Relic density of WIMPs

- Assume a new stable (very long-lived) neutral weakly-interacting particle
- Will be in thermal equilibrium when T of Universe much larger than its mass
- Equilibrium abundance maintained by processses

$$\chi\bar{\chi} \rightarrow e^+e^-, \mu^+\mu^-, \tau^+\tau^-, q\bar{q}, W^+W^-, ZZ$$

- As well as reverse processes, inverse reaction proceeds with equal rate

- Number density of dilute weakly interacting particle

$$n_\chi = \frac{g}{(2\pi)^3}\int f(\mathrm{p})d^3\mathrm{p}$$

- g: number of internal degrees of freedom

$$f(\mathrm{p}) = exp\left(\frac{E-\mu}{T}\pm 1\right)^{-1}$$

- μ: chemical potential, $E^2=p^2+m^2$

$$n_\chi = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}}{exp\left((E-\mu)/T\right)\pm 1}EdE$$

- In relativistic limit (T>>m, T>>μ)

$$n_\chi \quad = \quad \frac{3}{4}\frac{\zeta(3)}{\pi^2}gT^3 \quad \text{Fermi}$$

- n ~T^3, as many χ than photons

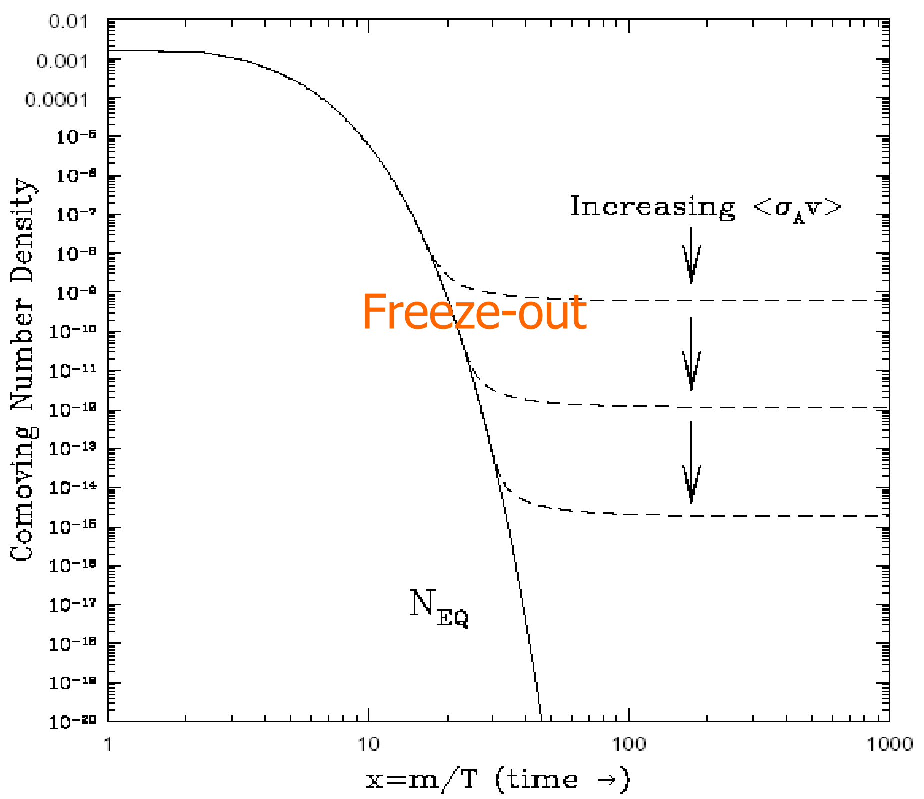
- In non relativistic limit m>>T (also T>>μ)

$$n_{\chi}^{eq} \approx g(m_{\chi}T/2\pi)^{\frac{3}{2}}exp(-m_{\chi}/T).$$

- The number density is Boltzmann suppressed.
- Iφ expansion of the Universe was so slow that thermal equilibrium was maintained -> number of WIMPs today would be exponentially suppressed (no Wimps today)
- T>μ Wimps abundant, rapidly annihilating in SM particles (vice-versa)
- Universe expands T drops below m, n drops exponentially, rate of annihilation drops below expansion rate Γ< H
- When not enough χ for annihilation - > fall out of equilibrium and freeze-out (production of wimps ceases) T_{FO}~m/20

