Disorder and chaos in quantum systems II.

Lecture 1.

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Physics Department, Columbia University

Regional School on Physics at the Nanoscale:

Theoretical and Computational Aspects

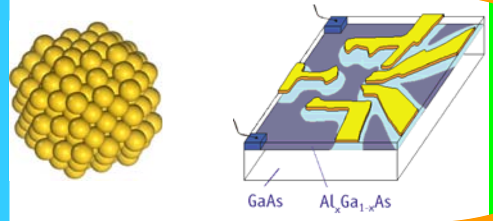
14-25 December 2009, Hanoi, Vietnam

Meso

.
.
.
Nuclei
Atoms
Small
molecules

Fluids
Crystalls
Glasses
.
.
.

Nanoobjects



Nano means **Big** !?

Nanoscale objects do not fully belong to the microcosm

Many atoms, electron, etc., are involved

↓

Number of degrees of freedom is large

Micro	Few degrees of freedom
Macro	# of the degrees of freedom tends to infinity
Meso	Large but finite number

Mesoscopic systems

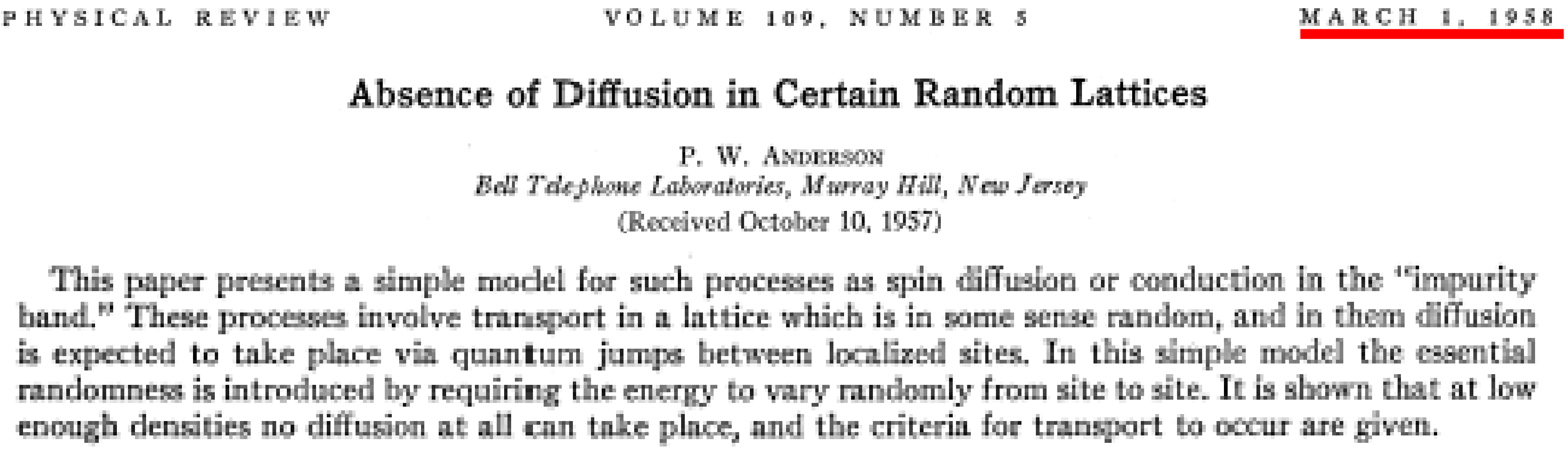
1. Too big to be analyzed individually

2. Two small to neglect sample-to-sample (ensemble) fluctuations

Lecture1.

1.Introduction

>50 years of Anderson Localization

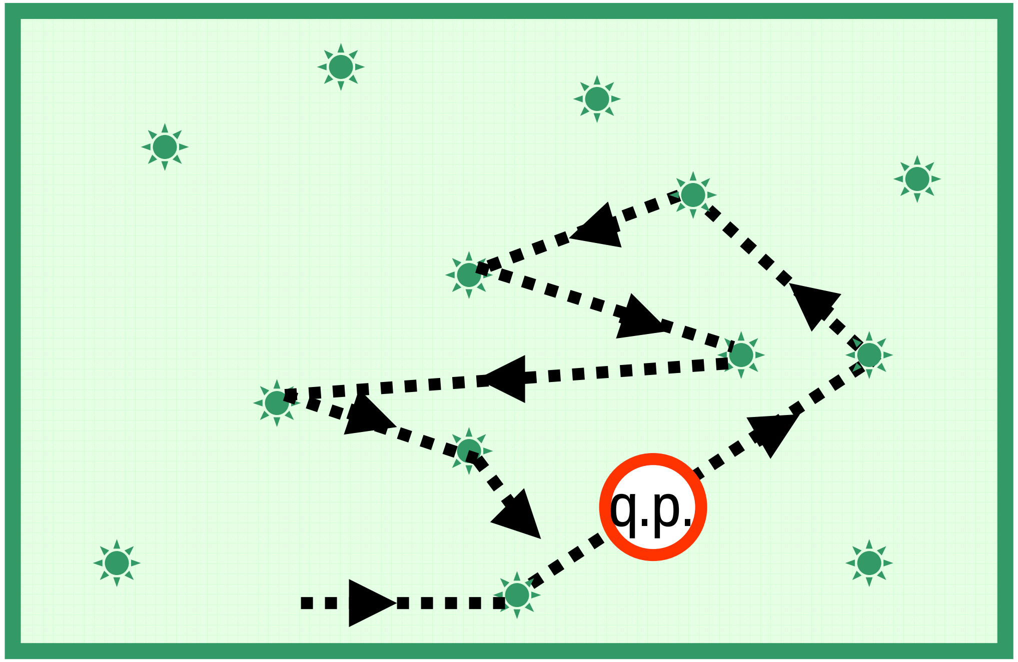


PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

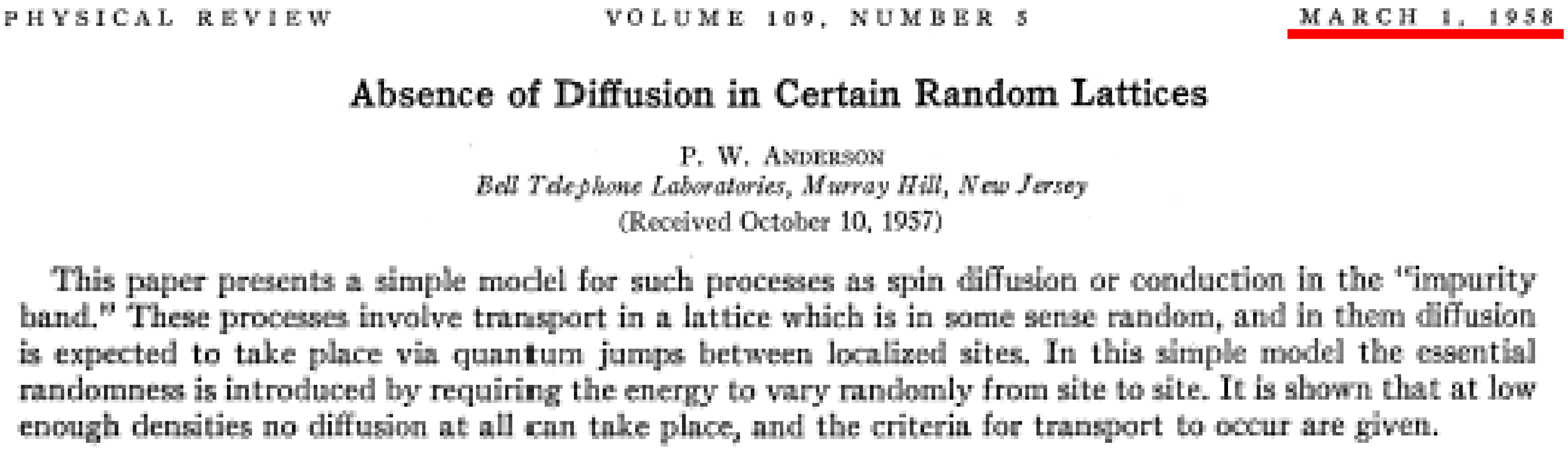
Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.



>50 years of Anderson Localization



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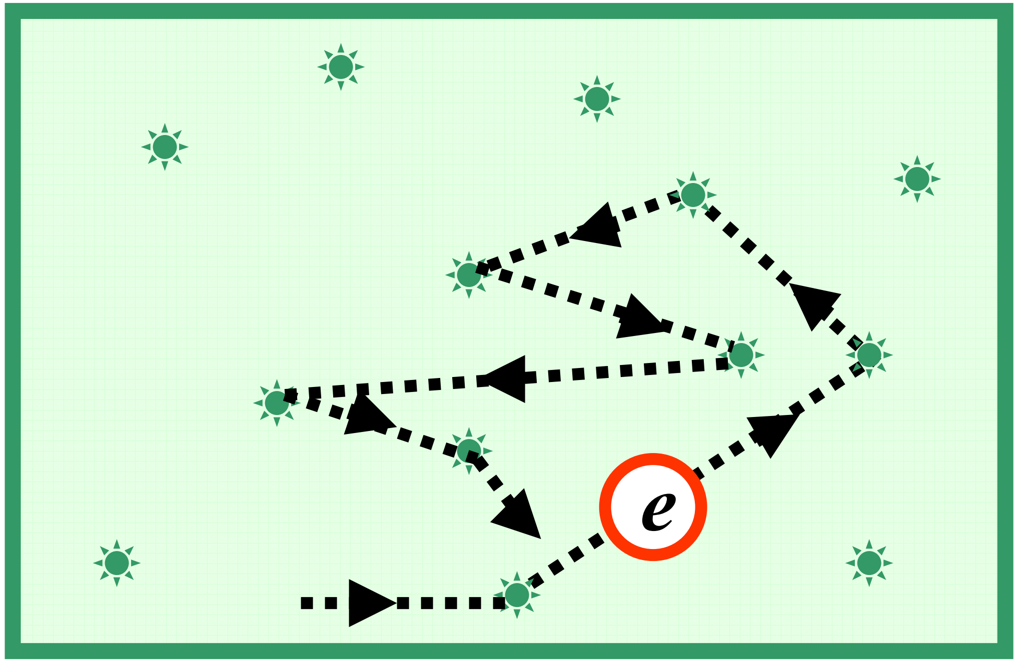
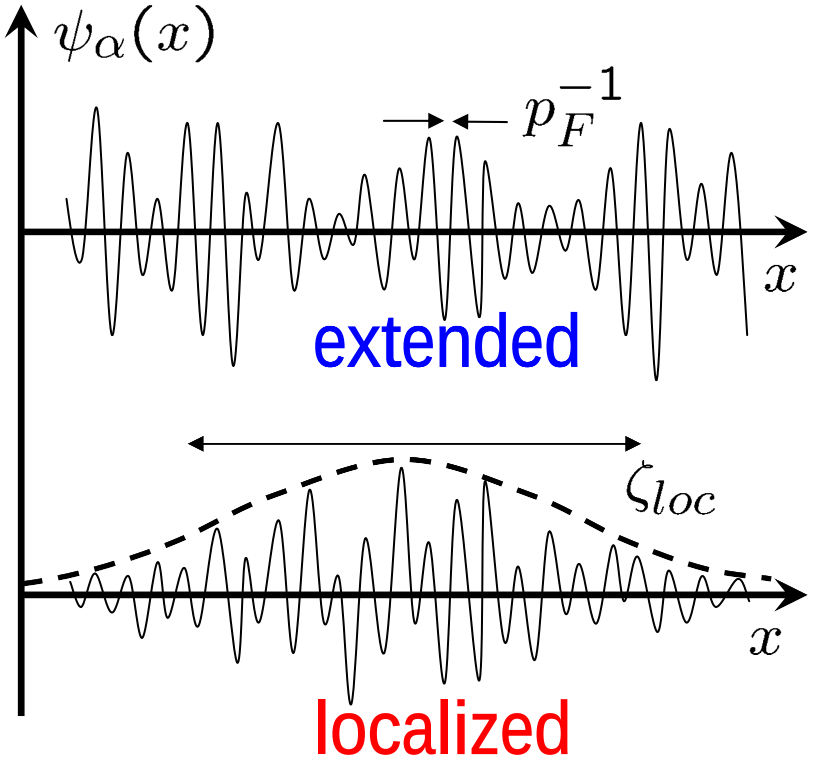


