QCD@LHC for beginners

Lesson 3

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Outline

- Lesson 3
  - Asymptotic Free
    - β-function and Renormalization Equation
    - α_s determination
    - Scale dependence/R-ratio

UV divergence

- Lesson 4
  - Around "IR divergence"
    - IR divergence in QCD/KLN theorem
    - Factorization
    - DGLAP Equation/PDF

IR divergence

Outline

- Lesson 3
  - Asymptotic Free
    - β-function and Renormalization Equation
    - α_s determination
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GOAL: Understanding

(1) running α_s

(2) Λ_{QCD}

(3) μ^2(renormalization scale) and Q^2 (energy scale of the process)

Renormalization Equation/ β-function

$$\frac{g_0^2}{(4\pi)^{d/2}} \equiv \mu^{2\epsilon} \frac{\alpha_s(\mu)}{4\pi} Z_\alpha e^{\gamma_E \epsilon}$$

$$Z_\alpha = 1 - \left(\frac{11}{3} C_A - \frac{4}{3} T_F n_f \right) \frac{\alpha_s}{4\pi\epsilon} + \mathcal{O}(\alpha_s^2)$$

Now we are ready to discuss running α_s and asymptotic free!

Renormalization Equation/ β-function

- What is β-function?

$$\frac{d}{d\ln\mu^2}\left(\frac{g_0^2}{(4\pi)^{d/2}} \equiv \mu^{2\epsilon}\frac{\alpha_s(\mu)}{4\pi}Z_\alpha e^{\gamma_E\epsilon}\right)$$

$$0 = \left(\epsilon + \frac{d\ln\alpha_s}{d\ln\mu^2} + \frac{d\ln Z_\alpha}{d\ln\mu^2}\right)\mu^{2\epsilon}\frac{\alpha_s(\mu)}{4\pi}Z_\alpha e^{\gamma_E\epsilon}$$

$$\frac{d\alpha_s}{d\ln\mu^2} = -\epsilon\alpha_s - \alpha_s\frac{d\ln Z_\alpha(\alpha_s(\mu))}{d\ln\mu^2}$$

$\varepsilon \to 0$

$$\frac{d\alpha_s}{d\ln\mu^2} = -\beta(\alpha_s) = -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 - \beta_2\alpha_s^4 - \beta_3\alpha_s^5 + \cdots$$

Renormalization Equation/ β-function

- What is β-function?

$$\frac{g_0^2}{(4\pi)^{d/2}} \equiv \mu^{2\epsilon}\frac{\alpha_s(\mu)}{4\pi}Z_\alpha e^{\gamma_E\epsilon}$$

$\frac{d}{d\ln\mu^2}$

$$0 = \left(\epsilon + \frac{d\ln\alpha_s}{d\ln\mu^2} + \frac{d\ln Z_\alpha}{d\ln\mu^2}\right)\mu^{2\epsilon}\frac{\alpha_s(\mu)}{4\pi}Z_\alpha e^{\gamma_E\epsilon}$$

$$\frac{d\alpha_s}{d\ln\mu^2} = -\epsilon\alpha_s - \alpha_s\frac{d\ln Z_\alpha(\alpha_s(\mu))}{d\ln\mu^2}$$

$$\alpha_s\frac{d\ln Z_\alpha(\alpha_s(\mu))}{d\ln\mu^2} \equiv \beta(\alpha_s)$$

$\varepsilon \to 0$

$$\frac{d\alpha_s}{d\ln\mu^2} = -\beta(\alpha_s) = -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 - \beta_2\alpha_s^4 - \beta_3\alpha_s^5 + \cdots$$

Renormalization Equation/ β-function

$$Z_\alpha = 1 - \left(\frac{11}{3}C_A - \frac{4}{3}T_F n_f\right)\frac{\alpha_s}{4\pi\epsilon} + \mathcal{O}(\alpha_s^2)$$

$$\beta_0 = \frac{11C_A - 4T_F n_f}{12\pi}$$

$$\begin{cases} C_A = 3 \\ T_F = 1/2 \end{cases}$$

$\beta_0 > 0$ when $n_f < 16$

$$\frac{d\alpha_s}{d\ln\mu^2} = -\beta_0\alpha_s^2$$

Asymptotic free !

Renormalization Equation/ β-function

$$\frac{d\alpha_s}{d\ln\mu^2} = -\beta(\alpha_s) = -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 - \beta_2\alpha_s^4 - \beta_3\alpha_s^5 + \cdots$$

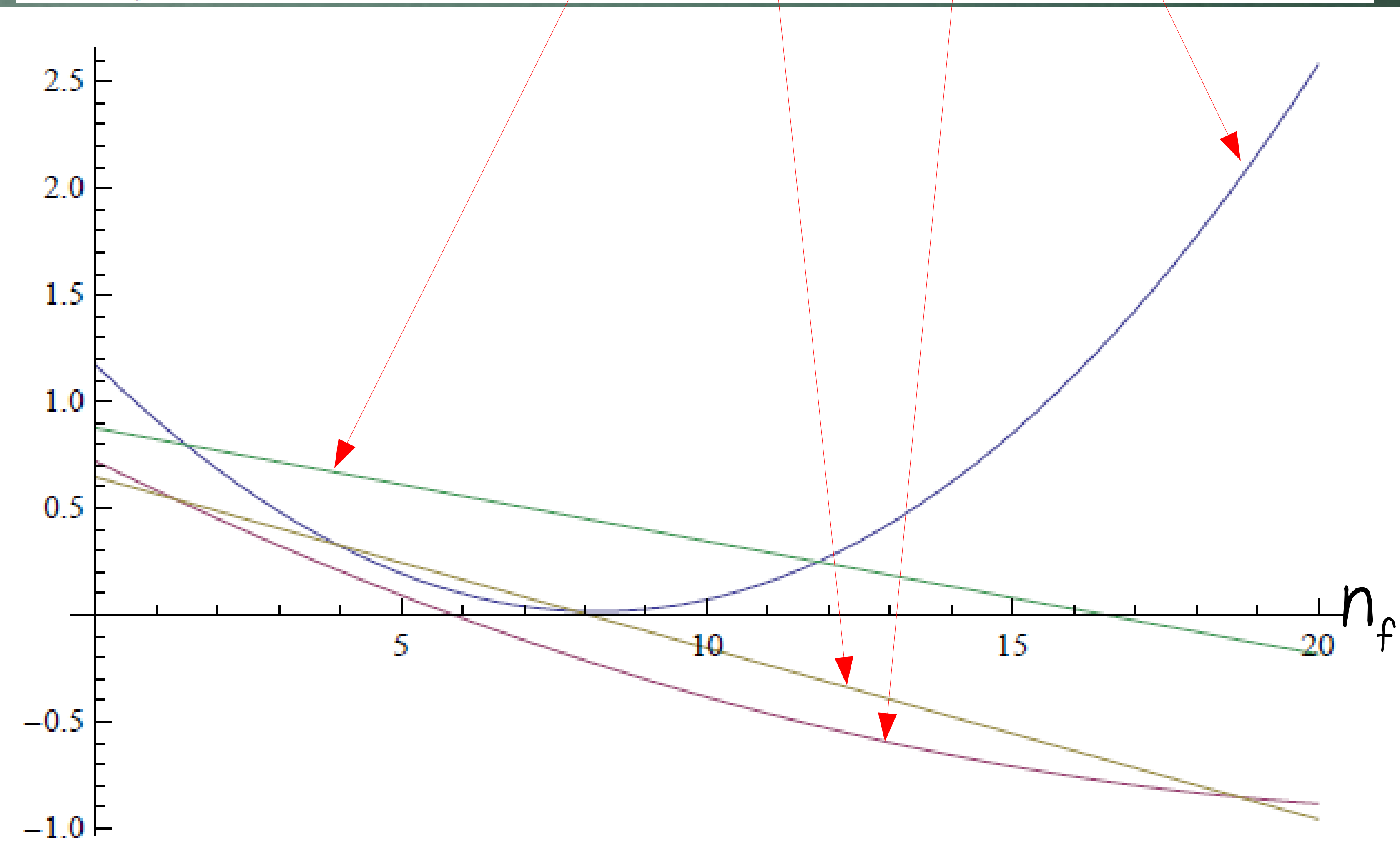
$$\beta_1 = \frac{153 - 19n_f}{24\pi^2} ,$$

