QCD@LHC for beginners

Lesson 4

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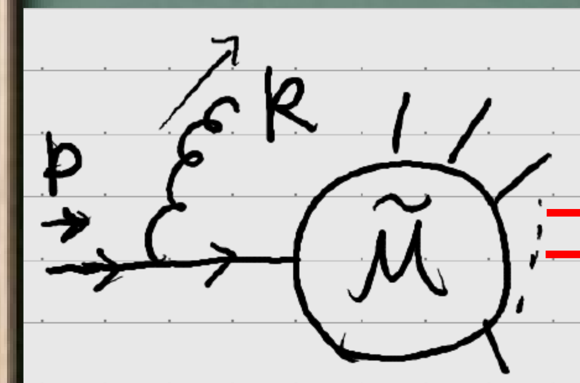
Outline

- Lesson 4
  - Around "IR divergence"
    - IR divergence in QCD/KLN theorem
    - Factorization
    - DGLAP Equation/PDF

GOAL : Understanding
(1) IR divergence structure.
(2) what is PDF.

IR divergence/Structure

- For QED case:



$k_0 \doteqdot 0$ Soft or collinear $\vec{k} \,//\, \vec{p}$

$$= \tilde{\mathcal{M}} \frac{i}{\not p - \not k - m_e} i g \gamma^\mu u(p) \rightarrow \tilde{\mathcal{M}} g \frac{\xi p^\mu}{p \cdot k} u(p)$$

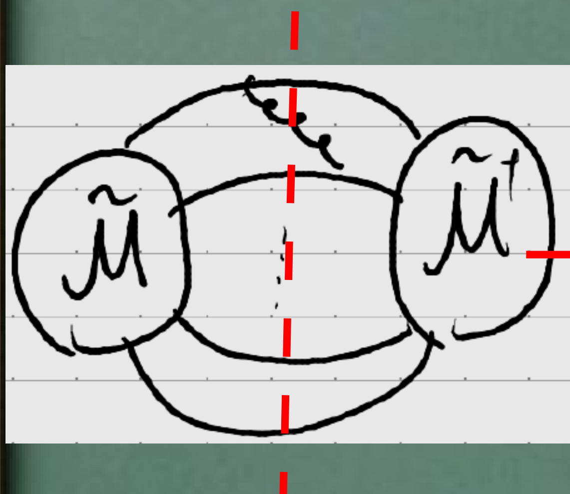
$$\left.\begin{aligned}(\not p - m_e)u(p) &= 0 \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}\end{aligned}\right\}$$

$$p - k \approx \xi p$$

$$p \cdot k = |\vec{k}|(E_p - |\vec{p}| \cos\theta)$$

IR divergence/Structure

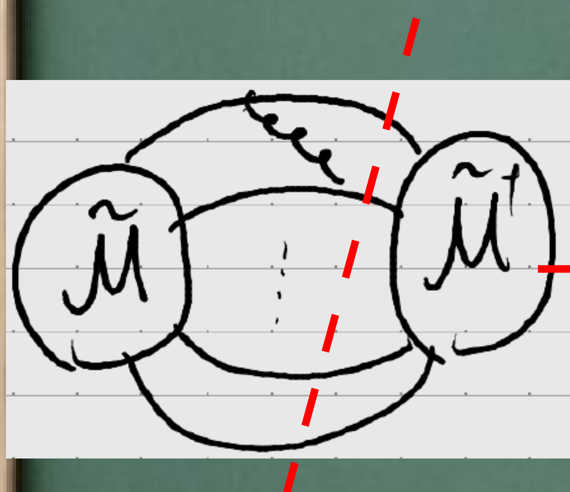
- For QED case:



$$\int d|\vec{k}|^2 d\cos\theta^{(\prime)} \frac{\xi\xi' p\cdot p'}{|\vec{k}|^2(E_p-|\vec{p}|\cos\theta)(E_{p'}-|\vec{p}'|\cos\theta')}$$

d=4+2 ε_{IR}

same IR structure



$$-\int \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|} \frac{\xi\xi' p\cdot p'}{|\vec{k}|^2(E_p-|\vec{p}|\cos\theta)(E_{p'}-|\vec{p}'|\cos\theta')}$$

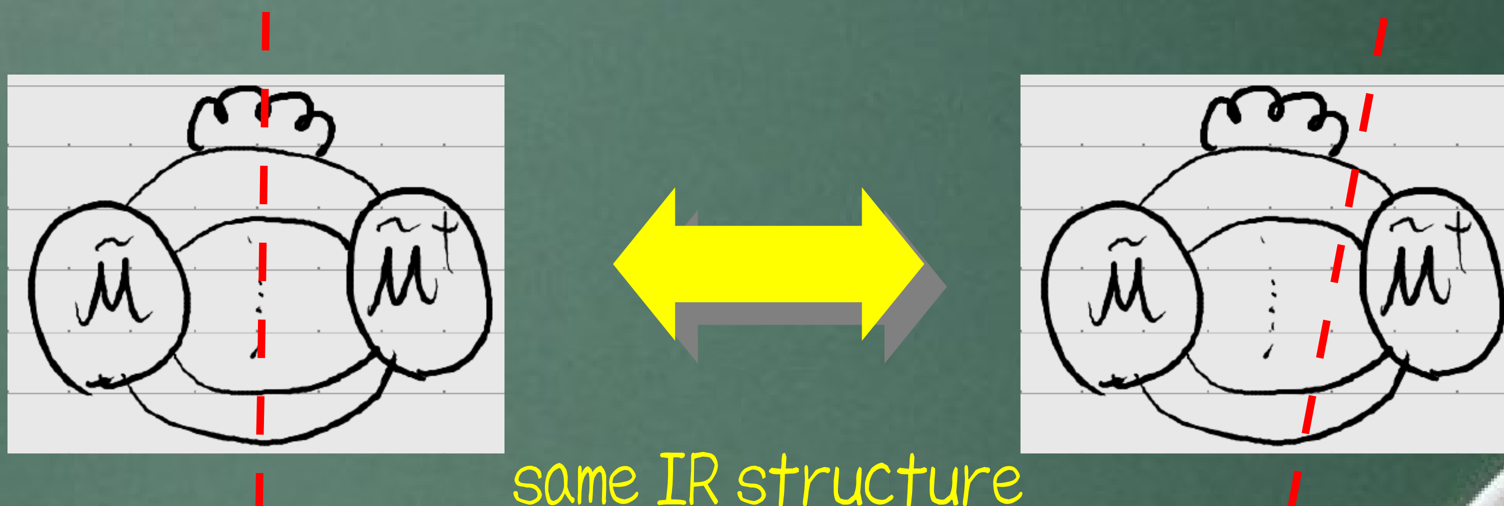
$1/\varepsilon_{IR}-\gamma_E+\ln(4\pi)$, $\ln(m_e^2/s)$

