

QCD@LHC for beginners

Lesson 4.5

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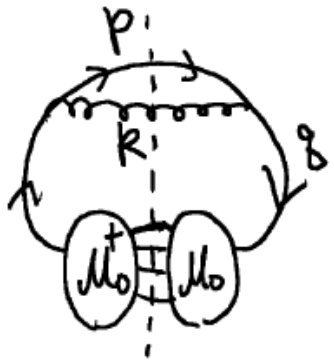
Outline

- Lesson 4
 - Around "IR divergence"
 - IR divergence in QCD/KLN theorem
 - Factorization
 - DGLAP Equation/PDF

GOAL: Understanding
(1) IR divergence structure.
(2) what is PDF.



IR divergence/DIGLAP Eq.

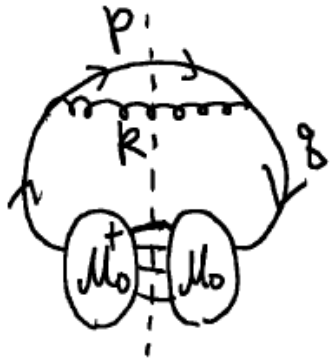


$$= g^2 \text{Tr} \left[\bar{u}(p) \not{\epsilon}^*(k) \frac{\sum u(q) \bar{u}(q)}{q^2} |\tilde{M}_0|^2 \frac{\sum u(q) \bar{u}(q)}{q^2} \not{\epsilon}(k) u(p) \right]$$

$$q = p + k$$

IR divergence/DIGLAP Eq.

$$|M_0|^2$$

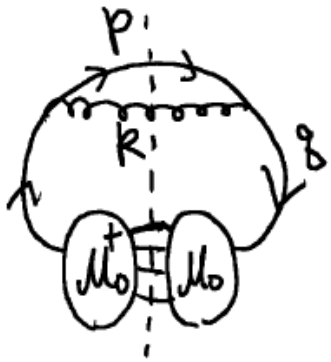


$$= g^2 \text{Tr} \left[\bar{u}(p) \not{\epsilon}^*(k) \frac{\sum u(q) \bar{u}(q)}{q^2} |\tilde{M}_0|^2 \frac{\sum u(q) \bar{u}(q)}{q^2} \not{\epsilon}(k) u(p) \right]$$

$$= |M_0|^2 \frac{g^2}{(q^2)^2} \frac{1}{2} \sum_{\lambda} \text{Tr} [\not{p} \not{\epsilon}_{\lambda}(k) \not{p} \not{\epsilon}_{\lambda}(k)]$$

$$q = p + k$$

IR divergence/DIGLAP Eq.



$$\begin{aligned}
 &= g^2 \text{Tr} \left[\bar{u}(p) \not{\epsilon}^*(k) \frac{\sum u(q) \bar{u}(q)}{q^2} |\tilde{M}_0|^2 \frac{\sum u(q) \bar{u}(q)}{q^2} \not{\epsilon}(k) u(p) \right] \\
 &= |M_0|^2 \frac{g^2}{(q^2)^2} \frac{1}{2} \sum_{\lambda} \text{Tr} [\not{p} \not{\epsilon}_{\lambda}(k) \not{q} \not{\epsilon}_{\lambda}(k)] \equiv R_I \quad q=p+k
 \end{aligned}$$

$|M_0|^2$

$$\sum_{\lambda} \epsilon_{\lambda}^{\mu}(k) \epsilon_{\lambda}^{\nu}(k) = -g^{\mu\nu} + \frac{k^{\mu} n^{\nu} + k^{\nu} n^{\mu}}{k \cdot n}$$

$$R_I = 2g_s^2 \frac{k_I}{x(1-x)} (P(x) + (1-x)\epsilon_{IR})$$

$$P(x) = \frac{1+x^2}{1-x}$$

IR divergence/DIGLAP Eq.

Factorization: Coll. Approx.