$$\beta_2 = \frac{77139 - 15099n_f + 325n_f^2}{3456\pi^3} ,$$

$$\beta_3 \approx \frac{29243 - 6946.3n_f + 405.089n_f^2 + 1.49931n_f^3}{256\pi^4}$$

Renormalization Equation/ β-function

$$\frac{d\alpha_s}{d\ln\mu^2} = -\beta(\alpha_s) = -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 - \beta_2\alpha_s^4 - \beta_3\alpha_s^5 + \cdots$$



Renormalization Equation/ β-function

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Asymptotic free !

Renormalization Equation/ β-function

Let's solve renormalization equation!

$$\frac{d\alpha_s}{d\ln\mu^2} = -\beta_0 \alpha_s^2$$

Boundary condition: $1/\alpha_s(\mu^2)=0$ @ $\mu^2=\Lambda^2$

$$\int_{\alpha_s(\Lambda)}^{\alpha_s(Q)} \frac{d\alpha_s}{\alpha_s^2} = -\beta_0 \int_{\Lambda^2}^{Q^2} \frac{d\mu^2}{\mu^2}$$

$$\left(-\frac{1}{\alpha_s(Q)} + \frac{1}{\alpha_s(\Lambda)}\right) = -\beta_0 \ln\left(\frac{Q^2}{\Lambda}\right)$$

0

$$\alpha_s(Q) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}$$

Renormalization Equation/ β-function

Let's solve renormalization equation!

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$$\int_{\alpha_s(\Lambda)}^{\alpha_s(Q)} \frac{d\alpha_s}{\alpha_s^2} = -\beta_0 \int_{\Lambda^2}^{Q^2} \frac{d\mu^2}{\mu^2}$$

0

$$\left(-\frac{1}{\alpha_s(Q)} + \frac{1}{\alpha_s(\Lambda)}\right) = -\beta_0 \ \ln\left(\frac{Q^2}{\Lambda}\right)$$

Renormalization Equation/ β-function

Let's solve renormalization equation!

0

$$\left(-\frac{1}{\alpha_s(Q)}+\frac{1}{\alpha_s(\Lambda)}\right)=-\beta_0\ \ln\left(\frac{Q^2}{\Lambda}\right)$$

$$\alpha_s(Q)=\frac{1}{\beta_0\ln(Q^2/\Lambda^2)}$$

Renormalization Equation/ β-function

Meaning of the boundary condition

Boundary condition: $1/\alpha_s(\mu^2)=0$ @ $\mu^2=\Lambda^2$

$$\alpha_s(Q) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}$$

↓ $Q=\mu$

$$\alpha_s(\mu) = \frac{1}{\beta_0 \ln\left(\mu^2/\Lambda^2\right)}$$

↓ Solve w.r.t Λ

$$\Lambda = \mu \exp[-1/(2\beta_0\alpha_s(\mu))]$$

Renormalization Equation/ β-function

Meaning of the boundary condition

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↓ $Q=\mu$

$$\alpha_s(\mu) = \frac{1}{\beta_0 \ln\left(\mu^2/\Lambda^2\right)}$$

↓ Solve w.r.t. Λ

Λ_{QCD}

$$\Lambda = \mu \exp[-1/(2\beta_0\alpha_s(\mu))]$$

$\alpha_s(\Lambda_{QCD}) = \infty$: Lower limit where p-QCD can be applied!

Renormalization Equation/ β-function

Exercise

$$\Lambda = \mu \exp[-1/(2\beta_0 \alpha_s(\mu))]$$

$$\beta_0 = \frac{11C_A - 4T_F n_f}{12\pi}$$

$M_z = 91.1786$ GeV

$\alpha_s(M_z) = 0.1184$

$n_f = 5$

$C_A = 3$

$T_F = 1/2$

$\Lambda_{QCD} = ?$

Renormalization Equation/ β-function

Higher order corrections

$$\alpha_s(Q) = \frac{1}{\beta_0 L} - \frac{1}{\beta_0^3 L^2}\ln L + \frac{1}{\beta_0^3 L^3}\left(\frac{\beta_1^2}{\beta_0^2}\left(\ln^2 L - \ln L - 1\right) + \frac{\beta_2}{\beta_0}\right)$$

LO

$$L = \ln(Q^2/\Lambda^2)$$

Renormalization Equation/ β-function

Higher order corrections

$$\alpha_s(Q) = \frac{1}{\beta_0 L} - \frac{1}{\beta_0^3 L^2} \ln L + \frac{1}{\beta_0^3 L^3} \left(\frac{\beta_1^2}{\beta_0^2} \left(\ln^2 L - \ln L - 1 \right) + \frac{\beta_2}{\beta_0} \right)$$

NLO

$$L = \ln(Q^2/\Lambda^2)$$

Λ_{QCD} @NLO=200~300 MeV

$$m_u = 2 - 3\text{MeV}、m_d = 4 - 6\text{MeV}、m_s = 80 - 130\text{MeV}$$

Massless quarks

Massive quarks

$$m_c \sim 1.3\text{GeV}、m_b \sim 4.2\text{GeV}、m_t = 170 - 175\text{GeV}$$

Renormalization Equation/ β-function

Higher order corrections

$$\alpha_s(Q) = \frac{1}{\beta_0 L} - \frac{1}{\beta_0^3 L^2}\ln L + \frac{1}{\beta_0^3 L^3}\left(\frac{\beta_1^2}{\beta_0^2}\left(\ln^2 L - \ln L - 1\right) + \frac{\beta_2}{\beta_0}\right)$$

$$L = \ln(Q^2/\Lambda^2)$$

NNLO

Λ_{QCD} @NLO=200~300 MeV

Renormalization Equation/ α_s meas.

$$\alpha_s(Q) = \frac{1}{\beta_0 L} - \frac{1}{\beta_0^3 L^2}\ln L + \frac{1}{\beta_0^3 L^3}\left(\frac{\beta_1^2}{\beta_0^2}\left(\ln^2 L - \ln L - 1\right) + \frac{\beta_2}{\beta_0}\right)$$