IR divergence/Structure

- For QED case:

$$\int d|\vec{k}|^2 d\cos\theta^{(\prime)} \frac{\xi\xi' p\cdot p'}{|\vec{k}|^2 (E_p - |\vec{p}|\cos\theta)(E_{p'} - |\vec{p}'|\cos\theta')}$$

remove '
$p\cdot p = m_e^2$



$d=4+2\varepsilon_{IR}$

$$-\int \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|} \frac{\xi\xi' p\cdot p'}{|\vec{k}|^2 (E_p - |\vec{p}|\cos\theta)(E_{p'} - |\vec{p}'|\cos\theta')}$$

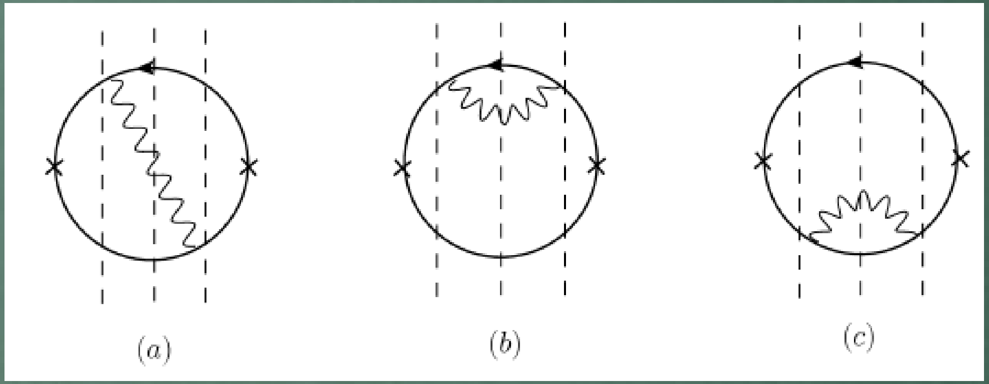
$1/\varepsilon_{IR} - \gamma_E + \ln(4\pi)$, $\alpha/(2\pi)\ln(Q^2/m_e^2)$, $\alpha/(2\pi)\ln(Q^2/\mu_F^2)$

IR divergence/Structure

- For QED case: Bloch-Nordsieck Theorem

$$d\sigma^{\text{virt.}} + d\sigma^{\text{soft real}} = \text{finite}$$

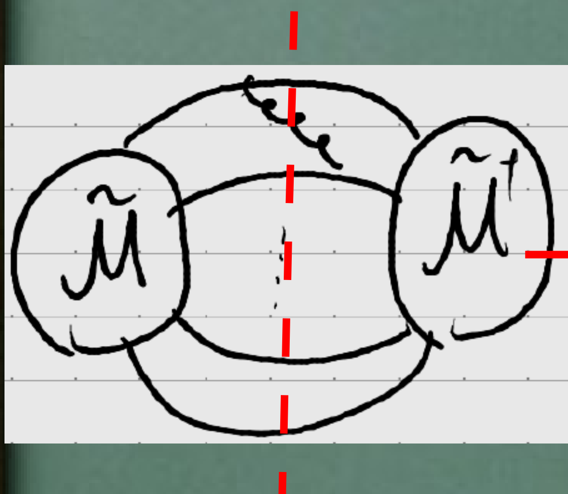
Summing up all possible graphs and cuts.



This theorem is proved for all order of QED.

IR divergence/Structure

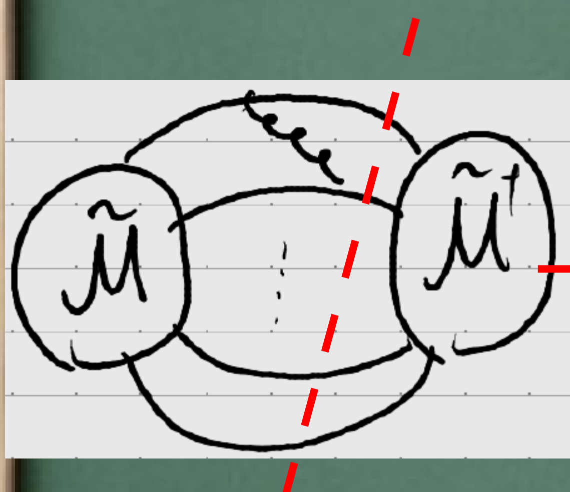
- For QCD case: massless quarks $E_p = |\vec{p}|$



$$\int d|\vec{k}|^2 d\cos\theta^{(\prime)} \frac{\xi\xi' p\cdot p'}{|\vec{k}|^2(E_p - |\vec{p}|\cos\theta)(E_{p'} - |\vec{p}'|\cos\theta')}$$

$d=4+2\varepsilon_{IR}$

same IR structure



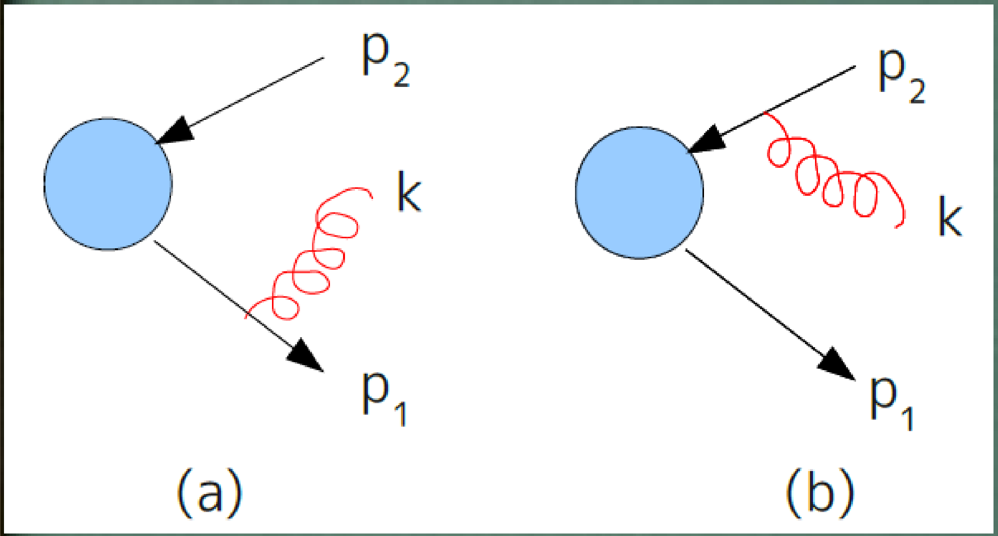
$$-\int \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|} \frac{\xi\xi' p\cdot p'}{|\vec{k}|^2(E_p - |\vec{p}|\cos\theta)(E_{p'} - |\vec{p}'|\cos\theta')}$$