Relic density of WIMPs

- In early universe WIMPs are present in large number and they are in thermal equilibrium
- As the universe expanded and cooled their density is reduced through pair annihilation
- Eventually density is too low for annihilation process to keep up with expansion rate
  - Freeze-out temperature
- LSP decouples from standard model particles, density depends only on expansion rate of the universe



Boltzmann equation

- Time evolution of the number density of Wimps

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma v \rangle \left((n_\chi)^2 - (n_\chi^{eq})^2 \right)$$

Depletion of χ due to annihilation

Creation of χ from inverse process

$$H = \dot{R}/R$$

H: Hubble expansion rate
R: scale factor of the Universe

Solving Boltzmann equation

- Y: ratio of number density to entropy density, s

$$\frac{dY}{dt} = \frac{d}{dt}\left(\frac{n}{s}\right) = \frac{dn}{dt}\frac{1}{s} - \frac{n}{s^2}\frac{ds}{dt}$$

- $R^3 s$ is constant in absence of entropy production

$$\frac{ds}{dt} = -3Hs \qquad\qquad \frac{dY}{dt} = \frac{dn}{dt}\frac{1}{s} + 3H\frac{n}{s}$$

- Evolution eq. $$\frac{dY}{dt} = -s\langle\sigma v\rangle\left(Y^2 - Y_{eq}^2\right)$$

- RHS depends only on T $$\frac{ds}{dT}\frac{dT}{dt} = -3Hs$$

$$\frac{dY}{dT} = \frac{1}{3H}\frac{ds}{dT}\langle\sigma v\rangle\left(Y^2 - Y_{eq}^2\right)$$

- Change of variable $x = m/T$

$$\frac{dY}{dx} = -\frac{m}{x^2}\frac{1}{3H}\frac{ds}{dT}\langle\sigma v\rangle\left(Y^2 - Y_{eq}^2\right)$$

- In radiation dominated universe in FRW cosmology

$$H^2 = \frac{8\pi G\rho}{3}$$

- Energy and entropy density parametrized with eff. Degrees of freedom geff, heff

$$\rho = g_{eff}(T)\frac{\pi^2}{30}T^4 \quad s = h_{eff}(T)\frac{2\pi^2}{45}T^3$$

$$\frac{dY}{dx} = -\sqrt{\frac{\pi g_*(T)}{45G}}\frac{m}{x^2} < \sigma v > (Y^2 - Y_{eq}^2)$$

$$\frac{dY}{dx} = -\sqrt{\frac{\pi g_*(T)}{45G}}\frac{m}{x^2} < \sigma v > (Y^2 - Y_{eq}^2)$$