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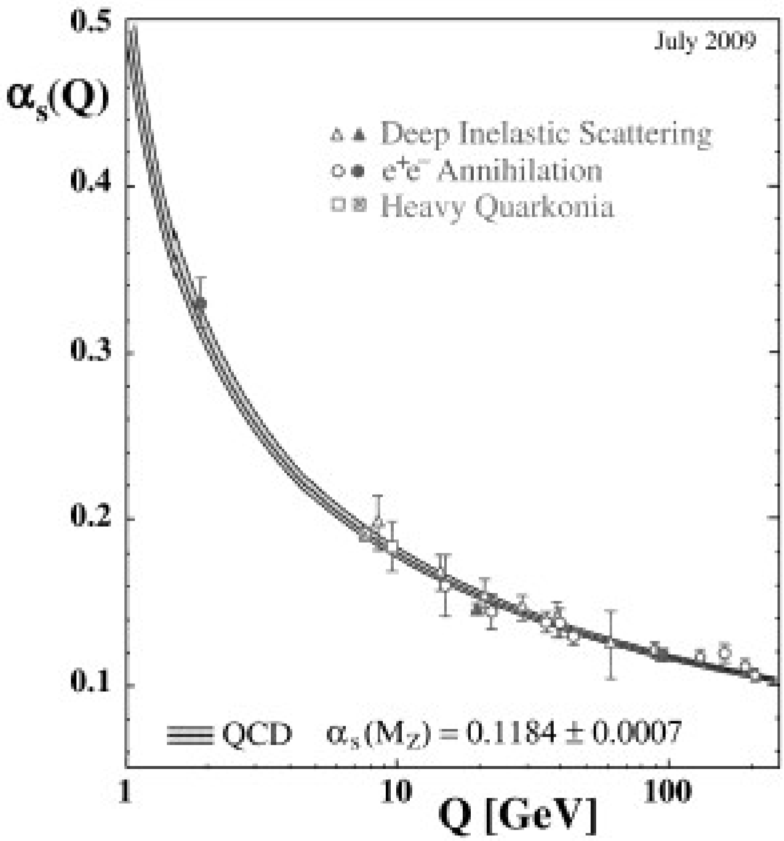

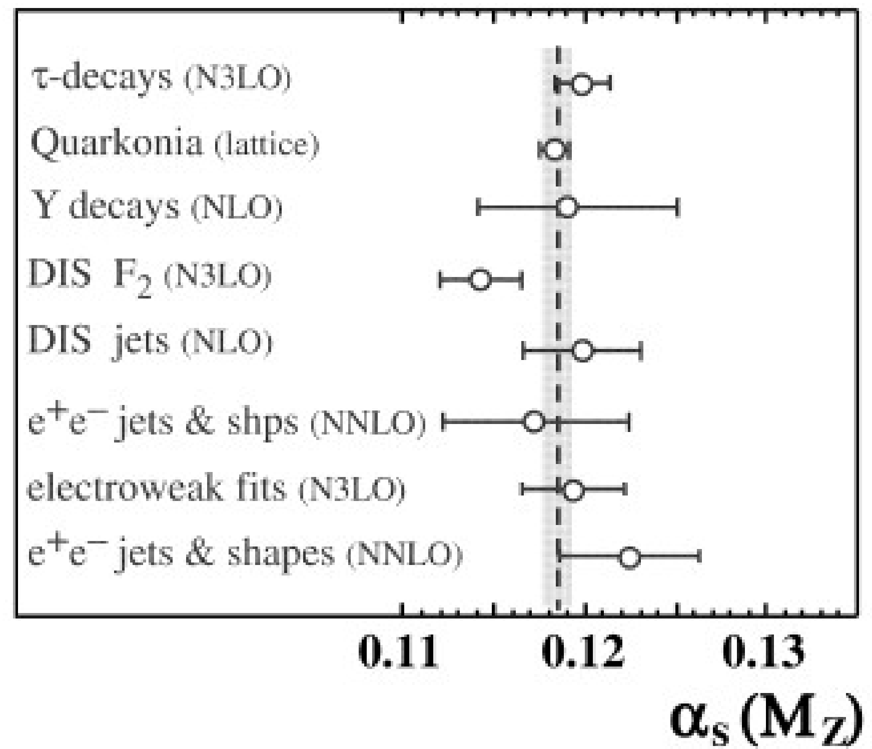

World Average '09 $\alpha_s(M_z) = 0.1184 \pm 0.0007$

Renormalization Equation/R-ratio

- General discussion
  - Physical observable: P ← Energy scale independent

$P(\tau = \ln(Q^2/\mu^2), \alpha_s) = c_1(\tau)\alpha_s + c_2(\tau)\alpha_s^2 + \cdots$

$$\frac{dP(\tau,\alpha_s)}{d\ln\mu^2} = \left(\frac{\partial}{\partial\ln\mu^2} + \frac{\partial\alpha_s}{\partial\ln\mu^2}\frac{\partial}{\partial\alpha_s}\right)P(\tau,\alpha_s) = 0$$

$$\left(-\frac{\partial}{\partial\tau} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)P(\tau,\alpha_s) = 0$$

Renormalization Equation/R-ratio

- General discussion

Introduce a new function $\alpha_s(\tau)$:

$$\tau = \int_{\alpha_s}^{\alpha_s(\tau)} \frac{d\alpha}{\beta(\alpha)}$$

boundary condition:

$$\alpha_s(\tau = 0) = \alpha_s(\mu^2) = \alpha_s$$

Show $\alpha_s(\tau)$ satisfying following equation.

$$\left(-\frac{\partial}{\partial\tau} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\alpha_s(\tau) = 0$$

Renormalization Equation/R-ratio

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Show $\alpha_s(\tau)$ satisfying following equation.

$$\left(-\frac{\partial}{\partial \tau} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \alpha_s(\tau) = 0$$

Exercise: Show above equation.

Renormalization Equation/R-ratio

- General discussion

Introduce a new function $\alpha_s(\tau)$:

$$\tau = \int_{\alpha_s}^{\alpha_s(\tau)} \frac{d\alpha}{\beta(\alpha)}$$

boundary condition:

$$\alpha_s(\tau = 0) = \alpha_s(\mu^2) = \alpha_s$$

$$\left(-\frac{\partial}{\partial \tau} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\alpha_s(\tau) = 0$$

$$\left(-\frac{\partial}{\partial \tau} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)P(\tau, \alpha_s) = 0$$

