Quantisation

We repeat the bosonic arguments, except for the fact that we have now anti-commutation relations between electron and positron creation-annihilation operators

$$\widehat{\psi}_\alpha(t,\vec{x}) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[u_\alpha(\vec{k},s)\,\widehat{b}(\vec{k},s)e^{-i\omega_{\vec{k}}t+i\vec{k}\cdot\vec{x}} + v_\alpha(\vec{k},s)\,\widehat{d}^{\dagger}(\vec{k},s)e^{i\omega_{\vec{k}}t-i\vec{k}\cdot\vec{x}}\right].$$

$$\{\widehat{\psi}_\alpha(t,\mathbf{x}),\widehat{\psi}^\dagger_\beta(t,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$$

$$\{b(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\} = (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},$$
$$\{b(\mathbf{k},s),b(\mathbf{k}',s')\} = \{b^\dagger(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\} = 0.$$
$$\{d(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\} = (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},$$
$$\{d(\mathbf{k},s),d(\mathbf{k}',s')\} = \{d^\dagger(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\} = 0.$$

$$\widehat{H} = \frac{1}{2}\sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\left[b^\dagger(\mathbf{k},s)b(\mathbf{k},s) - d(\mathbf{k},s)d^\dagger(\mathbf{k},s)\right].$$

$$\widehat{H} = \sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[\omega_{\vec{k}}b^\dagger(\vec{k},s)b(\vec{k},s) + \omega_{\vec{k}}d^\dagger(\vec{k},s)d(\vec{k},s)\right] - 2\int d^3k\,\omega_{\vec{k}}\delta(\vec{0}).$$

We have a conserved charge and current

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0 \qquad Q = e\int d^3x\, j^0$$

The two-point function or Feynman propagator is:

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|T\left[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right]|0\rangle$$
$$T\left[\psi_\alpha(x)\overline{\psi}_\beta(y)\right] = \theta(x^0-y^0)\psi_\alpha(x)\overline{\psi}_\beta(y) - \theta(y^0-x^0)\overline{\psi}_\beta(y)\psi_\alpha(x).$$

Introducing gauge fields

The canonical gauge field is the electromagnetic field. The first one that was understood as a gauge field. For some time this symmetry sounded like a luxury. In fact the classical theory can be formulated exclusively in terms of the E,B field that are manifestly gauge invariant. This is not so in the quantum theory, where we need to use the vector and scalar potentials. There are new, non-local observables. They are responsible for the Bohm-Aharonov effect and the quantisation of electric charge (if there is a single monopole in the Universe, (Dirac)).

What we have learned is that all fundamental interactions known to us are mediated by suitable generalisations of the EM field. They are gauge theories. In fact it seems as though Nature abhors global symmetries. It appears that all the known global symmetries are just low-energy accidents. All symmetries in the UV should be local.

We do not know why this should be so. String Theory is the only theory where this fact finds an explanation. Unfortunately there is no evidence for it at this moment...

E&M in Quantum Mechanics

Classical EM

$$\nabla\cdot\mathbf{E}=0$$
$$\nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla\times\mathbf{B}=\frac{\partial}{\partial t}\mathbf{E}$$

$$\mathbf{E}=-\nabla\varphi-\frac{\partial\mathbf{A}}{\partial t}$$
$$\mathbf{B}=\nabla\times\mathbf{A}.$$

$$\partial_\mu F^{\mu\nu}=j^\mu \qquad j^\mu=(\rho,\mathbf{j})$$
$$\varepsilon^{\mu\nu\sigma\eta}\partial_\nu F_{\sigma\eta}=0, \qquad A^\mu=(\varphi,\mathbf{A})$$
$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

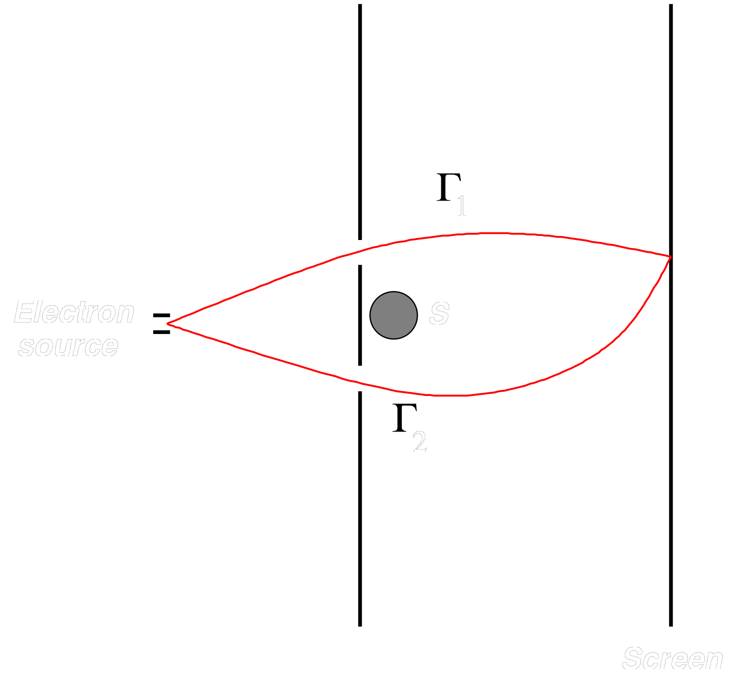
Classical EM in relativistic form

Coupling to QM requires the gauge potentials and a non-trivial transformation of the wave function, this gives subtle consequences to gauge symmetry

$$i\frac{\partial}{\partial t}\Psi=\left[-\frac{1}{2m}(\nabla-ie\mathbf{A})^2+e\varphi\right]\Psi$$
$$\Psi(t,\mathbf{x})\longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x})$$
$$\varphi(t,\mathbf{x})\to\varphi(t,\mathbf{x})+\frac{\partial}{\partial t}\varepsilon(t,\mathbf{x}), \qquad \mathbf{A}(t,\mathbf{x})\to\mathbf{A}(t,\mathbf{x})+\nabla\varepsilon(t,\mathbf{x})$$
$$A_\mu\longrightarrow A_\mu+\partial_\mu\varepsilon$$

