarks u and d have masses that are much smaller than the natural scale of the theory, Λ_{QCD}. To see this we estimate the kinetic energy of these quarks using Heisenberg's uncertainty principle. Since they are confined inside a hadron of typical size $\Lambda_{\text{QCD}}^{-1}$, the uncertainty in their momenta can be estimated to be $\Delta p \sim \Lambda_{\text{QCD}}$. Moreover, using isotropy we can assume that the average momentum of the quarks is equal to zero, $\langle \mathbf{p} \rangle = 0$. Then, $(\Delta p)^2 = \langle \mathbf{p}^2 \rangle$ and we finally conclude that

$$\langle \mathbf{p}^2 \rangle \sim \Lambda_{\text{QCD}}^2. \tag{10.42}$$

So far we have not made any hypothesis as to the mass of the quarks. Let us now assume that we are dealing with *light quarks*. They are defined as those whose masses satisfy $m_q \ll \Lambda_{\text{QCD}}$. This is the case of the u and d quarks that make up most of the matter that we see around us. In this case, Eq. (10.42) can be recast as

$$\langle \mathbf{p}^2 \rangle \gg m_q^2 \quad (q = u, d). \tag{10.43}$$

This means that light quarks inside hadrons are relativistic. What is more important, Eq. (10.42) implies that the typical energy of these quarks if of order Λ_{QCD} and therefore we are in regime where QCD is strongly coupled.

There are two conclusions to be extracted from this discussion. The first is that we have found the reason behind the technical problems in calculating the masses of hadrons such as protons or neutrons from first principles: we would have to deal with a theory in a regime where perturbation theory does not work. Hence, we have to resort to numerical approaches such as lattice field theory.

The second lesson we learn is that the Brout–Englert–Higgs mechanism actually contributes very little to explaining the mass we see around us. In fact, most of the mass of an atom comes from the nucleus (from about 99.95% for hydrogen to 99.9997% for uranium) that is made of protons and neutrons. What we have argued is that the quark mass parameters m_q generated by electroweak spontaneous symmetry breaking contribute very little to the mass of these hadrons: most of the mass of protons and neutrons, and therefore of the world we see, come from Λ_{QCD}.

That all difficulties in computing hadron masses come from having light quarks can be seen in a toy model due to Howard Georgi [9]. He imagines a world essentially identical to our own but with a single crucial difference: the masses of the u and d quarks satisfy

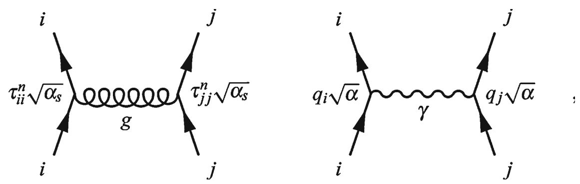
$$m_u \simeq m_d \simeq \frac{1}{3} m_{\text{proton}} \gg \Lambda_{\text{QCD}}. \tag{10.44}$$

Therefore $\langle \mathbf{p}^2 \rangle \ll m_q^2$ and the quarks can be treated nonrelativistically. Thus, the typical energy of the processes inside the proton is m_q, and the condition (10.44) implies that the theory at this scale is weakly coupled. Tuning m_q/Λ_{QCD} we can even make

$$\alpha_s(m_q) \equiv \frac{g(m_q)^2}{4\pi} \simeq \frac{1}{137}. \tag{10.45}$$

This sets $\Lambda_{\text{QCD}} \sim 10^{-42} m_q$.

Given all this, it should be possible to study the bound state of the three quarks in the proton using the techniques of atomic physics. Since the theory is in a coupling regime where perturbation theory can be used, the static potential between the quarks is obtained from the diagram where the two quarks interchange a gluon. In fact we do not even have to compute the diagram. It suffices to compare the corresponding processes in QCD and QED



where q_i, q_j are the charges of the corresponding quarks. A look at the Feynman rules for nonabelian Yang–Mills theories listed in Chap. 6 shows that the only difference between the contribution of the two previous diagrams comes from the presence of the SU(3) generators in the vertices of the former. This means that the first diagram is obtained from the second by the replacement

$$\alpha q_i q_j \longrightarrow \alpha_s(m_q) \sum_{n=1}^{8} \tau_{ii}^n \tau_{jj}^n, \tag{10.46}$$

where $\tau^n = \frac{1}{2}\lambda_n$, with λ_n the Gell–Mann matrices shown in Eq. (B.16). Making this replacement in the Coulomb potential, we find the chromostatic potential between two quarks in the proton to be

$$V_{qq}(r) = C_F \frac{\alpha_s(m_q)}{r}, \tag{10.47}$$

where C_F is the color factor on the right-hand side of (10.46).