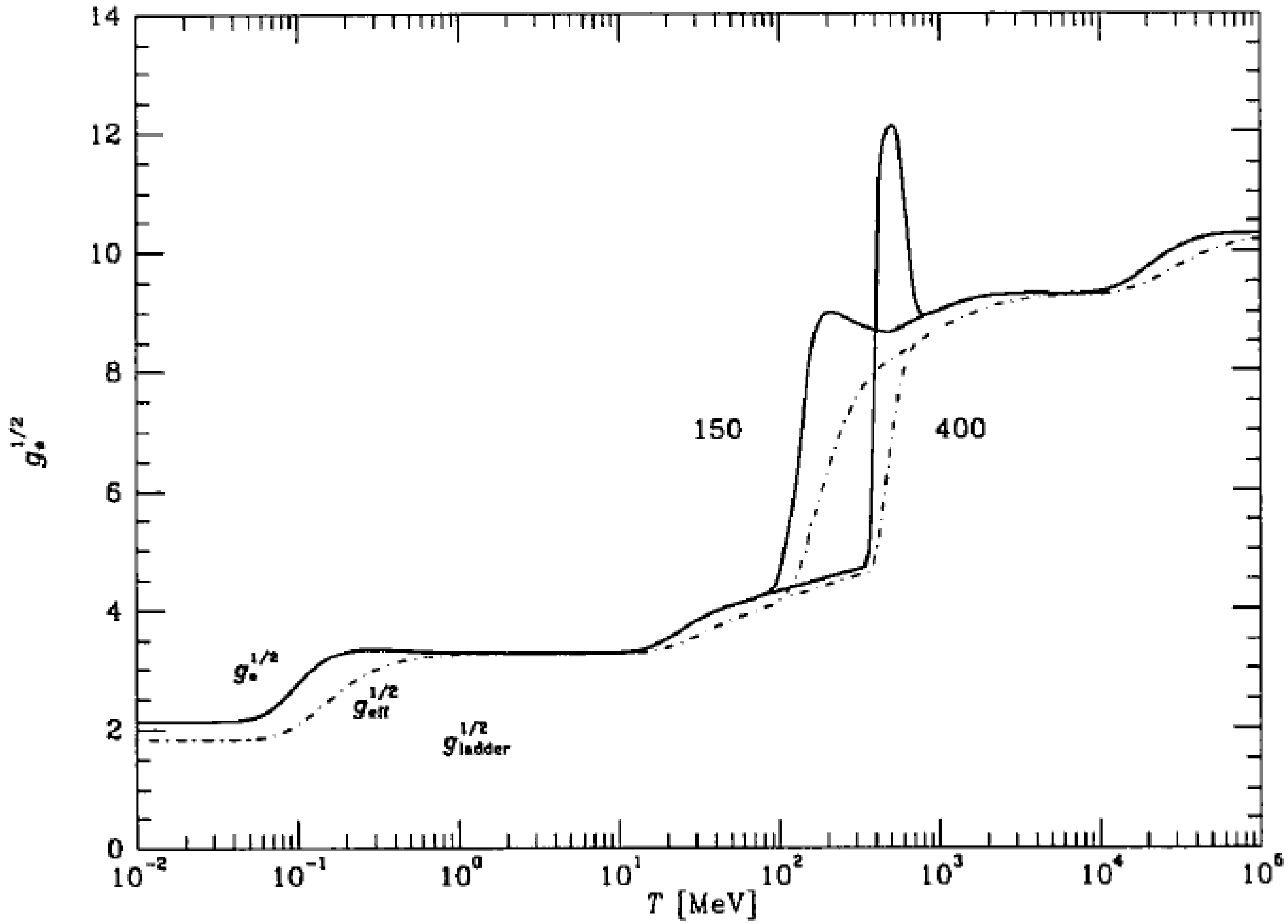
- $g^*(T)$ degree of freedom parameter derived from thermodynamics describing state of universe

$$g_*^{1/2} = \frac{h_{\text{eff}}}{g_{\text{eff}}^{1/2}}\left(1 + \frac{1}{3}\frac{T}{h_{\text{eff}}}\frac{dh_{\text{eff}}}{dT}\right)$$

- $Y_{eq}(T)$: thermal equilibrium abundance

$$Y_{eq}(T) = \frac{n_{eq}}{s} = \frac{45}{4\pi^4 h_{eff}(T)}\, g\frac{m^2}{T^2}K_2\left(\frac{m}{T}\right)$$

- Y_{eq} falls rapidly as temperature decreases
- Equation is valid under the condition that annihilation processes are in thermal equilibrium and chemical potential negligible



Effective degrees of freedom as function of temperature
From Gondolo, Gelmini, Nucl.Phys. B360(91) 145

$$\frac{dY}{dx} = -\sqrt{\frac{\pi g_*(T)}{45G}}\frac{m}{x^2} < \sigma v > (Y^2 - Y_{eq}^2)$$

- Integrating from T=inf. to $T=T_0$ (photon temperature of the Universe today) gives Y_0
- Relic density at present

$$\Omega_\chi = \frac{m_\chi n_\chi}{\rho_{\rm crit}} = \frac{m_\chi s_0 Y_0}{\rho_{\rm crit}}$$