Non-local observables

$$\Psi = e^{ie\int_{\Gamma_1}\mathbf{A}\cdot d\mathbf{x}}\Psi_1^{(0)} + e^{ie\int_{\Gamma_2}\mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}$$
$$= e^{ie\int_{\Gamma_1}\mathbf{A}\cdot d\mathbf{x}}\left[\Psi_1^{(0)} + e^{ie\oint_{\Gamma}\mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}\right]$$



$$U = \exp\left[ie\oint_{\Gamma}\mathbf{A}\cdot d\mathbf{x}\right]$$

This is the Aharonov-Bohm effect. The phase factor, and its non-abelian generalisation are known as “Wilson loops” or holonomies of the gauge field. Note that classically there would be no effect. The Lorentz force equation only involves E,B hence the electrons would not see the solenoid at all!!

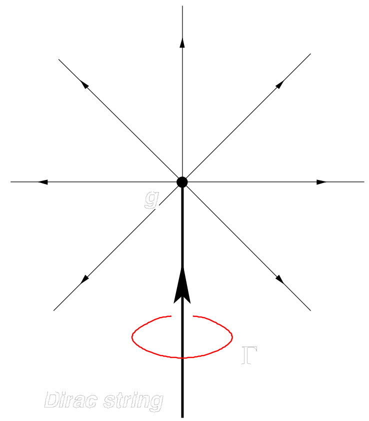
$$m\frac{du^{\mu}}{d\tau} = eF^{\mu\nu}u_{\nu}$$

Magnetic monopoles:
Dirac and charge quantisation

$$\nabla\cdot\mathbf{E} = 0$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla\times\mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}$$

$$\mathbf{E} - i\mathbf{B} \longrightarrow e^{i\theta}(\mathbf{E} - i\mathbf{B})$$

For angle = 90 E and B get exchanged



The symmetry extend to matter if we have magnetic sources:

$$\rho - i\rho_m \longrightarrow e^{i\theta}(\rho - i\rho_m), \qquad \mathbf{j} - i\mathbf{j}_m \longrightarrow e^{i\theta}(\mathbf{j} - i\mathbf{j}_m).$$

Consider a magnetic pole:

$$\nabla\cdot\mathbf{B} = g\,\delta(\mathbf{x}). \qquad B_r = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|^2}, \qquad B_\varphi = B_\theta = 0$$

$$A_\varphi = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|}\tan\frac{\theta}{2}, \qquad A_r = A_\theta = 0.$$

The Dirac string can be changed by gauge transformations, in doing QM it has to be unobservable. Then we can do a “A-B” like argument (Dirac did it 20 years earlier). We should not forget the fact that there is a factor of $\hbar c$

$$e^{ieg} = 1 \qquad eg = 2\pi n$$

$$q_1 g_2 - q_2 g_1 = 2\pi n.$$

Electromagnetic Fields and Photons

Ignoring sources, the E&M field is a "free field"

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

The electric field is the momentum p and the vector potential the "coordinate" q

$$\mathscr{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

$$\partial_\mu F^{\mu\nu} = 0 \qquad 0 = \partial_\mu \partial^\mu A^\nu - \partial_\nu\left(\partial_\mu A^\mu\right) = \partial_\mu \partial^\mu A^\nu$$

To be able to invert, we need to fix the gauge: $\partial_\mu A^\mu = 0.$

$$\varepsilon_\mu(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}}$$
$$k^\mu \varepsilon_\mu(\mathbf{k},\lambda) = 0$$

As usual, we look for plane wave solutions Residual gauge transformation used to fully fix the gauge

$$\epsilon_\mu(\mathbf{k},\lambda) \to \epsilon_\mu(\mathbf{k},\lambda) + k_\mu\,\chi(\mathbf{k}), \;\; k^2 = 0$$
$$k^2 = k_\mu k^\mu = (k^0)^2 - \mathbf{k}^2 = 0$$

Now, as usual we expand the field in oscillator and apply CCR. After fully fixing the gauge there are only two physical polarisations. Gauge invariance seems more a redundancy rather than a symmetry in the description of the theory

$$\widehat{A}_\mu(t,\mathbf{x}) = \sum_{\lambda=\pm1}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}\left[\varepsilon_\mu(\mathbf{k},\lambda)\widehat{a}(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + \varepsilon_\mu(\mathbf{k},\lambda)^*\,\widehat{a}^\dagger(\mathbf{k},\lambda)e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\right].$$

$$[\widehat{a}(\mathbf{k},\lambda),\widehat{a}^\dagger(\mathbf{k}',\lambda')] = (2\pi)^3(2|\mathbf{k}|)\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$$