Localization of single-electron wave-functions:

$$\left[-\frac{\nabla^2}{2m}+U(\boldsymbol{r})-\epsilon_F\right]\psi_\alpha(\boldsymbol{r})=\xi_\alpha\psi_\alpha(\boldsymbol{r})$$

Disorder



Ohmic Conductance:

$$G=\left.\frac{I}{V}\right|_{V\to 0}=\begin{cases}\sigma\frac{L_xL_y}{L_z}; & \text{extended}\\ \propto\exp\left(-L_z/\zeta_{loc}\right); & \text{localized}\end{cases}$$

Philip W. Anderson

The Nobel Prize in Physics 1977

Nobel Lecture

Nobel Lecture, December 8, 1977

Local Moments and Localized States

I was cited for work both. in the field of magnetism and in that of disordered systems, and I would like to describe here one development in each held which was specifically mentioned in that citation. The two theories I will discuss differed sharply in some ways. The theory of local moments in metals was, in a sense, easy: it was the condensation into a simple mathematical model of ideas which. were very much in the air at the time, and it had rapid and permanent acceptance because of its timeliness and its relative simplicity. What mathematical difficulty it contained has been almost fully- cleared up within the past few years.

Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it .

Experiment

Spin Diffusion

Feher, G., Phys. Rev. 114, 1219 (1959); Feher, G. & Gere, E. A., Phys. Rev. 114, 1245 (1959).

Light

Wiersma, D.S., Bartolini, P., Lagendijk, A. & Righini R. “Localization of light in a disordered medium”, *Nature* 390, 671-673 (1997).

Scheffold, F., Lenke, R., Tweer, R. & Maret, G. “Localization or classical diffusion of light”, *Nature* 398,206-270 (1999).

Schwartz, T., Bartal, G., Fishman, S. & Segev, M. “Transport and Anderson localization in disordered two dimensional photonic lattices”. *Nature* 446, 52-55 (2007).

C.M. Aegerter, M.Störzer, S.Fiebig, W. Bührer, and G. Maret : JOSA A, 24, #10, A23, (2007)

Microwave

Dalichaouch, R., Armstrong, J.P., Schultz, S.,Platzman, P.M. & McCall, S.L. “Microwave localization by 2-dimensional random scattering”. *Nature* 354, 53, (1991).

Chabanov, A.A., Stoytchev, M. & Genack, A.Z. Statistical signatures of photon localization. *Nature* 404, 850, (2000).

Pradhan, P., Sridar, S, “Correlations due to localization in quantum eigenfunctions od disordered microwave cavities”, PRL 85, (2000)

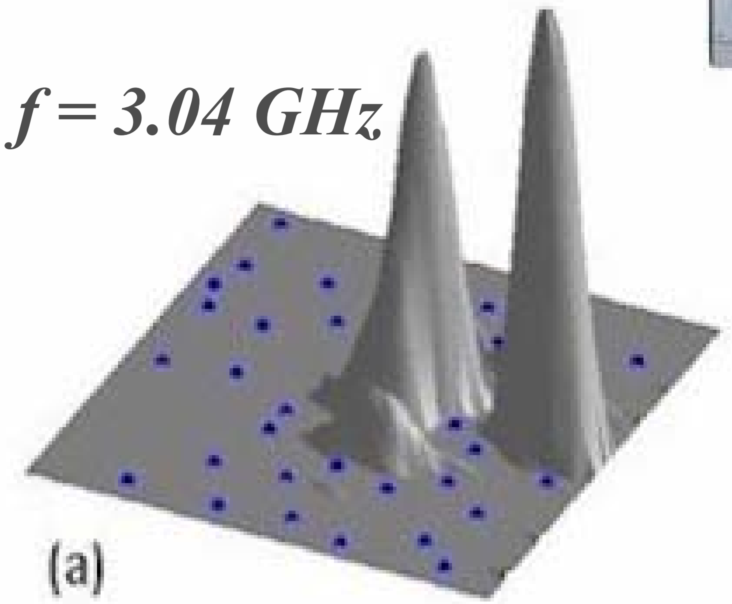
Sound

Weaver, R.L. Anderson localization of ultrasound. *Wave Motion* 12, 129-142 (1990).

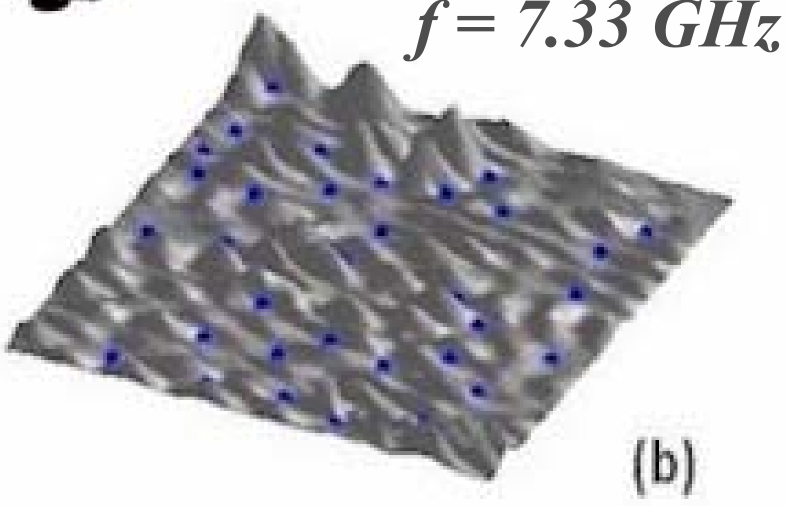
VOLUME 85, NUMBER 11 PHYSICAL REVIEW LETTERS 11 SEPTEMBER 2000

Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)



Localized State
Anderson Insulator

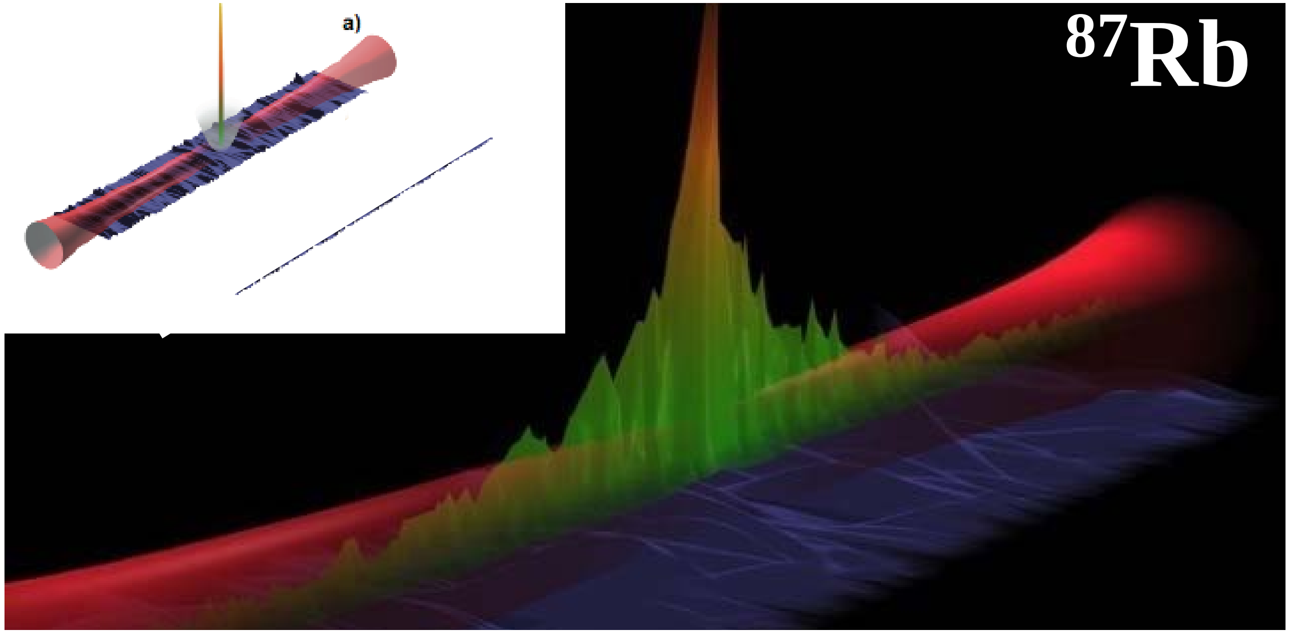


Extended State
Anderson Metal

Experiment

Cold Atoms

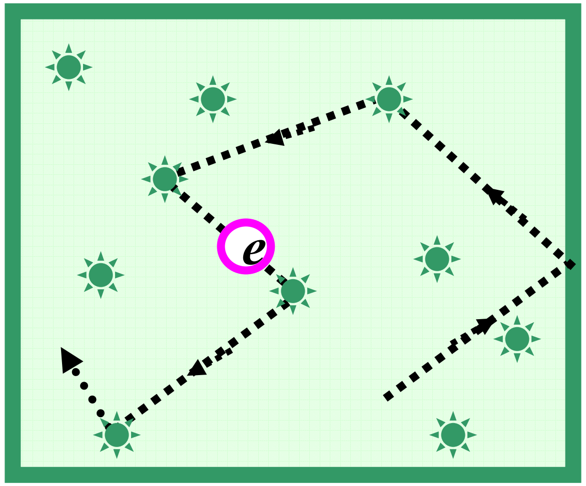
J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D.Clément, L.Sanchez-Palencia, P. Bouyer & A. Aspect, "**Direct observation of Anderson localization of matter-waves in a controlled Disorder" *Nature* 453, 891-894 (12 June 2008)**



L. Fallani, C. Fort, M. Inguscio: "Bose-Einstein condensates in disordered potentials" arXiv:0804.2888

Q: What about electrons ?

