Proc. Natl. Conf. Theor. Phys. **36** (2011), pp. 14-22

WARD-TAKAHASHI IDENTITY FOR VERTEX FUNCTIONS OF SQED

H. T. HUNG, L. T. HUE, H. N. LONG
Institute of Physics, VAST, P. O. Box 429, Bo Ho, Hanoi 10000, Vietnam

Abstract. *Ward-Takahashi identity is an useful tool for calculating amplitude of scattering processes. In the high-order perturbative theory of sQED, propagators and vertex functions include many high-order corrections. By using Ward-Takahashi identity, each vertex function is separated into two parts: "longitudinal" and "transverse". The longitudinal part can be directly calculated from Ward-Takahashi identity. The transverse part depends on the expanding of specific orders of the theory. This paper will present one method based on the Ward-Takahashi identity, to calculate parts of vertex functions at the one-loop order in arbitrary gauge and dimensions in sQED.*

I. INTRODUCTION

We introduce a method which use Ward-Takahashi identity to decompose the vertex into longitudinal part and transverse part. This form of vertex satisfies two conditions: (i) has no kinematics singularities in both two parts, (ii) the longitudinal part of a vertex has fixed scalar coefficient that.

II. PROPAGATORS AND VERTEX FUNCTIONS OF SQED IN BARE PERTURBATION

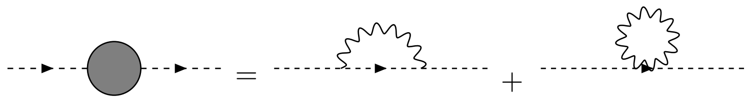
In the scalar Quantum Electrodynamics Dynamics (sQED), propagators and vertex function in any gauge ξ are determined as follow:

$\Delta^0_{\mu\nu} = \frac{-g_{\mu\nu}p^2+(1-\xi)p_\mu p_\nu}{p^4}$; arbitrary gauge ξ $\qquad$ $S^0(p) = \frac{1}{p^2-m^2}$

Fig. 1. Propagators of sQED in bare perturbative theory

$\Gamma^0_\mu = (k+p)^\mu$ $\qquad$ $e^2\Gamma^0_{\mu\nu} = e^2 g_{\mu\nu}$

Fig. 2. Vertex functions of sQED in bare perturbative theory



$-i\Sigma(p^2) \quad = \quad \Sigma_1(p^2) \quad + \quad \Sigma_2(p^2)$

Fig. 3. Propagator of complex scalar particle at one-loop.

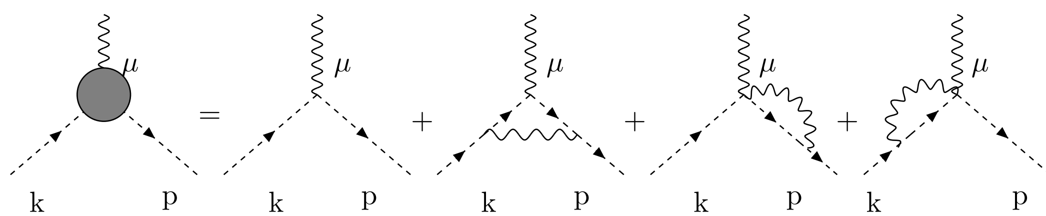
III. WARD-TAKAHASHI IDENTITY WITH 3-POINT VERTEX FUNCTION OF SQED

Propagators of scalar particles at one-loop order :

In regular dimension the second diagram (tadpole) vanishes. The one-loop propagator is given by:

$$\begin{aligned} S^{-1} &= \frac{-e^2}{m^2}(\frac{m^2}{4\pi})^l\Gamma(1-l)\{1-\frac{2(m^2+p^2)}{m^2}{}_2F_1(2-l,1;l;\frac{p^2}{m^2}) \\ &+ (1-\xi)\frac{(m^2-p^2)^2}{m^4})_2F_1(3-l,2;l;\frac{p^2}{m^2})\} \end{aligned} \tag{1}$$

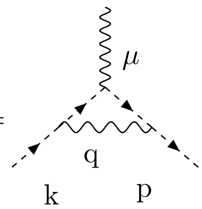
The 3-point function of sQED at one-loop:

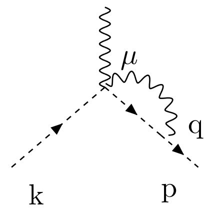


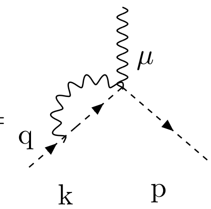
In term of mathematical language, we have

$$\Gamma^\mu(k,p) = (k+p)^\mu + \Gamma^\mu_1(k,p) + \Gamma^\mu_2(p) + \Gamma^\mu_2(k) \tag{2}$$

In which:

$\Gamma_1^\mu(k,p) =$


$\Gamma_2^\mu(p) =$


$\Gamma_2^\mu(k) =$


For vertex functions:

$$\begin{aligned}
\Gamma^1_\mu &= \frac{-ie^2}{(2\pi)^{2l}}\{4(kp)(k+p)_\mu J^0 + [-8(kp)g^\nu_\mu - 2(k+p)_\mu(k+p)^\nu]J^1_\nu + 4(k+p)^\nu J^2_{\mu\nu} \\
&+ (k+p)_\mu K^0 - 2K^1_\mu + (\xi-1)[(k+p)_\mu K^0 + 4(k+p)_\mu P^\alpha K^\beta I^2_{\alpha\beta} - 8p^\alpha K^\beta I^3_{\mu\alpha\beta} \\
&- 2(k+p)_\mu(k+p)^\alpha J^1_\alpha + 4(k+p)^\alpha J^2_{\mu\alpha} - 2K^1_\mu]\} \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma^\mu_2(p) &= \frac{e^2p^2p^\mu}{(2\pi)^{2l}}\{[3+\frac{m^2}{p^2}]Q_1(p) - \frac{\pi^{l-2}}{p^2}\Gamma(1-l)(m^2)^{l-1} \\
&+ (\xi-1)\frac{p^2-m^2}{p^2}[Q_1(p)+(p^2-m^2)Q_3(p)]\} \qquad (4)
\end{aligned}$$

Ward-Takahashi identity for the 3-point vertex function:

$$q_\mu\Gamma^\mu(k,p) = S^{-1}(k) - S^{-1}(p) \qquad (5)$$

in higher correlative orders we introduce:

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p) \qquad (6)$$

Longitudinal component and the transverse component is

$$\Gamma^\mu_L(k,p) = \frac{S^{-1}(k)-S^{-1}(p)}{k^2-p^2}(k+p)^\mu; \Gamma^\mu_T(k,p) = \tau(k^2,p^2,q^2)T^\mu(k,p) \qquad (7)$$

Where

$$T^\mu(k,p) = pqk^\mu - kqp^\mu = \frac{1}{2}[q^\mu(k^2-p^2) - (k+p)^\mu q^2] \qquad (8)$$

The condition of $\Gamma^\mu_T(k,p)$ is:

$$q_\mu\Gamma^\mu_T(k,p) = 0; \Gamma^\mu_T(p,p) = 0 \qquad (9)$$

The function $\tau(k^2,p^2,q^2)$ is reduced as follow:

$$\begin{aligned}\tau(k^2,p^2,q^2) &= \frac{e^2\pi^2}{2(2\pi)^d\Delta^2}\{(k^2-2m^2+p^2-4kp)[-K_0+(m^2+kp)J_0]\\&+\frac{2Q_1(p)}{k^2-p^2}[p^2(p^2-3kp)+k^2(kp-3p^2)-2m^2(p^2+kp)]\\&-\frac{2Q_1(k)}{k^2-p^2}[k^2(k^2-3kp)+p^2(kp-3k^2)-2m^2(k^2+kp)]\\&+(\xi-1)(m^2-k^2)(m^2-p^2)[J_0-(kp+m^2)I_0-\\&\frac{2Q_3(p)}{k^2-p^2}(kp+p^2)-\frac{2Q_3(k)}{k^2-p^2}(kp+k^2)]\}\end{aligned} \tag{10}$$