Matrix Element

$$\left| \mathcal{M}_{N+1}^{(d)} \right|^2 = \left| \mathcal{M}_N^{(4)} \left(q \rightarrow \sum_{i=1}^N q_i \right) \right|^2 \frac{16\pi}{s\mu^{2\varepsilon_{IR}}} f_c \frac{\alpha_s}{2\pi} P(x) \frac{1}{k_T^2} \left(\frac{1-x}{x} \right)$$
$$q^\mu = p_1^\mu + p_2^\mu - k^\mu,$$

Phase Space

$$d\Phi_{N+1}^{(d)} = d\Phi_N^{(4)} \left(q \rightarrow \sum_{i=1}^N q_i \right)$$
$$\times \frac{1}{16\pi^2 \Gamma(1 + \varepsilon_{IR})} \left(\frac{k_T^2}{4\pi x^2} \right)^{\varepsilon_{IR}} \frac{1}{1-x} dx dk_T^2$$

IR divergence/DIGLAP Eq.

Factorization: Coll. Approx.

Splitting functions:

$$P_{qq}^{(0)}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right],$$

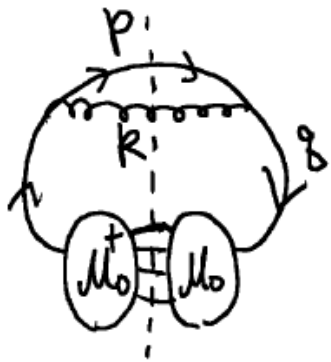
$$P_{qg}^{(0)}(z) = T_F [z^2 + (1-z)^2],$$

$$P_{gq}^{(0)}(z) = C_F \frac{1+(1-z)^2}{z},$$

$$P_{gg}^{(0)}(z) = 2C_A \left[\frac{z}{(1-z)_+} - \frac{1-z}{z} + z(1-z) \right] + 2\pi\beta_0 \delta(1-z),$$

IR divergence/DIGLAP Eq.

Factorization: Coll. Approx. Cross Section



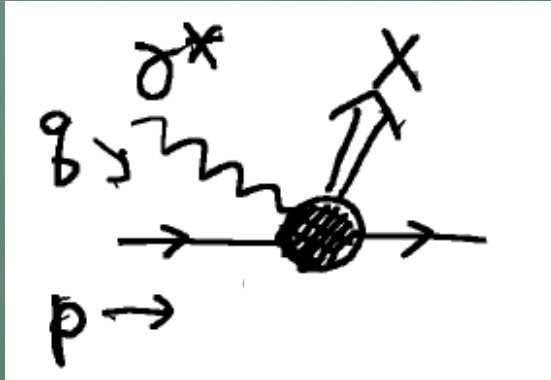
$$\begin{aligned}
 &= \frac{1}{(2p_1^0)(2p_2^0)v_{rel}} \int_{\Omega_{full}} d\Phi_{N+1}^{(d)} \left| \mathcal{M}_{N+1}^{(d)} \right|^2, \\
 &= \left(\frac{s}{4\pi\mu^2} \right)^{\epsilon_{IR}} \frac{B(\epsilon_{IR}, \epsilon_{IR})}{2\Gamma(1 + \epsilon_{IR})} f_c \frac{\alpha_s}{2\pi} \int_0^1 dx \sigma_0(xs) P(x) \left(\frac{1-x}{x} \right)^{2\epsilon_{IR}}
 \end{aligned}$$

$$\frac{e^{-\gamma\epsilon} P(x) B(\epsilon, \epsilon) \left(\frac{s}{\mu^2} \right)^\epsilon}{2\Gamma(\epsilon + 1)}$$

$$\frac{P(x)}{\epsilon} + P(x) \log \left(\frac{s}{\mu^2} \right) + O(\epsilon^1)$$

IR divergence/DIGLAP Eq.