- s_0 : today's entropy at T=2.726K s_0=2889.2 cm^{-3}

- H= 100 h km/s/Mpc
- P= 1.88X 10^{-29} h^2 g cm^{-3}

$$\Omega_\chi h^2 = 2.755 \times 10^8 \frac{m_\chi}{GeV} Y_0$$

Solving for Y

- Equation for Y can be solved numerically or use freeze-out approximation
- High T WIMP are close to equilibrium $Y \sim Y_{eq}$
- $d(Y-Y_{eq})/dT$ is negligible
- At freeze-out Y will be almost constant and Y_{eq} decreases significantly

$$\frac{dln(Y_{eq})}{dT} = \sqrt{\frac{\pi g_*(T)}{45G}} < \sigma v > Y_{eq}\delta(\delta+2)$$

- When $Y \gg Y_{eq}$ can neglect Y_{eq} completely

$$\frac{1}{Y(0)} = \frac{1}{Y_f} + \sqrt{\frac{\pi}{45G}} \int_{T_0}^{T_f} g_*^{1/2}(T) < \sigma v > dT$$

- Solve iteratively and match the two solutions at freeze-out

Solving for Y

- These solutions are implemented in numerical codes that solve for the relic density of dark matter in supersymmetry (DarkSUSY, Isarelic, SuperISO) and in other extensions of the SM (micrOMEGAs)
- The complexity is in computing sigma v in a given model (many processes can contribute depending on the details of the model) – more in next lectures

Approximate solution

- χ freeze out at T~m/20 or 25, particles are non relativistic when FO.
- Expand $\sigma v = a + bv^2$
- Thermal average $\langle \sigma v \rangle = a + 6bT/m$
- After neglecting $1/Y_f$

$$Y_0^{-1} = \sqrt{\frac{\pi}{45G}} \int_{T_0}^{T_f} g_*^{1/2}(T) < \sigma v > dT$$

$$\begin{aligned}
\Omega h^2 &= 2.755 \times 10^8 GeV^{-1} x_F \sqrt{\frac{45}{\pi}} \frac{x_F}{M_{Pl} g_*^{1/2} \langle \sigma v \rangle} \\
&= \frac{2.755 \times 10^8 GeV^{-1}}{1.2 \times 10^1 9 GeV} \sqrt{\frac{45}{\pi}} 0.389 GeV^2 mb \times 10^{-27} cm^2/mbc \left(\frac{x_F}{g_*^{1/2} \langle \sigma v \rangle} \right) \\
&= 1.07 \times 10^{-27} \left(\frac{x_F}{g_*^{1/2} \langle \sigma v \rangle} \right) \approx \frac{3 \times 10^{-27} \mathrm{cm}^3 \mathrm{s}^{-1}}{\langle \sigma v \rangle} \qquad (2.
\end{aligned}$$

Thermally averaged cross section

$$\langle \sigma v \rangle = \frac{\int d^3p_1 d^3p_2 f(E_1) f(E_2) \sigma v}{\int d^3p_1 d^3p_2 f(E_1) f(E_2)}$$

- At T=m/20 $\quad f(E) \propto exp^{-E/T}$

$$d^3p_1 d^3p_2 = 4\pi p_1 dE_1 4\pi p_2 dE_2 \frac{1}{2} d\cos\theta$$

- Change of variables, $E_+ = E_1 + E_2 \quad E_- = E_1 - E_2$

$$s = 2m^2 + 2E_1E_2 - 2p_1p_2\cos\theta$$

$$d^3p_1 d^3p_2 = 2\pi^3 E_1 E_2 dE_+ dE_- ds$$

The integration regions $(E_1 >, E_2 > m - 1 < \cos\theta < 1)$ transforms to

$$|E_1| \leq \sqrt{1 - \frac{4m^2}{s}} \sqrt{E_+^2 - s} \quad E_+ \geq \sqrt{s} \quad s \geq 4m^2$$

$$
\begin{aligned}
Num &= 2\pi^2 \int dE_+ \int dE_- \int ds\sigma v E_1 E_2 exp^{-E_+/T} \\
&= 4\pi^2 \int ds\sigma F \sqrt{1-\frac{4m^2}{s}} \int dE_+ exp^{-E_+/T} \sqrt{E_+^2 - s} \\
&= 2\pi^2 T \int ds\sigma (s-4m^2)\sqrt{s} K_1\left(\sqrt{s}/T\right)
\end{aligned}
$$

