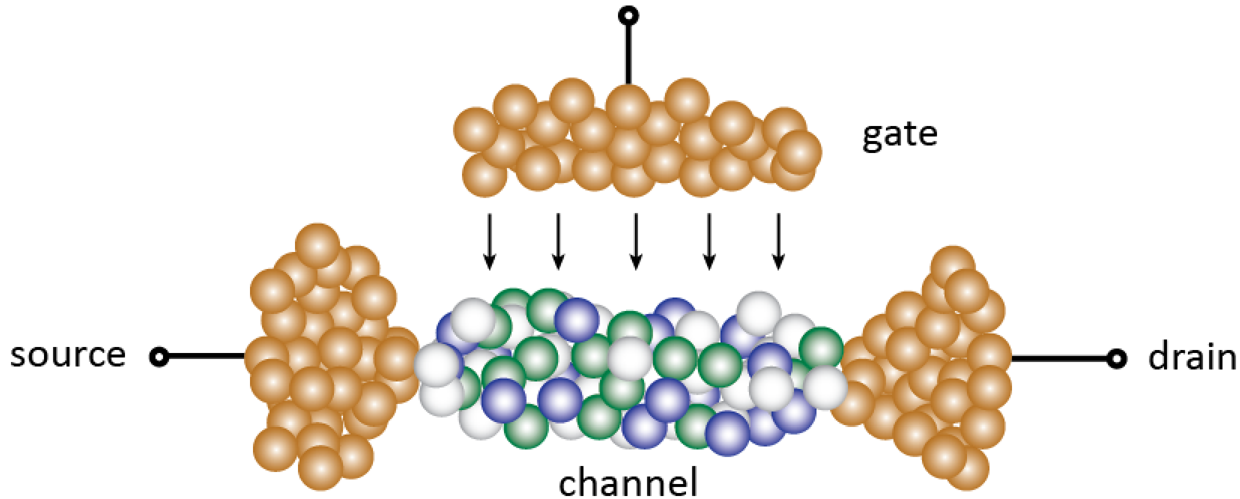
Quantum transport in nanojunctions with time-varying components

Eduardo C. Cuansing
Institute of Mathematical Sciences and Physics
University of the Philippines, Los Baños

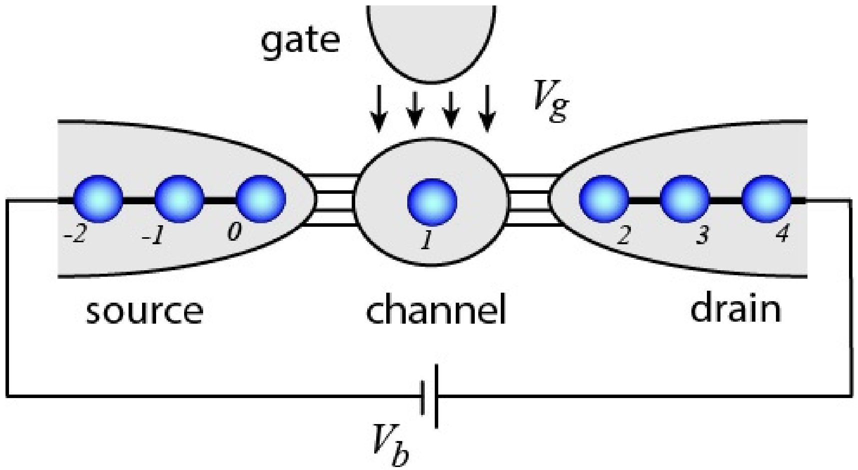
Electronics at the nanoscale:



Nanoscale device with a source-channel-drain configuration

- Channel can be a molecule, quantum dot, impurity, graphene nanoribbon
- Electrons propagate from the source to the drain
- Presence of time-varying components such as a time-dependent gate voltage, temperature gradient, incident laser, incoming electromagnetic signal, mechanical deformations
- Current is dynamic: transient regime, long-time regime
- Theory: quantum + many-body + nonequilibrium

Single-site channel with a time-varying gate:



Gate exerts a time-dependent potential on the channel

Step-function gate is switched on at $t = 0$

Chemical potentials:

$$\mu_L = E_F = 0 \qquad \mu_R = E_F - V_b$$

Temperatures: $T^{\mathrm{L}} = T^{\mathrm{R}} = 300$ K

Current flowing out of the left lead:

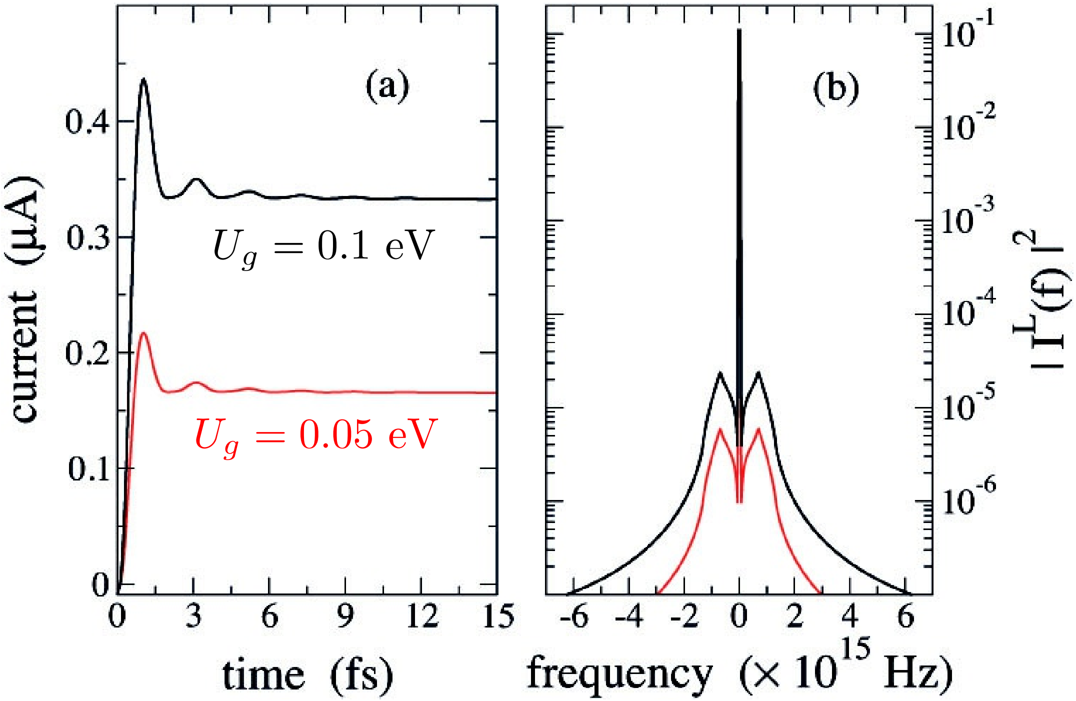
No source-drain bias voltage: $V_b = 0$

Current response to the gate:

- not instantaneous
- overshoots
- oscillates
- eventually settles down to a steady value

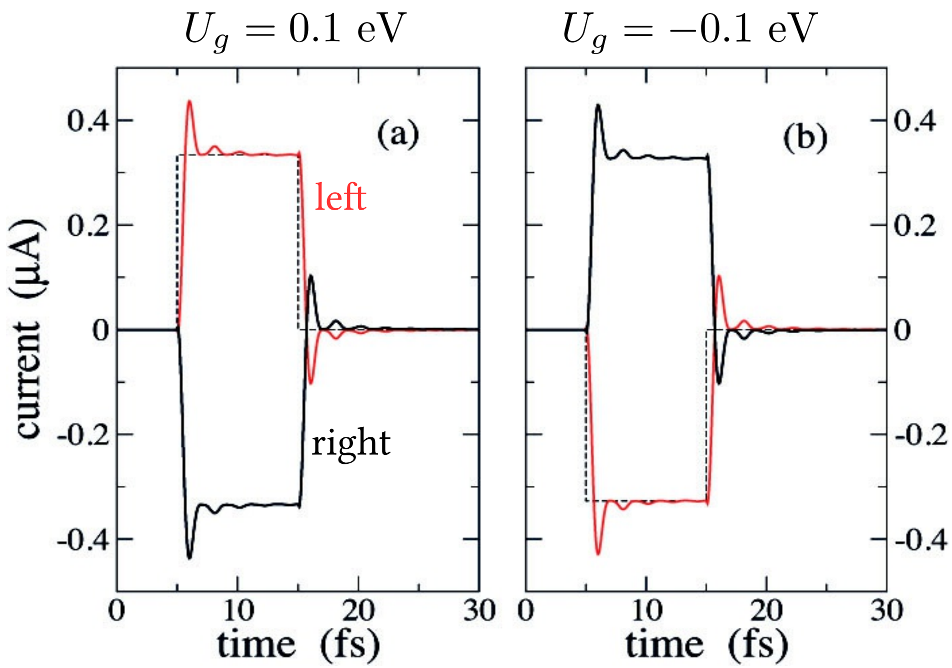
Frequency spectrum:

- dc component
- transient oscillation at $f = 0.7 \times 10^{15}$ Hz



ECC, Int. J. Mod. Phys. B **31**, 1750105 (2017)

Current when the gate potential is a pulse:



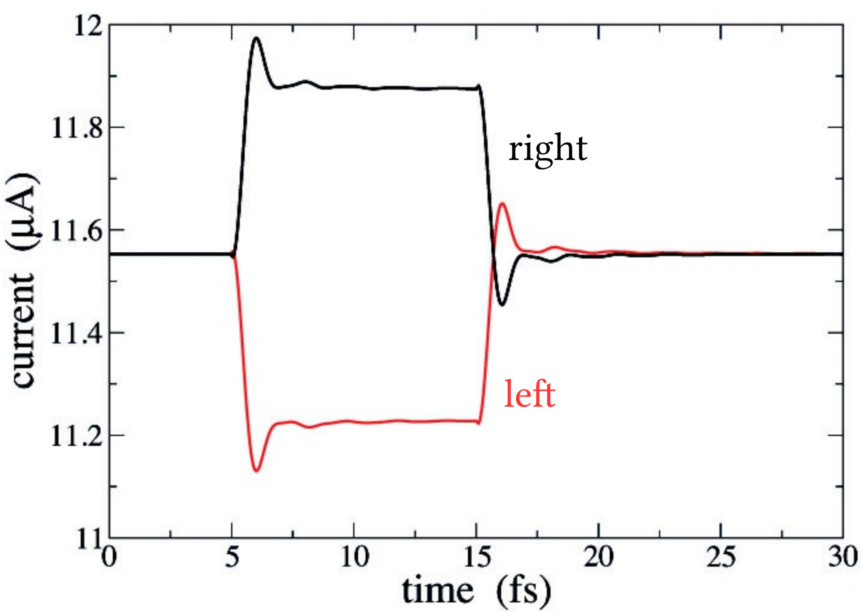
No source-drain bias voltage: $V_b = 0$

Gate potential pulse from 5 fs to 15 fs

Direction of current flow reverses with the sign of the gate potential

No net current in the channel:

- continuity equation



Bias voltage: $V_b = 0.3$ eV

Pulse gate potential: $U_g = -0.1$ eV

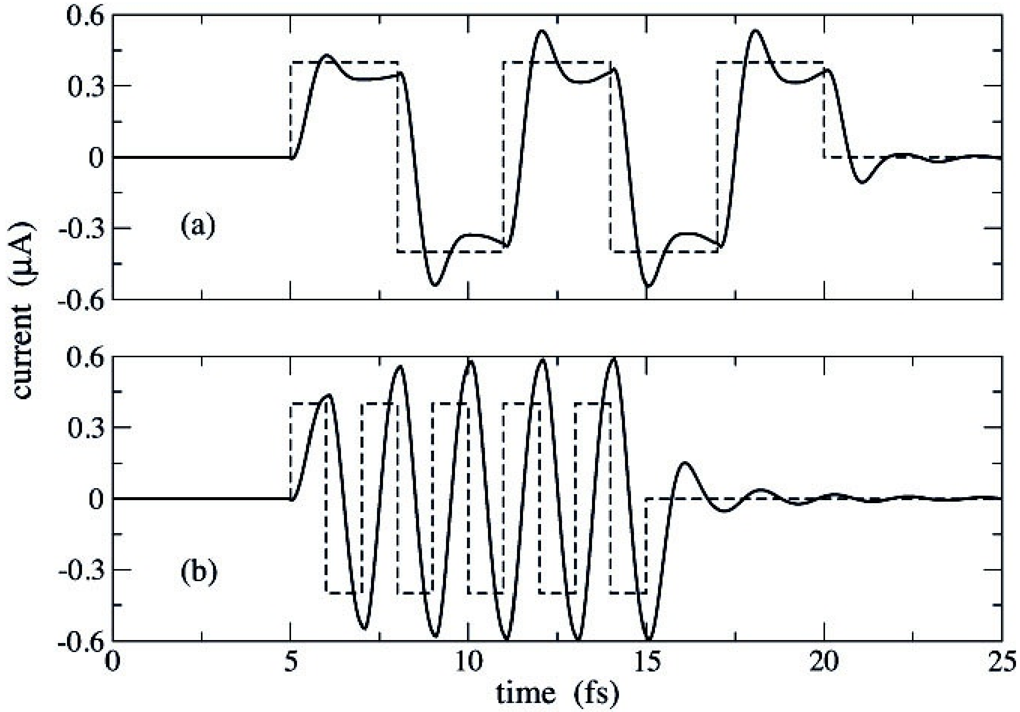
Right current is amplified

Left current is attenuated

Steady value is the same as the Landauer formula value

ECC, Int. J. Mod. Phys. B **31**, 1750105 (2017)

Potential pulses and ramps:



3 fs pulses

1 fs pulses

Gate potential: $U_g = \pm 0.3$ eV

No source-drain bias potential:

$V_b = 0$

Current flowing into the right lead

$I^{\mathrm{R}}(t)$

varying the ramp width

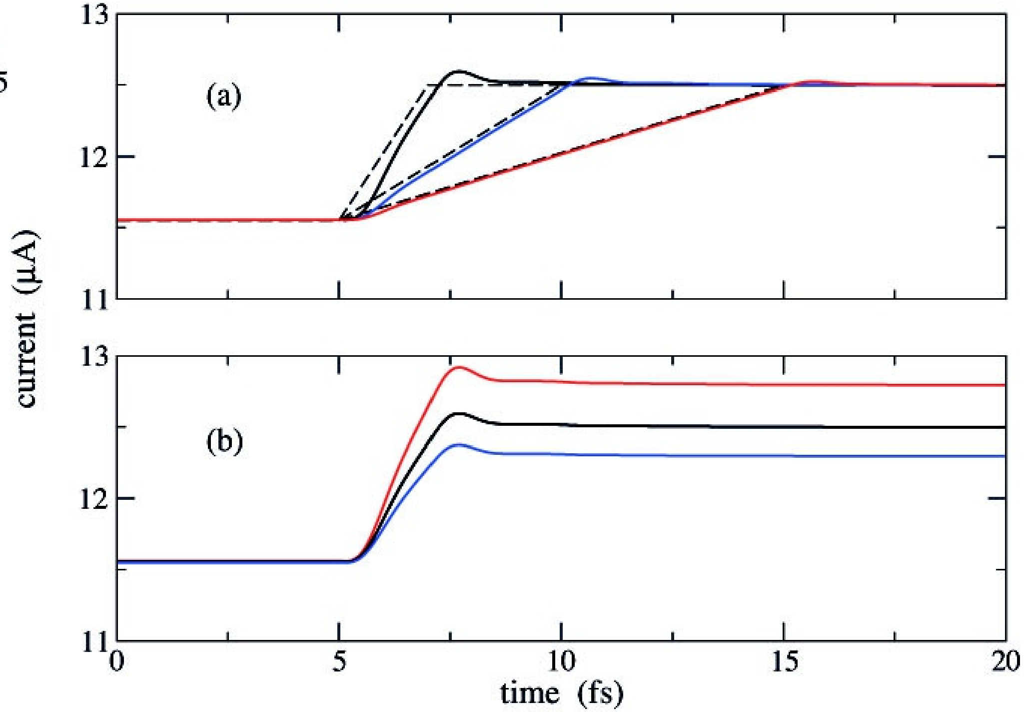
source-drain bias potential: $V_b = 0.3$ eV