If we keep all four polarisation by partial gauge fixing, then we get negative probabilities (Gupta-Bleuler, BRST)

$$\delta_{\lambda,\lambda'} \to -\eta_{\lambda,\lambda'}$$

Coupling matter

We imitate the coupling in the Schrödinger equation, this is what used to be called minimal coupling. We make derivatives covariant with respect to space-time dependent changes of phases in the wave-function

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi \qquad D_\mu\left[e^{ie\varepsilon(x)}\psi\right] = e^{ie\varepsilon(x)}D_\mu\psi.$$

$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x}) \qquad D_\mu = \partial_\mu - ieA_\mu.$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon$$

The rigid phase rotation invariance of the Dirac Lagrangian for electrons is transformed into local phase rotations, a physically more satisfactory concept. This defines the coupling of the electron to the E&M field:

$$\mathscr{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi, \qquad \mathscr{L}^{(\rm int)}_{\rm QED} = -eA_\mu\overline{\psi}\gamma^\mu\psi.$$

$$\psi \longrightarrow e^{ie\varepsilon(x)}\psi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon(x).$$

This is QED, the best tested theory in the history of science, an example is the gyromagnetic ratio of the electron,

$$g\frac{e}{8m}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}$$

$$g/2 = 1.00115965218085(76)$$
$$\alpha^{-1} = 137.035999070(98)$$

$$\vec{\mu} = g_\mu\frac{e\hbar}{2m_\mu c}\vec{s}, \qquad \underbrace{g_\mu}_{\rm Dirac} = 2(1+a_\mu)$$

Group Theory reminder

For the SM all group we will need are:

$$G: \quad U(1),\ SU(2),\ SU(3) \qquad [T^a, T^b] = i f^{abc} T^c \qquad G_{SM} = SU(3)\times SU(2)\times U(1)$$

$$g \in G \qquad g = e^{i\epsilon^a T^a} \qquad \mathrm{tr}(T^a T^b) = T_2(R)\,\delta^{ab}$$

$$\det g = 1 \Rightarrow \mathrm{tr}\, T^a = 0$$ (for SU(2), SU(3) not for U(1) of course)

U(1) is of course the simplest, just phase multiplication, i.e. as in QED

SU(2): angular momentum, isospin, and also weak isospin

$$[T^a, T^b] = i\varepsilon^{abc} T^c. \qquad T^{\pm} = \frac{1}{\sqrt{2}}(T^1 \pm i T^2), \qquad T^3$$

$$[T^3, T^{\pm}] = \pm T^{\pm}. \qquad [T^+, T^-] = T^3$$

$$T^a = \frac{1}{2}\sigma^a$$ For spin ½ $$\mathrm{tr}\frac{\sigma^a}{2}\frac{\sigma^a}{2} = \frac{1}{2}\delta^{ab} \qquad a,b = 1,2,3$$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ For spin 1

For SU(3) the generators are the eight Gell-Mann 3x3 traceless hermitean matrices chosen to satisfy:

$$\mathrm{tr}\frac{\lambda^a}{2}\frac{\lambda^a}{2} = \frac{1}{2}\delta^{ab}; \quad a,b = 1,\ldots,8$$

SU(3) of color, an exact gauge symmetry, also flavor SU(3), which is global (see later)

More about SU(3)

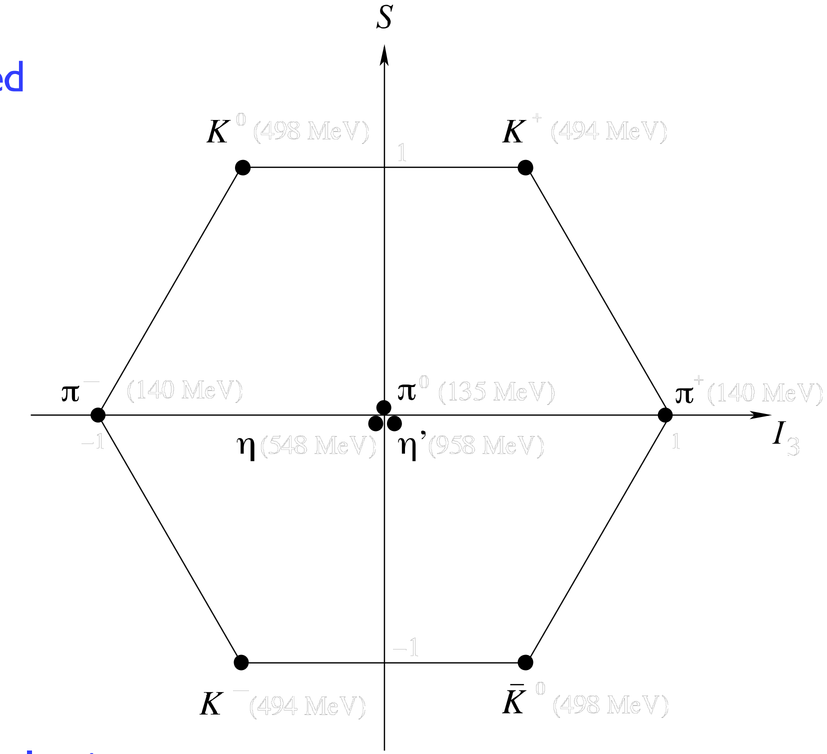
There are very few representations we will need for color SU(3):