- Note that $\sigma F = \sigma v E_1 E_2$ is a function of s only

$$F = 1/2\sqrt{s(s-4m^2)}.$$

- Denominator $\int d^3p_1 d^3p_2 f(E_1) f(E_2) = \left[4\pi m^2 T K_2(m/T)\right]^2$

$$\langle \sigma v \rangle = \frac{1}{8m^4 T K_2^2(m/T)} \int_{4m^2}^{\infty} \sigma (s-4m^2)\sqrt{s} K_1(\sqrt{s}/T) ds$$

- This result is general and valid near thresholds and resonances where often used approximation $a+bv^2$ fails
- Resonance -> strong increase of cross section at $s=4m^2$

Coannihilation

- Take N non-standard particles, mass mi, gi degrees of freedom
- Assume discrete symmetry guarantees that lightest particle is stable
- Relic abundance in general determined not only by annihilation of LSP but also annihilation of heavier particles. These all decay into LSP/
- Relevant only when small mass splitting between LSP and other particles

- Reactions that change number densities

$$\chi_i \chi_j \to XY \qquad \chi_i X \to \chi_j Y \qquad \chi_j \to \chi_i XY$$

X,Y: SM particles

- Set of N Boltzmann equations

$$\begin{aligned}\frac{dn_i}{dt} &= -3Hn_i - \sum_{i,j=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{eq} n_j^{eq} \right) \\ &- \sum_{j \neq i} \langle \sigma'_{Xij} v_{ij} \rangle \left(n_i n_X - n_i^{eq} n_X^{eq} \right) - \sigma'_{Xji} v_{ij} \rangle \left(n_j n_X - n_j^{eq} n_X^{eq} \right) \\ &- \sum_{j \neq i} \left(\Gamma_{ij} (n_i - n_i^{eq}) - \Gamma_{ji} (n_j - n_j^{eq}) \right)\end{aligned}$$

- Annihilations

$$\sigma_{ij} = \sum_{SM} \sigma(\chi_i \chi_j \to X_{SM} X_{SM})$$

- Scattering off cosmic thermal background

$$\sigma'_{Xij} = \sum_{X,Y} \sigma(\chi_i X_{SM} \to \chi_j Y_{SM})$$

- Decays

$$\Gamma_{ij} = \sum_X \Gamma(\chi_i \to \chi_j X_{SM})$$

- Thermally averaged cross section

$$\langle \sigma_{ij} v_{ij} \rangle = \frac{\int d^3p_i d^3p_j f(E_i) f(E_j) \sigma_{ij} v_{ij}}{\int d^3p_i d^3p_j f(E_i) f(E_j)} \qquad v_{ij} = \frac{\left((p_i \cdot p_j)^2 - m_i^2 m_j^2\right)^{1/2}}{E_i E_j}$$

- Decay rate of χ_i other than LSP much faster than decay rate of Universe, all decay into LSP. LSP abundance sum of abundances

$$n = \sum_{i=1}^{N} n_i$$

- Generalisation of equation for number density

$$\frac{dn}{dt} = -3Hn - \sum_{i,j=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{eq} n_j^{eq} \right)$$

Scattering rate for χ_i on SM is much faster than annihilation rate. SM particles assumed to be light hence relativistic their densities is much larger (T^3) than those of non-relativistic particles

$$n_i n_j \sigma_{ij} \propto T^3 m_i^{3/2} m_j^{3/2} \sigma_{ij} exp^{-(m_i+m_j)/T}$$

$$n_i n_X \sigma'_{Xij} \propto T^{9/2} m_i^{3/2} \sigma'_{ij} exp^{-m_i/T}$$

$$n_X/n_j \propto (T/m_j)^{3/2} exp^{m_j/T} \approx 10^9$$

- χ_i particles remain in thermal equilibrium, ratio of densities = equilibrium one before,during,after FO

$$\frac{n_i}{n} \approx \frac{n_i^{eq}}{n^{eq}}$$

- Define

$$r_i = \frac{n_i^{eq}}{n^{eq}} = \frac{g_i(1+\Delta_i)^{3/2} exp(-x_i \Delta_i)}{g_{eff}}$$