In which:

$$\Delta^2=(kp)^2-k^2p^2=(kq)^2-k^2q^2 \tag{11}$$

It is convenient to present $\tau(k^2,p^2,q^2)$ in terms of propagators of scalar particle:

$$\begin{aligned}\tau(k^2,p^2,q^2) &= \frac{1}{4\Delta^2}\frac{[S^{-1}(k,\xi=1)-S^{-1}(p,\xi=1)]}{[(m^2+k^2)Q_1(k)-(m^2+p^2)Q_1(p)]}\{(k^2-2m^2+p^2-4kp)\\&\times[-K_0+(m^2+kp)J_0]\frac{2Q_1(p)}{k^2-p^2}[p^2(p^2-3kp)+k^2(kp-3p^2)\\&-2m^2(p^2+kp)]-\frac{2Q_1(k)}{k^2-p^2}[k^2(k^2-3kp)+p^2(kp-3k^2)-2m^2(k^2+kp)]\}\\&+\frac{1}{2\Delta^2}\frac{[S^{-1}(k,\xi-1)-S^{-1}(p,\xi-1)]}{[(m^2-k^2)Q_3(k)-(m^2-p^2)Q_3(p)]}(m^2-k^2)(m^2-p^2)\\&\times\{J_0-(kp+m^2)I_0-\frac{2Q_3(p)}{k^2-p^2}(kp+p^2)-\frac{2Q_3(k)}{k^2-p^2}(kp+k^2)\}\end{aligned} \tag{12}$$

We define

$$q^\mu=(k-p)^\mu;P^\mu=(k+p)^\mu$$

W-T identity for three-point function can be written in the form of

$$q_\mu\Gamma_\nu-q_\nu\Gamma_\mu=(q_\mu P_\nu-q_\nu P_\mu)\left[\frac{S^{-1}(k)-S^{-1}(p)}{k^2-p^2}+\frac{q^2}{2}\tau(k^2,p^2,q^2)\right] \tag{13}$$

$$P_\mu\Gamma_\nu-P_\nu\Gamma_\mu=(P_\mu q_\nu-P_\nu q_\mu)\left[\frac{k^2-p^2}{2}\right]\tau(k^2,p^2,q^2) \tag{14}$$

IV. WARD-TAKAHASHI IDENTITY WITH 4-POINT VERTEX FUNCTION OF SQED

Ward-Takahashi identity of 4-point function relates with 3-point vertex function by:

$$\begin{aligned}k'^\mu\Gamma_{\nu\mu}(p',k';p,k)&=\Gamma_\nu(p+k,p)-\Gamma_\nu(p',p'-k)\\k^\mu\Gamma_{\nu\mu}(p',k';p,k)&=\Gamma_\nu(p',p'+k')-\Gamma_\nu(p-k',p)\end{aligned} \tag{15}$$

Following Eq. (**IV**) we can determine the longitudinal component of 4-point vertex function based on 3-point vertex functions. We denote:

$$\begin{aligned} Q_\mu &= k'_\mu(p+p')k - kk'(p+p')_\mu; R_\mu = k_\mu k'k - k'^2 k_\nu \\ Q'_\nu &= k_\nu(p+p')k' - kk'(p+p')_\nu; R'_\nu = k'_\nu k'k - k'^2 k_\nu \end{aligned} \quad (16)$$

Then 4-point vertex function is written by:

$$\begin{aligned} \Gamma_{\mu\nu} &= \Gamma^L_{\mu\nu} + \Gamma^T_{\mu\nu} = Ag_{\mu\nu} + B_{11}(kk'g_{\mu\nu} - k_\nu k'_\mu) + B_{12}Q'_\nu k'_\mu + B_{13}R'_\nu k'_\mu \\ &+ B_{21}k_\nu Q_\mu + B_{22}Q'_\nu Q_\mu + B_{23}R'_\nu Q_\mu + B_{31}k_\nu R_\mu + B_{32}Q'_\nu R_\mu + B_{33}R'_\nu R_\mu \end{aligned} \quad (17)$$