Deep Inelastic Cross Section:



$$= \sigma_{DIS}(p^2, Q^2 = -q^2)$$

$$\sigma_{DIS}(p^2, Q^2) = 1 + ag^2 \ln \left(\frac{Q^2}{p^2} \right) + \dots$$

$$= 1 + ag^2 \left(\ln \left(\frac{\mu_F^2}{p^2} \right) + \ln \left(\frac{Q^2}{\mu_F^2} \right) \right) + \dots$$

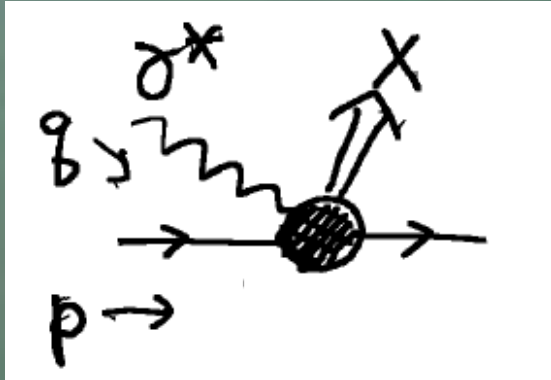
$$= \left[1 + ag^2 \ln \left(\frac{\mu_F^2}{p^2} \right) + \dots \right] \left[1 + ag^2 \ln \left(\frac{Q^2}{\mu_F^2} \right) + \dots \right]$$

PDF

Hard Part

IR divergence/DIGLAP Eq.

Deep Inelastic Cross Section:



$$= \sigma_{DIS}(p^2, Q^2 = -q^2)$$

Cross section must be scale independent:

$$\frac{d\sigma_{DIS}}{d \ln \mu_F^2} = 0$$

$$\sigma_{DIS}(p, Q^2) = \int dx f(x, \mu_F^2) \sigma_H(x \cdot s, \mu_F^2)$$

$$\frac{d\sigma_{DIS}}{d \ln \mu_F^2} = \int dx \frac{df(x, \mu_F^2)}{d \ln \mu_F^2} \sigma_H(x \cdot s, \mu_F^2) + \int dx f(x, \mu_F^2) \frac{d\sigma_H(x \cdot s, \mu_F^2)}{d \ln \mu_F^2}$$

IR divergence/DIGLAP Eq.

Deep Inelastic Cross Section:

$$\frac{d\sigma_{DIS}}{d\ln\mu_F^2} = \int dx \frac{df(x, \mu_F^2)}{d\ln\mu_F^2} \sigma_H(x \cdot s, \mu_F^2) + \int dx f(x, \mu_F^2) \frac{d\sigma_H(x \cdot s, \mu_F^2)}{d\ln\mu_F^2}$$

$$\sigma_H \propto \frac{\alpha_s}{2\pi} \left(\frac{P(x)}{\epsilon} + P(x) \ln \left(\frac{Q^2}{\mu_F^2} \right) + O(\epsilon) \right)$$

$$\frac{d\sigma_H(x \cdot s, \mu_F^2)}{d\ln\mu_F^2} = -\frac{\alpha_s}{2\pi} P(x)$$

IR divergence/DIGLAP Eq.

Deep Inelastic Cross Section:

$$\frac{d\sigma_{DIS}}{d\ln\mu_F^2} = \int dx \frac{df(x, \mu_F^2)}{d\ln\mu_F^2} \sigma_H(x \cdot s, \mu_F^2) + \int dx f(x, \mu_F^2) \frac{d\sigma_H(x \cdot s, \mu_F^2)}{d\ln\mu_F^2}$$

$$\sigma_H \propto \frac{\alpha_s}{2\pi} \left(\frac{P(x)}{\epsilon} + P(x) \ln \left(\frac{Q^2}{\mu_F^2} \right) + O(\epsilon) \right)$$

$$\frac{d\sigma_{DIS}}{d\ln\mu_F^2} = 0$$

$$\frac{d\sigma_H(x \cdot s, \mu_F^2)}{d\ln\mu_F^2} = -\frac{\alpha_s}{2\pi} P(x)$$

$$\frac{df(x, \mu_F^2)}{d\ln\mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi} \int \frac{dz}{z} P(z) f\left(\frac{x}{z}, \mu_F^2\right)$$

IR divergence/DGLAP Equation

DGLAP Equation

Splitting function ← pQCD

$$\frac{dD(x, Q^2)}{d \ln Q^2} = \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} P_+(x/y) D(y, Q^2)$$

$$D(x, Q^2) = \Pi(Q^2, Q_s^2) D(x, Q_s^2) + \frac{\alpha}{2\pi} \int_{Q_s^2}^{Q^2} \frac{dK^2}{K^2} \Pi(Q^2, K^2) \int_x^{1-\epsilon} \frac{dy}{y} P(y) D(x/y, K^2)$$