$$\Delta_i = \frac{m_i - m_1}{m_1}$$

$$g_{eff} = \sum_{i=1}^{N} g_i(1+\Delta_i)^{3/2} exp(-x\Delta_i)$$

- Boltzmann eq. including coannihilation

$$\frac{dn}{dt} = -3Hn - \langle \sigma_{eff} v \rangle \left(n^2 - n_{eq}^2\right)$$

$$\langle \sigma_{eff} v \rangle = \sum_{ij} \langle \sigma_{ij} v_{ij} \rangle \frac{n_i^{eq}}{n^{eq}} \frac{n_j^{eq}}{n^{eq}}$$

$$\begin{aligned} \sigma_{eff} &= \sum_{ij}^{N} \sigma_{ij} r_i r_j \\ &= \sum_{ij}^{N} \sigma_{ij} \frac{g_i g_j}{g_{eff}^2} (1+\Delta_i)^{3/2} (1+\Delta_j)^{3/2} exp^{-x(\Delta_i + \Delta_j)} \end{aligned}$$

- Following same steps as before

Exp(- ΔM)/T

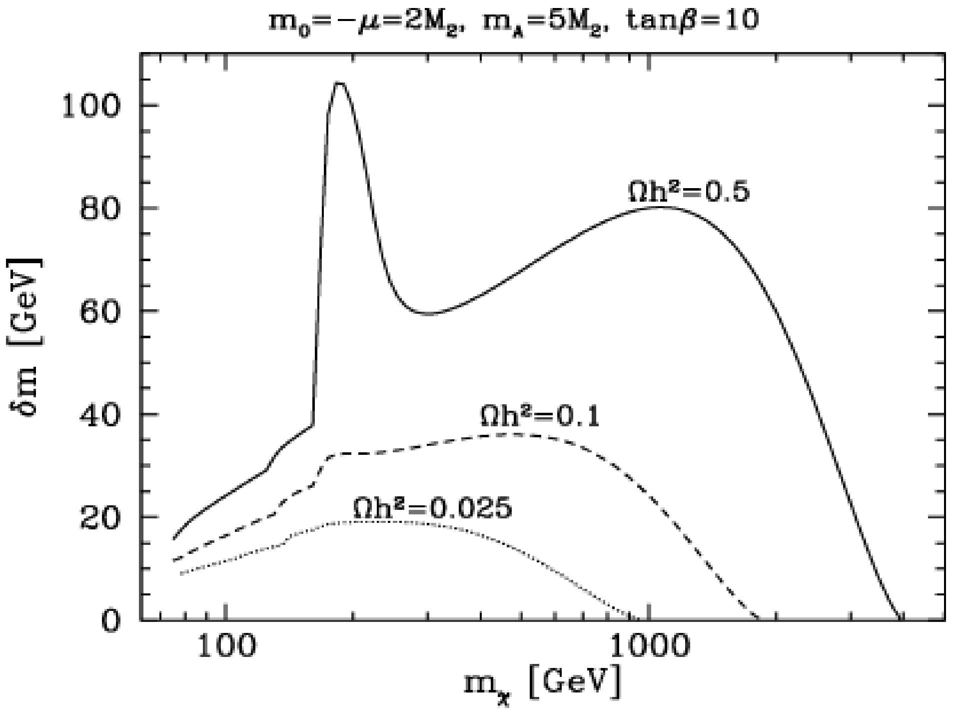
$$< \sigma_{eff} v > = \frac{\sum_{i,j} g_i g_j \int_{(m_i+m_j)^2} ds \sqrt{s} K_1(\sqrt{s}/T) p_{ij}^2 \sigma_{ij}(s)}{2T\left(\sum_i g_i m_i^2 K_2(m_i/T)\right)^2}$$

- One can proceed as for the case of only LSP to solve for abundance and obtain relic density

- Coannihilation processes are strongly suppressed if large mass difference NLSP-LSP
- For $\chi_i \chi_1$->XY with 20% mass difference

$$exp(-20(m_j - m_1)/m_1) \approx 0.02$$

- Larger mass differences can be relevant if coan process has much larger cross section than ann.