$1/\varepsilon^2_{IR}$, $1/\varepsilon_{IR}$, $\alpha_s/(4\pi)\ln(Q^2/\mu_F^2)$

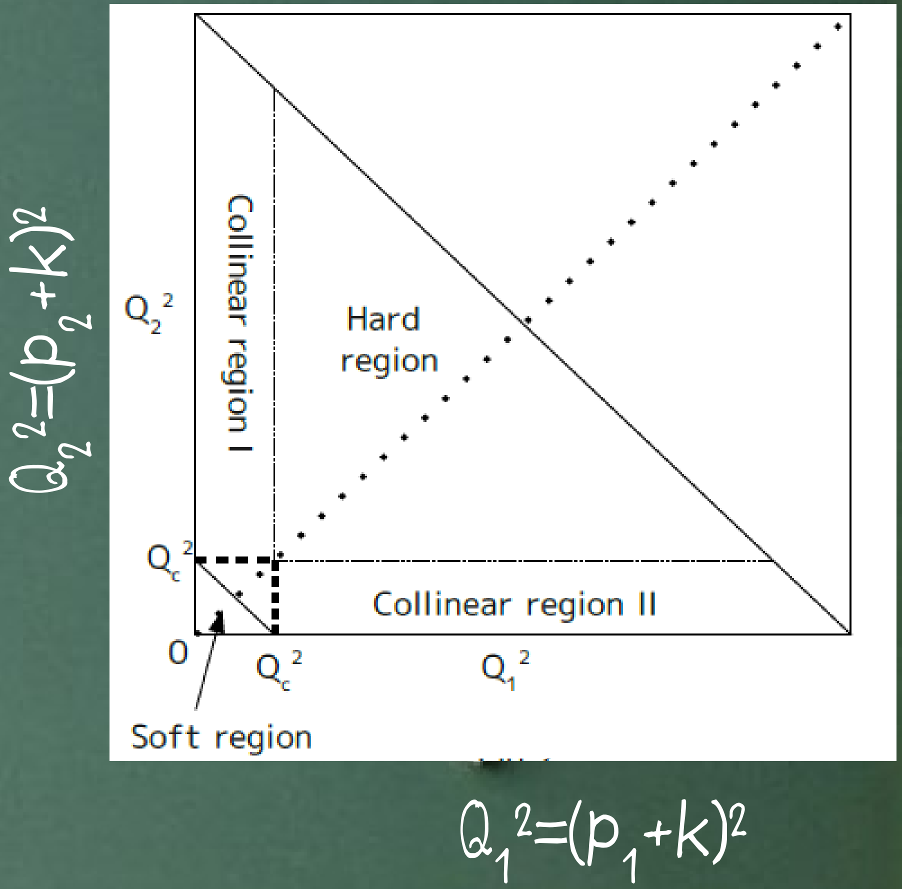
IR divergence/Structure

- For QCD case: massless quarks

Final state radiation



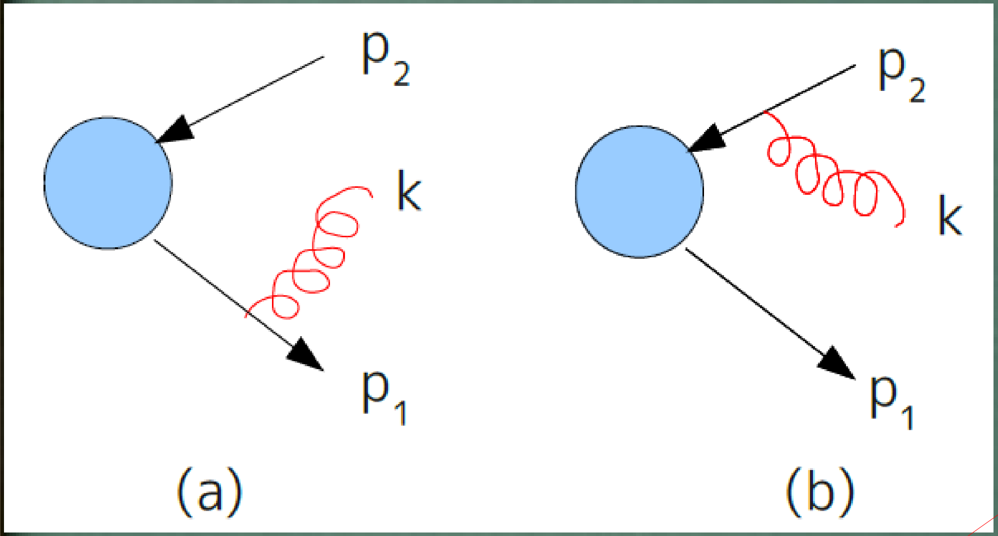
Daritz Plot



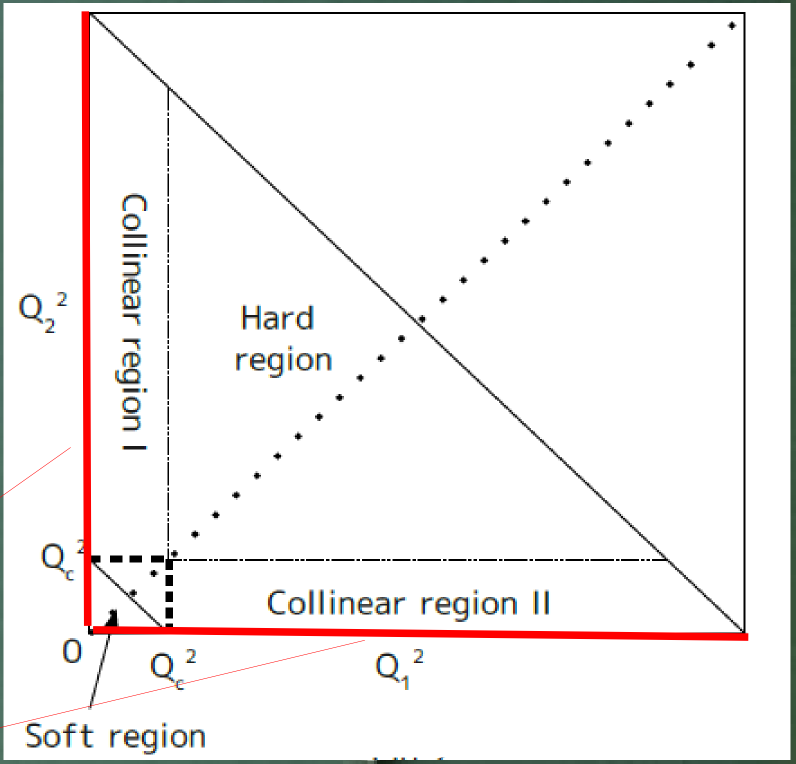
IR divergence/Structure

- For QCD case: massless quarks

Final state radiation



Daritz Plot



$Q_2^2=(p_2+k)^2$