A: Yes,… but electrons interact with each other



✹ *Scattering centers,*
e.g., impurities

Models of disorder:

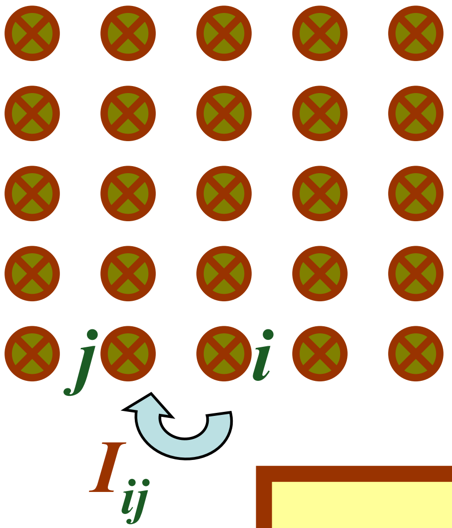
Randomly located impurities

White noise potential

Lattice models

- Anderson model
- Lifshits model

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i *- random*
- *Hopping matrix elements* I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$$I_c = f(d)*W$$

$$I < I_c$$

Insulator

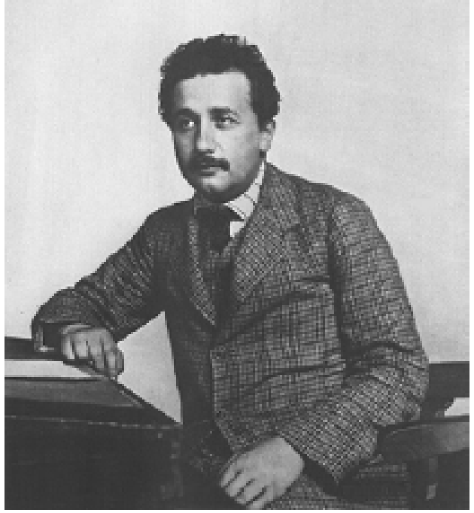
All eigenstates are localized

Localization length ξ

$$I > I_c$$

Metal

There appear states extended all over the whole system



Einstein (1905):

Random walk

⇓

always diffusion

as long as the system
has no memory
(Marcovian process)

$$\langle r^2 \rangle = Dt$$

diffusion constant

$$\boxed{conductivity \propto D}$$

Einstein relation

Anderson(1958):

For quantum
particles

⇓

not always!

It might be that

$$\langle r^2 \rangle \xrightarrow[t\to\infty]{} const$$

$$D = 0$$

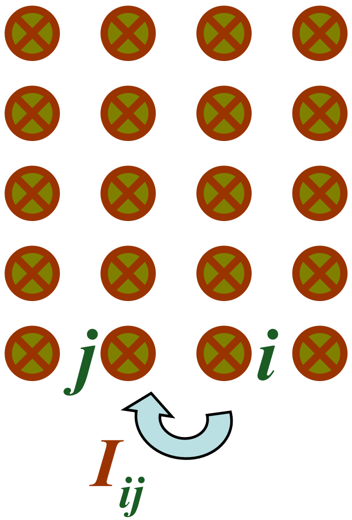
⇓

$$\boxed{conductivity = 0}$$

Anderson insulator

Quantum interference ⇒ memory

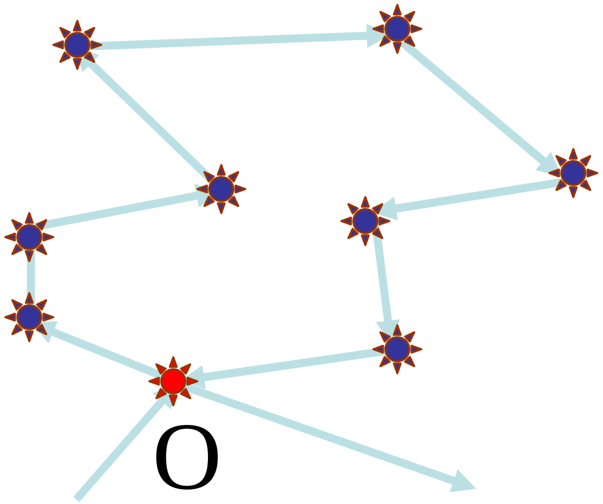
Q: Why arbitrary weak hopping I is not sufficient for the existence of the diffusion ?



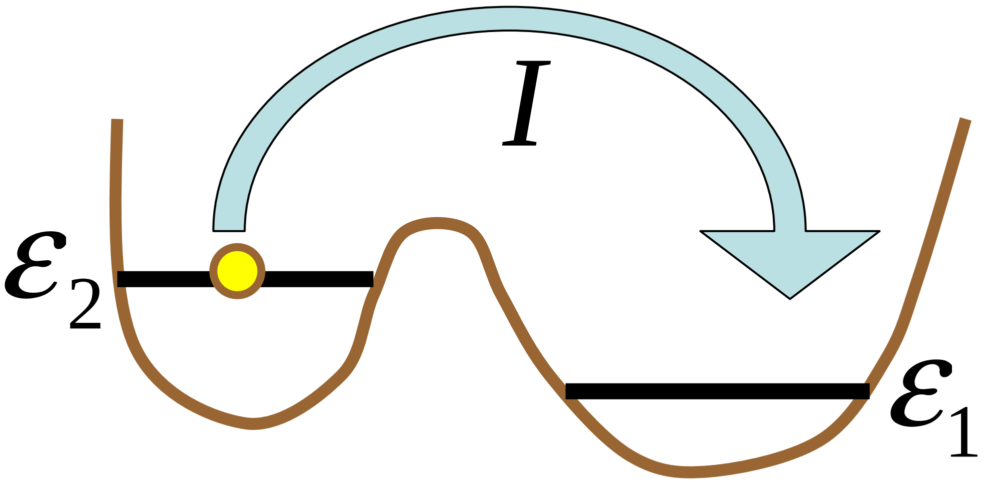
Einstein (1905): Marcovian (no memory) process → diffusion

Quantum mechanics is not marcovian
There is memory in quantum propagation !

Why ?



Quantum interference – long time memory



Hamiltonian

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2}$$

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{array}{ll} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{array}$$

$$\Downarrow$$

von Neumann & Wigner

“noncrossing rule”

Level repulsion

v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

Eigenfunctions

$$\phi_1, \varepsilon_1; \phi_2, \varepsilon_2 \quad \Leftarrow \quad \psi_1, E_1; \psi_2, E_2$$

$$\varepsilon_2 - \varepsilon_1 >> I$$

$$\psi_{1,2} = \varphi_{1,2} + O\left(\frac{I}{\varepsilon_2 - \varepsilon_1}\right)\varphi_{2,1}$$

Off-resonance

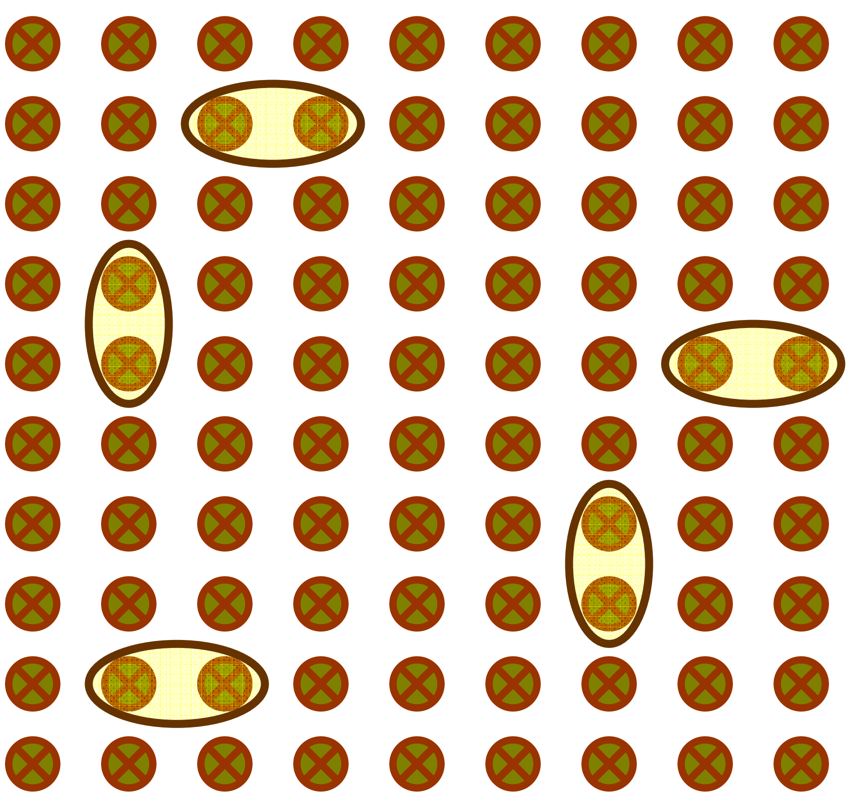
Eigenfunctions are close to the original on-site wave functions