- An example of how coannihilation can reduce the relic density (this case if for SUSY – more later)

If σv for coan >> σv ann , NLSP annihilates into SM faster -> reduce number of LSP

Summary

- Relic density of Wimp is obtained after solving eq.

$$\frac{dY}{dx} = -\sqrt{\frac{\pi g_*(T)}{45G}}\frac{m}{x^2} < \sigma v > (Y^2 - Y_{eq}^2)$$

- Depends only on effective annihilation cross section – calculable in specific particle physics model.

$$\Omega h^2 \approx \frac{3 \times 10^{-27} \mathrm{cm}^3 \mathrm{s}^{-1}}{\langle \sigma v \rangle}$$

- Typical $\sigma v = 3\ 10^{-26}\ \mathrm{cm}^3/\mathrm{s}$
- In this eq. all processes involving annihilation and coannihilation should be included (in supersymmetric model can be over 3000 processes)

Extra Notes

Deriving Boltzmann equation

- Evolution of phase space density f(E,t) of a particle

L[f] = C[f]

- L : Liouville operator gives rate of change in time of particle phase space density
- C collision operator: number of particles per phase space volume lost or gained after collision with other particles

$$L[f] = \frac{\partial f}{\partial t} - H\frac{|p|^2}{E}\frac{\partial f}{\partial E}$$

- Botzmann equation integrated over p and summed over spin
- After in $g_1 \int L[f_1] \frac{d^3p_1}{(2\pi)^3} = \frac{1}{R^3}\frac{d}{dt}(R^3 n_1) = \dot{n}_1 + 3Hn_1$ EdE=pdp

- Collision term in case of 1+2->3+4

$$g_1 \int C[f_1] \frac{d^3p_1}{(2\pi)^3} = -\sum_{spins} \int \Big(f_1 f_2 (1 \pm f_3)(1 \pm f_4) |\mathcal{M}_{12\to 34}|^2 - f_3 f_4 (1 \pm f_1)(1 \pm f_2) |\mathcal{M}_{34\to 12}|^2 \Big) (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \Pi_1 \Pi_2 \Pi_3 \Pi_4$$

$$\Pi_i = d^3p_i/((2\pi)^3 2E_i)$$

- Assume annihilation products go quickly into equilibrium with thermal background (em interactions with photons) f3f4->$f_3^{eq} f_4^{eq}$ and 1+f_i ~1
- δ– function $f_3^{eq}, f_4^{eq} = exp(-(E_3 + E_4)/T$ -> $f_1^{eq}, f_2^{eq} = exp(-(E_1 + E_2)/T$

- Unpolarized cross section

$$\sigma_{12;34} = \frac{1}{4Fg_1g_2}\sum_{spins}\int |\mathcal{M}_{12\to34}|^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$F = [(p_1\cdot p_2)^2 - m_1^2 m_2^2]^{1/2}.$$

- Collision term

$$g_1\int C[f_1]\frac{d^3p_1}{(2\pi)^3} = -\int \sigma v g_1 g_2 \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}(f_1 f_2 - f_1^{eq} f_2^{eq})$$

$$\sigma = \sum\nolimits_{XY}\sigma_{12\to XY} \qquad v = F/E_1E_2 \qquad v = \left[|v_1 - v_2|^2 - |v_1\times v_2|^2\right]^{1/2}$$

- vn_1n_2 : invariant under Lorentz transformation
- Thermally averaged cross section

$$\langle\sigma v\rangle = \frac{\int d^3p_1 d^3p_2 f(E_1)f(E_2)\sigma v}{\int d^3p_1 d^3p_2 f(E_1)f(E_2)}$$

- Collision term

$$g_1 \int C[f_1] \frac{d^3p_1}{(2\pi)^3} = - \int \sigma v g_1 g_2 \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} (f_1 f_2 - f_1^{eq} f_2^{eq})$$

$$g_1 \int C[f_1] \frac{d^3p_1}{(2\pi)^3} = -\langle \sigma v \rangle \left(n_1 n_2 - n_1^{eq} n_2^{eq} \right)$$

- Boltzmann equation (same expresssion for n_2 and $n=n_1=n_2$ if two particles are identical)

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma v \rangle \left((n_\chi)^2 - (n_\chi^{eq})^2 \right)$$