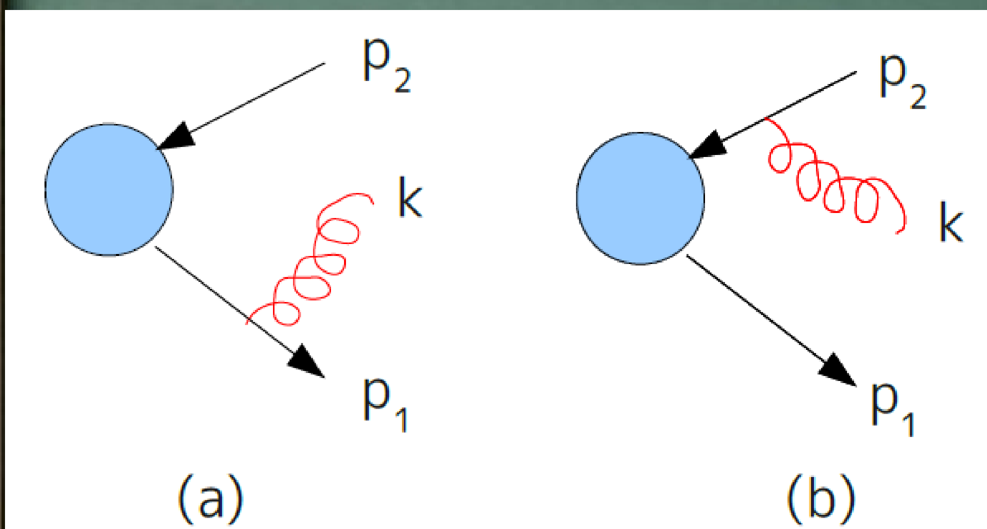
$Q_1^2=(p_1+k)^2$

Collinear divergence: $1/\varepsilon_{IR}$

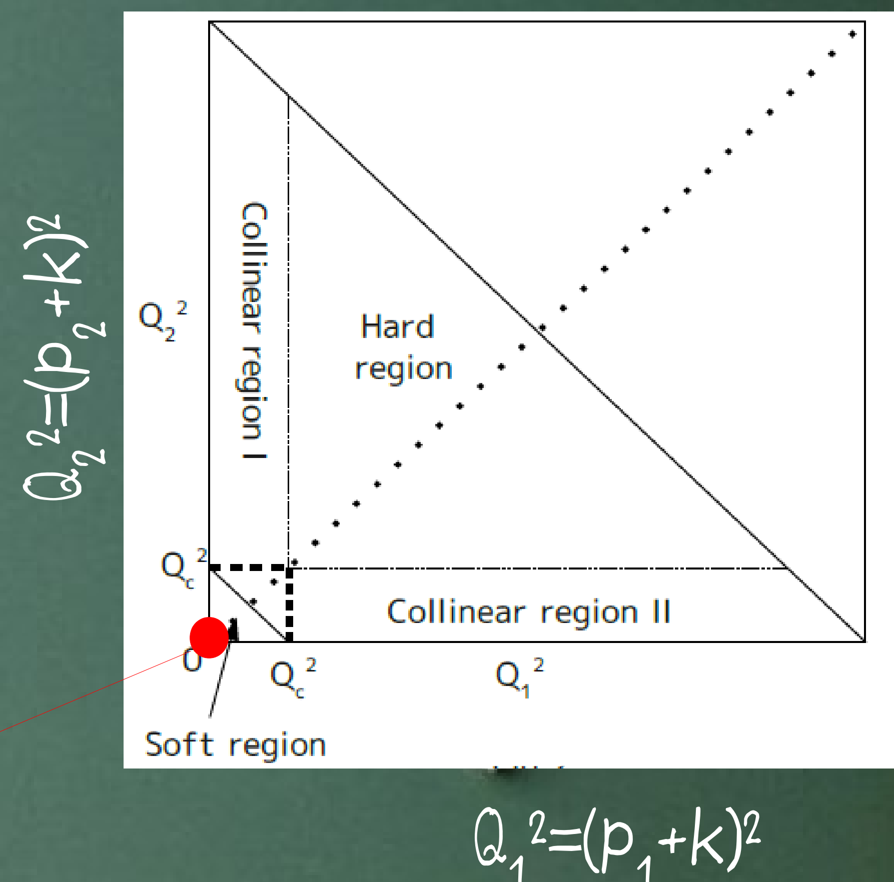
IR divergence/Structure

- For QCD case: massless quarks

Final state radiation



Daritz Plot

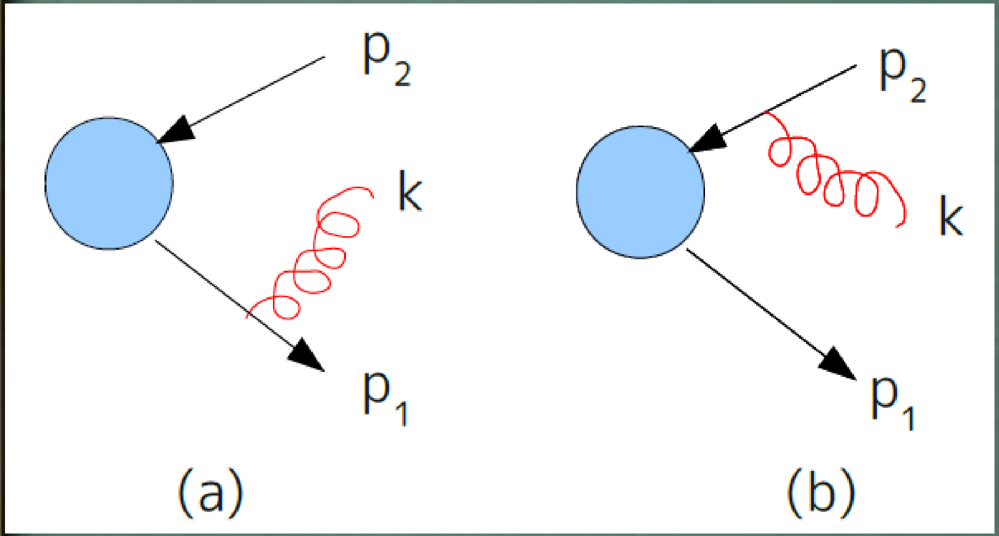


Soft divergence: $1/\varepsilon_{IR}$

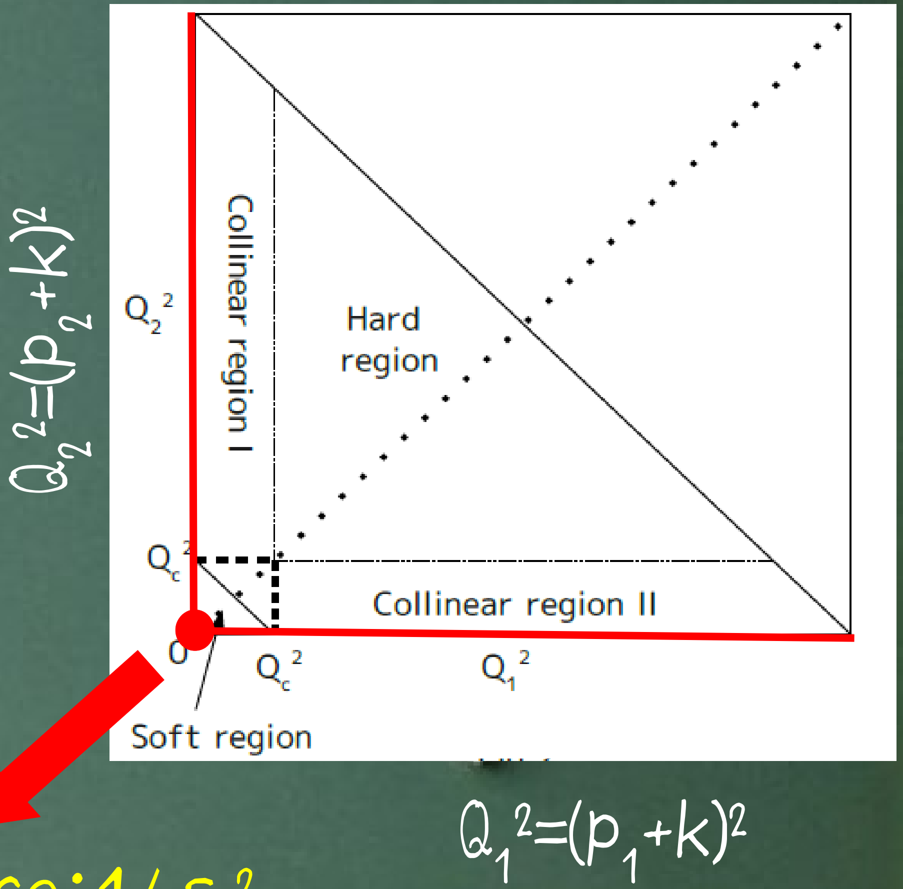
IR divergence/Structure

- For QCD case: massless quarks