$$\varepsilon_2 - \varepsilon_1 << I$$

$$\psi_{1,2} \approx \varphi_{1,2} \pm \varphi_{2,1}$$

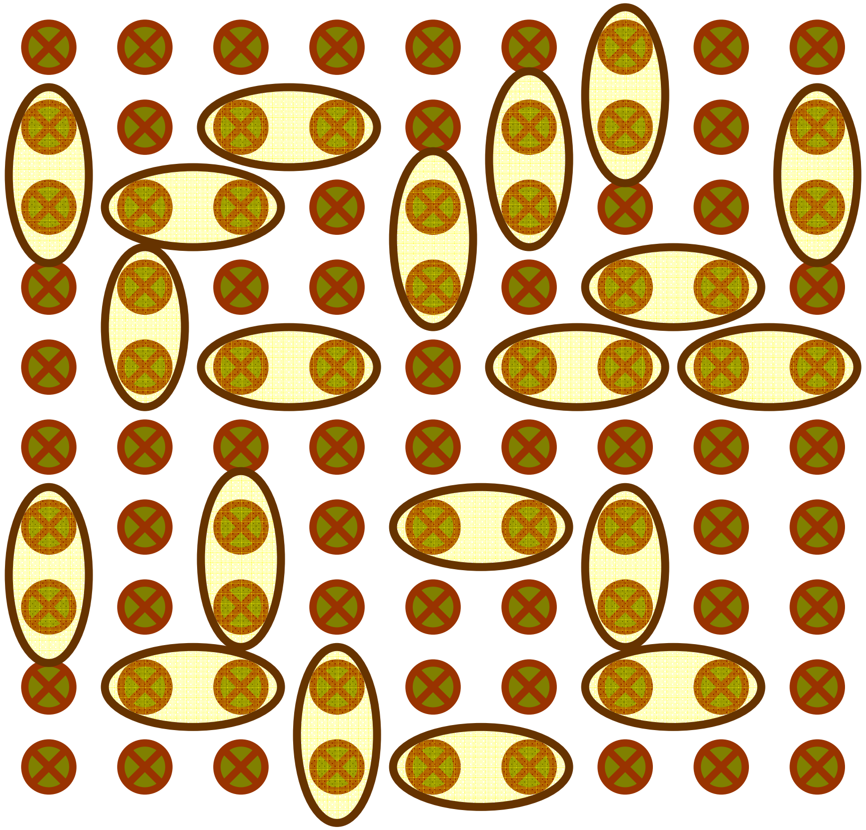
Resonance

In both eigenstates the probability is equally shared between the sites



Anderson insulator

Few isolated resonances



Anderson metal

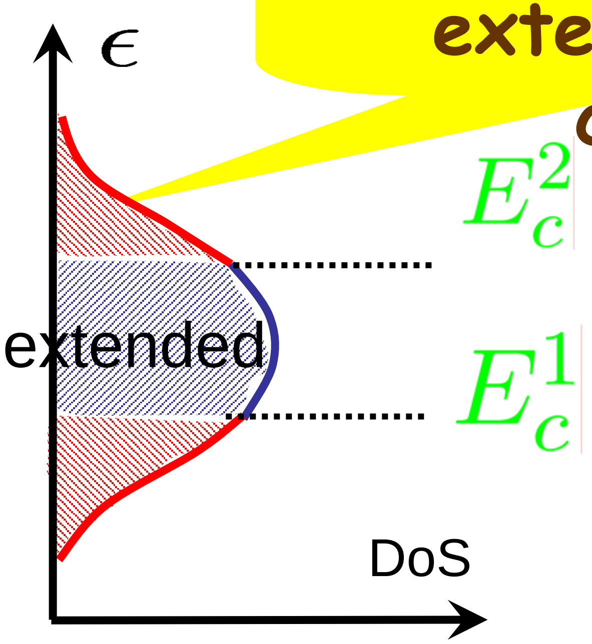
There are many resonances
and they overlap

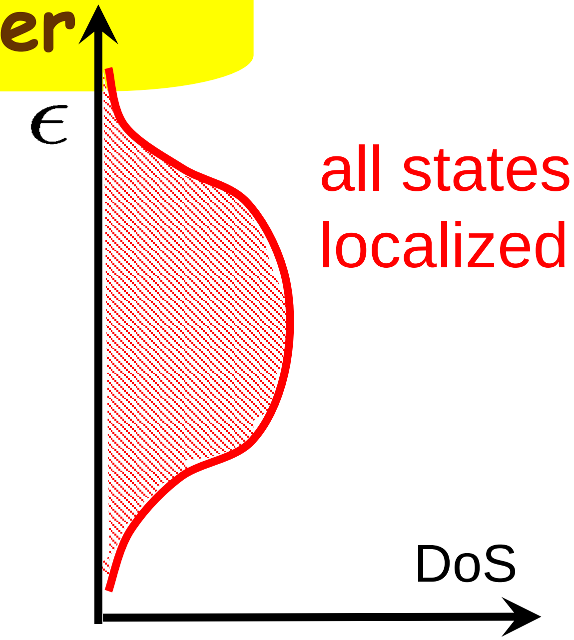
Anderson Transition

$I > I_c$

$I < I_c$

localized and extended never coexist!

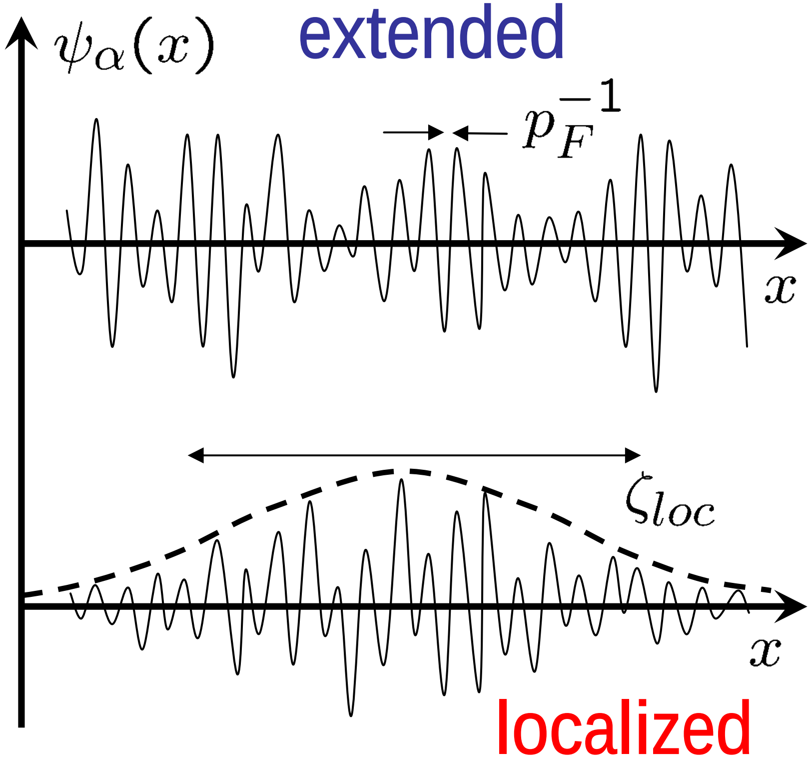




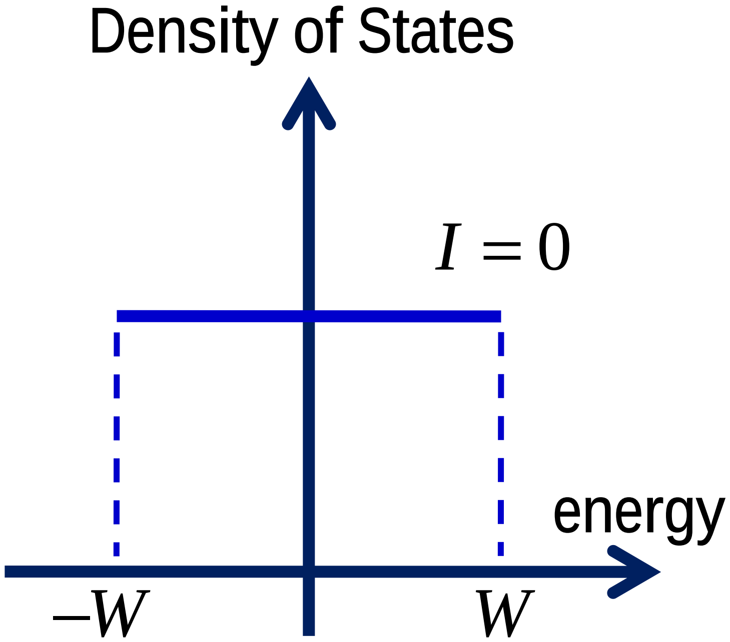
all states are localized

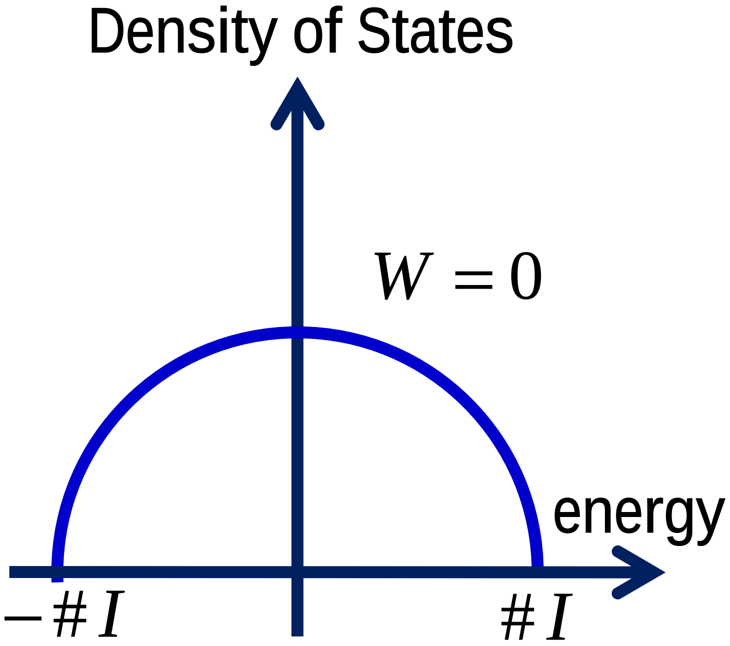
E_c - mobility edges (one particle)