And now we can determine longitudinal and transverse component of 4-point vertex function as follow:

$$\Gamma^L_{\mu\nu} = Ag_{\mu\nu} + B_{12}Q'_\nu k'_\mu + B_{13}R'_\nu k'_\mu + B_{21}k_\nu Q_\mu + B_{31}k_\nu R_\mu \quad (18)$$

$$\Gamma^T_{\mu\nu} = B_{11}(kk'g_{\mu\nu} - k_\nu k'_\mu) + B_{22}Q'_\nu Q_\mu + B_{23}R'_\nu Q_\mu + B_{32}Q'_\nu R_\mu + B_{33}R'_\nu R_\mu \quad (19)$$

Factors of longitudinal component of 4-point vertex fuction is:

$$\begin{aligned} A &= -\frac{1}{kk'}\left\{S^{-1}(p+k) - S^{-1}(p) + S^{-1}(p'-k) - S^{-1}(p')\right\} \\ B_{21} &= -\frac{1}{kk'}\{\frac{S^{-1}(p+k) - S^{-1}(p)}{(p+k)^2 - p^2} - \frac{S^{-1}(p') - S^{-1}(p'-k)}{p'^2 - (p'-k)^2} \\ &- k^2\left[\Gamma_T(p+k,p) - \Gamma_T(p',p'-k)\right]\} \\ B_{12} &= -\frac{1}{kk'}\{\frac{S^{-1}(p'-k') - S^{-1}(p')}{(p'-k')^2 - p'^2} - \frac{S^{-1}(p) - S^{-1}(p+k')}{p^2 - (p+k')^2} \\ &- k'^2\left[\Gamma_T(p'-k',p') - \Gamma_T(p,p+k')\right]\} \\ B_{31} &= -\frac{1}{(kk')^2}\left\{[(p+k)^2 - p^2]\Gamma_T(p+k,p) + [(p'-k)^2 - p'^2]\Gamma_T(p',p'-k)\right\} \\ B_{13} &= -\frac{1}{(kk')^2}\left\{[(p'-k')^2 - p'^2]\Gamma_T(p'-k',p') + [(p+k')^2 - p^2]\Gamma_T(p,p-+k')\right\} \end{aligned} \quad (20)$$

In which Γ_T is the transverse component of 3-point vertex function and it is determined as follow:

$$\Gamma_\mu(p+k,p) = (2p+k)_\mu \frac{S^{-1}(p+k) - S^{-1}(p)}{(p+k)^2 - p^2} + 2(k_\mu pk - k^2 p_\mu)\Gamma_T(p+k,p) \quad (21)$$

Factors B_{11} , B_{22} , B_{23}, B_{32} and B_{33} of transverse component of 4-point vertex function will be calculate according to perturbative -orders of the theory. We will introduce technique to calculate the transverse component of 4-point vertex function at one-loop. The corresponding Feynman diagrams for this function are:

For sake of simplicity, we denote $q = k + p$ and $q' = k' + p$, then results of Feynman diagrams at one-loop are given by

$$\Gamma^{\mu\nu}_{D_1}(q) = -\frac{2ie^2}{(2\pi)^D}\left\{-g^{\mu\nu}K(q) + (1-\xi)L^{\mu\nu}(q)\right\} \tag{22}$$

$$\Gamma^{\mu\nu}_{D_4}(p,q) = \frac{ie^2}{(2\pi)^D}\left\{-2\frac{(p+q)^\mu q^\nu}{q^2-m^2}K(q) + \frac{(p+q)^\mu}{q^2-m^2}K^\nu(q) + (1-\xi)(p+q)^\mu L^\nu(q)\right\} \tag{23}$$

$$\begin{aligned}\Gamma^{\mu\nu}_{D_8}(p,q) &= -\frac{ie^2}{(2\pi)^D}g^{\mu\nu}\Big\{\tilde{K}(p-p') - 2(p+p')^\mu I_\mu(p,p') + 4pp'I(p,p') \\ &+ (\xi-1)\tilde{K}(p-p') - 2(p+p')^\mu I_\mu(p.p') + 4p'^\mu p^\nu J_{\mu\nu}(p,p')\Big\}\end{aligned} \tag{24}$$