Final state radiation



Daritz Plot



Soft/Collinear divergence: $1/\varepsilon^2_{IR}$

IR divergence/Example

$$\sigma_{n+1} = \frac{1}{(flux)} \int d\Phi_{n+1} |\mathcal{M}|^2$$

$$\approx \frac{1}{(flux)} \int d\Phi_{n+1} |\mathcal{M}_0|^2 \frac{R_I}{(q_1^2)^2}$$

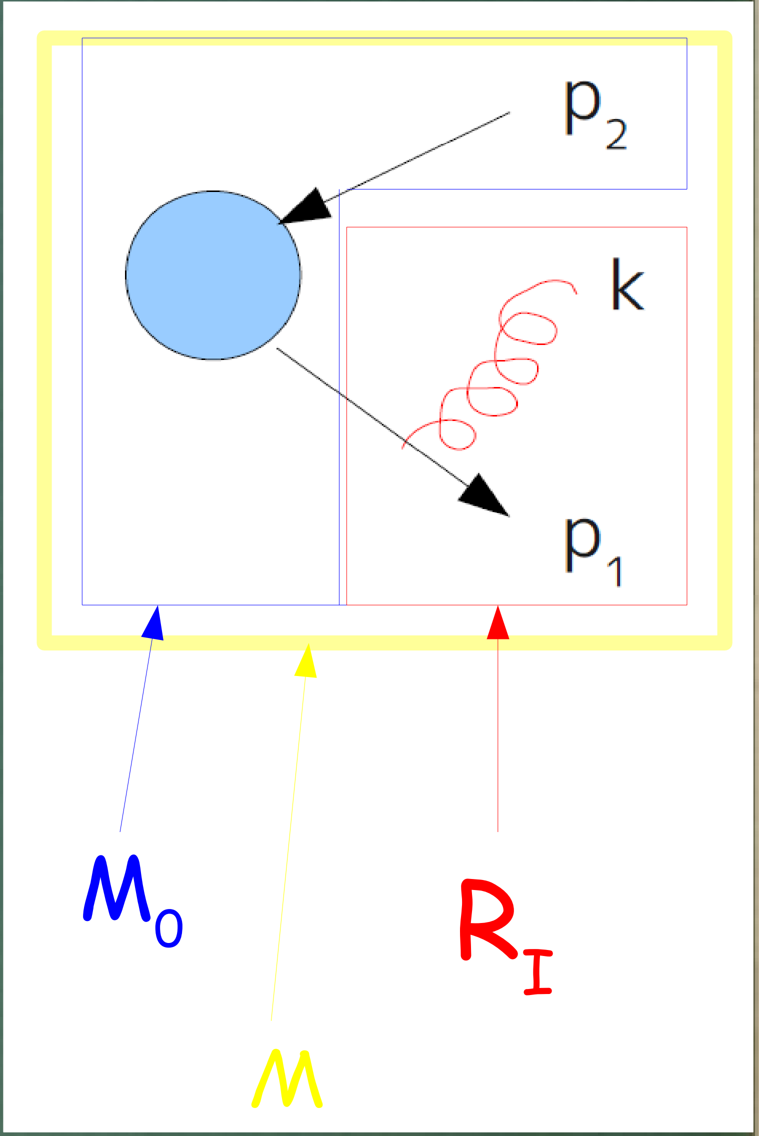
$$= \frac{1}{(flux)} \int d\Phi_n |\mathcal{M}_0|^2 \int \frac{dq_1^2}{2\pi (q_1^2)^2} \int d\Phi_2 R_I$$