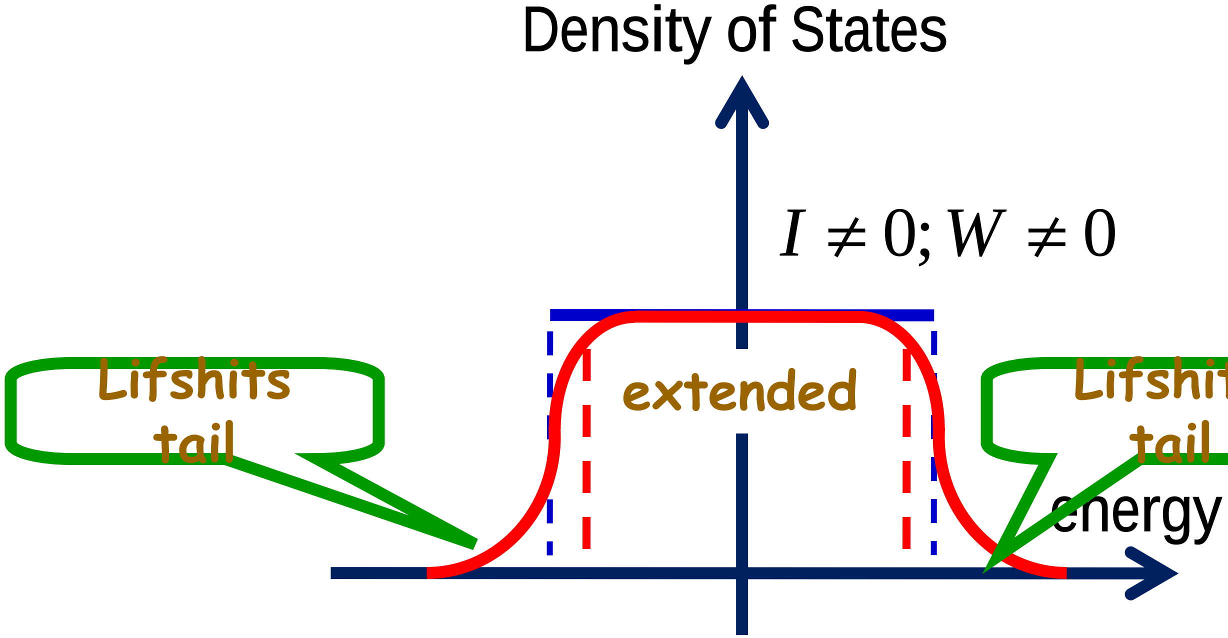
Eigenfunctions



Q: Is anything interesting happening with the spectrum?


Density of States
$I = 0$
energy
$-W$
W

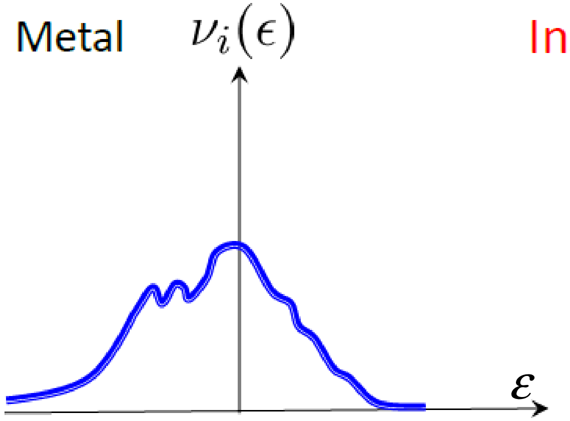

Density of States
$W = 0$
energy
$-\# I$
$\# I$

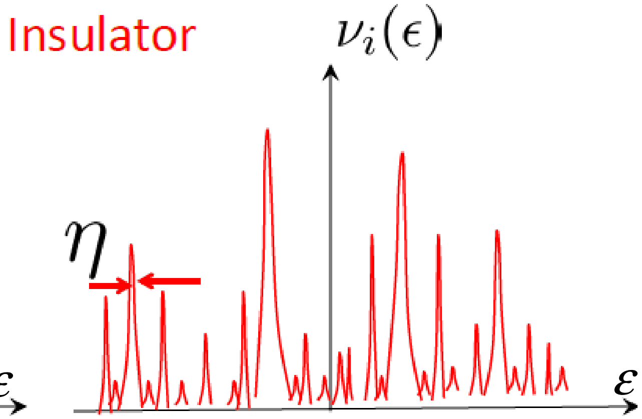

Density of States
$I \neq 0; W \neq 0$
extended
Lifshits tail
Lifshits tail
energy

Density of States is not singular at the Anderson transition

This applies only to the **average** Density of States !

Fluctuations ?





Lecture1.

2. Spectral statistics and Localization

RANDOM MATRIX THEORY

Spectral statistics

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

E_α - spectrum (set of eigenvalues)

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle$ - mean level spacing

$\langle ...... \rangle$ - ensemble averaging

$s \equiv \frac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

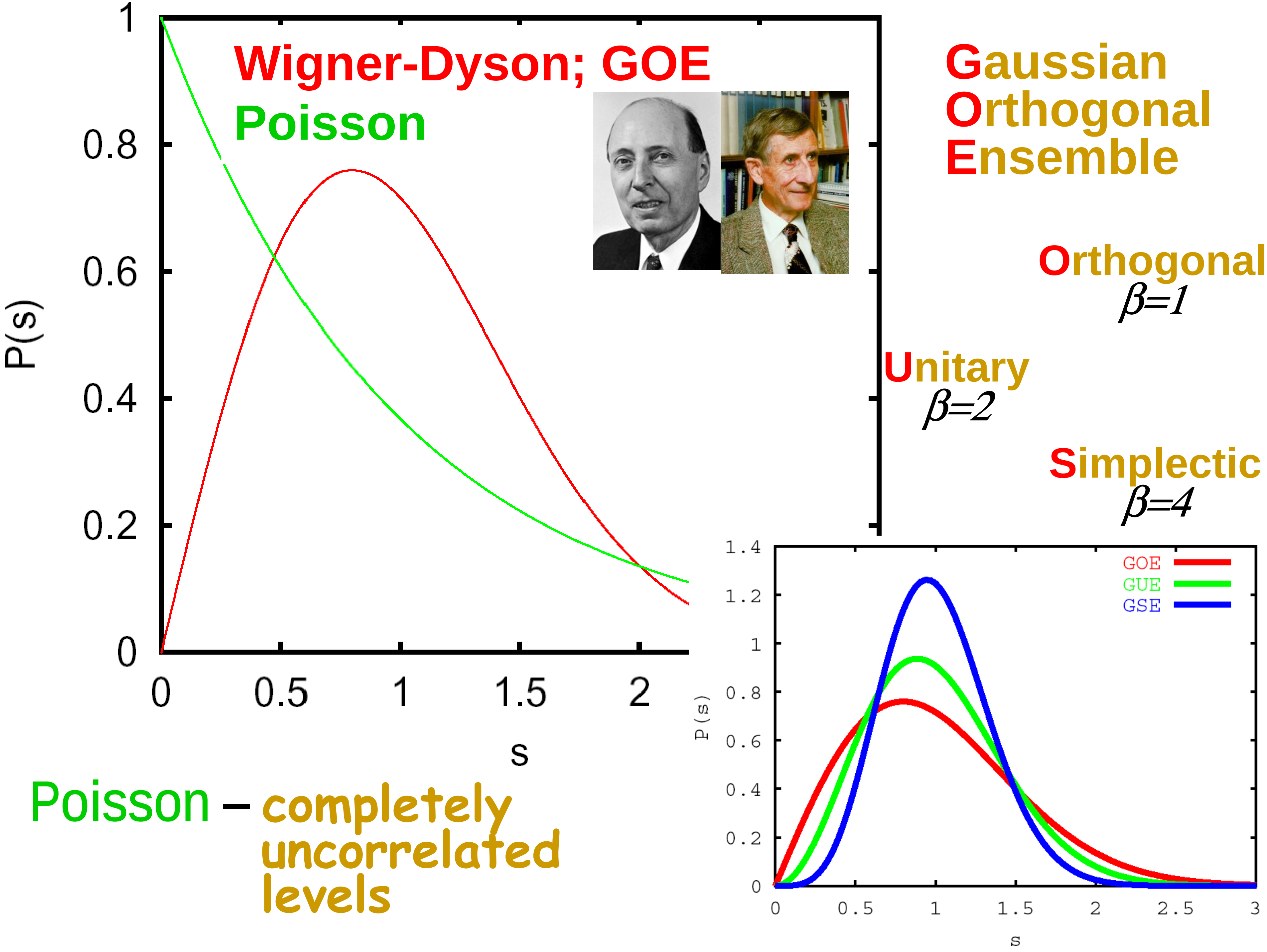
$P(s)$ - distribution function of nearest neighbors spacing between

Spectral Rigidity

Level repulsion

$$P(s=0)=0$$

$$P(s<<1) \propto s^\beta \quad \beta=1,2,4$$



Gaussian
Orthogonal
Ensemble

Orthogonal
$\beta=1$

Unitary
$\beta=2$

Simplectic
$\beta=4$

Poisson – completely uncorrelated levels

RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \to \infty$

Dyson Ensembles

Matrix elements	Ensemble	β	realization
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

Reason for $P(s) \rightarrow 0$ when $s \rightarrow 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

$$\underbrace{E_2 - E_1}_{\text{small}} = \sqrt{(\underbrace{H_{22} - H_{11}}_{\text{small}})^2 + |\underbrace{H_{12}}_{\text{small}}|^2}$$

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If H_{12} is real (orthogonal ensemble), then for s to be small two statistically independent variables ($(H_{22}$- $H_{11})$ and H_{12}) should be small and thus $P(s) \propto s \qquad \beta = 1$

$$\iint d\left(H_{11}-H_{22}\right)dH_{12}\,\delta\left(E_2-E_1-\sqrt{}\right)P\left(H_{11}-H_{22}\right)P\left(H_{12}\right)$$

Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(H_{22} - H_{11})^2 + |H_{12}|^2}$$

small

small

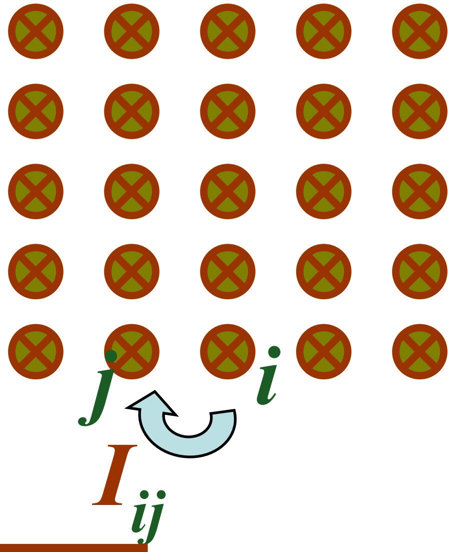
small

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If H_{12} is real (orthogonal ensemble), then for s to be small two statistically independent variables ((H_{22}- H_{11}) and H_{12}) should be small and thus $P(s) \propto s$ $\beta = 1$

3. Complex H_{12} (unitary ensemble) ⇒ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent ⇒ three independent random variables should be small ⇒ $P(s) \propto s^2$ $\beta = 2$

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

Is there much in common between Random Matrices and Hamiltonians with random potential ?