$$\begin{aligned}\Gamma^{\mu\nu}_{D_9}(p,q) &= -\frac{ie^2}{(2\pi)^D}\{2(p+q)^\mu p^\nu I(p,q) - (p+q)^\mu I^\nu(p,q) - 4p^\nu I^\mu(p+q) + 2I^{\mu\nu}(p,q) \\ &- (1-\xi)[(p^2-m^2)(p+q)^\mu]J^\nu(p,q) - 2(p^2-m^2)J^{\mu\nu}(p,q) + 2L^{\mu\nu}(q) \\ &- (p+q)^\mu L^\nu(q)\}\end{aligned} \tag{25}$$

$$\Gamma^{\mu\nu}_{D_{15}}(p,q,p') = \frac{ie^2}{(2\pi)^D}\left\{-2p'^\nu K(p') + K^\nu(p') + (1-\xi)(p'^2-m^2)L^\nu(p')\right\} \tag{26}$$

$$\begin{aligned}\Gamma^{\mu\nu}_{D_{19}}(p,q,p') &= \frac{ie^2}{2(2\pi)^D}(p+q)^\mu(p'+q)^\nu\left\{\left(\frac{2}{q^2-m^2} + \frac{4m^2}{(q^2-m^2)^2}\right)K(q)\right. \\ &- \left.\frac{T}{(q^2-m^2)^2} - \frac{(1-\xi)}{q^2-m^2}L(q)\right\}\end{aligned} \tag{27}$$

$$\begin{aligned}\Gamma^{\mu\nu}_{D_{20}}(p,q,p') &= -\frac{ie^2}{2(2\pi)^D}\frac{(p'+q')^\mu(p+q')^\nu}{(q'^2-m^2)^2}\Big\{-4q'^2K(q') - T + 4q'_\alpha K^\alpha(q') \\ &+ (1-\xi)\left[T - 4q'_\alpha K^\alpha(q') + 4q'_\alpha q'_\beta L^{\alpha\beta}(q')\right]\Big\}\end{aligned} \tag{28}$$

$$\begin{aligned}\Gamma^{\mu\nu}_{D_{23}}(p,q,p') &= \frac{ie^2}{2(2\pi)^D}\left\{(p+q)^\mu(p'+q)^\nu\left[\left(\frac{4p'q}{q^2-m^2} + \frac{m^2-p'^2}{q^2-m^2} - 1\right)I(p',q)\right.\right. \\ &- \left.\frac{\tilde{K}(p'-q)}{q^2-m^2} + \frac{K(p')}{q^2-m^2} + \frac{K(q)}{q^2-m^2}\right] - 2(p+q)^\mu\left[(\frac{4p'q}{q^2-m^2} + \frac{m^2-p'^2}{q^2-m^2} - 1)\right. \\ &\times \left. I^\nu(p',q) - \frac{(p'+q)^\nu}{2(q^2-m^2)}\tilde{K}(p'-q) + \frac{K^\nu(p')}{q^2-m^2} + \frac{K^\nu(q)}{q^2-m^2}\right] \\ &- (1-\xi)\left[(p+q)^\mu(p'+q)^\nu[(p'^2-m^2)J(p',q) - L(q) - \frac{p'^2-m^2}{q^2-m^2}L(p')]\right. \\ &- \left.\left. 2(p+q)^\mu[(p'^2-m^2)J^\nu(p',q) - L^\nu(q) - \frac{p'^2-m^2}{q^2-m^2}L^\nu(p')]\right]\right\}\end{aligned} \tag{29}$$

$$\begin{aligned}\Gamma^{\mu\nu}_{D_{27}}(p,q,p') &= \frac{ie^2}{2(2\pi)^D}\left\{(k+2p)^\mu(k+6p+p')^\nu[(-2m^2+(p'-p)^2-2pp')U(p.p'.q)\right.\\ &+ \tilde{I}(q-p,p'-p)-I(p,q)-I(p',q)]-2(k+p+p'^\nu[(-2m^2+(p'-p)^2\\ &- \left.2pp')U^\mu(p,p',q)]) + \tilde{I}^\mu(q-p,p'-p)\right\} \qquad (30)\end{aligned}$$