$$= \frac{1}{(flux)} \int d\Phi_n |\mathcal{M}_0|^2 \otimes \delta_{col} ,$$

$$\delta_{col} \equiv \int \frac{dq_1^2}{2\pi (q_1^2)^2} \int d\Phi_2 R_I$$

$$R_I = 2 g_s^2 \frac{k_T}{x(1-x)} \left(P(x) + (1-x) \epsilon_{IR} \right)$$

$$P(x) = \frac{1+x^2}{1-x}$$



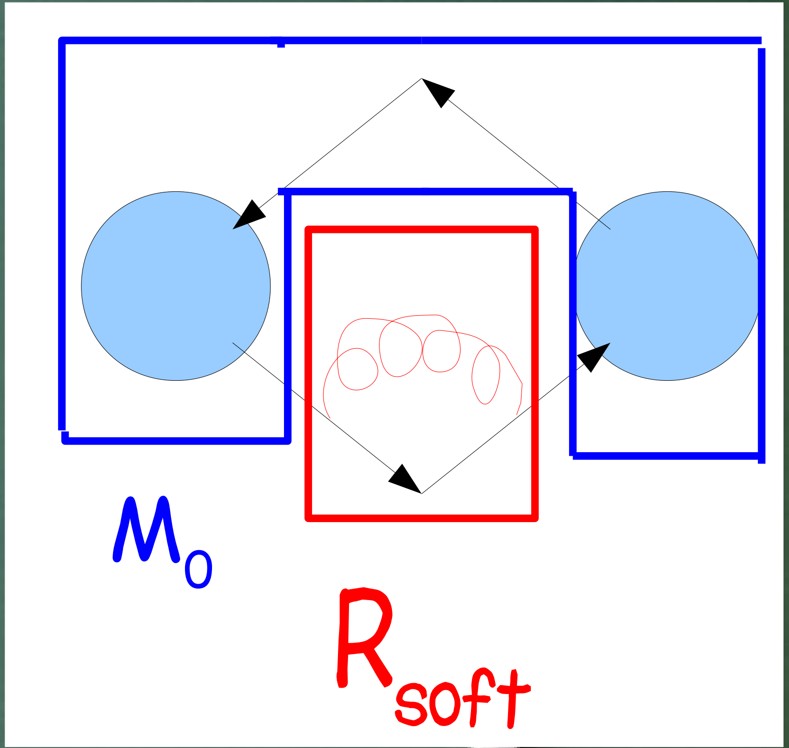
IR divergence/Example

Soft Part

$$\sigma_{n+1} = \frac{1}{(flux)} \int d\Phi_{n+1} |M|^2$$

$$\approx \frac{1}{(flux)} \int d\Phi_n |M_0|^2 \otimes \delta_{soft},$$

$$\delta_{soft} \equiv \int \frac{dk_0}{\pi} \int d\Phi_2 \frac{s R_{soft}}{q_1^2 q_2^2}$$



$$R_{soft} = -4\, p_1 \cdot p_2 (1 + \epsilon_{IR})$$

IR divergence/Example

Collinear Part:

$$\delta_{col} = \frac{\alpha_s}{\pi}\frac{1}{\epsilon_{IR}^2} + \frac{\alpha_S}{4\pi}\left(-3 + 4\log\left(Q_c^2/\mu_F^2\right)\right)\frac{1}{\epsilon_{IR}}$$

$$-\frac{\alpha_s}{4\pi}\left(-7 + \pi^2 + 3\log\left(Q_c^2/\mu_F^2\right) - \log^2\left(Q_c^2/\mu_F^2\right)\right)$$

Soft Part:

$$\delta_{soft} = -\frac{\alpha_s}{\pi}\frac{1}{\epsilon_{IR}^2} - \frac{\alpha_S}{\pi}\left(\log\left(Q_c^2/s\right) + \log\left(Q_c^2/\mu_F^2\right)\right)\frac{1}{\epsilon_{IR}}$$

$$+\frac{\alpha_s}{12\pi}\left(\pi^2 - 6\left(\log\left(Q_c^2/s\right) + \log\left(Q_c^2/\mu^2\right)\right)^2\right)$$

IR divergence/Example

Examples $Z \to d\bar{d}$

$$\Gamma_{Z\to d\bar{d}} = \Gamma_0(1+\delta_{\alpha s}),$$
$$\delta_{\alpha s} = \delta_v + 2\delta_{col} + \delta_{soft}$$

Virtual correction:

$$\delta_v = -\frac{\alpha_s}{\pi}\frac{1}{\epsilon_{IR}^2} + \frac{\alpha_S}{2\pi}\left(3-2\log(s/\mu_F^2)\right)\frac{1}{\epsilon_{IR}} + \frac{\alpha_s}{2\pi}\left(-8+\frac{7}{6}\pi^2+3\log(s/\mu_F^2)-\log^2(s/\mu_F^2)\right)$$

Result:

$$\delta_{\alpha s} = \frac{\alpha_s}{2\pi}\left(-1+\pi^2/3-3\log(Q_c^2/s)-2\log^2(Q_c^2/s)\right)$$

IR divergence/Example

Examples $Z \to d\bar{d}$

$$\Gamma_{Z\to d\bar{d}} = \Gamma_0(1+\delta_{\alpha s}),$$
$$\delta_{\alpha s} = \delta_v + 2\delta_{col} + \delta_{soft}$$

Virtual correction

$$\delta_v = -\frac{\alpha_s}{\pi}\frac{1}{\epsilon_{IR}^2} + \frac{\alpha_S}{2\pi}\left(3 - 2\log(s/\mu_F^2)\right)\frac{1}{\epsilon_{IR}} + \frac{\alpha_s}{2\pi}\left(-8 + \frac{7}{6}\pi^2 + 3\log(s/\mu_F^2) - \log^2(s/\mu_F^2)\right)$$

Result:

$$\delta_{\alpha s} = \frac{\alpha_s}{2\pi}\left(-1 + \pi^2/3 - 3\log(Q_c^2/s) - 2\log^2(Q_c^2/s)\right)$$

+Hard Corr. →

$$\Gamma_{\alpha s} = \Gamma_0 \frac{\alpha_s}{\pi} = 0.01445 \quad GeV$$

IR divergence/KLN Theorem

- For QCD case: massless quarks

Kinoshita-Lee-Nauenberg (KLN) Theorem:

After summing up contributions from all possible degenerate stats, the s-matrix has no IR-divergence.