Renormalization Equation/R-ratio

- General discussion

$$P\left(\tau, \alpha_s(\tau)\right) = P\left(0, \alpha_s(\tau)\right)$$

$\tau = \ln(Q^2/\mu^2)$

All τ (or μ) dependences can be encapsulate in $\alpha_s(\tau)$

Renormalization Equation/R-ratio

- General discussion

$$P(\tau, \alpha_s(\tau)) = P(0, \alpha_s(\tau))$$

$\tau = \ln(Q^2/\mu^2)$

All τ (or μ) dependences can be encapsulate in $\alpha_s(\tau)$

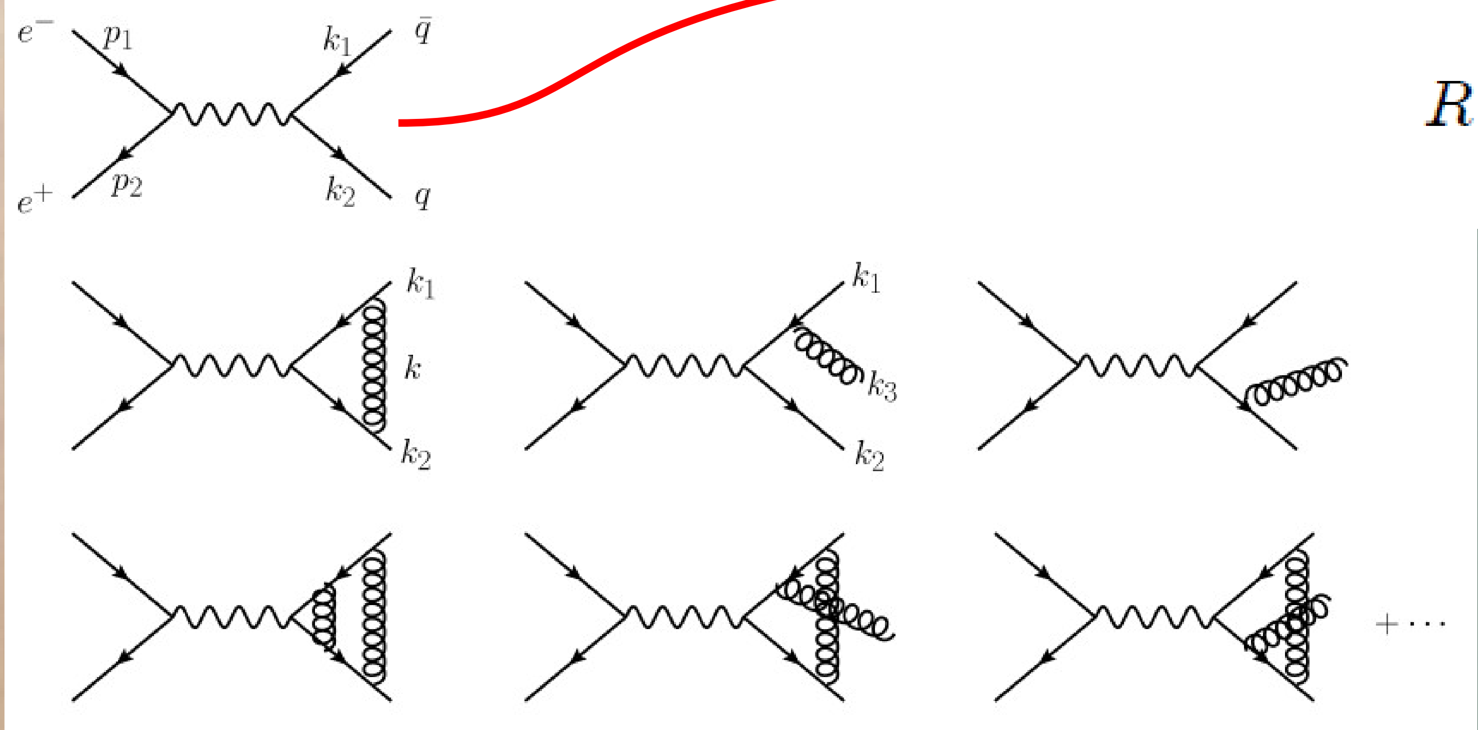
- Example: R-ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Renormalization Equation/R-ratio