$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8}$$

quarks antiquarks gluons

For flavor SU(3) more needed: mesons, baryons

$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8},\ \mathbf{10},\ \overline{\mathbf{10}},\ \mathbf{27}\ldots$$



pseudo-scalar meson octet

$$Q = I_3 + \frac{B+S}{2},$$

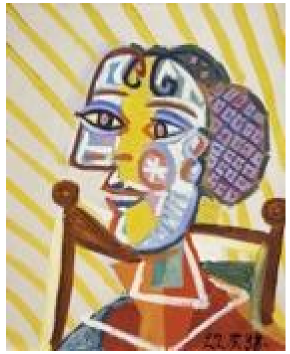
A remarkable fact about the SM and QCD in particular is the fact that once we write the most general Lagrangian compatible with color gauge symmetry, flavor appears as an approximate global symmetry of the problem, although it was theorised earlier.

$$|\Delta^{++}; s_z = \tfrac{3}{2}\rangle = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle \equiv |u\uparrow, u\uparrow, u\uparrow\rangle.$$

$$|uud\rangle_S = \frac{1}{\sqrt{6}}\Big(|uud\rangle + |udu\rangle - 2|duu\rangle\Big), \qquad |\Uparrow\rangle_S = \frac{1}{\sqrt{6}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\Big), \qquad |p\uparrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Uparrow\rangle_A + |uud\rangle_A \otimes |\Uparrow\rangle_S\Big),$$

$$|uud\rangle_A = \frac{1}{\sqrt{2}}\Big(|uud\rangle - |udu\rangle\Big). \qquad |\Uparrow\rangle_A = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big). \qquad |p\downarrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Downarrow\rangle_A + |uud\rangle_A \otimes |\Downarrow\rangle_S\Big).$$

Gauge theories and their quantisation



Imagine we have a theory with a global symmetry

$$\psi \to g\,\psi \qquad \bar\psi \to \bar\psi g^\dagger \qquad \mathcal{L} = \bar\psi i\partial\!\!\!/\,\psi$$

Imitating electromagnetism:

$$\partial_\mu \to D_\mu\psi = \left(\partial_\mu + ieA_\mu^a T^a\right)\psi \equiv (\partial_\mu + ieA_\mu)\psi \qquad D_\mu\psi \to gD_\mu\psi$$

We can read off the gauge field transformations

$$A_\mu \to \frac{1}{ie} g\,\partial_\mu g^{-1} + gA_\mu\, g^{-1}$$

$$g \approx 1+\epsilon \qquad A_\mu \to A_\mu + \frac{1}{ie} D_\mu\epsilon \qquad D_\mu\epsilon + ie[A_\mu,\epsilon]$$

$$[D_\mu, D_\nu] = ieT^a F^a_{\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ef^{abc}A^b_\mu A^c_\nu$$

$$F_{\mu\nu} \equiv T^a F^a_{\mu\nu} \to gF_{\mu\nu}\,g^{-1}$$

Nonabelian gauge fields have self-couplings unlike photons. This is responsible for confinement, among other things

General Gauge Theory

General gauge theory Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + i\overline{\psi}\not{D}\psi + (D_\mu\phi)^\dagger D^\mu\phi$$
$$-\overline{\psi}\left[M_1(\phi) + i\gamma_5 M_2(\phi)\right]\psi - V(\phi).$$

We need to provide the gauge group and the matter representations for bosons and fermions and off we go

Quantising a gauge theory is no joke. There are plenty of subtleties. We give you just a taste

We can define chromoelectric and magnetic fields as in QED

$$F^a_{0i} = \partial_0 A^a_i - \partial_i A^a_0 - ief^{abc}A^b_0 A^c_i \equiv E^a_i$$
$$F^a_{ij} = \epsilon_{ijk}B^a_k, \qquad F^a_{0i} = \partial_0 A^a_0 - D_i A^a_0$$

The canonical variables are

$$\mathbf{A}^a, \mathbf{E}^a$$

$$\mathcal{L} = \mathbf{E}^a\,\partial_0\mathbf{A}^a - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - A^a_0\,(\mathbf{D}\cdot\mathbf{E})^a$$

A^a_0 implements a constraint

We can read off the Hamiltonian density

General Gauge Theory

$$H = \int d^3x \left(\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + A_0^a \, (\mathbf{D}\cdot\mathbf{E})^a \right)$$

$$[A_i^a(\mathbf{x},0), E_j^b(\mathbf{y},0)] = i\,\delta_{ij}\,\delta^{ab}\delta(\mathbf{x}-\mathbf{y})$$

We can fix the gauge A_0=0 so that we only have time-independent gauge transformations in the Hamiltonian theory, but we are missing one of the equations of motion, Gauss' law that has to be implemented as a constraint.

$$(\mathbf{D}\cdot\mathbf{E})^a = 0$$

Cannot be implemented at the operator level. It generates gauge transformations

$$[Q(\epsilon), A_i^a] = i(D\epsilon)^a \qquad U(\epsilon) = \exp(i\int d^3x\,\epsilon^a(\mathbf{x})\,(\mathbf{D}\cdot\mathbf{E})^a), \qquad U\,H\,U^{-1} = H$$

Gauss' law becomes a condition on the physical states:

$$U(\epsilon)|\text{phys}\rangle = |\text{phys}\rangle$$

$$\mathbf{D}\cdot\mathbf{E}\,|\text{phys}\rangle = 0$$

Each gauge configuration sits in an orbit and we need choose only one element, this is done by "fixing" the gauge for the t-independent gauge transf.