The remain Feynman diagrams is determined according to:

$$\Gamma^{\mu\nu}_{D_2}=\Gamma^{\mu\nu}_{D_1}(q');\Gamma^{\mu\nu}_{D_5}=\Gamma^{\mu\nu}_{D_4}(p,q');\Gamma^{\mu\nu}_{D_6}=\Gamma^{\mu\nu}_{D_4}(p',q');\Gamma^{\mu\nu}_{D_7}=\Gamma^{\mu\nu}_{D_4}(p',q) \qquad (31)$$

$$\Gamma^{\mu\nu}_{D_{10}}=\Gamma^{\mu\nu}_{D_9}(p,q');\Gamma^{\mu\nu}_{D_{11}}=\Gamma^{\mu\nu}_{D_9}(p',q');\Gamma^{\mu\nu}_{D_{12}}=\Gamma^{\mu\nu}_{D_9}(p',q);\Gamma^{\mu\nu}_{D_{16}}=\Gamma^{\mu\nu}_{D_{15}}(p,q',p') \qquad (32)$$

$$\Gamma^{\mu\nu}_{D_{17}}=\Gamma^{\mu\nu}_{D_{15}}(p',q',p);\Gamma^{\mu\nu}_{D_{18}}=\Gamma^{\mu\nu}_{D_{15}}(p',q,p);\Gamma^{\mu\nu}_{D_{24}}=\Gamma^{\mu\nu}_{D_{23}}(p,q',p');\Gamma^{\mu\nu}_{D_{25}}=\Gamma^{\mu\nu}_{D_{23}}(p',q',p) \qquad (33)$$

$$\Gamma^{\mu\nu}_{D_{26}}=\Gamma^{\mu\nu}_{D_{23}}(p',q,p);\Gamma^{\mu\nu}_{D_{28}}=\Gamma^{\mu\nu}_{D_{27}}(k,p,q',p') \qquad (34)$$

Diagrams $\Gamma^{\mu\nu}_{D_3};\Gamma^{\mu\nu}_{D_{13}};\Gamma^{\mu\nu}_{D_{14}};\Gamma^{\mu\nu}_{D_{21}}$ v$\Gamma^{\mu\nu}_{D_{22}}$ vanish, so they do not contribute. And the calculating follow us to arrange terms of total 4-point vertex function in term of:

$$\begin{aligned}\Gamma_{\mu\nu} = C_0g_{\mu\nu}+C_1k_\mu k_\nu+C_2k_\mu p_\nu+C_3p_\mu k_\nu+C_4k_\mu p'_\nu+C_5p'_\mu k_\nu\\ +C_6p_\mu p'_\nu+C_7p'_\mu p_\nu+C_8p'_\mu p'_\nu+C_9p_\mu p_\nu,\end{aligned} \qquad (35)$$

in which factors C_i can be computed according to one-loop diagrams. Now factors of transverse component of 4-point vertex function can be determined based on C_i as follow:

$$\begin{aligned}B_{11} &= -\frac{k^2}{kk'}C_1+\frac{k^2(kp+p^2-pp')}{(kk')^2}C_2-\frac{kp}{kk'}C_3+\frac{k^2(-p'^2+pp'+kp')}{(kk')^2}C_4\\ &+\frac{kp(-p'^2+pp'+kp')}{(kk')^2}C_6-\frac{kp'}{kk'}C_5+\frac{kp'(kp+p^2-pp')}{(kk')^2}C_7\\ &+\frac{kp'(-p'^2+pp'+kp')}{(kk')^2}C_8+\frac{kp(kp+p^2-pp')}{(kk')^2}C_9\\ B_{22} &= \frac{1}{4}\frac{C_6+C_7+C_8+C_9}{(kp')(-2kp+kp')+(kp)^2+k^2(k^2+2kp-2kp')}\\ B_{23} &= \frac{1}{4}\frac{C_6-C_7+C_8-C_9}{(kp')(-2kp+kp')+(kp)^2+k^2(k^2+2kp-2kp')}\\ B_{32} &= \frac{1}{4}\frac{2C_2+2C_4-C_6+C_7+C_8-C_9}{(kp')(-2kp+kp')+(kp)^2+k^2(k^2+2kp-2kp')}\\ B_{33} &= -\frac{1}{4}\frac{2C_2-2C_4+C_6+C_7-C_8-C_9}{(kp')(-2kp+kp')+(kp)^2+k^2(k^2+2kp-2kp')}\end{aligned} \qquad (36)$$