Q: What are the spectral statistics of a finite size Anderson model ?

Anderson Transition

Strong disorder

$I < I_c$

Insulator

All eigenstates are localized

Localization length ξ

The eigenstates, which are localized at different places will not repel each other

Poisson spectral statistics

Weak disorder

$I > I_c$

Metal

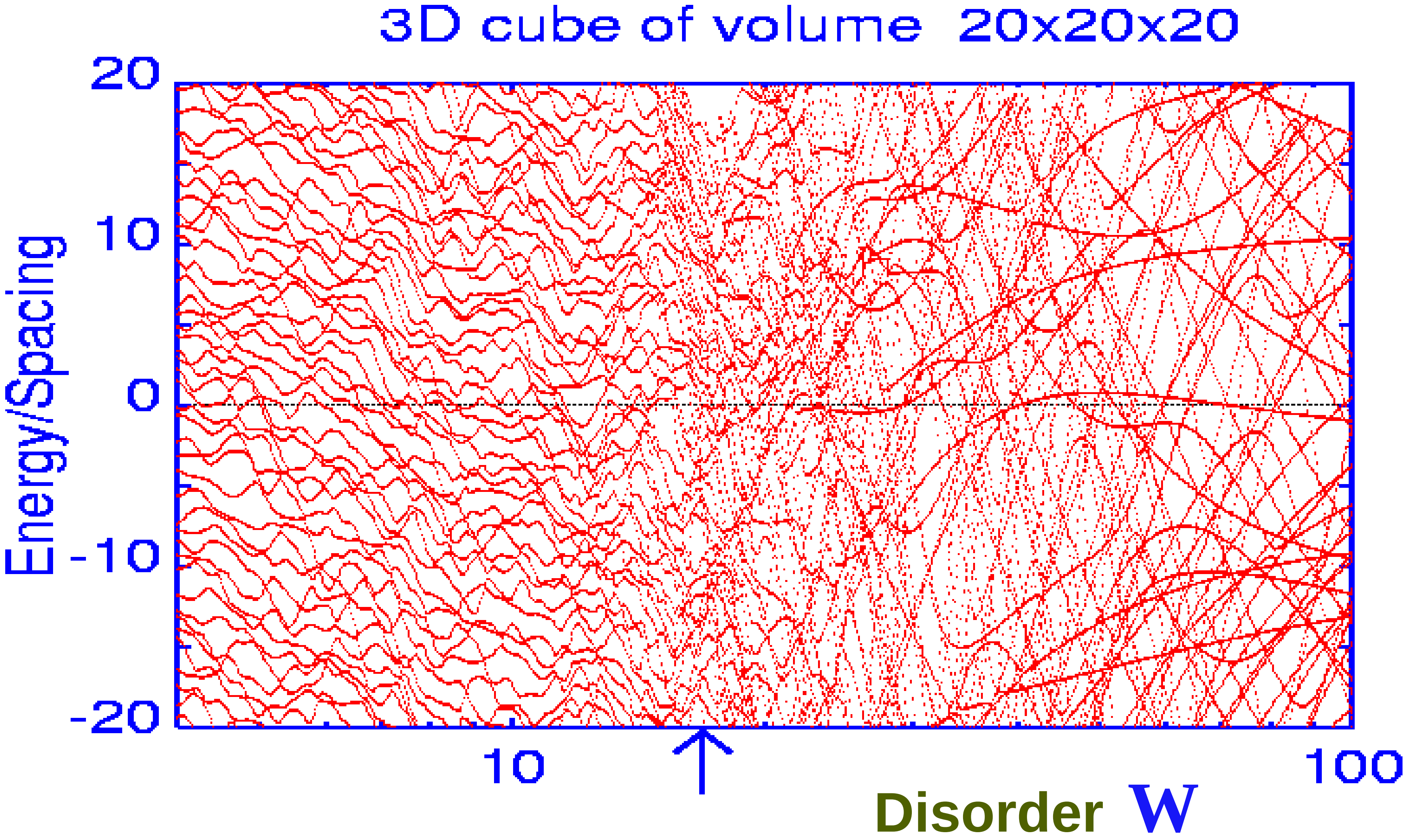
There appear states extended all over the whole system

Any two extended eigenstates repel each other

Wigner – Dyson spectral statistics

Zharekeschev & Kramer.

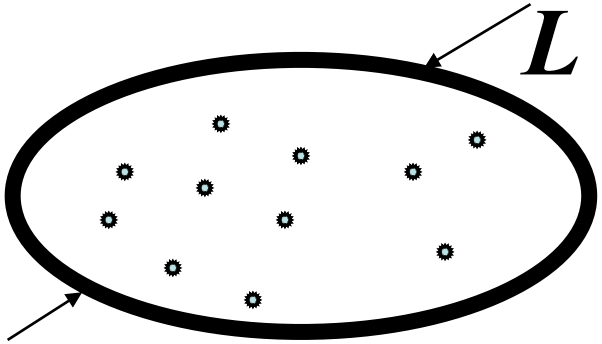
Exact diagonalization of the Anderson model

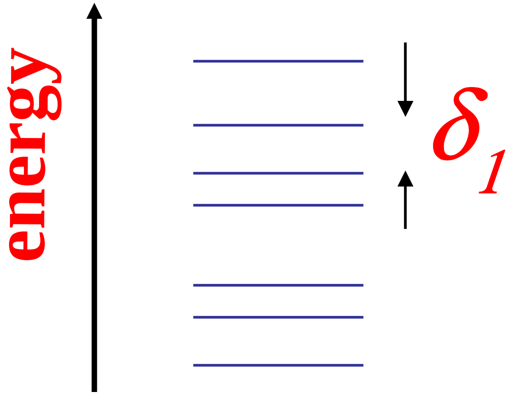


Energy scales in the localization problem. (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L is the system size;

d is the number of dimensions

This scale exists in the Random Matrix theory

Energy scales in the localization problem. (*Thouless, 1972*)

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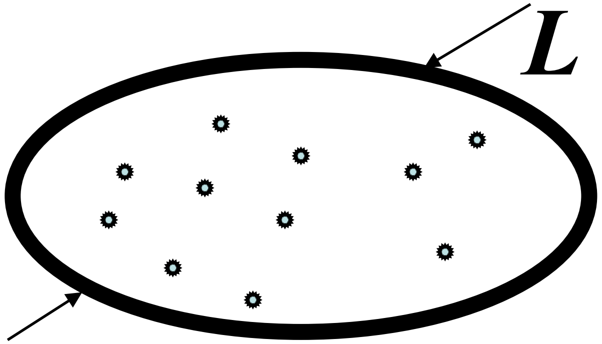
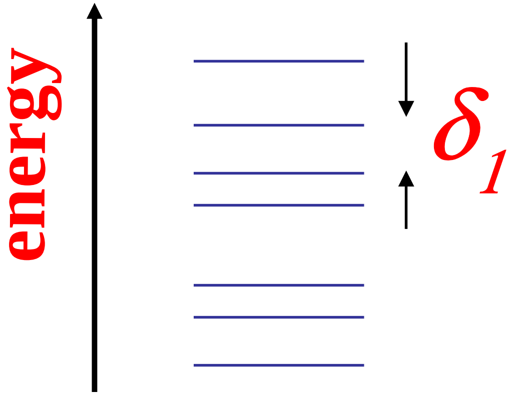
$$\delta_1 = 1/\nu \times L^d$$





L is the system size;

d is the number of dimensions

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

E_T has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)

$$g = E_T / \delta_1$$

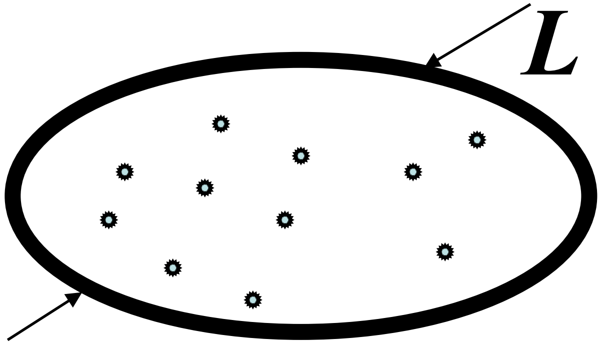
dimensionless
Thouless
conductance

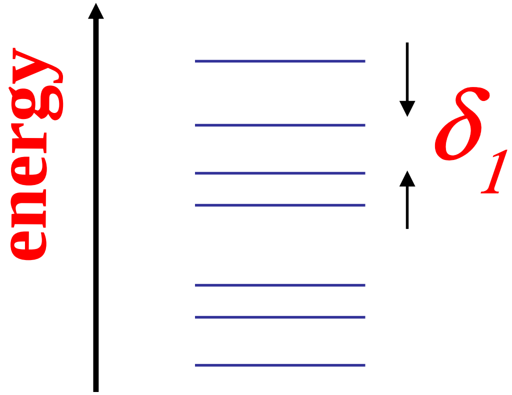
$$g = Gh/e^2$$

Energy scales in the localization problem. (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L is the system size;

d is the number of dimensions

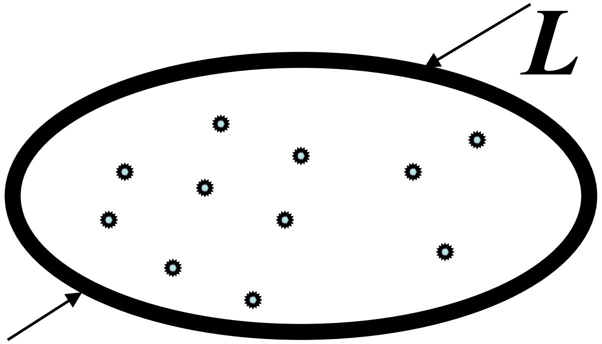
1. This energy scale exists in the Random Matrix theory.