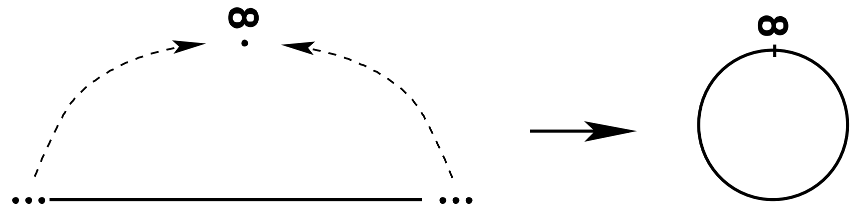
WE HAVE 2-DIM G PHYSICAL DEGREES OF FREEDOM

Some remarks

❖ Gauge symmetry is more a redundant description of the d.o.f.

❖ Gauss' law implements gauge invariance under gauge t. connected to the identity. Consider finite-E configurations

$$g(\mathbf{x}) = e^{i\alpha(\mathbf{x})} \to 1 \qquad |\mathbf{x}| \to \infty$$
$$\alpha(\mathbf{x}) \to 0 \qquad |\mathbf{x}| \to \infty$$



There are others, and Gauss' law cannot impose invariance

$$g(\mathbf{x}) : S^3 \to G, \quad g(\infty) = 1 \qquad \pi_3(G) = Z \text{ the integers}$$

$$g : S^1 \longrightarrow U(1), \qquad g(x) = e^{i\alpha(x)}$$

$$\alpha(2\pi) = \alpha(0) + 2\pi n$$

$$\oint_{S^1} g(x)^{-1} dg(x) = 2\pi n$$

$$n = \frac{1}{24\pi^2} \int_{S^3} d^3x\, \varepsilon_{ijk} \mathrm{Tr}\left[(g^{-1}\partial_i g)(g^{-1}\partial_i g)(g^{-1}\partial_i g) \right]$$

You cannot comb a sphere

A surprise: CP violation

- Gauge invariance only requires that under non-trivial transformations, a phase is generated. This is a vacuum angle! In fact it violates CP.
- It can be measured by looking for an edm of the neutron. So far no result:
- The strong CP problem, axions, invisible axions, axion cosmology, dark matter...

$$g_1 \in \mathcal{G}/\mathcal{G}_0 \ \text{the generator}$$

$$\mathscr{U}(g_1)|\text{phys}\rangle = e^{i\theta}|\text{phys}\rangle.$$

$$S = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu a} - \frac{\theta g^2_{\rm YM}}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$$

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}F^{\sigma\lambda a} \qquad F^a_{\mu\nu}\tilde{F}^{\mu\nu a} = 4\,\mathbf{E}^a\cdot\mathbf{B}^a$$

$$\frac{g^2_{\rm YM}}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu a} = \frac{1}{24\pi^2}\int d^3x\,\varepsilon_{ijk}\mathrm{Tr}\left[(g\partial_i g^{-1})(g_+\partial_j g^{-1})(g_+\partial_k g^{-1})\right].$$

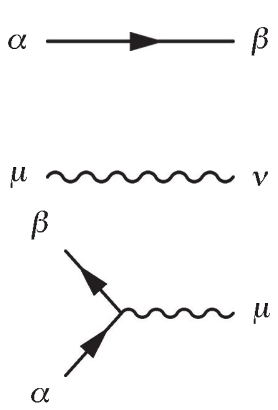
Computational tools

- There are two general procedures to obtain computational rules in QFT: The canonical formalism and the Path Integral formulation.

- You may recall that one used the Interaction Representation, Wick's theorem, T-products, Gaussian integrations…

- In the end we get a collection of well-defined rules that allow us to compute the probability amplitude associates to a given scattering process, out of which we can evaluate the decay width, differential and total cross section and many other quantities that can be observed for instance in collider experiments. The next few pages provide simply a reminder

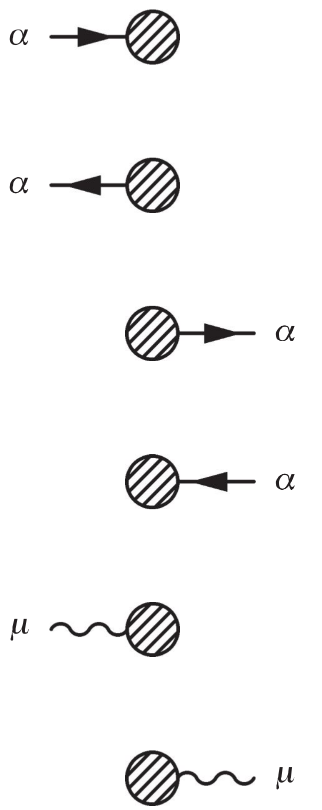
QED Feynman rules



$$\Longrightarrow \left(\frac{i}{\not p - m + i\varepsilon}\right)_{\beta\alpha}$$

$$\Longrightarrow \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon}$$

$$\Longrightarrow -ie\gamma^{\mu}_{\beta\alpha}(2\pi)^4\delta^{(4)}(p_1 + p_2 + p_3).$$