- Example:R-ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



$$R = R_0$$

Renormalization Equation/R-ratio

- Example:R-ratio

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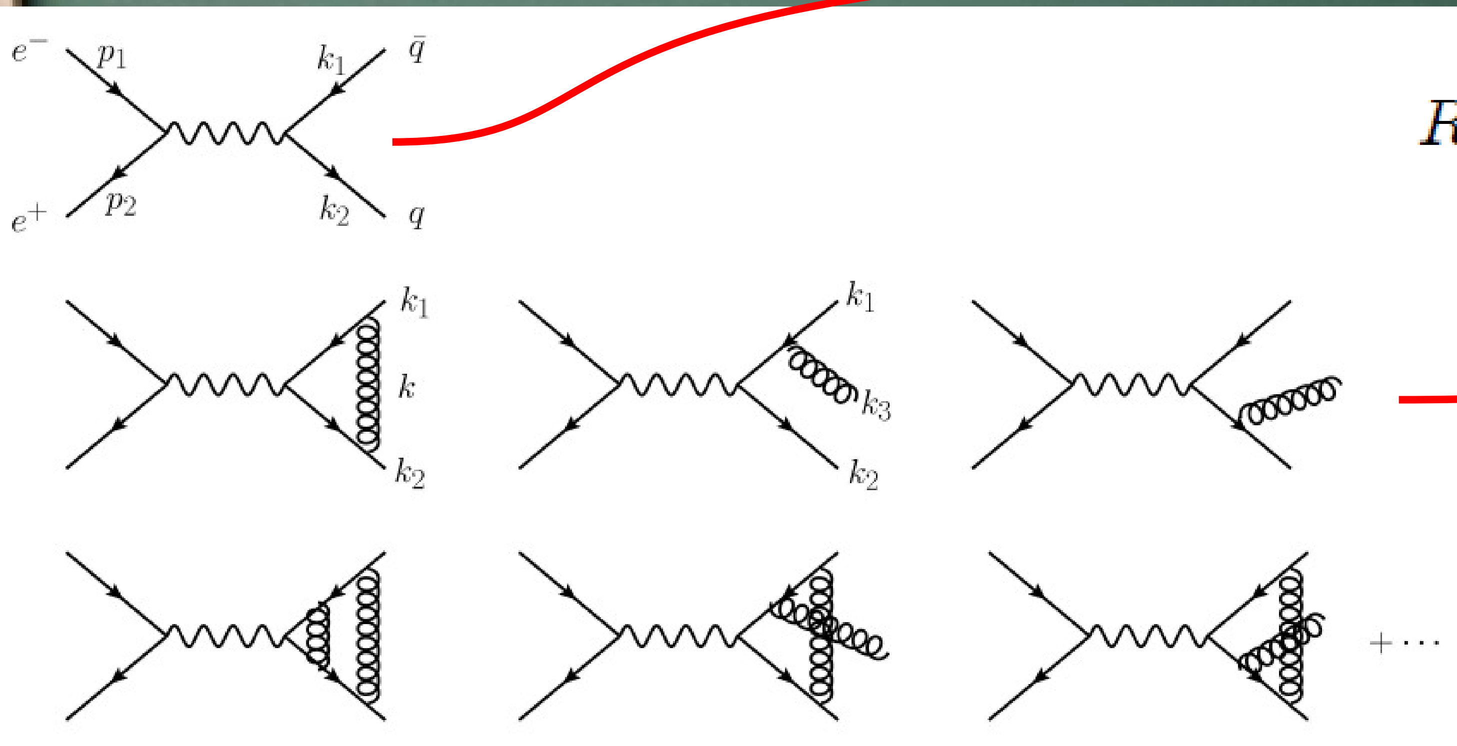
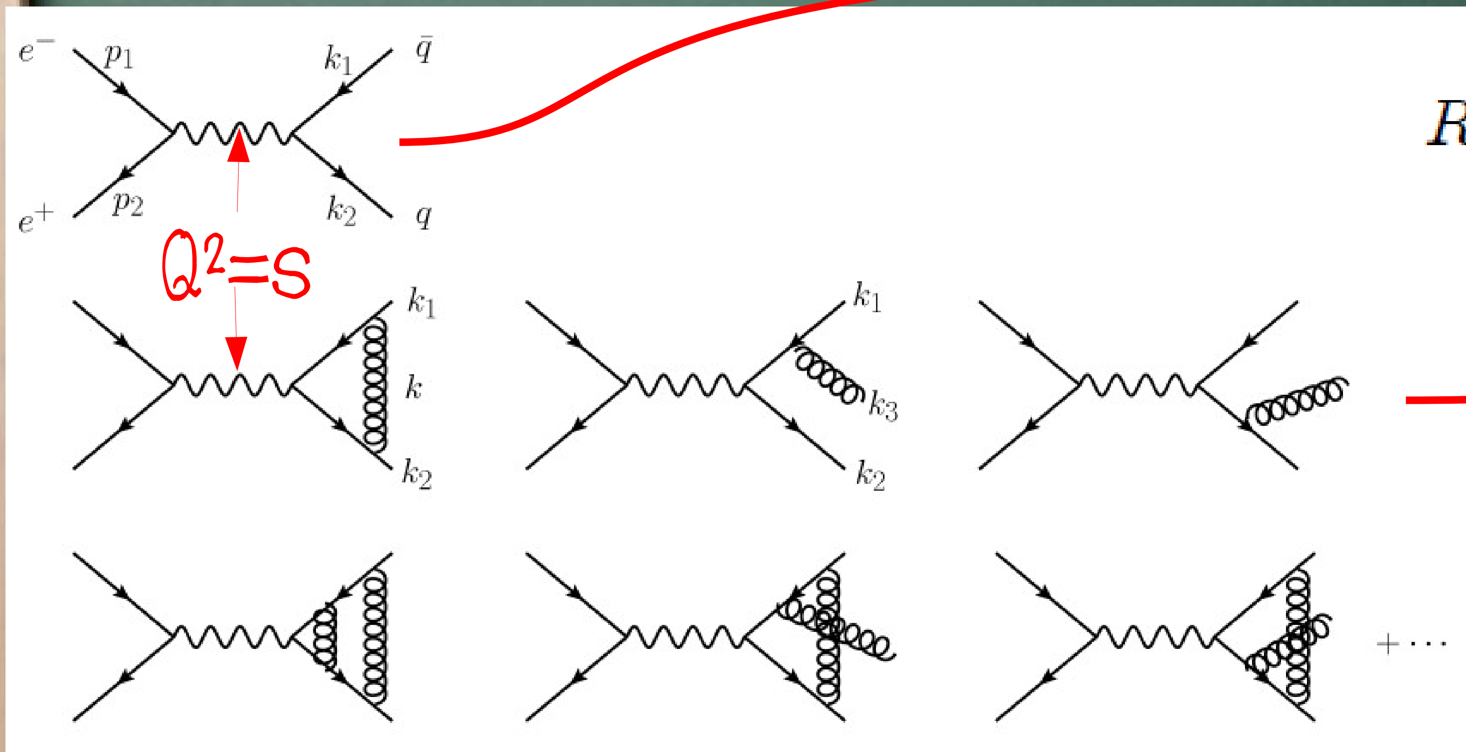


$$R = R_0\left(1 + \frac{\alpha_s(\mu)}{\pi}\right)$$

Renormalization Equation/R-ratio

- Example: R-ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

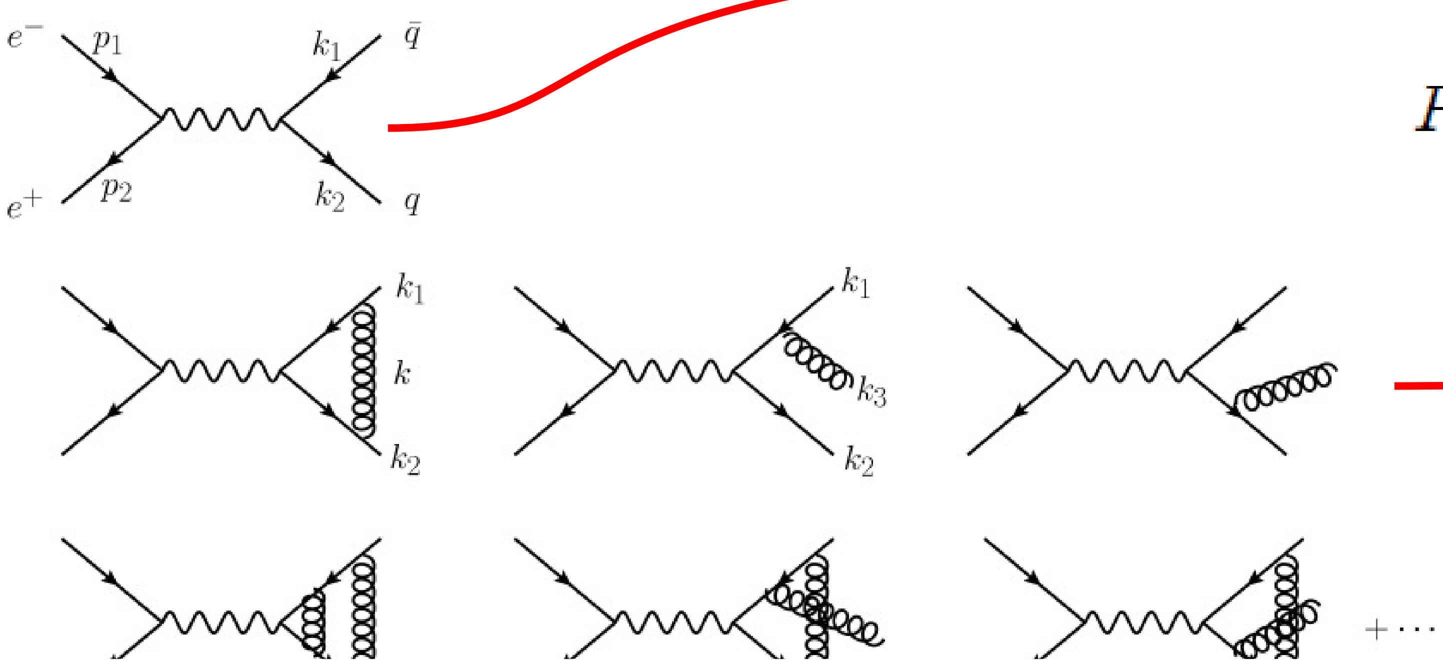


$$R = R_0\left(1 + \frac{\alpha_s(Q^2)}{\pi}\right)$$

Renormalization Equation/R-ratio

- Example: R-ratio

$$R = \frac{\sigma(e^+e^- \to \mathrm{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