V. CONCLUSION

Using Ward Takahashi Identity to present 3 and and 4-point vertex functions as the sum of two parts. First, Longitudinal part of 3-point vertex can be written in terms of complete scalar propagator while for 4-point vertex, this part is presented in terms of scalar propagators and transverse part of 3-point vertex function. Second, transverse parts are not presented in term of fix components which depend on specific orders of the perturbative theory. The thirst, this method can used to derived transverse part of 3-point and 4-point vertex functions at higher order of the perturbative theory.

REFERENCES

[1] A. Bashir, Y. Concha-Sanchez, R. Delbourgo, *Phys. Rev. D* **76** (2007) 068009.
[2] A. Bashir, Y. Concha-Sanchez, R. Delbourgo, *Phys. Rev. D* **80** (2009) 045007.
[3] H. N. Long, *Basic of Particle Physics*, 2006 Statistic Press.
[4] P. V. Dong, H. N. Long, *The Economical* $SU(3) \times SU(3) \times U(1)$ *Gauge Model–Series of Monographs Basic Reseach*, 2008 Vietnam Academy of Science and Technology.

Received 30-08-2011.

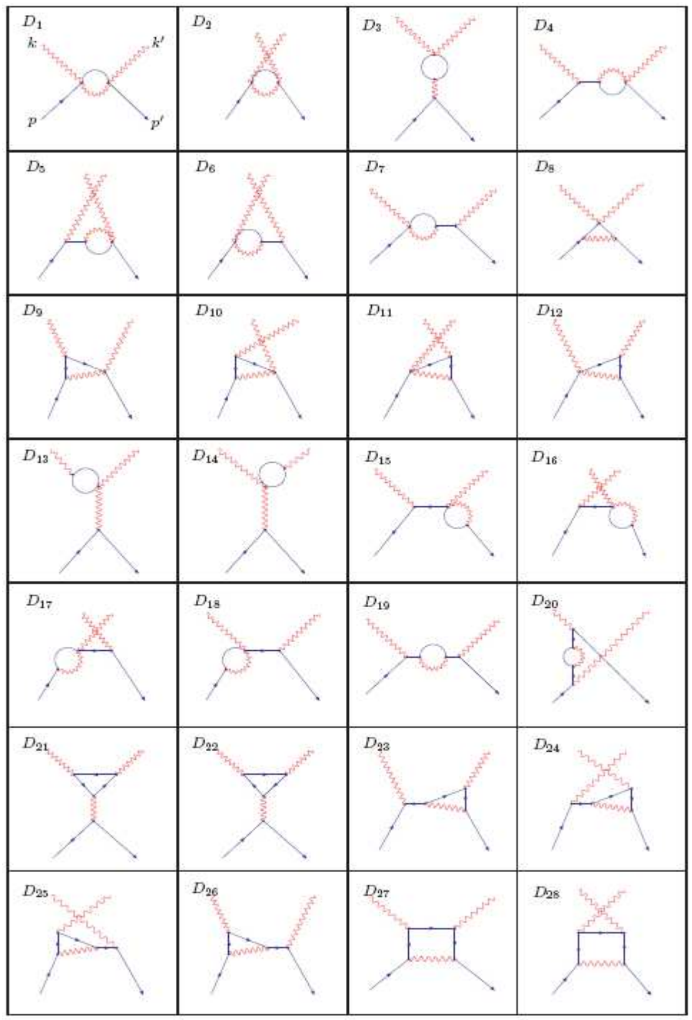

D_1
k
k'
p
p'
D_2
D_3
D_4
D_5
D_6
D_7
D_8
D_9
D_{10}
D_{11}
D_{12}
D_{13}
D_{14}
D_{15}
D_{16}
D_{17}
D_{18}
D_{19}
D_{20}
D_{21}
D_{22}
D_{23}
D_{24}
D_{25}
D_{26}
D_{27}
D_{28}

Fig. 4. giando