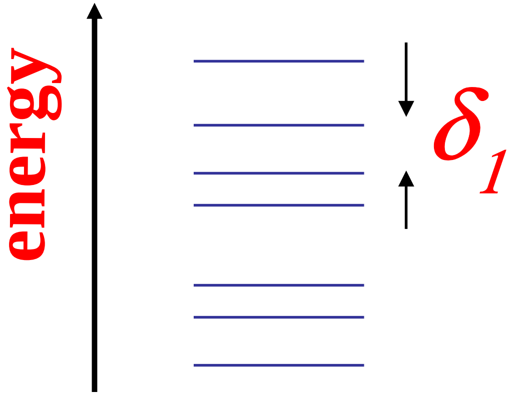
2. This is the only energy scale in the RM theory

Energy scales in the localization problem. (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





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D is the diffusion const

E_T has a meaning of the **inverse diffusion time** of the traveling through the system or the **escape rate** (for open systems)

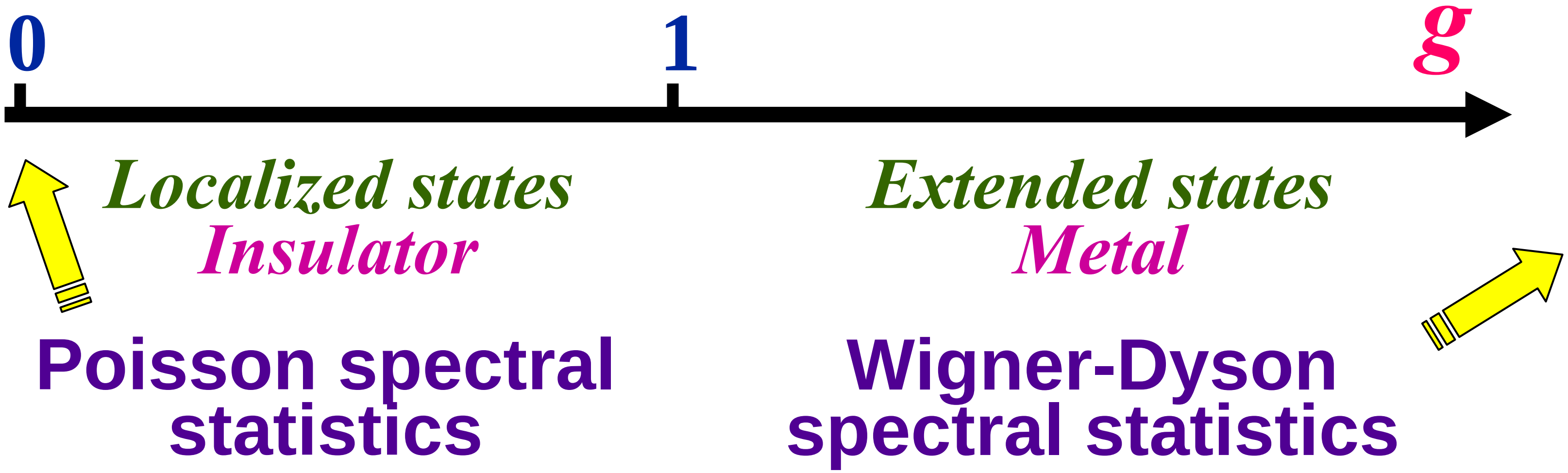
In the Random Matrix theory this energy scale is absent

$$g = \frac{E_T}{\delta_1}$$

dimensionless **Thouless** conductance

$$g = \frac{h}{e^2} G$$

Thouless Conductance and One-particle Spectral Statistics



Localized states
Insulator

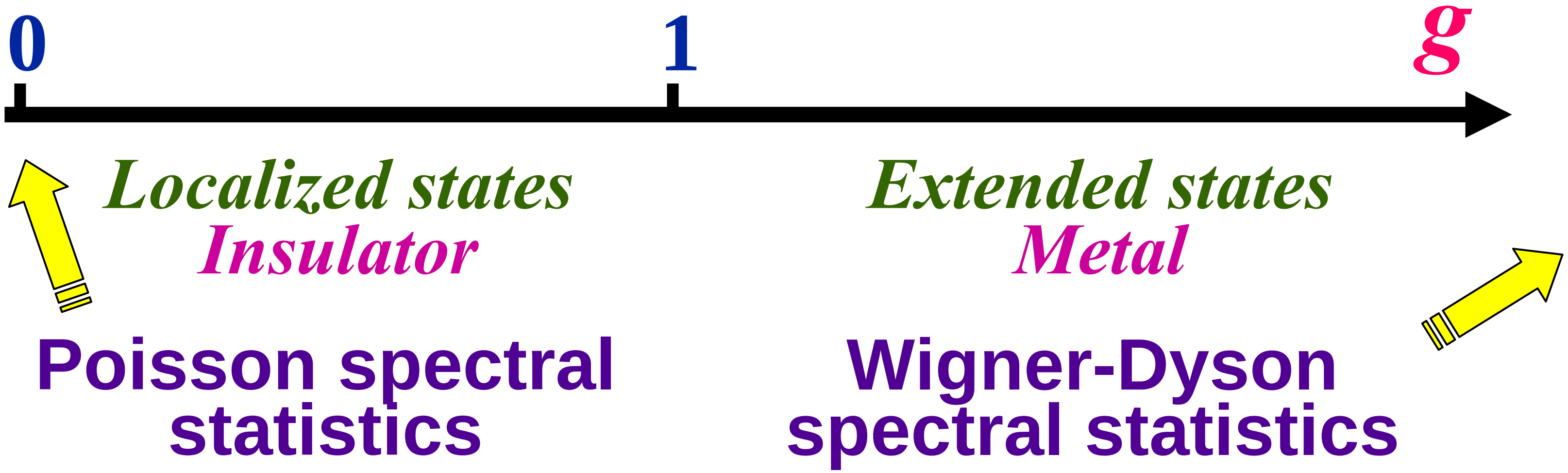
Poisson spectral statistics

Extended states
Metal

Wigner-Dyson spectral statistics

Gaussian Invariant Random Matrix Ensembles describe well those “complex” quantum systems, which are characterized by large Thouless conductance.