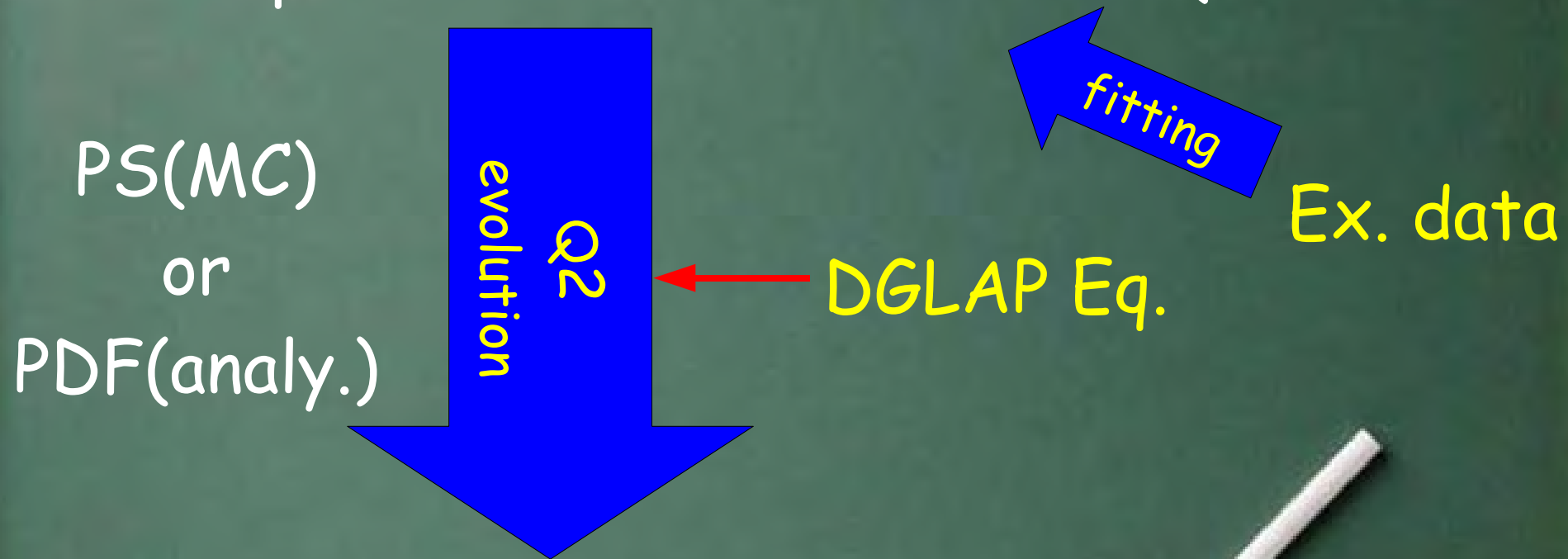
$$\Pi(Q^2, Q_s^2) = \exp\left(-\frac{\alpha}{2\pi} \int_{Q_s^2}^{Q^2} \frac{dK^2}{K^2} \int_0^{1-\epsilon} dx P(x)\right)$$