How different is Georgi's toy world from our own? In fact, we are not so far off. Because the quarks are nonrelativistic, the binding energy can be estimated from a formula analog to the one for the ground state energy of the hydrogen atom

$$\Delta E \sim \alpha_s(m_q)^2 m_q \simeq 16\,\text{keV}. \tag{10.48}$$

Therefore $\Delta E \ll m_q$ and the mass of the proton is essentially the sum of the masses of the quarks. This means that we can fine tune m_q to have $m_{\text{proton}} = 938\,\text{MeV}$ while preserving (10.44). As for the proton size, it is set by the corresponding Bohr radius

$$R_{\text{proton}} \sim \frac{1}{m_q \alpha_s(m_q)} \simeq 90\,\text{fm}. \tag{10.49}$$

Although almost two orders of magnitude above the real proton radius, it is still about 500 times smaller than the radius of the hydrogen atom. Thus we can expect the electronic structure of the atoms not to be radically changed. Notice that now the size of the hadron is dictated by perturbative effects, as opposed to real hadrons where the relevant physics is nonperturbative and their size is determined by the length scale $\Lambda_{\text{QCD}}^{-1}$.

The main advantage of this toy model is that in it, unlike in the real world, QCD computations are "easy". In particular, the length scale at which confinement takes place is macroscopic. With $\Lambda_{\text{QCD}} \sim 10^{-42} m_q$ and $m_q \sim 300$ MeV we find that

$$\Lambda_{\text{QCD}}^{-1} \sim 10^3\text{Mpc} \tag{10.50}$$

and free quarks could be observed! In fact, in this imaginary world the constituent masses are the physical quark masses and the nonrelativistic quark model is the correct QCD description of hadrons.

With this example we wanted to make an important point: confinement itself is not at the bottom of the difficulties with QCD, but the fact that quarks are much lighter than the energy scale at which confinement occurs. This is illustrated also in "real" QCD with heavy quarks, those whose mass is much larger than Λ_{QCD}. This is the case of the *b, c* and *t* quarks, although the short lifetime of the latter prevents it from forming hadrons. Then the strong coupling constant is small at the quark mass scale and the bound state of heavy quarks is amenable to QCD perturbation theory. Moreover, applying (10.42) we have

$$\langle \mathbf{p}^2 \rangle \ll m_q^2 \qquad \text{(heavy quarks)}, \tag{10.51}$$

and therefore heavy quarks inside hadrons are nonrelativistic.

References

1. Djouadi, A.: The anatomy of electroweak symmetry breaking. Tome I: the Higgs boson in the standard model. Phys. Rept. **457**, 1 (2008)
2. Grojean, C.: New approaches to electroweak symmetry breaking. Phys. Usp. **50**, 1 (2007)
3. Stueckelberg, E.C.G.: Die Wechselwirkungs Kräfte in der Elektrodynamik und in der Feldtheorie der Kernkräfte. Helv. Phys. Acta **11**, 225 (1938)
4. Ruegg H., Ruiz-Altaba M.: The Stückelberg field. Int. J. Mod. Phys. **A19**, 3265 (2004) [arXiv:hep-th/0304245]
5. Hooft, G.'t: Renormalizable Lagrangians for massive Yang–Mills fields. Nucl. Phys. **B35**, 167 (1971)
6. Hooft, G.'t, Veltman, M.J.G.: Regularization and renormalization of gauge fields. Nucl. Phys. **B44**, 189 (1972)
7. Hooft, G.'t.: Nobel lecture: a confrontation with infinity. Rev. Mod. Phys. **72**, 333 (2000)
8. Veltman, M.J.G.: Nobel lecture: from weak interactions to gravitation. Rev. Mod. Phys. **72**, 341 (2000)
9. Georgi, H.: Weak Interactions and Modern Particle Theory (revised and updated edition). Dover, New York (2009)