Luis Álvarez-Gaumé · Miguel Á. Vázquez-Mozo

An Invitation to Quantum Field Theory

Springer

Contents

Chapter 1
Why Do We Need Quantum Field Theory After All?

Quantum field theory is the basic tool to understand the physics of the elementary constituents of matter (see [1–15] for an incomplete list of textbooks in the subject). It is both a very powerful and a very precise framework: using it we can describe physical processes in a range of energies going from the few millions electrovolts typical of nuclear physics to the thousands of billions of the Large Hadron Collider (LHC). And all this with astonishing precision.

In this first chapter our aim is to explain why quantum mechanics is not enough and how quantum field theory is forced upon us by special relativity. We will review a number of riddles that appear in the attempt to extend the results of quantum mechanics to systems where relativistic effects cannot be ignored. Their resolution requires giving up the quantum mechanical description of a single particle to allow for the creation and annihilation of particles. As we will see, quantum fields provide the right tool to handle this.

1.1 Relativistic Quantum Mechanics

In spite of the impressive success of quantum mechanics in describing atomic physics, it was immediately clear after its formulation that its relativistic extension was not free of difficulties. These problems were clear already to Schrödinger, whose first guess for a wave equation of a free relativistic particle was the Klein–Gordon equation[1]

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(t, \mathbf{x}) = 0. \tag{1.1}$$

This equation follows directly from the relativistic "mass-shell" identity $E^2 = \mathbf{p}^2 + m^2$ using the correspondence principle

[1] We use natural units $\hbar = c = 1$. A summary of the units and conventions used in the book can be found in Appendix A.

L. Álvarez-Gaumé and M. Á. Vázquez-Mozo, *An Invitation to Quantum Field Theory*,
Lecture Notes in Physics 839, DOI: 10.1007/978-3-642-23728-7_1,

$$\begin{aligned}E &\to i\frac{\partial}{\partial t},\\ \mathbf{p} &\to -i\nabla.\end{aligned} \tag{1.2}$$

Plane wave solutions to the wave equation (1.1) are readily obtained

$$\psi(t,\mathbf{x}) = e^{-ip_\mu x^\mu} = e^{\mp iE_\mathbf{p}t + i\mathbf{p}\cdot\mathbf{x}} \tag{1.3}$$

with

$$E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}. \tag{1.4}$$

In order to have a complete basis of functions, we must include both signs in the exponent. The probability density is read from the time component of the conserved current

$$j_\mu = \frac{i}{2}\Big(\psi^*\partial_\mu\psi - \partial_\mu\psi^*\psi\Big), \tag{1.5}$$

Since $j^0 = E$, we find that it is not positive definite.

A complete, properly normalized, continuous basis of solutions of the Klein–Gordon equation (1.1) labelled by the momentum $\mathbf{p}$ is given by

$$\begin{aligned}f_p(t,\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2E_\mathbf{p}}}e^{-iE_\mathbf{p}t + i\mathbf{p}\cdot\mathbf{x}},\\ f_{-p}(t,\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2E_\mathbf{p}}}e^{iE_\mathbf{p}t - i\mathbf{p}\cdot\mathbf{x}}.\end{aligned} \tag{1.6}$$

Defining the inner product

$$\langle\psi_1|\psi_2\rangle = i\int d^3x\Big(\psi_1^*\partial_0\psi_2 - \partial_0\psi_1^*\psi_2\Big),$$

the states (1.6) form an orthonormal basis

$$\begin{aligned}\langle f_p|f_{p'}\rangle &= \delta(\mathbf{p}-\mathbf{p}'),\\ \langle f_{-p}|f_{-p'}\rangle &= -\delta(\mathbf{p}-\mathbf{p}'),\end{aligned} \tag{1.7}$$

$$\langle f_p|f_{-p'}\rangle = 0. \tag{1.8}$$

The wave functions $f_p(t,\mathbf{x})$ describe states with momentum $\mathbf{p}$ and energy $E_\mathbf{p} = \sqrt{\mathbf{p}^2+m^2}$. On the other hand, the wave functions $f_{-p}(t,\mathbf{x})$ not only have negative scalar product but they correspond to negative energy states

$$i\frac{\partial}{\partial t}f_{-p}(t,\mathbf{x}) = -\sqrt{\mathbf{p}^2+m^2}\,f_{-p}(t,\mathbf{x}). \tag{1.9}$$

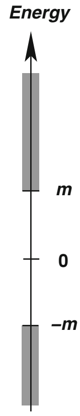

Fig. 1.1 Spectrum of the Klein–Gordon wave equation

Therefore the energy spectrum of the theory satisfies $|E| > m$ and is unbounded from below (see Fig. 1.1). Although in the case of a free theory the absence of a ground state is not necessarily a fatal problem, once the theory is coupled to the electromagnetic field this is the source of all kinds of disasters, since nothing can prevent the decay of any state by the emission of electromagnetic radiation.