↓

If one looks at some specific initial state, some collinear divergence is still remaining.

↓

Parton Distribution Function

IR divergence/Initial Radiation

Factorization: Coll. Approx.

Matrix Element

$$\left|\mathcal{M}_{N+1}^{(d)}\right|^2 = \left|\mathcal{M}_N^{(4)}\left(q \to \sum_{i=1}^{N} q_i\right)\right|^2 \frac{16\pi}{s\mu^{2\varepsilon_{IR}}} f_c \frac{\alpha_s}{2\pi} \mathrm{P}(x) \frac{1}{k_T^2}\left(\frac{1-x}{x}\right)$$

$$q^\mu = p_1^\mu + p_2^\mu - k^\mu,$$

Phase Space

$$d\Phi_{N+1}^{(d)} = d\Phi_N^{(4)}\left(q \to \sum_{i=1}^{N} q_i\right)$$

$$\times \frac{1}{16\pi^2\Gamma(1+\varepsilon_{IR})}\left(\frac{k_T^2}{4\pi x^2}\right)^{\varepsilon_{IR}} \frac{1}{1-x} dx dk_T^2$$

IR divergence/Initial Radiation

Factorization: Coll. Approx.

Splitting functions:

$$
\begin{aligned}
P_{qq}^{(0)}(z) &= C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right] , \\
P_{qg}^{(0)}(z) &= T_F[z^2+(1-z)^2] , \\
P_{gq}^{(0)}(z) &= C_F\frac{1+(1-z)^2}{z} , \\
P_{qq}^{(0)}(z) &= 2C_A\left[\frac{z}{(1-z)_+}-\frac{1-z}{z}+z(1-z)\right]+2\pi\beta_0\delta(1-z) ,
\end{aligned}
$$

IR divergence/Initial Radiation

Factorization: Coll. Approx.

Cross Section

$$\sigma = \frac{1}{(2p_1^0)(2p_2^0)v_{rel}} \int_{\Omega_{full}} d\Phi_{N+1}^{(d)} \left| \mathcal{M}_{N+1}^{(d)} \right|^2,$$

$$= \left(\frac{s}{4\pi\mu^2} \right)^{\varepsilon_{IR}} \frac{B(\varepsilon_{IR}, \varepsilon_{IR})}{2\Gamma(1+\varepsilon_{IR})} f_c \frac{\alpha_s}{2\pi} \int_0^1 dx \sigma_0(xs) \mathrm{P}(x) \left(\frac{1-x}{x} \right)^{2\varepsilon_{IR}}$$

$1/\varepsilon_{IR}$

$$\frac{1}{\varepsilon_{IR}} f_c \frac{\alpha_s}{2\pi} \mathrm{P}(x)$$

PDF

IR divergence/Initial Radiation

Factorization: Coll. Approx.

Cross Section

No IR-divergence

$$----_{NLO}=[----_{tree}(1+\overset{\frown}{\cdot}_{V}+\overset{\frown}{\cdot}_{s/c})+----_{vis}]\otimes \text{PDF/PS}$$

$1/\varepsilon^2_{IR}, 1/\varepsilon_{IR}$

cancellation

$$\frac{1}{\varepsilon_{IR}} f_c \frac{\alpha_s}{2\pi} \mathrm{P}(x)$$

P(x): Splitting function

Space/time dimension: $d=4+2\varepsilon_{IR}$

IR divergence/DGLAP Equation

DGLAP Equation

Splitting function ← pQCD

$$\frac{dD(x,Q^2)}{d\ln Q^2}=\frac{\alpha}{2\pi}\int_x^1\frac{dy}{y}P_+(x/y)D(y,Q^2)$$

$$D(x,Q^2)=\Pi(Q^2,Q_s{}^2)D(x,Q_s{}^2)+\frac{\alpha}{2\pi}\int_{Q_s{}^2}^{Q^2}\frac{dK^2}{K^2}\Pi(Q^2,K^2)\int_x^{1-\epsilon}\frac{dy}{y}P(y)D(x/y,K^2)$$

$$\Pi(Q^2,Q'^2)=\exp\left(-\frac{\alpha}{2\pi}\int_{Q'^2}^{Q^2}\frac{dK^2}{K^2}\int_0^{1-\epsilon}dxP(x)\right)$$