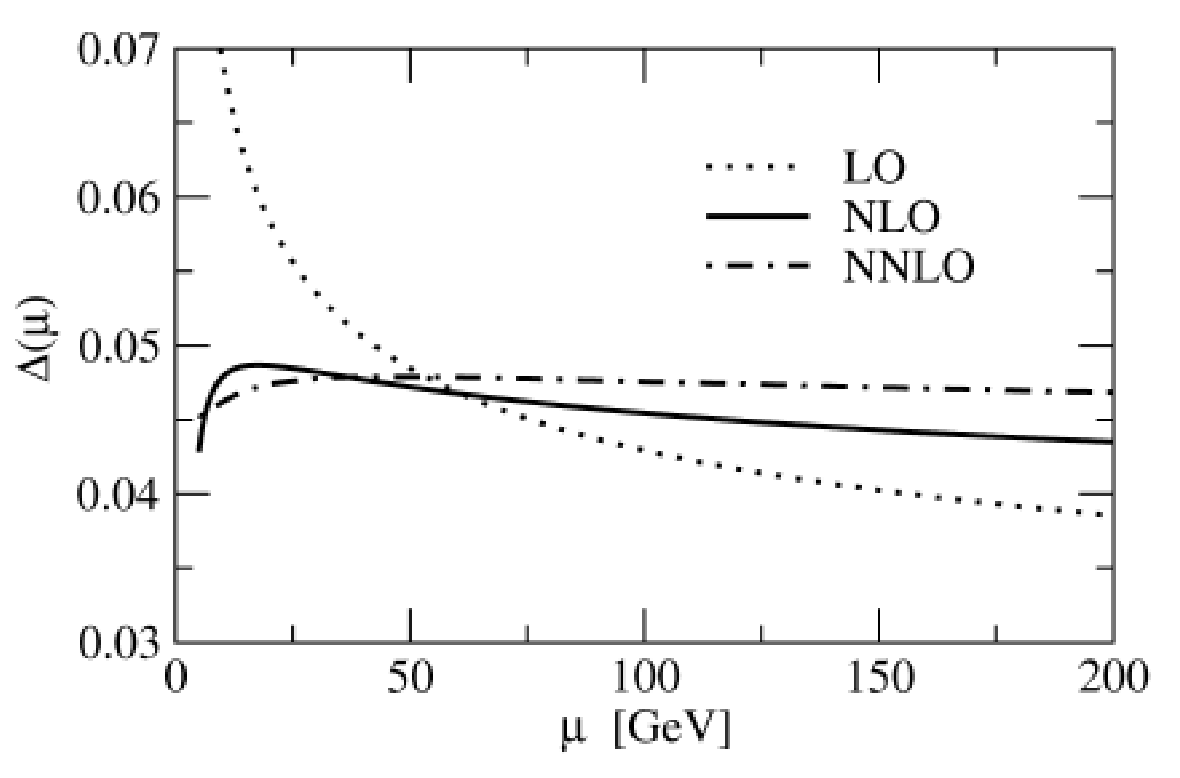
$$R = R_0\left(1 + \frac{\alpha_s(Q^2)}{\pi}\right)$$

Q^2 dependence

$$R = R_0\left\{1 + \left(\frac{\alpha_s}{\pi}\right) + 1.411\left(\frac{\alpha_s}{\pi}\right)^2 - 12.8\left(\frac{\alpha_s}{\pi}\right)^3 + \cdots\right\}$$

Renormalization Equation/R-ratio

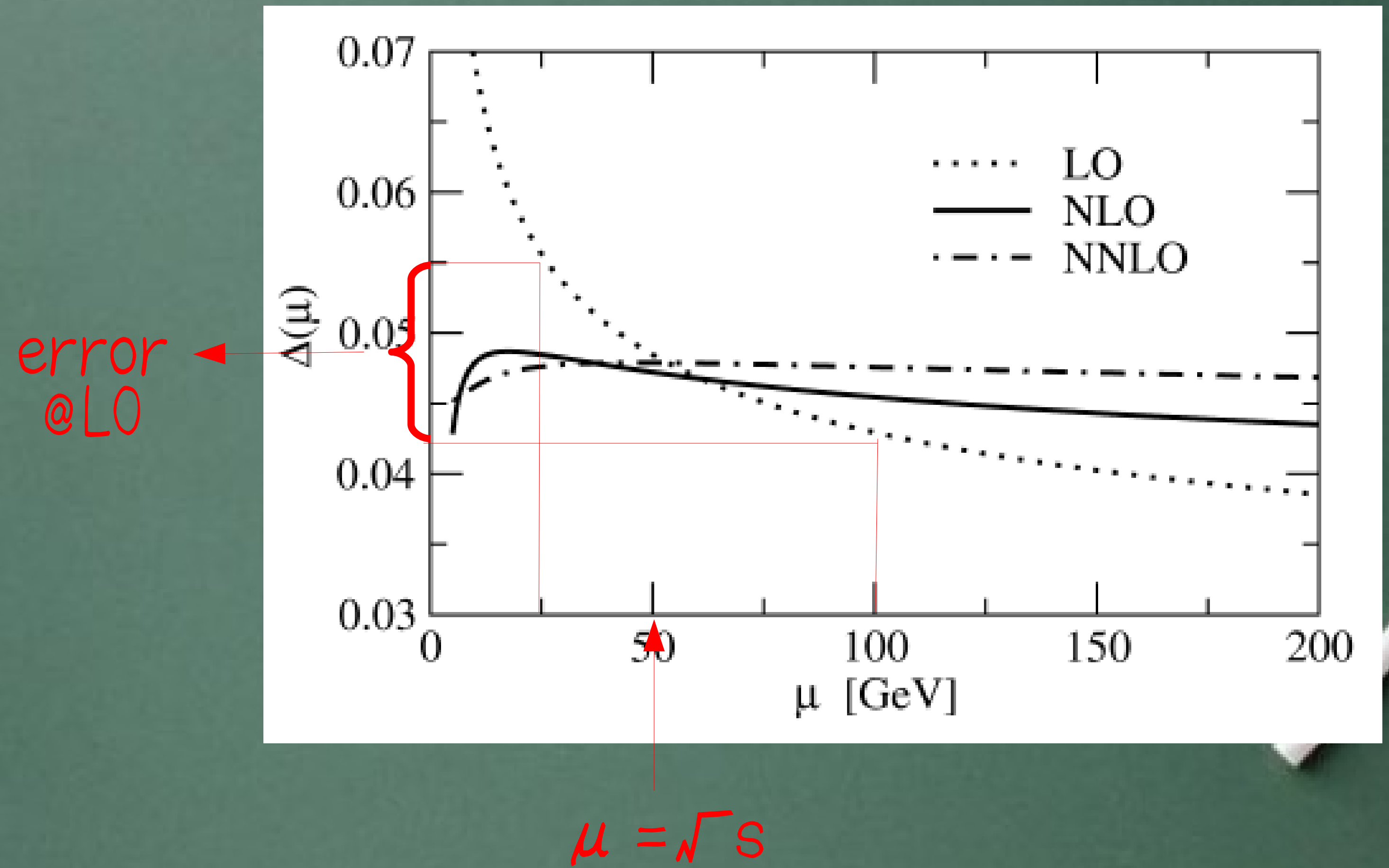
- Example: R-ratio



$$R = R_0 \left\{ 1 + \left(\frac{\alpha_s}{\pi}\right) + 1.411 \left(\frac{\alpha_s}{\pi}\right)^2 - 12.8 \left(\frac{\alpha_s}{\pi}\right)^3 + \cdots \right\}$$