The problem of the instability of the "first-quantized" relativistic wave equation can be heuristically tackled in the case of spin-$\frac{1}{2}$ particles, described by the Dirac equation

$$\left(-i\beta\frac{\partial}{\partial t} + \alpha\cdot\nabla - m\right)\psi(t,\mathbf{x}) = 0, \tag{1.10}$$

where α and β are 4×4 matrices

$$\alpha^i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{1.11}$$

with σ_i the Pauli matrices (see Appendix A) and the wave function $\psi(t,\mathbf{x})$ has four components: it is a Dirac spinor, an object that will be studied in more detail in Chap. 3. The wave equation (1.10) can be thought of as a kind of "square root" of the Klein–Gordon equation (1.1), since the latter can be obtained as

$$\begin{aligned}&\left(-i\beta\frac{\partial}{\partial t} + \alpha\cdot\nabla - m\right)^{\dagger}\left(-i\beta\frac{\partial}{\partial t} + \alpha\cdot\nabla - m\right)\psi(t,\mathbf{x})\\ &= \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(t,\mathbf{x}).\end{aligned} \tag{1.12}$$

An analysis of Eq. (1.10) along the lines of the one presented for the Klein–Gordon equation leads again to the existence of negative energy states and

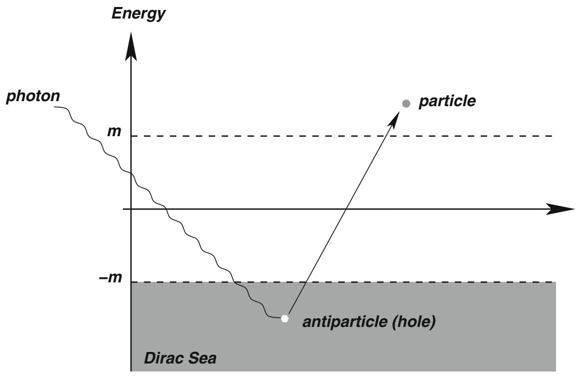

Fig. 1.2 Creation of a particle-antiparticle pair in the Dirac sea picture

a spectrum unbounded from below as in Fig. 1.1. Dirac, however, solved the instability problem by pointing out that now the particles are fermions and therefore they are subject to Pauli's exclusion principle. Hence, each state in the spectrum can be occupied by at most one particle, so the states with $E = m$ can be made stable if we assume that *all* the negative energy states are filled.

If Dirac's idea restores the stability of the spectrum by introducing a stable vacuum where all negative energy states are occupied, the so-called Dirac sea, it also leads directly to the conclusion that a single-particle interpretation of the Dirac equation is not possible. Indeed, a photon with enough energy ($E > 2m$) can excite one of the electrons filling the negative energy states, leaving behind a "hole" in the Dirac sea (see Fig. 1.2). This hole behaves as a particle with equal mass and opposite charge that is interpreted as a positron, so there is no escape to the conclusion that interactions will produce particle-antiparticle pairs out of the vacuum.

1.2 The Klein Paradox

In spite of the success of the heuristic interpretation of negative energy states in the Dirac equation this is not the end of the story. In 1929 Oskar Klein stumbled into an apparent paradox when trying to describe the scattering of a relativistic electron by a square potential using Dirac's wave equation [16] (for pedagogical reviews see [17–19]). In order to capture the essence of the problem without entering into unnecessary complication we will study Klein's paradox in the context of the Klein–Gordon equation.