Incoming fermion: $\Longrightarrow u_\alpha(\mathbf{p}, s)$

Incoming antifermion: $\Longrightarrow \overline{v}_\alpha(\mathbf{p}, s)$

Outgoing fermion: $\Longrightarrow \overline{u}_\alpha(\mathbf{p}, s)$

Outgoing antifermion: $\Longrightarrow v_\alpha(\mathbf{p}, s)$

Incoming photon: $\Longrightarrow \varepsilon_\mu(\mathbf{p})$

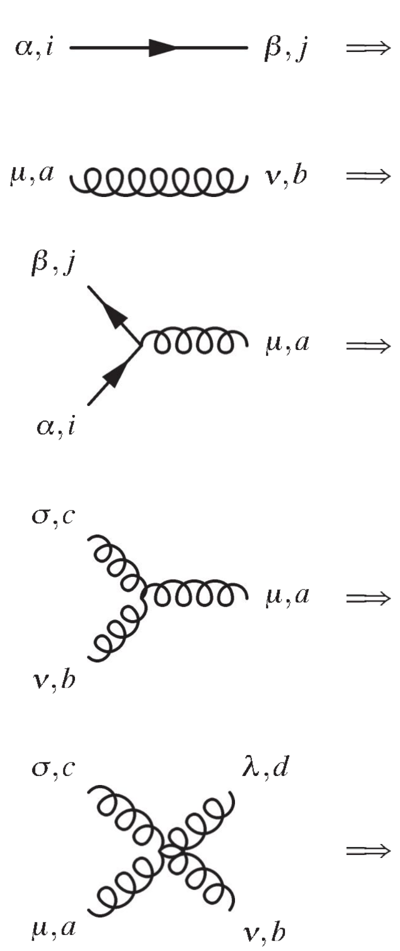
Outgoing photon: $\Longrightarrow \varepsilon_\mu(\mathbf{p})^*$

Integrate over loop momenta

$$\int \frac{d^d p}{(2\pi)^4};$$

A minus sign has to be included for every fermion loop and for every positron line that goes from the initial to the final state. With some extra effort we can derive the Feynman rules for QCD-like theories. They appear in the next page. The quark and anti-quark factors are similar to the electron positron ones, except that we need to include color quantum numbers. The real difference comes with the gluon or non-abelian vector bosons interactions, the are quite involved and contain a large amount of interesting physics perturbatively and specially non-perturbatively.

Standard Model Feynman rules



$$\left(\frac{i}{\not p - m + i\varepsilon}\right)_{\beta\alpha}\delta_{ij}$$

$$\frac{-i\eta_{\mu\nu}}{p^2+i\varepsilon}\delta^{ab}$$

$$-ig\gamma^{\mu}_{\beta\alpha}t^{a}_{ij}$$

$$g\, f^{abc}\Big[\eta^{\mu\nu}(p_1^{\sigma}-p_2^{\sigma})\text{permutations}\Big]$$

$$-ig^2\Big[f^{abe}f^{cde}\left(\eta^{\mu\sigma}\eta^{\nu\lambda}-\eta^{\mu\lambda}\eta^{\nu\sigma}\right) + \text{permutations}\Big]$$

Although the rules seem to be those for QCD, notice that we could always include in the group theory factors t^a-{ij} chiral projectors and make the group not simple but semi-simple as in the case of the SM: SU(3)xSU(2)xU(1). If we work in nice renormalizable gauges, the only difference is that we have to include the Feynman rules for the couplings of the scalar sector. Something we will do later.

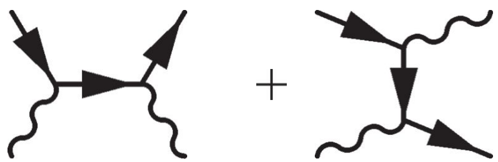
$$t^a_{ij} \to t^a_{ij}\,\frac{1}{2}(1\pm\gamma_5)$$

With this simple trick the hard part, which is the coupling of the W,Z, and photons can be read simply from the rules in the LHS

One example: Thomson Scattering

We work in the NR approximation for simplicity but keeping explicitly the dependence on the photon polarisations. We can guess that the answer has to be a pure number times the classical electron radius

$$\gamma(k,\varepsilon)+e^-(p,s)\longrightarrow\gamma(k',\varepsilon')+e^-(p',s')$$



$$= (ie)^2\overline{u}(\mathbf{p}',s')\not\varepsilon'(\mathbf{k}')^*\frac{\not p+\not k+m_e}{(p+k)^2-m_e^2}\not\varepsilon(\mathbf{k})u(\mathbf{p},s)$$
$$+ (ie)^2\overline{u}(\mathbf{p}',s')\not\varepsilon(\mathbf{k})\frac{\not p-\not k'+m_e}{(p-k')^2-m_e^2}\not\varepsilon'(\mathbf{k}')^*u(\mathbf{p},s)$$

$$p^2=m_e^2=p'^2 \qquad |\mathbf{p}|,|\mathbf{k}|,|\mathbf{p}'|,|\mathbf{k}'|\ll m_e \qquad \not a\not b=-\not b\not a+2(a\cdot b)\mathbf{1}$$

$$k^2=0=k'^2 \qquad (p+k)^2-m_e^2\approx 2m_e|\mathbf{k}|, \qquad (p-k')^2-m_e^2\approx -2m_e|\mathbf{k}'|$$

$$(\not k-m)u(k,s)=0.$$

$$\overline{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s)=2k^\mu$$

Thomson Scattering, continued

$$\langle f|\hat{S}|i\rangle = \langle f|i\rangle + (2\pi)^4\delta^{(4)}\left(\sum_{\text{final}} p'_i - \sum_{\text{initial}} p_j\right) i\mathcal{M}_{i\to f}$$

$$d\sigma = \frac{|\mathcal{M}_{i\to f}|^2}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|}(2\pi)^4\,\delta^{(4)}\left(p_1+p_2-\sum_{j=1}^{n} p'_j\right) d\Phi_k$$

$$F_{\text{coll}} = 4E_1E_2|\mathbf{v}_1-\mathbf{v}_2| = 4E_1E_2\left|\frac{\mathbf{p}_1}{E_1}-\frac{\mathbf{p}_2}{E_2}\right|$$
$$= 4|E_2\mathbf{p}_1 - E_1\mathbf{p}_2| = 4\left(E_2|\mathbf{p}_1| + E_1|\mathbf{p}_2|\right)$$
$$= 4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}.$$