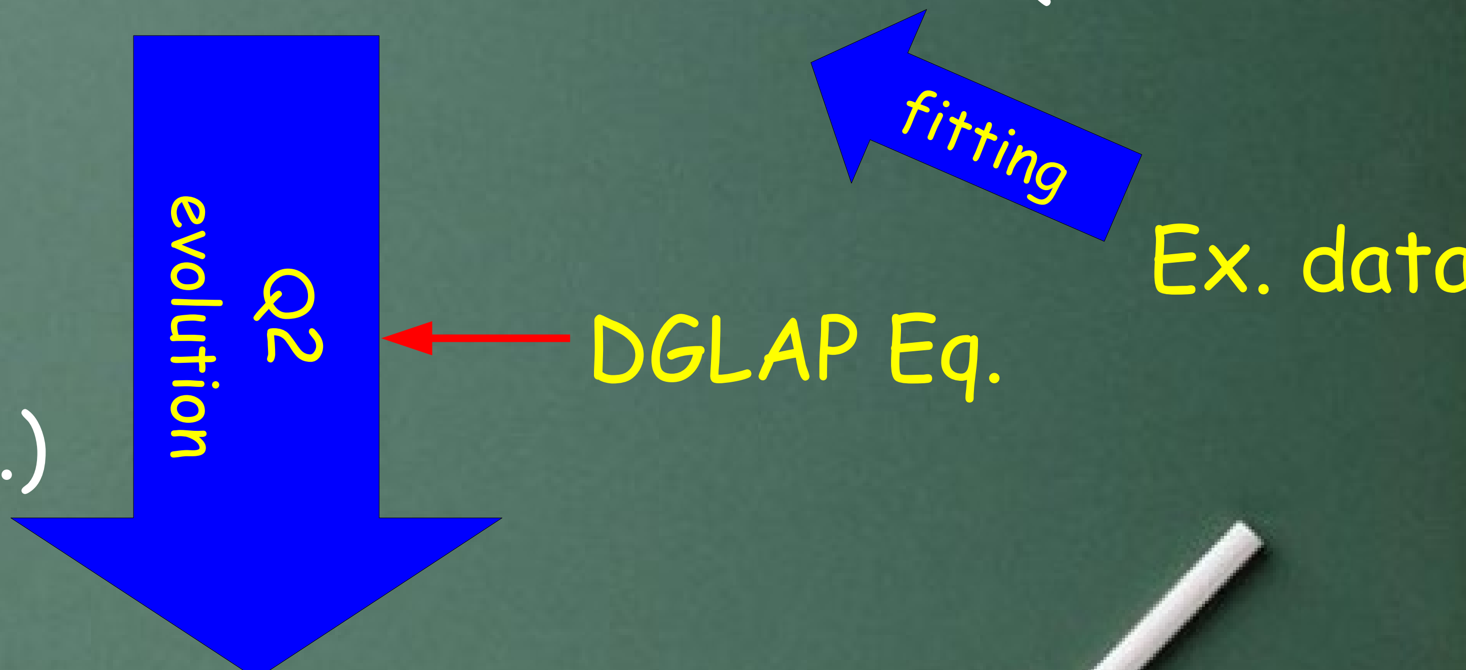
Sudakov Factor

non-branch provability

IR divergence/DGLAP Equation

Initial parton distribution @ Low Q^2

PS(MC)
or
PDF(analy.)



Parton distribution @ High Q^2

IR divergence/DGLAP Equation

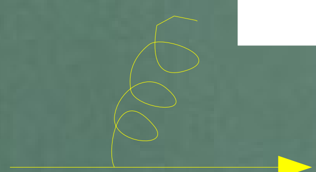
- PDF:
  - Initial distribution: data fitting
  - Q^2 evolution: Analytic solution of DGLAP Eq.
  - No kinematical information
- PS:
  - Initial distribution: from PDF
  - Q^2 evolution: MC method to solve DGLAP Eq.
  - generate Pt distribution

IR divergence/DGLAP Equation

DGLAP Eq. (Integral-differential equation)

$$\frac{d\,D(x,Q^2)}{d\log(Q^2)}=\frac{\alpha}{2\pi}\int_0^x\frac{dy}{y}P(x/y)D(y,Q^2)$$

q→q+g : $$P(x)=\left(\frac{1+x^2}{1-x}\right)\to\theta(1-\epsilon-x)-\delta(1-x)\int_0^{1-\epsilon}dy\,P(y)$$



$$\frac{d\,D}{d\,K^2}K^2+\frac{\alpha}{2\pi}c(\epsilon)D=\frac{\alpha}{2\pi}P\sqrt{5}\;D$$

$$c(\epsilon)=\int_0^{1-\epsilon}dz\,p(z)$$

IR divergence/DGLAP Equation

$$\frac{d\,\Pi}{d\,K^2}K^2+\frac{\alpha}{2\pi}c\,\Pi=0$$

Sudakov form factor

↓

non-branching prob.

Inhomogeneous Eq.

$$\frac{d\,D}{d\,K^2}K^2+\frac{\alpha}{2\pi}c\,D=\frac{\alpha}{2\pi}P\sqrt{5}\;D$$

$$D(x,K^2)=\hat{D}(x,K^2)\,\Pi(K^2,Q^2)$$

↓

$$\frac{d\,\hat{D}}{d\,K^2}K^2=\frac{\alpha}{2\pi}P\sqrt{5}\hat{D}$$

$$\hat{D}=\frac{\alpha}{2\pi}\int P\sqrt{5}\hat{D}\,\frac{d\,K^2}{K^2}+c$$

IR divergence/DGLAP Equation

Integral equation

$$D(x,Q^2)=\Pi(Q^{2,}Q_s^2)\,D(x,Q_s^2)+\frac{\alpha}{2\pi}P\sqrt{5}\;D(x,K^2)\frac{dK^2}{K^2}$$

$$\Pi(Q_{1,}^2Q_0^2)=-\exp\left(\int_{Q_0^2}^{Q_1^2}\frac{\alpha}{2\pi}\frac{dK^2}{K^2}\int_0^{1-\epsilon}dx\,P(x)\right)$$

Method of successive substitution

Equation

$$\phi(x)=f(x)+\lambda\int_a^x K(x,y)\phi(y)\,dy$$

0th approx.

$$\phi_0(x)=f(x)$$

1st

$$\phi_1(x)=f(x)+\lambda\int_a^x K(x,y)\phi_0(y)\,dy$$

2nd

$$\phi_2(x)=f(x)+\lambda\int_a^x K(x,y)\phi_1(y)\,dy$$

IR divergence/DGLAP Equation

Method of successive substitution

No emission

1 parton emission

$$\phi_2 = f(x) + \lambda \int_a^x K(x, y)\phi(y)dy$$
$$+\lambda^2 \int_{y_1}^x dy \int_a^{y_1} dy_1 K(x, y_1)\phi(y_1) K(y_1, y)\phi(y)$$

2 parton emission

IR divergence/DGLAP Equation

Method of successive substitution

$$\phi(x)=f(x)+\lambda\int_a^x K(x,y)\phi(y)dy$$

$$\phi_2=f(x)+\lambda\int_a^x K(x,y)\phi(y)dy$$
$$+\lambda^2\int_{y_1}^x dy\int_a^{y_1} dy_1 K(x,y_1)\phi(y_1)K(y_{1,}y)\phi(y)$$

$$\phi_n(x)=f(x)+\sum_{l=1}^n \lambda^l\int_a^x K_l(x,y)f(y)dy$$

$$K_l(x,y)=\int\cdots\int dy_1\cdots dy\, K(x,y_1)\cdots K(y_{l-1},y)$$

IR divergence/DGLAP Equation