Let us consider a square potential with height $V_0 > 0$ of the type showed in Fig. 1.3. A solution to the wave equation in regions I and II is given by

$$\begin{aligned} \psi_I(t,x) &= e^{-iEt+ip_1x} + Re^{-iEt-ip_1x}, \\ \psi_{II}(t,x) &= Te^{-iEt+ip_2x}, \end{aligned} \tag{1.13}$$

where the mass-shell condition implies

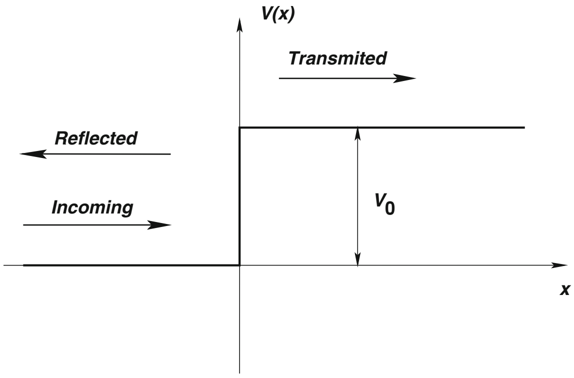

Fig. 1.3 Illustration of the Klein paradox

$$p_1 = \sqrt{E^2 - m^2}, \quad p_2 = \sqrt{(E - V_0)^2 - m^2}. \tag{1.14}$$

The constants R and T are computed by matching the two solutions across the boundary $x = 0$. The conditions $\psi_I(t, 0) = \psi_{II}(t, 0)$ and $\partial_x \psi_I(t, 0) = \partial_x \psi_{II}(t, 0)$ imply that

$$T = \frac{2p_1}{p_1 + p_2}, \quad R = \frac{p_1 - p_2}{p_1 + p_2}. \tag{1.15}$$

At first sight one would expect a behavior similar to the one encountered in the nonrelativistic case. If the kinetic energy is bigger than V_0 both a transmitted and reflected wave are expected, whereas when the kinetic energy is smaller than V_0 one only expects to find a reflected wave, the transmitted wave being exponentially damped within a distance of a Compton wavelength inside the barrier.

This is indeed what happens if $E - m > V_0$. In this case both p_1 and p_2 are real and we have a partly reflected, and a partly transmitted wave. In the same way, if $V_0 - 2m < E - m < V_0$ then p_2 is imaginary and there is total reflection.

However, in the case when $V_0 > 2m$ and the energy is in the range $0 < E - m < V_0 - 2m$ a completely different situation arises. In this case one finds that both p_1 and p_2 are real and therefore the incoming wave function is partially reflected and partially transmitted across the barrier. This is a shocking result, since it implies that there is a nonvanishing probability of finding the particle at any point across the barrier with negative kinetic energy ($E - m - V_0 < 0$)! This weird result is known as Klein's paradox.

As with the negative energy states, the Klein paradox results from our insistence in giving a single-particle interpretation to the relativistic wave function. In fact, a multiparticle analysis of the paradox [17] shows that what happens when $0 < E - m < V_0 - 2m$ is that the reflection of the incoming particle by the barrier is accompanied by the creation of particle-antiparticle pairs out of the energy of the barrier (notice that the condition implies that $V_0 > 2m$, the threshold for the creation of a particle-antiparticle pair).

This particle creation can be understood by noticing that the sudden potential step in Fig. 1.3 localizes the incoming particle with mass m in distances smaller than its Compton wavelength $\lambda = 1/m$. This can be seen by replacing the square potential by another one where the potential varies smoothly from 0 to $V_0 > 2m$ in distance scales larger than $1/m$. This case was worked out by Sauter shortly after Klein pointed out the paradox [20]. He considered a situation where the regions with $V = 0$ and $V = V_0$ are connected by a region of length d with a linear potential $V(x) = V_0 x/d$. When $d > 1/m$ he found that the transmission coefficient is exponentially small.[2]

1.3 From Wave Functions to Quantum Fields

The creation of particles is impossible to avoid whenever one tries to localize a particle of mass m within its Compton wavelength. Indeed, from the Heisenberg uncertainty relation we find that if $\Delta x \sim 1/m$, the fluctuations in the momentum will be of order $\Delta p \sim m$ and fluctuations in the energy of order

$$\Delta E \sim m \tag{1.16}$$

can be expected. Therefore, in a relativistic theory, the fluctuations of the energy are enough to allow for the creation of particles out of the vacuum. In the case of a spin-$\frac{1}{2}$ particle, the Dirac sea picture shows clearly how, when the energy fluctuations are of order m, electrons from the Dirac sea can be excited to positive energy states, thus creating electron–positron pairs.