Square the amplitude, sum over final electron polarisations, and sum over the initial ones. We will consider unpolarised incoming photons and study how the outgoing photons can gain some degree of polarisation

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\bar{u}_\beta(\mathbf{k},s) = (\not{k}+m)_{\alpha\beta};$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_e^2}\overline{|i\mathcal{M}_{i\to f}|^2} = \left(\frac{e^2}{4\pi m_e}\right)^2 \left|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*\right|^2$$

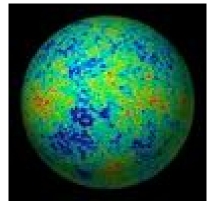
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}\sigma_T\left|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*\right|^2;$$

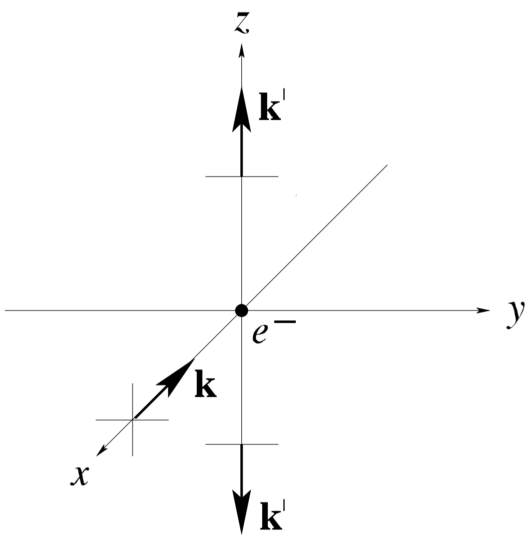
$$\sigma_T = \frac{e^4}{6\pi m_e^2} = \frac{8\pi}{3}r_{\text{cl}}^2$$

$$\frac{1}{2}\sum_{a=1,2}\left|\varepsilon(\mathbf{k},a)\cdot\varepsilon'(\mathbf{k}')^*\right|^2 = \frac{1}{2}\left(\delta_{ij}-\frac{k_ik_j}{|\mathbf{k}|^2}\right)\varepsilon'_j(\mathbf{k}')\varepsilon'_i(\mathbf{k}')^*$$
$$= \frac{1}{2}\left[1-|\hat{\mathbf{k}}\cdot\varepsilon'(\mathbf{k}')|^2\right],$$

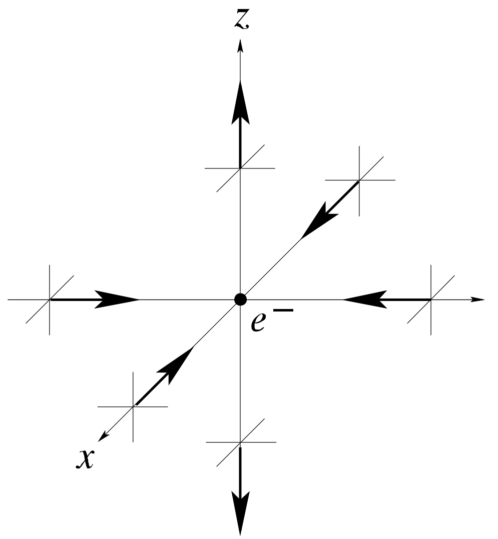
We want to monitor the polarisation of the outgoing photons even when the incoming ones are not polarised

Thomson and CMB Polarisation

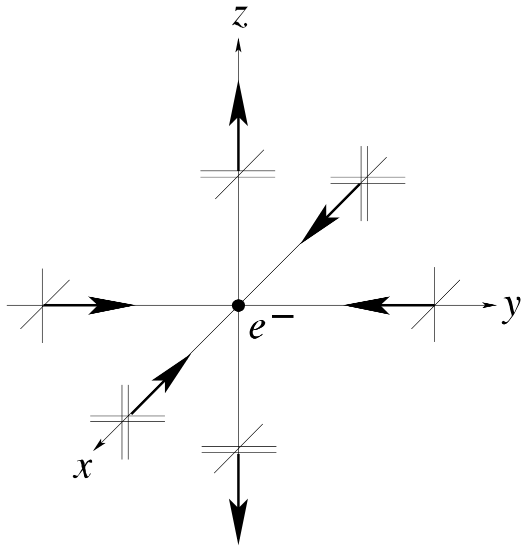




How we can get polarised light



An isotropic incoming distribution of light does not generate polarisation



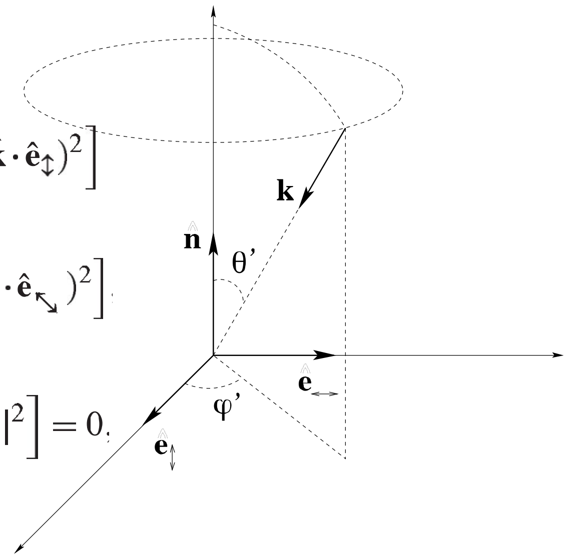
A incoming light with a quadrupole perturbation generates net polarisation