Sudakov Factor

non-branch provability

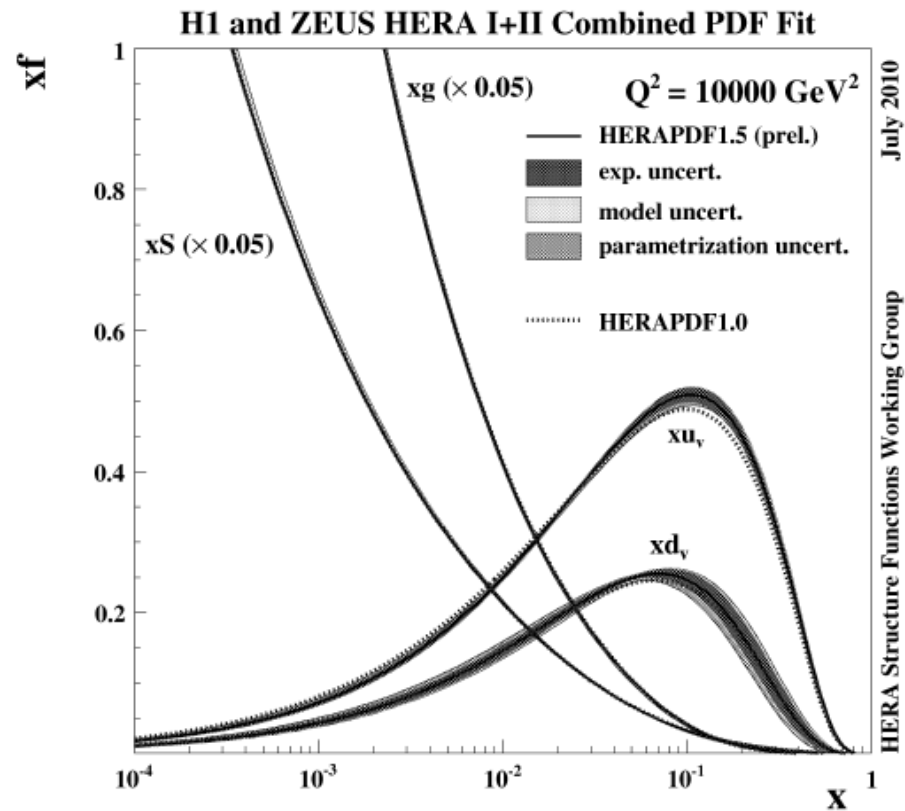
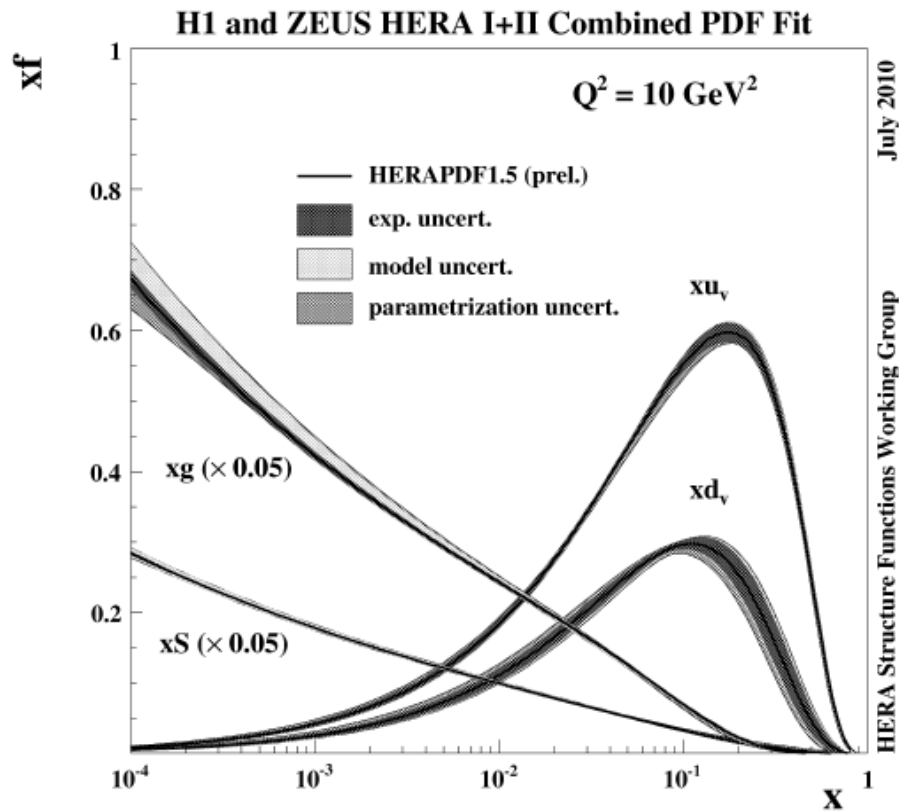
IR divergence/DGLAP Equation