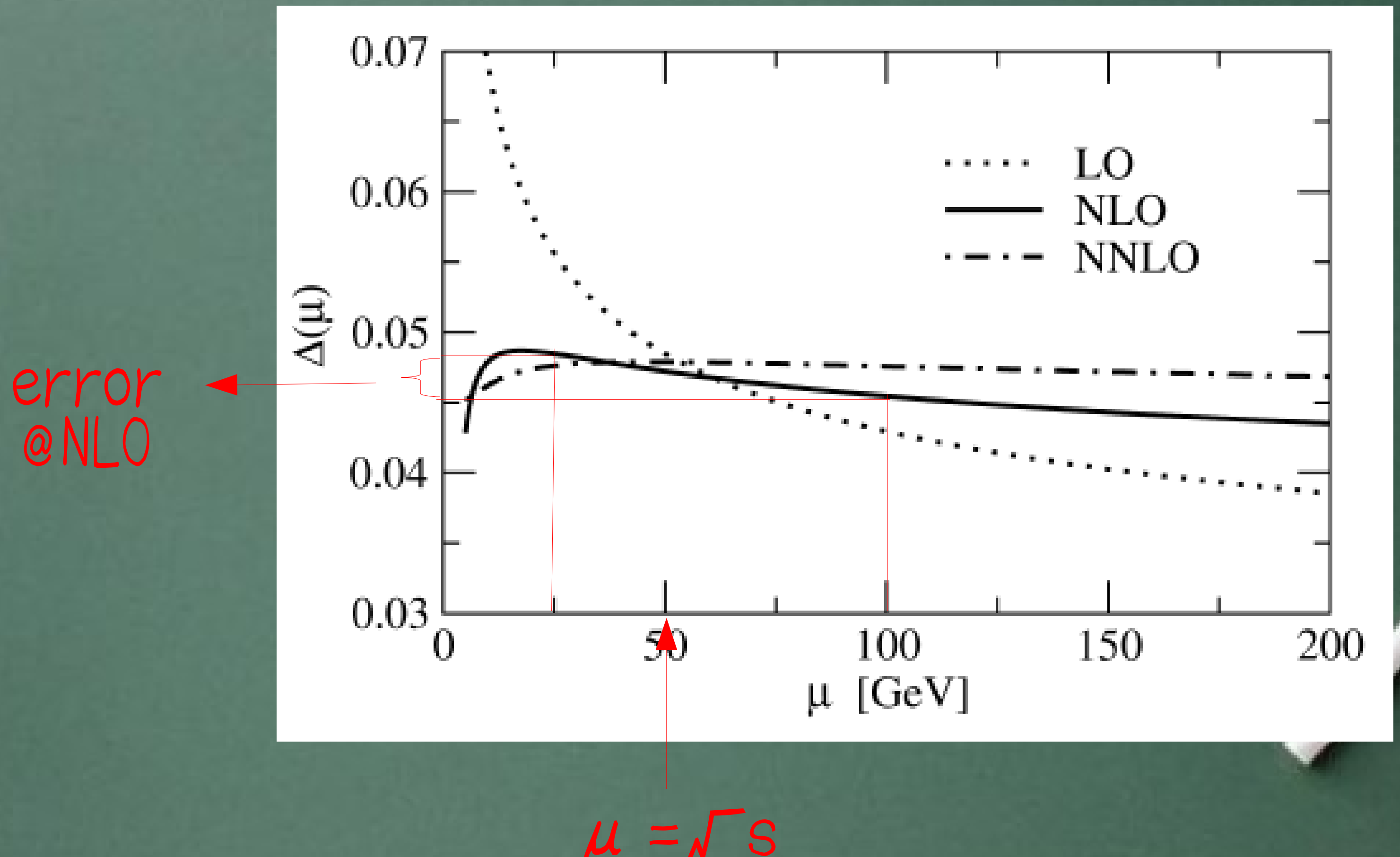
Renormalization Equation/R-ratio

- Example: R-ratio Systematic error from theoretical prediction



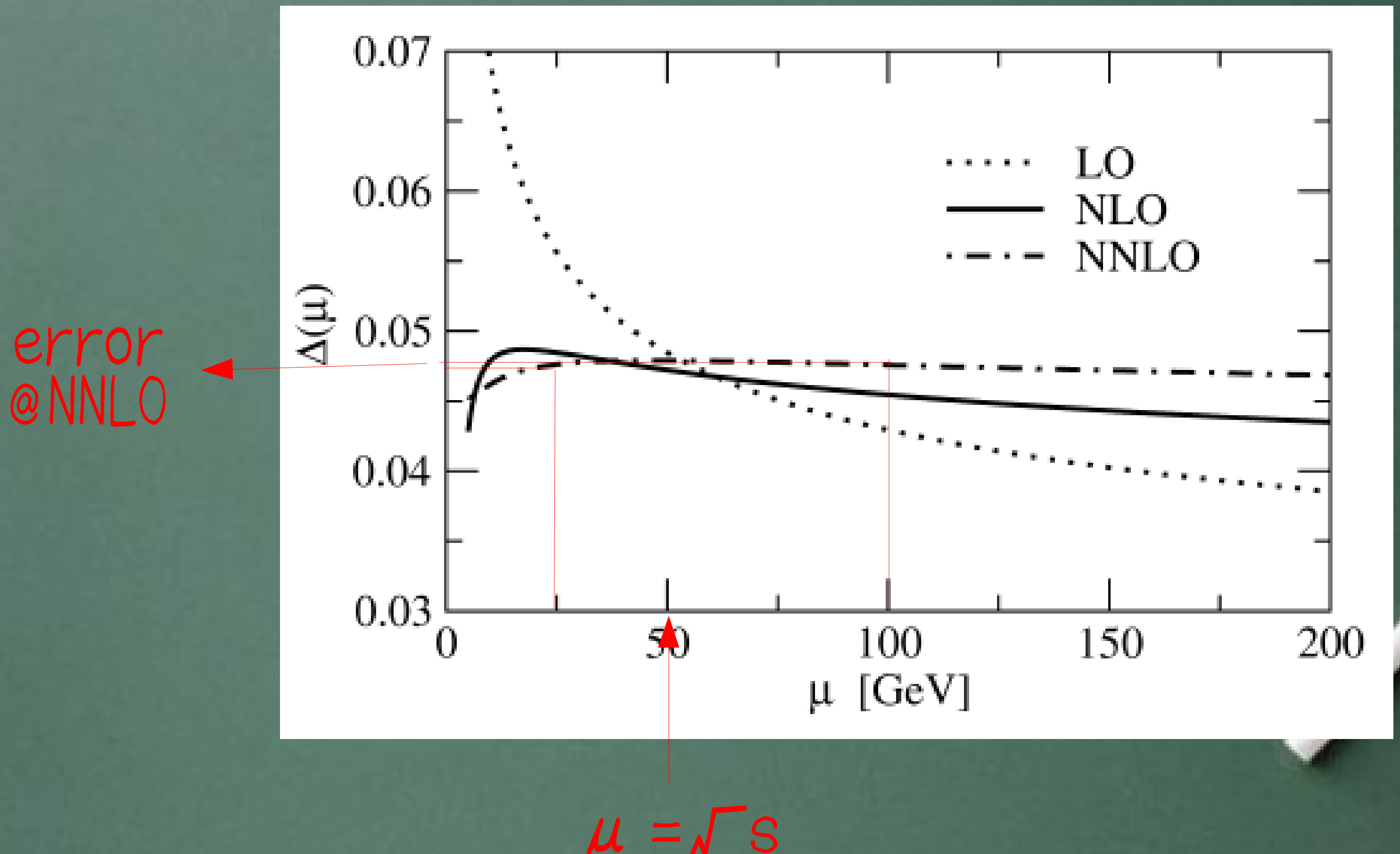
Renormalization Equation/R-ratio

- Example: R-ratio Systematic error from theoretical prediction



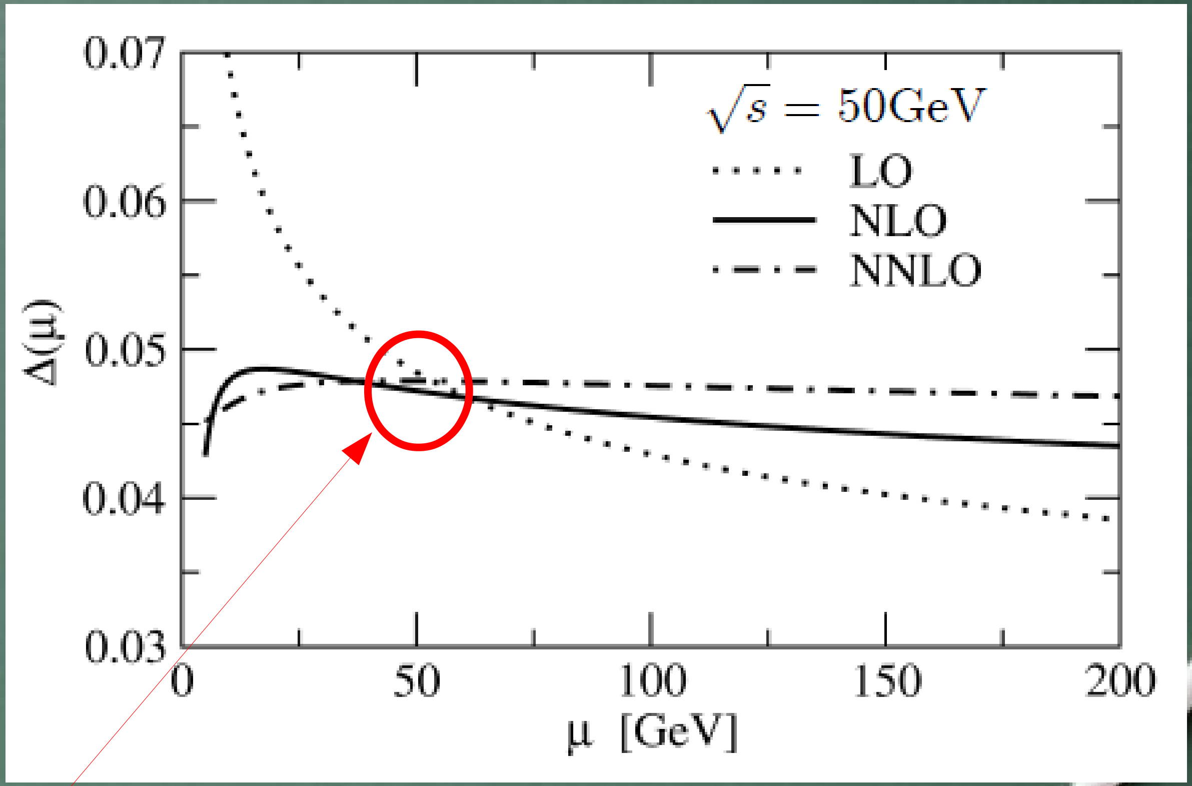
Renormalization Equation/R-ratio

- Example: R-ratio Systematic error from theoretical prediction



Renormalization Equation/R-ratio

- Example: R-ratio Systematic error from theoretical prediction



Correction is very stable @ $\mu = \sqrt{s}$

Renormalization Equation/LL

- General Structure of HO corrections

Typical loop correction

$$I = \frac{\alpha_s(\mu)}{4\pi} \frac{(\mu^2 e^{\gamma_E})^\epsilon}{\Gamma(1-\epsilon)} \int_0^\infty dk^2 \frac{(k^2)^{-\epsilon}}{k^2+Q^2}$$

$$= \frac{\alpha_s(\mu)}{4\pi} \left(\frac{\mu^2 e^{\gamma_E}}{Q^2}\right)^\epsilon \Gamma(\epsilon) = \frac{\alpha_s(\mu)}{4\pi} \left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{Q^2} + \mathcal{O}(\epsilon)\right)$$

$a \in \mathbb{R},\ a > 0$

ε -expansion: $a^\varepsilon = 1 + c_1 \varepsilon + c_2 \varepsilon^2 + \cdots$

$c_i = ??$

Renormalization Equation/LL