It is possible to see how the multiparticle interpretation is forced upon us by relativistic invariance. In non-relativistic quantum mechanics observables are represented by self-adjoint operator that in the Heisenberg picture depend on time. Therefore measurements are localized in time but are global in space. The situation is radically different in the relativistic case. Since no signal can propagate faster than the speed of light, measurements have to be localized both in time and space. Causality demands then that two measurements carried out in causally-disconnected regions of space–time cannot interfere with each other. In mathematical terms this means that if $\mathscr{O}_{R_1}$ and $\mathscr{O}_{R_2}$ are the observables associated with two measurements localized in two causally-disconnected regions R_1, R_2 (see Fig. 1.4), they satisfy

$$[\mathscr{O}_{R_1}, \mathscr{O}_{R_2}] = 0, \quad \text{if}(x_1 - x_2)^2 < 0, \quad \text{for all} \quad x_1 \in R_1, \quad x_2 \in R_2. \tag{1.17}$$

Hence, in a relativistic theory, the basic operators in the Heisenberg picture must depend on the space–time position x^μ. Unlike the case in non-relativistic quantum mechanics, here the position $\mathbf{x}$ *is not* an observable, but just a label, similarly to the case of time in ordinary quantum mechanics. Causality is then imposed microscopically by requiring

---

2 In Sect. 13.1 we will see how, in the case of the Dirac field, this exponential behavior can be associated with the creation of electron–positron pairs due to a constant electric field (Schwinger effect).

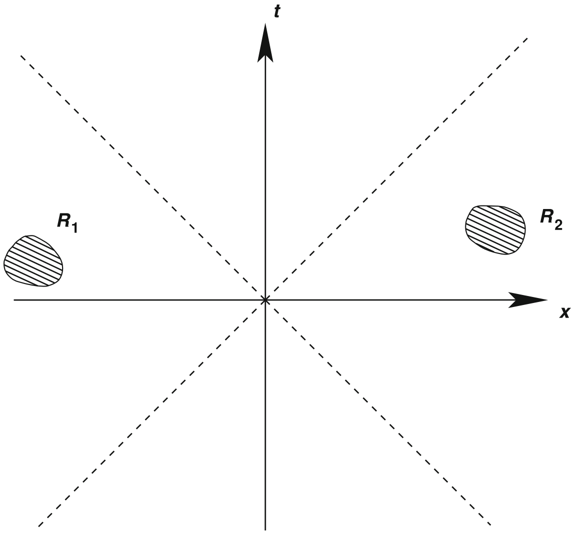

Fig. 1.4 Two regions R_1, R_2 that are causally disconnected

$$[\mathscr{O}(x), \mathscr{O}(y)] = 0, \quad \text{if } (x-y)^2 < 0. \tag{1.18}$$

A smeared operator $\mathscr{O}_R$ over a space–time region R can then be defined as

$$\mathscr{O}_R = \int d^4x\, \mathscr{O}(x) f_R(x) \tag{1.19}$$

where $f_R(x)$ is the characteristic function associated with R,

$$f_R(x) = \begin{cases} 1 & x \in R \\ 0 & x \notin R \end{cases}. \tag{1.20}$$

Equation (1.17) follows now from the microcausality condition (1.18).

Therefore, relativistic invariance forces the introduction of quantum fields. It is only when we insist in keeping a single-particle interpretation that we crash against causality violations. To illustrate the point, let us consider a single particle wave function $\psi(t, \mathbf{x})$ that initially is localized in the position $\mathbf{x} = 0$

$$\psi(0, \mathbf{x}) = \delta(\mathbf{x}). \tag{1.21}$$

Evolving this wave function using the Hamiltonian $H = \sqrt{-\nabla^2 + m^2}$ we find that the wave function can be written as

$$\psi(t, \mathbf{x}) = e^{-it\sqrt{-\nabla^2+m^2}}\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x} - it\sqrt{k^2+m^2}}. \tag{1.22}$$