Initial parton distribution @ Low Q^2



Parton distribution @ High Q^2

IR divergence/DGLAP Equation




IR divergence/DGLAP Equation

• PDF:

- Initial distribution: data fitting
- Q^2 evolution: Analytic solution of DGLAP Eq.
- No kinematical information

• PS:

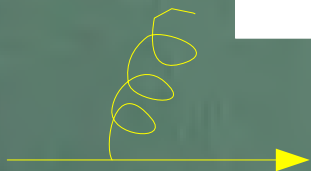
- Initial distribution: from PDF
 - Q^2 evolution: MC method to solve DGLAP Eq.
 - generate P_t distribution
- 

IR divergence/DGLAP Equation

DGLAP Eq. (Integral-differential equation)

$$\frac{d D(x, Q^2)}{d \log(Q^2)} = \frac{\alpha}{2\pi} \int_0^x \frac{dy}{y} P(x/y) D(y, Q^2)$$

$q \rightarrow q+g$: $P(x) = \left(\frac{1+x^2}{1-x} \right) \rightarrow \theta(1-\epsilon-x) - \delta(1-x) \int_0^{1-\epsilon} dy P(y)$



$$\frac{d D}{d K^2} K^2 + \frac{\alpha}{2\pi} c(\epsilon) D = \frac{\alpha}{2\pi} P \otimes D$$

$$c(\epsilon) = \int_0^{1-\epsilon} dz p(z)$$



IR divergence/DGLAP Equation


$$\frac{d \Pi}{d K^2} K^2 + \frac{\alpha}{2\pi} c \Pi = 0$$


Sudakov form factor
↓
non-branching prob.

Inhomogeneous Eq.

$$\frac{d D}{d K^2} K^2 + \frac{\alpha}{2\pi} c D = \frac{\alpha}{2\pi} P \otimes D$$

$$D(x, K^2) = \hat{D}(x, K^2) \Pi(K^2, Q^2)$$


$$\frac{d \hat{D}}{d K^2} K^2 = \frac{\alpha}{2\pi} P \otimes \hat{D}$$

$$\hat{D} = \frac{\alpha}{2\pi} \int P \otimes \hat{D} \frac{d K^2}{K^2} + c$$


IR divergence/DGLAP Equation

Integral equation

$$D(x, Q^2) = \Pi(Q^2, Q_s^2) D(x, Q_s^2) + \frac{\alpha}{2\pi} P \otimes D(x, K^2) \frac{dK^2}{K^2}$$

$$\Pi(Q_1^2, Q_0^2) = -\exp\left(\int_{Q_0^2}^{Q_1^2} \frac{\alpha}{2\pi} \frac{dK^2}{K^2} \int_0^{1-\epsilon} dx P(x)\right)$$

Method of successive substitution

Equation $\phi(x) = f(x) + \lambda \int_a^x K(x, y) \phi(y) dy$

0th approx.

$$\phi_0(x) = f(x)$$

1st

$$\phi_1(x) = f(x) + \lambda \int_a^x K(x, y) \phi_0(y) dy$$

2nd

$$\phi_2(x) = f(x) + \lambda \int_a^x K(x, y) \phi_1(y) dy$$

IR divergence/DGLAP Equation

Method of successive substitution

No emission

1 parton emission

$$\begin{aligned}\phi_2 = & f(x) + \lambda \int_a^x K(x, y) \phi(y) dy \\ & + \lambda^2 \int_{y_1}^x dy \int_a^{y_1} dy_1 K(x, y_1) \phi(y_1) K(y_1, y) \phi(y)\end{aligned}$$

2 parton emission



IR divergence/DGLAP Equation

Method of successive substitution

$$\phi(x) = f(x) + \lambda \int_a^x K(x, y) \phi(y) dy$$

$$\begin{aligned} \phi_2 = & f(x) + \lambda \int_a^x K(x, y) \phi(y) dy \\ & + \lambda^2 \int_{y_1}^x dy \int_a^{y_1} dy_1 K(x, y_1) \phi(y_1) K(y_1, y) \phi(y) \end{aligned}$$

$$\phi_n(x) = f(x) + \sum_{l=1}^n \lambda^l \int_a^x K_l(x, y) f(y) dy$$

$$K_l(x, y) = \int \cdots \int dy_1 \cdots dy_{l-1} K(x, y_1) \cdots K(y_{l-1}, y)$$