Thouless Conductance and One-particle Spectral Statistics



0

1

g

Localized states

Insulator

Poisson spectral statistics

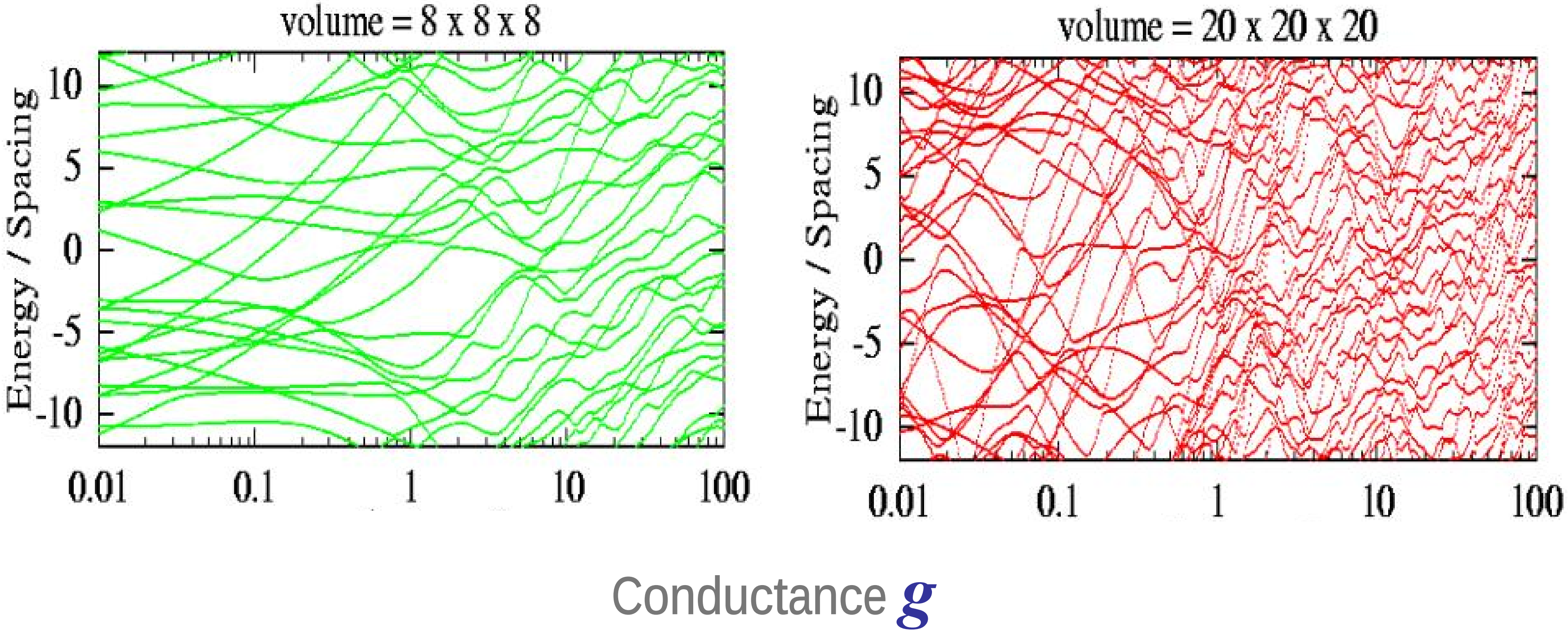
Extended states

Metal

Wigner-Dyson spectral statistics

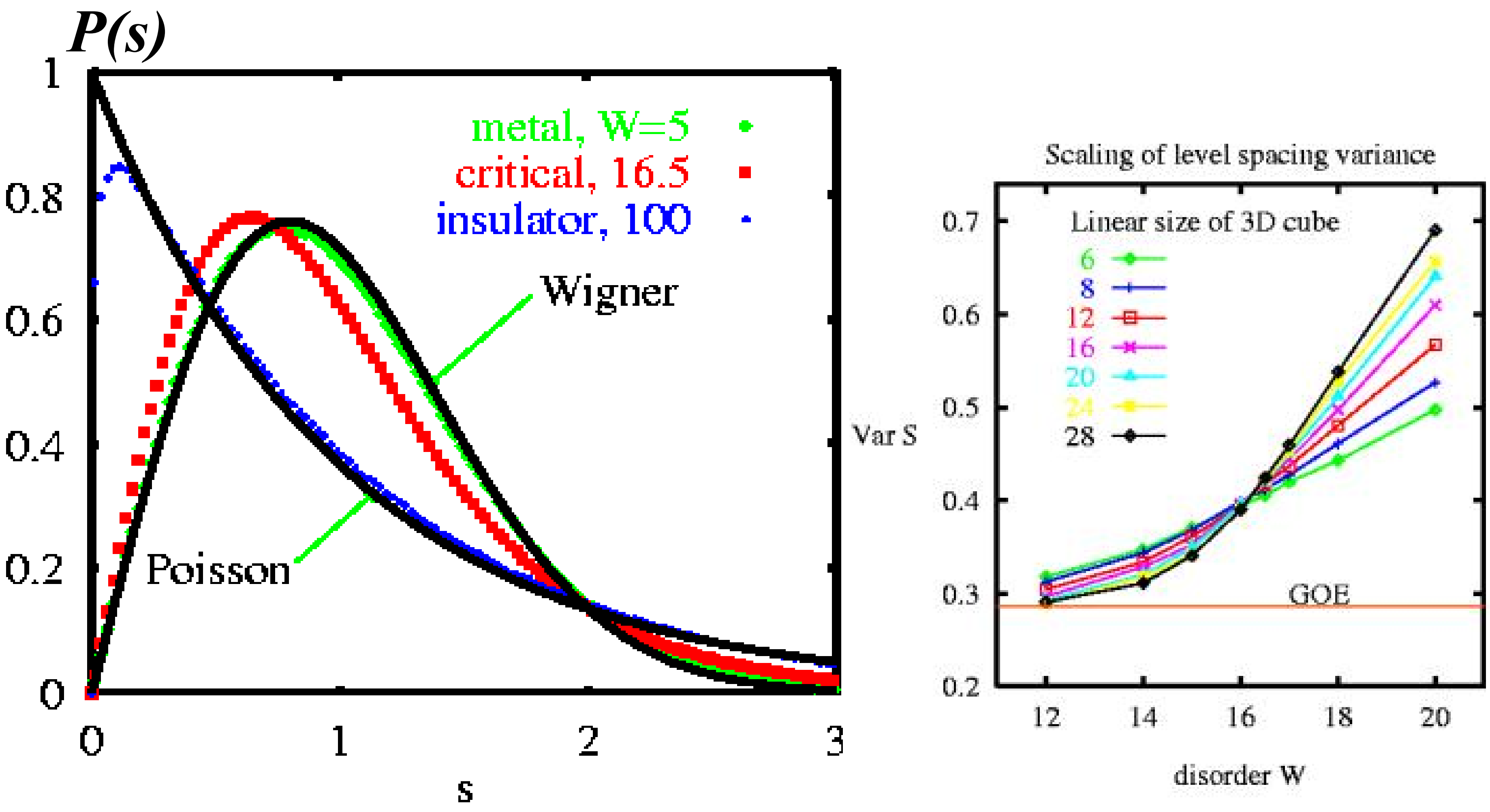
Transition at g~1.

Is it sharp?



The bigger the system the sharper the transition

Anderson transition in terms of pure level statistics



Lecture1.

3. Quantum Chaos, Integrability and Localization

ATOMS Main goal is to classify the eigenstates in terms of the quantum numbers

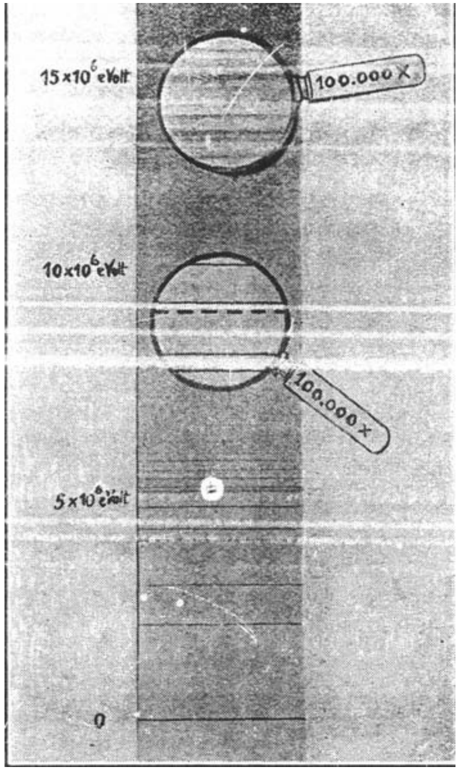
NUCLEI For the nuclear excitations this program does not work

Wigner: Study spectral **statistics** of a **particular** quantum system – a given nucleus

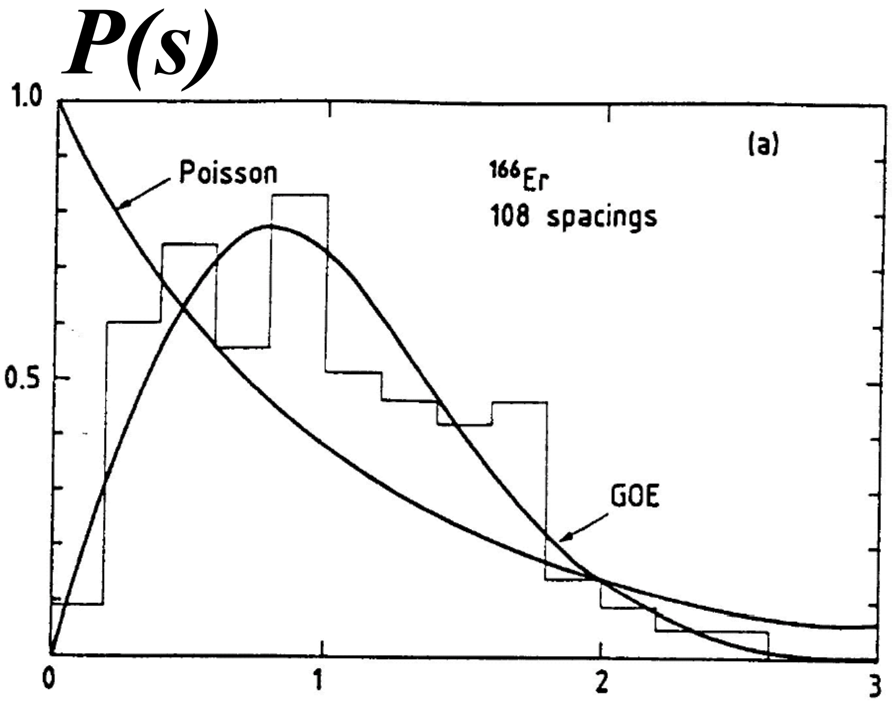
Spectra: $\{E_\alpha\}$

Random Matrices	Atomic Nuclei
• *Ensemble* • *Ensemble averaging*	•*Spectral averaging (over α)* •*Particular quantum system*

Nevertheless Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics

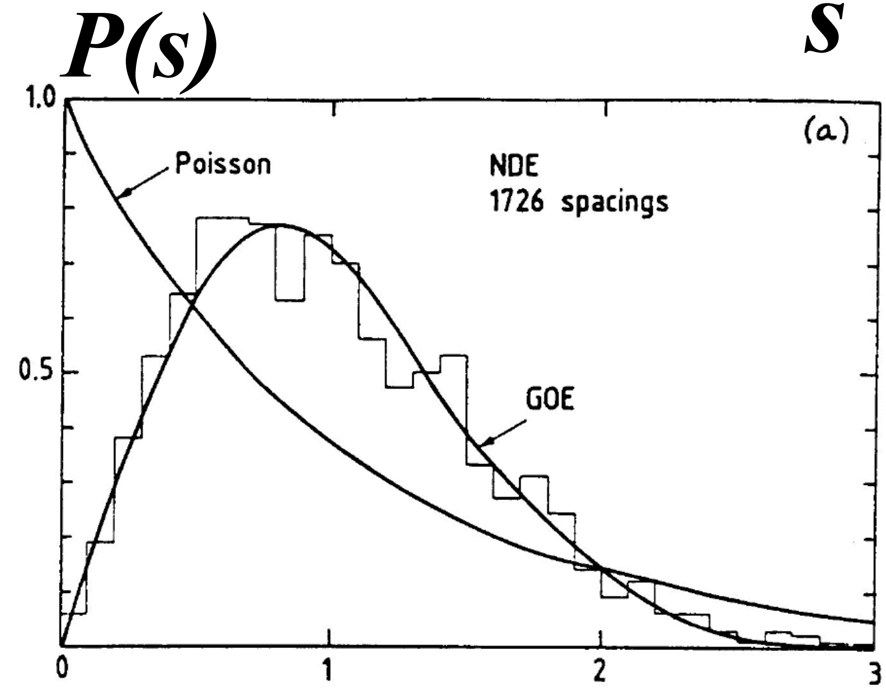


N. Bohr, Nature 137 (1936) 344.



Particular nucleus

^{166}Er



Spectra of several nuclei combined (after spacing) rescaling by the mean level

Q: ***Why the random matrix theory (RMT) works so well for nuclear spectra*** **?**

Original answer:

These are systems with a large number of degrees of freedom, and therefore the “complexity” is high

Q: *Why the random matrix theory (RMT) works so well for nuclear spectra* ?

Original answer:

These are systems with a large number of degrees of freedom, and therefore the "complexity" is high

Later it became clear that

there exist very "simple" systems with as many as 2 degrees of freedom (d=2), which demonstrate RMT - like spectral statistics