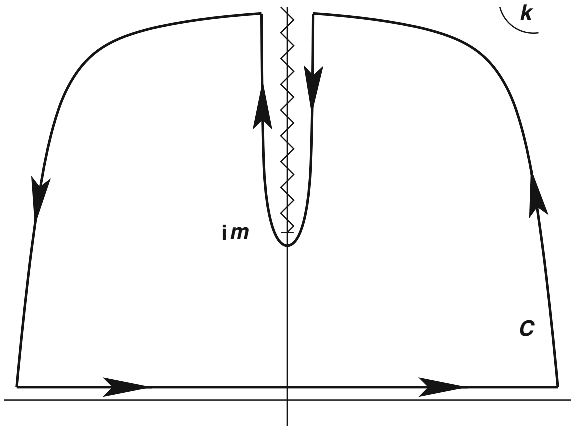

Fig. 1.5 Complex contour C for the computation of the integral in Eq. (1.23)

Integrating over the angular variables, the wave function can be recast in the form

$$\psi(t, \mathbf{x}) = \frac{-i}{4\pi^2|\mathbf{x}|} \int_{-\infty}^{\infty} k dk e^{ik|\mathbf{x}|} e^{-it\sqrt{k^2+m^2}}. \tag{1.23}$$

The resulting integral can be evaluated using the complex integration contour C shown in Fig. 1.5. The result is that, for any $t > 0$, $\psi(t, \mathbf{x}) \neq 0$ for any $\mathbf{x}$. If we insist in interpreting the wave function $\psi(t, \mathbf{x})$ as the probability density of finding the particle at the location $\mathbf{x}$ at the time t, the probability leaks out of the light cone, thus violating causality.

The bottom line of the analysis of this chapter is clear: a fully relativistic quantum theory must give up the idea of describing the system in terms of the wave function of a *single* particle. As a matter of fact, relativistic quantum mechanics is, at best, a narrow boundary area. It might be a useful tool to compute the first relativistic corrections in certain quantum systems. However it runs into serious trouble as soon as one tries to use it for a full-fledged relativistic description of the quantum phenomena. Next we will see how quantum field theory provides the right framework to handle these problems.

References

1. Bjorken, J.D, Drell, S.D: Relativistic Quantum Fields. McGraw-Hill, New York (1965)
2. Itzykson, C., Zuber, J.-B.: Quantum Field Theory. McGraw-Hill, New York (1980)
3. Aitchinson, I.J.R.: An Informal Introduction to Gauge Field Theories. Cambridge University Press, Cambridge (1982)
4. Mandl, F., Shaw, G.: Quantum Field Theory. Wiley, New York (1984)
5. Ramond, P.: Field Theory: A Modern Primer. Addison-Wesley, Reading (1990)

6. Sterman, G.: An Introduction to Quantum Field Theory. Cambridge University Press, Cambridge (1995)
7. Peskin, M.E, Schroeder, D.V: An Introduction to Quantum Field Theory. Addison Wesley, Reading (1995)
8. Weinberg, S.: The Quantum Theory of Fields. vol. 1–3, Cambridge University Press, Cambridge (1995)
9. Deligne, P. et al. (eds) Quantum Fields and Strings: A Course for Mathematicians. American Mathematical Society, Princeton (1999)
10. Gribov, V.N., Nyiri, J.: Quantum Electrodynamics. Cambridge University Press, Cambridge (2001)
11. Zee, A.: Quantum Field Theory in a Nutshell. 2nd edn. 2010 Princeton (2003)
12. DeWitt, B.S.: The Global Approach to Quantum Field Theory. vols. 1 and 2, Oxford University Press, Oxford (2003)
13. Nair, V.P.: Quantum Field Theory. A Modern Perspective. Springer, New York (2005)
14. Maggiore, M.: A Modern Introduction to Quantum Field Theory. Oxford University Press, Oxford (2005)
15. Banks, T.: Modern Quantum Field Theory. Cambridge University Press, Cambridge (2008)
16. Klein, O.: Die Reflexion von Elektronen an einem Potentialsprung nach der relativischen Dynamik von Dirac. Z. Phys. **53**, 157 (1929)
17. Holstein, B.R.: Klein's paradox. Am. J. Phys. **66**, 507 (1998)
18. Dombey, N., Calogeracos, A.: Seventy years of the Klein paradox. Phys. Rep. **315**, 41 (1999)
19. Dombey, N., Calogeracos, A.: History and physics of the Klein paradox. Contemp. Phys. **40**, 313 (1999) (quant-ph/9905076)
20. Sauter, F.: Zum Kleinschen Paradoxon. Z. Phys. **73**, 547 (1932)