IR divergence/DGLAP Equation

The diagram illustrates the DGLAP equation as a sum of terms representing different orders of gluon emission. Each term consists of a horizontal line with an arrow pointing right, representing a quark. Above the line, there are wavy lines representing gluons. The first term shows a quark with two gluons and an ellipsis, followed by an equals sign and a quark with one gluon. The second term is added to the first, showing a quark with two gluons and a quark with a more complex gluon structure, with a plus sign and the number 2 below. The third term is added, showing a quark with three gluons and an ellipsis, with a plus sign and the number 3! below. The final result is shown as an equals sign followed by the expression $\exp(\text{diagram})$ in red, where the diagram is a quark with one gluon.

$$\begin{aligned} & \text{[Diagram: quark with 2 gluons]} \dots \text{[Diagram: quark with 1 gluon]} = \text{[Diagram: quark with 1 gluon]} \\ & + \frac{\text{[Diagram: quark with 2 gluons]} + \text{[Diagram: quark with 2 gluons]}}{2} \\ & + \frac{\text{[Diagram: quark with 3 gluons]} + \dots}{3!} + \dots \\ & = \exp(\text{[Diagram: quark with 1 gluon]}) \end{aligned}$$

IR divergence/DGLAP Equation

The diagrammatic equation illustrates the DGLAP equation. On the left, a blue horizontal line with an arrow pointing right represents a quark. Above it, a series of diagrams shows the emission of gluons (represented by curly lines) at various points along the quark line. The first diagram shows a single gluon emission. The second shows two gluon emissions. The third shows three gluon emissions. The fourth shows a gluon emission followed by a self-energy loop on the quark line. The fifth shows a gluon emission followed by a gluon self-energy loop. The sixth shows a gluon emission followed by a gluon exchange between two quark lines. The seventh shows a gluon emission followed by a gluon exchange between two gluon lines. The eighth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The ninth shows a gluon emission followed by a gluon exchange between two gluon lines. The tenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The eleventh shows a gluon emission followed by a gluon exchange between two gluon lines. The twelfth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The thirteenth shows a gluon emission followed by a gluon exchange between two gluon lines. The fourteenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The fifteenth shows a gluon emission followed by a gluon exchange between two gluon lines. The sixteenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The seventeenth shows a gluon emission followed by a gluon exchange between two gluon lines. The eighteenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The nineteenth shows a gluon emission followed by a gluon exchange between two gluon lines. The twentieth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The twentieth diagram is followed by an ellipsis and a final diagram showing a single gluon emission. The sum of these diagrams is equal to the sum of the diagrams on the right. The right side of the equation shows a blue horizontal line with an arrow pointing right, representing a quark. Above it, a series of diagrams shows the emission of gluons (represented by curly lines) at various points along the quark line. The first diagram shows a single gluon emission. The second shows two gluon emissions. The third shows three gluon emissions. The fourth shows a gluon emission followed by a self-energy loop on the quark line. The fifth shows a gluon emission followed by a gluon self-energy loop. The sixth shows a gluon emission followed by a gluon exchange between two quark lines. The seventh shows a gluon emission followed by a gluon exchange between two gluon lines. The eighth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The ninth shows a gluon emission followed by a gluon exchange between two gluon lines. The tenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The eleventh shows a gluon emission followed by a gluon exchange between two gluon lines. The twelfth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The thirteenth shows a gluon emission followed by a gluon exchange between two gluon lines. The fourteenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The fifteenth shows a gluon emission followed by a gluon exchange between two gluon lines. The sixteenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The seventeenth shows a gluon emission followed by a gluon exchange between two gluon lines. The eighteenth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The nineteenth shows a gluon emission followed by a gluon exchange between two gluon lines. The twentieth shows a gluon emission followed by a gluon exchange between a quark and a gluon line. The twentieth diagram is followed by an ellipsis and a final diagram showing a single gluon emission.

Next Lesson!

$= \exp(\text{diagram})$