Stokes parameters:

$$Q(\hat{\mathbf{n}}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \left[|\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\leftrightarrow}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\updownarrow}|^2 \right] \quad -\frac{1}{2}\int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \left[(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\leftrightarrow})^2 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\updownarrow})^2 \right]$$

$$U(\mathbf{u}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \left[|\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\swarrow\!\nearrow}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\nwarrow\!\searrow}|^2 \right] = -\frac{1}{2}\int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \left[(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\swarrow\!\nearrow})^2 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\nwarrow\!\searrow})^2 \right]$$

$$V(\hat{\mathbf{u}}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \left[|\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{+}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{-}|^2 \right] = \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \left[|\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{+}|^2 - |\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{-}|^2 \right] = 0,$$



$$\hat{\mathbf{e}}_{\pm} = -\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_{\varphi} \pm i\hat{\mathbf{e}}_{\theta})$$

Quadrupole distribution

Finally we reach the punch line. No circular polarisation is generated by Thomson scattering, and we can write the combination:

$$Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}}) \sim -\int d\Omega(\theta',\varphi') f(\theta',\varphi';\hat{\mathbf{n}}) \sin^2\theta' e^{\pm 2i\varphi'}$$

$$Y_2^{\pm 2}(\theta',\varphi') = 3\sqrt{\frac{5}{96\pi}} \sin^2\theta' e^{\pm 2i\varphi'}$$

One of the obvious generators of quadrupole anisotropies are gravitational waves. Inflation predicts primordial gravitation waves, the measurement of polarisation in the CMB offers an amazing window to obtain this information. The simple computation of Thomson scattering has unexpected consequences

$$Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}}) = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left(E_{\ell m} \pm iB_{\ell m}\right) {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}})$$

$$\langle E^*_{\ell m} E_{\ell' m'}\rangle = C_\ell^{EE}\delta_{\ell\ell'}\delta_{mm'}, \qquad \langle B^*_{\ell m} B_{\ell' m'}\rangle = C_\ell^{BB}\delta_{\ell\ell'}\delta_{mm'}$$

Noether's Theorem

Quantum mechanical realisation of Symmetries (Wigner's theorem). In a QM theory physical symmetries are maps among states that preserve probability amplitudes (their modulus). They can be unitary or anti-unitary

$$|\alpha\rangle \longrightarrow |\alpha'\rangle, \qquad |\beta\rangle \longrightarrow |\beta'\rangle$$

$$|\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|.$$

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle$$ unitary

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle^*$$ anti-unitary T-reversal, CPT

For continuous symmetries we have Noether's celebrated theorem: If under infinitesimal transformations, AND WITHOUT USING THE EQUATIONS OF MOTION you can show that:

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu$$

then there is a conserved current in the theory

Noether's Theorem

In formulas:

$$\delta_\varepsilon \mathscr{L} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\varepsilon \phi + \frac{\partial \mathscr{L}}{\partial \phi} \delta_\varepsilon \phi$$
$$= \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \phi \right) + \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta_\varepsilon \phi$$
$$= \partial_\mu K^\mu .$$

$$\partial_\mu J^\mu = 0$$

$$\boxed{J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \phi - K^\mu}$$

With a conserved charge that generates the symmetry:

$$Q \equiv \int d^3x J^0(t, \mathbf{x})$$

$$\frac{dQ}{dt} = \int d^3x \, \partial_0 J^0(t, \mathbf{x}) = -\int d^3x \, \partial_i J^i(t, \mathbf{x}) = 0,$$

$$\delta \phi = i[\phi, Q].$$

Space-time translations -- Energy-Momentum
Lorentz transformation-- Angular momentum and CM motion
Phase rotation -- abelian and non-abelian charges

Useful examples

Massive Dirac fermions:

$$\mathscr{L} = i\overline{\psi}_j \partial\!\!\!/ \psi_j - m\overline{\psi}_j \psi_j \qquad \psi_i \longrightarrow U_{ij}\psi_j \qquad U \in U(N)\ \ N \text{the number of fermions}$$

$$U = \exp(i\alpha^a T^a)\,, \qquad (T^a)^\dagger = T^a$$

$$j^{\mu a} = \overline{\psi}_i T^a_{ij} \gamma^\mu \psi_j \qquad \partial_\mu j^\mu = 0 \qquad Q^a = \int d^3x\, \psi_i^\dagger T^a_{ij} \psi_j$$

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha) = e^{i\alpha^a Q^a}.$$

When U is the identity, we have fermion number, or charge

In the m=0 we have more symmetry: CHIRAL SYMMETRY, rotate L,R fermions independently

$$\mathcal{L} = i\bar{\psi}_{jL}\, \partial\!\!\!/\, \psi_{Lj} + i\bar{\psi}_{jR}\, \partial\!\!\!/\, \psi_{Rj}$$

$$\psi_{L,R} \to U_{L,R}\, \psi_{L,R} \qquad U(N)_L \times U(N)_R$$