- General Structure of HO corrections

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Typical correction term : Leading Log term

$$I^r = \frac{\alpha_s(\mu)}{4\pi} \left(\ln\frac{\mu^2}{Q^2} + \mathcal{O}(\epsilon)\right)$$

Renormalization Equation/LL

- General Structure of HO corrections

Typical loop correction

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Typical correction term : Leading Log term

$$I^r = \frac{\alpha_s(\mu)}{4\pi} \left(\ln\frac{\mu^2}{Q^2} + \mathcal{O}(\epsilon)\right)$$

→ 0 @ $\mu^2 = Q^2$

Renormalization Equation/Scheme Dependence

- Renormalization scheme dependence
  - Two schemes: Scheme A $\Leftrightarrow$ Scheme B

$$\alpha_s^0 = Z_A \alpha_s^A = Z_B \alpha_s^B$$

Expand α^B_s w.r.t. α^B_s.

$$\alpha_s^B = \alpha_s^A \left(1 + c_1 \alpha_s^A\right)$$

$$\Lambda = \mu \exp[-1/(2\beta_0 \alpha_s(\mu)]$$

$$\Lambda_B = \Lambda_A e^{c_1/(2\beta_0)}$$

Renormalization Equation/Scheme Dependence

$$\Lambda_{\overline{\mathrm{MS}}} = \Lambda_{\mathrm{MS}} e^{(\ln(4\pi)-\gamma_E)/2} = 2.66\Lambda_{\mathrm{MS}}$$

Renormalization Equation/Scheme Dependence

$$\Lambda_{\mathrm{MS}} e^{(\ln(4\pi)-\gamma_E)/2} = 2.66\Lambda_{\mathrm{MS}}$$

Next Lesson!