varying the leads-channel coupling

$v^{\mathrm{LC}} = v^{\mathrm{RC}} = 1.8$ eV

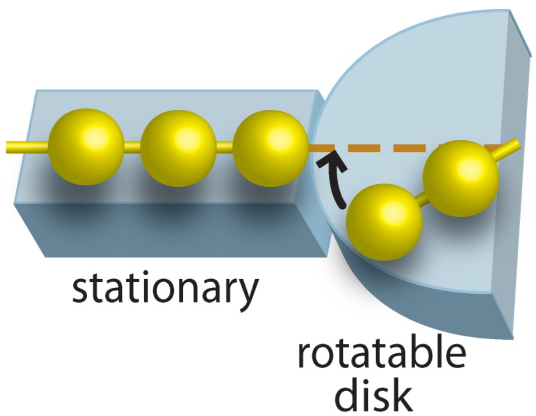
$v^{\mathrm{LC}} = v^{\mathrm{RC}} = 2.0$ eV

$v^{\mathrm{LC}} = v^{\mathrm{RC}} = 2.2$ eV



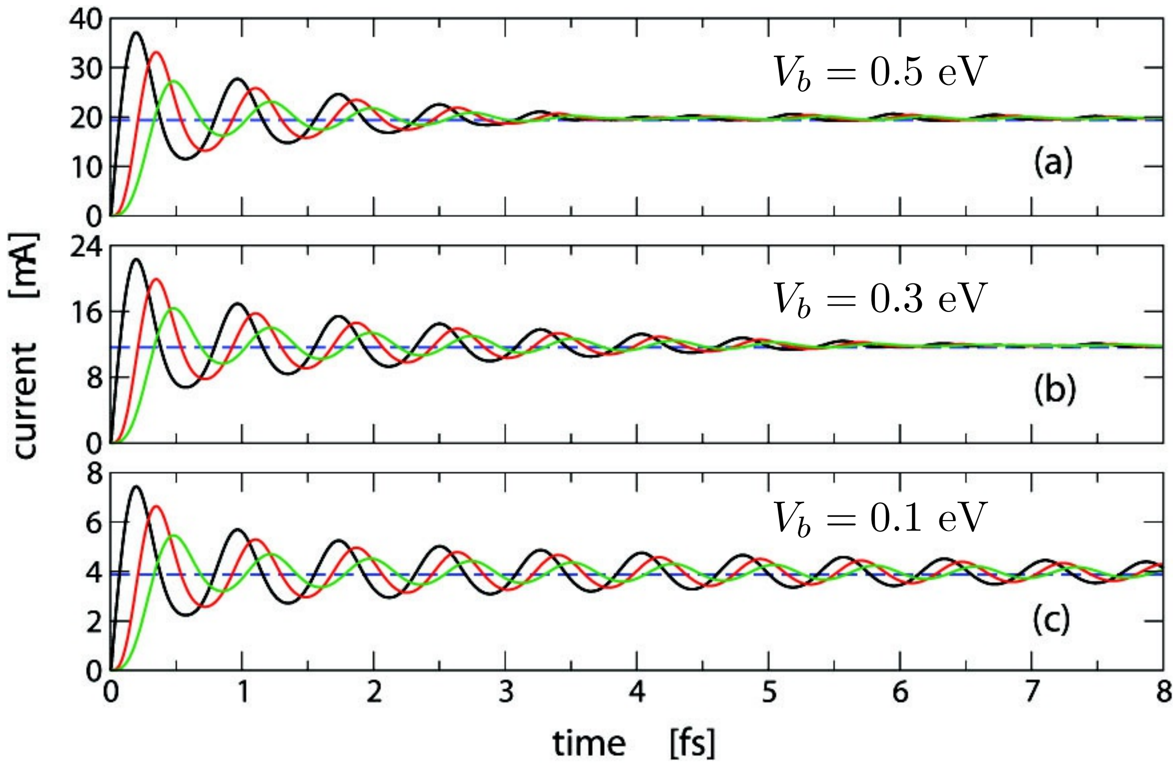
ECC, Int. J. Mod. Phys. B **31**, 1750105 (2017)

Nanoswitch:



Coupling between the two leads varies with time

- step function switch-on
- gradual
- modulated



Switch-on speeds:

Step function
Slow
Slowest

Form of the slow switch-on:

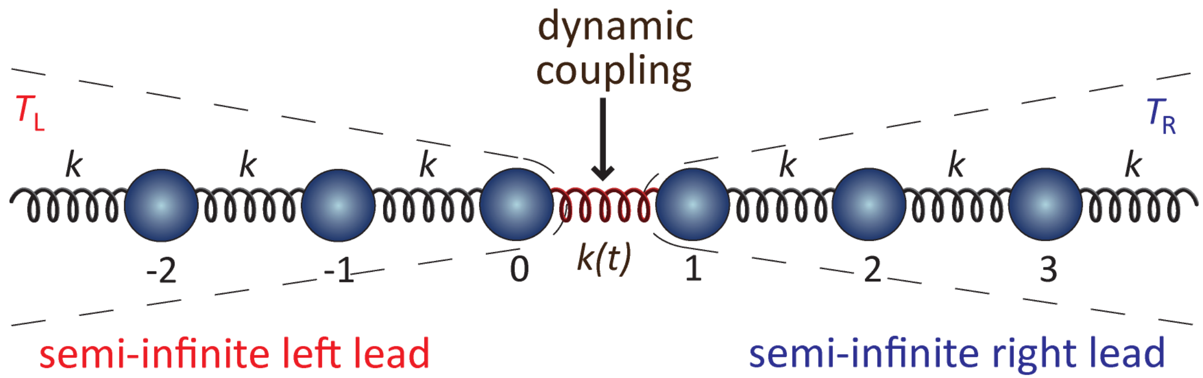
$$v^{\mathrm{LR}}(t) = v^{\mathrm{LR}} \tanh \omega_d t$$

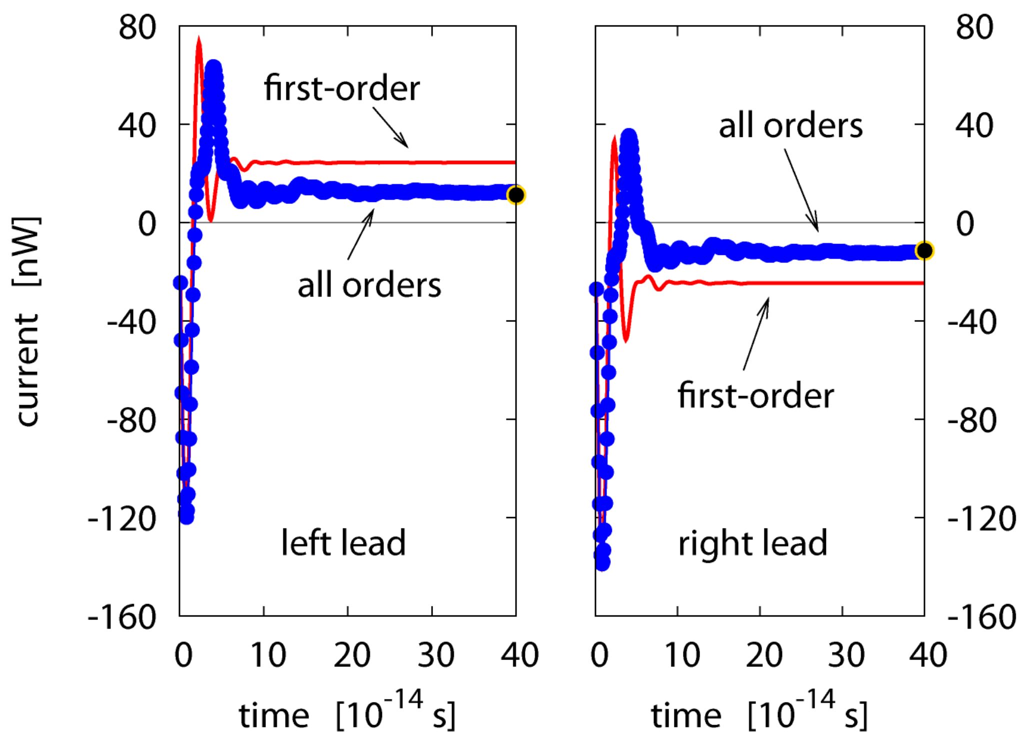
Power-law fit to the current:

$$I(t) = I_0 \left(\frac{t}{t_0}\right)^{-\alpha} \sin(\omega t + \phi) + I_0$$

ECC and G. Liang, J. Appl. Phys. **110**, 083704 (2011)

Thermal switch (phonon transport):





Heat bath temperatures:

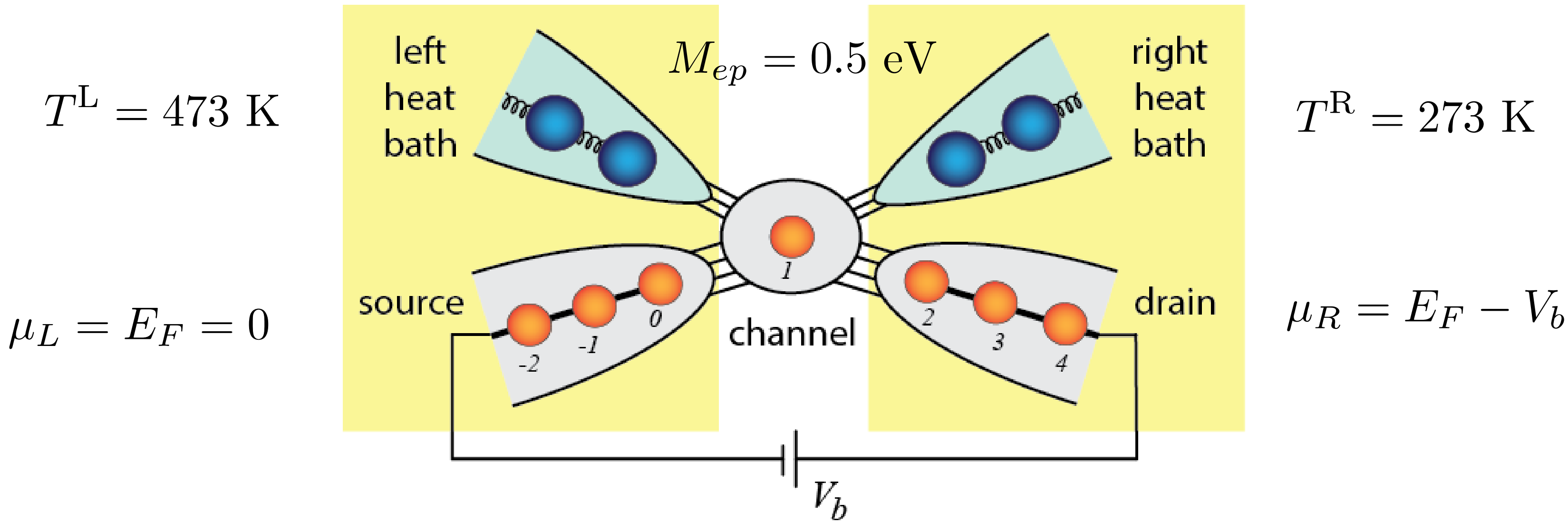
$T^{\mathrm{L}} = 310$ K

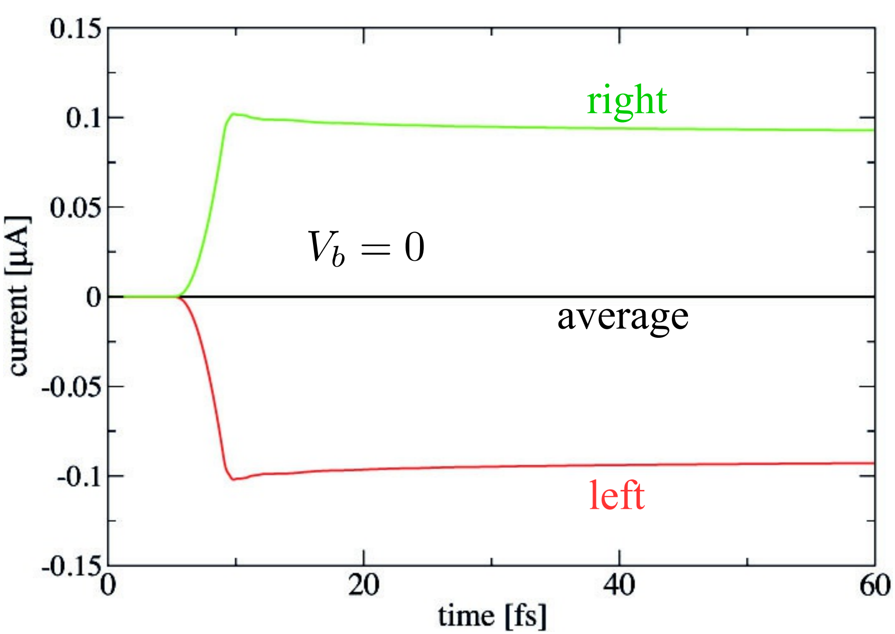
$T^{\mathrm{R}} = 290$ K

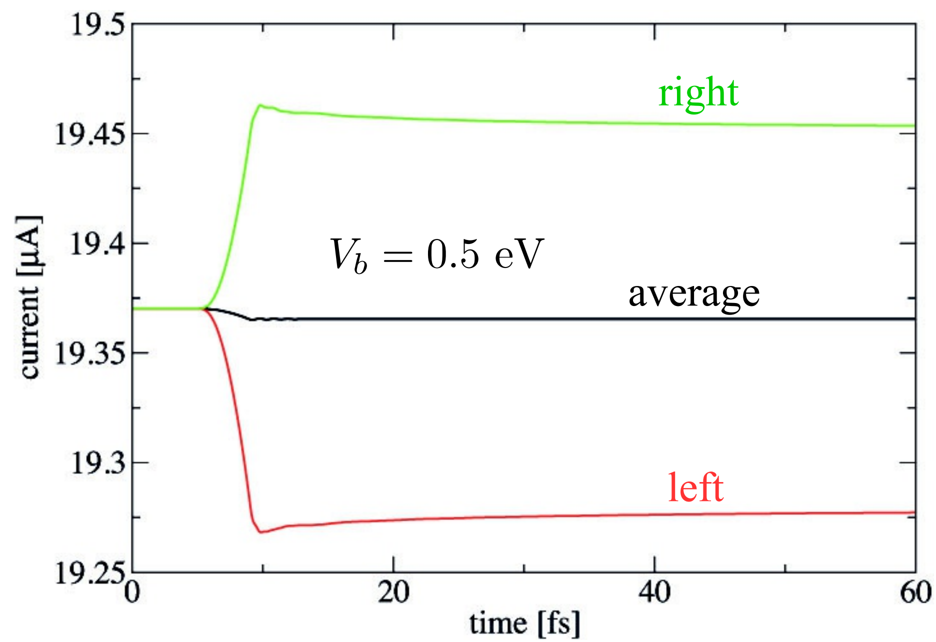
Oscillations in phonon transport is 10 times slower than in electron transport

ECC and J.-S. Wang, Phys. Rev. B **81**, 052302 (2010)
ECC and J.-S. Wang, Phys. Rev. E **82**, 021116 (2010)

With electron-phonon interactions:

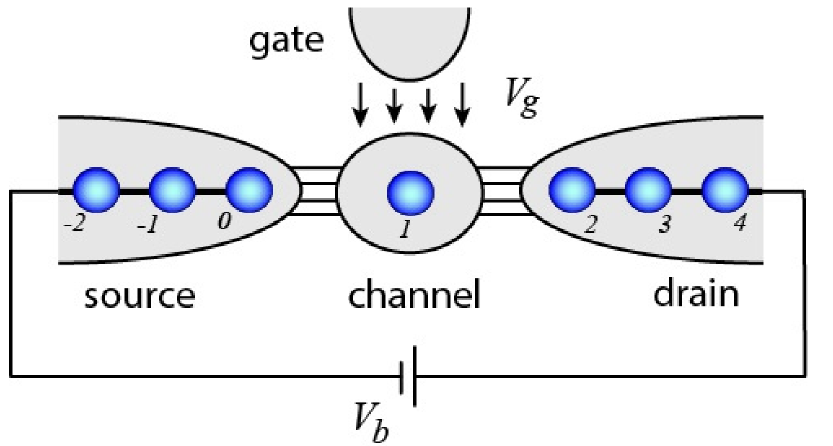






ECC, F.A. Bayocboc, and C.M. Laurio, AIP Conf. Proc. **1871**, 030003 (2017)

Calculating the time-dependent current: An example



Left Lead Hamiltonian:

$$H^{\mathrm{L}} = \sum_k \varepsilon_k^{\mathrm{L}}\, a_k^\dagger a_k + \sum_{kj} v_{kj}^{\mathrm{L}} \left(a_k^\dagger a_j + a_j^\dagger a_k \right)$$

Right Lead Hamiltonian:

$$H^{\mathrm{R}} = \sum_k \varepsilon_k^{\mathrm{R}}\, b_k^\dagger b_k + \sum_{kj} v_{kj}^{\mathrm{R}} \left(b_k^\dagger b_j + b_j^\dagger b_k \right)$$

Center Hamiltonian:

$$H_0^{\mathrm{C}} = \varepsilon_1^{\mathrm{C}}\, c_1^\dagger c_1$$

$$H_t^{\mathrm{C}} = U(t)\, c_1^\dagger c_1 \qquad U(t) = -q\, V_g(t)$$

$$H^{\mathrm{C}} = H_0^{\mathrm{C}} + H_t^{\mathrm{C}}$$

Couplings:

$$H^{\mathrm{LC}} = v_{01}^{\mathrm{LC}} \left(a_0^\dagger c_1 + c_1^\dagger a_0 \right)$$

$$H^{\mathrm{RC}} = v_{21}^{\mathrm{RC}} \left(c_1^\dagger b_2 + b_2^\dagger c_1 \right)$$

Total Hamiltonian: $H = H^{\mathrm{L}} + H^{\mathrm{R}} + H_0^{\mathrm{C}} + H^{\mathrm{LC}} + H^{\mathrm{RC}} + H_t^{\mathrm{C}}$

E.C. Cuansing, Int. J. Mod. Phys. B **31**, 17501015 (2017).

Current from electron flow and electron energy flow:

Current out of the Left Lead:

$$I^{\rm L}(t) = \left\langle -q\frac{dN^{\rm L}}{dt} \right\rangle = -\frac{iq}{\hbar}\left\langle \left[H, N^{\rm L}\right] \right\rangle = 2q\,\mathrm{Re}\left[v_{01}^{\rm LC}\, G_{10}^{\rm CL,<}(t,t)\right]$$

Current into the Right Lead:

$$I^{\rm R}(t) = \left\langle q\frac{dN^{\rm R}}{dt} \right\rangle = \frac{iq}{\hbar}\left\langle \left[H, N^{\rm R}\right] \right\rangle = -2q\,\mathrm{Re}\left[v_{21}^{\rm RC}\, G_{12}^{\rm CR,<}(t,t)\right]$$

positive current ⟶ flowing to the right

Lesser Nonequilibrium Green's Functions:

$$G_{10}^{\rm CL,<}(t_1,t_2) = \frac{i}{\hbar}\left\langle a_0^\dagger(t_2)c_1(t_1) \right\rangle \qquad G_{12}^{\rm CR,<}(t_1,t_2) = \frac{i}{\hbar}\left\langle b_2^\dagger(t_2)c_1(t_1) \right\rangle$$

Electronic Energy Current out of the Left Lead:

$$Q^{\rm L}(t) = \left\langle \frac{dH^{\rm L}}{dt} \right\rangle = 2\,\mathrm{Re}\left[\left(\varepsilon_0^{\rm L} v_{01}^{\rm LC} + v_{00}^{\rm L} v_{01}^{\rm LC}\right) G_{10}^{\rm CL,<}(t,t)\right]$$

Electronic Energy Current into the Right Lead:

$$Q^{\rm R}(t) = \left\langle \frac{dH^{\rm R}}{dt} \right\rangle = -2\,\mathrm{Re}\left[\left(\varepsilon_2^{\rm R} v_{21}^{\rm RC} + v_{22}^{\rm R} v_{21}^{\rm RC}\right) G_{12}^{\rm CR,<}(t,t)\right]$$

Nonequilibrium Green's Functions:

Perturbing Hamiltonian: $H_t^{\mathrm{C}} = U(t)\, c_1^\dagger c_1$

Contour-Ordered Green's Function:

Keldysh contour

$$G_{10}^{\mathrm{CL}}(\tau_1,\tau_2) = -\frac{i}{\hbar}\left\langle \mathrm{T}_c\, c_1(\tau_1) a_0^\dagger(\tau_2)\right\rangle = -\frac{i}{\hbar}\left\langle \mathrm{T}_c\, e^{-\frac{i}{\hbar}\int_c H_t^{\mathrm{C}} d\tau'} c_1(\tau_1) a_0^\dagger(\tau_2)\right\rangle_0$$

$$= G_{10,0}^{\mathrm{CL}}(\tau_1,\tau_2) + \int_c d\tau'\, G_{11,0}^{\mathrm{CC}}(\tau_1,\tau')\, U(\tau')\, G_{10}^{\mathrm{CL}}(\tau',\tau_2)$$

Use analytic continuation and Langreth's theorem:

Retarded CL Nonequilibrium Green's Function:

$$G_{10}^{\mathrm{CL},r}(t_1,t_2) = G_{10,0}^{\mathrm{CL},r}(t_1,t_2) + \int_0^t dt'\, G_{11,0}^{\mathrm{CC},r}(t_1,t')\, U(t')\, G_{10}^{\mathrm{CL},r}(t',t_2)$$

Advanced CL Nonequilibrium Green's Function:

$$G_{10}^{\mathrm{CL},a}(t_1,t_2) = G_{10,0}^{\mathrm{CL},a}(t_1,t_2) + \int_0^t dt'\, G_{11,0}^{\mathrm{CC},a}(t_1,t')\, U(t')\, G_{10}^{\mathrm{CL},a}(t',t_2)$$

Lesser CL Nonequilibrium Green's Function:

$$G_{10}^{\mathrm{CL},<}(t_1,t_2) = G_{10,0}^{\mathrm{CL},<}(t_1,t_2) + \int_0^t dt'\, G_{11}^{\mathrm{CC},r}(t_1,t')\, U(t')\, G_{10,0}^{\mathrm{CL},<}(t',t_2)$$
$$+ \int_0^t dt'\, G_{11,0}^{\mathrm{CC},<}(t_1,t')\, U(t')\, G_{10}^{\mathrm{CL},a}(t',t_2)$$
$$+ \int_0^t dt' \int_0^t dt''\, G_{11}^{\mathrm{CC},r}(t_1,t')\, U(t')\, G_{11,0}^{\mathrm{CC},<}(t',t'')\, U(t'')\, G_{10}^{\mathrm{CL},a}(t'',t_2)$$

Nonequilibrium Green's Functions: CC terms

CC Contour-Ordered Green's Function:

$$G_{11}^{\mathrm{CC}}(\tau_1,\tau_2) = -\frac{i}{\hbar}\left\langle \mathrm{T}_c\, c_1(\tau_1) c_1^\dagger(\tau_2)\right\rangle = -\frac{i}{\hbar}\left\langle \mathrm{T}_c\, e^{-\frac{i}{\hbar}\int_c H_t^{\mathrm{C}} d\tau'} c_1(\tau_1) c_1^\dagger(\tau_2)\right\rangle_0$$

$$= G_{11,0}^{\mathrm{CC}}(\tau_1,\tau_2) + \int_c d\tau'\, G_{11,0}^{\mathrm{CC}}(\tau_1,\tau')\, U(\tau')\, G_{11}^{\mathrm{CC}}(\tau',\tau_2)$$

Retarded CC Nonequilibrium Green's Function:

$$G_{11}^{\mathrm{CC},r}(t_1,t_2) = G_{11,0}^{\mathrm{CC},r}(t_1,t_2) + \int_0^t dt'\, G_{11,0}^{\mathrm{CC},r}(t_1,t')\, U(t')\, G_{11}^{\mathrm{CC},r}(t',t_2)$$

Advanced CC Nonequilibrium Green's Function:

$$G_{11}^{\mathrm{CC},a}(t_1,t_2) = G_{11,0}^{\mathrm{CC},a}(t_1,t_2) + \int_0^t dt'\, G_{11,0}^{\mathrm{CC},a}(t_1,t')\, U(t')\, G_{11}^{\mathrm{CC},a}(t',t_2)$$

Lesser CC Nonequilibrium Green's Function:

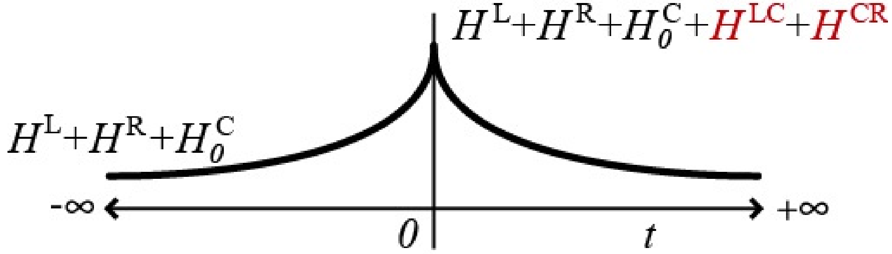
$$G_{11}^{\mathrm{CC},<}(t_1,t_2) = G_{11,0}^{\mathrm{CC},<}(t_1,t_2) + \int_0^t dt'\, G_{11}^{\mathrm{CC},r}(t_1,t')\, U(t')\, G_{11,0}^{\mathrm{CC},<}(t',t_2)$$

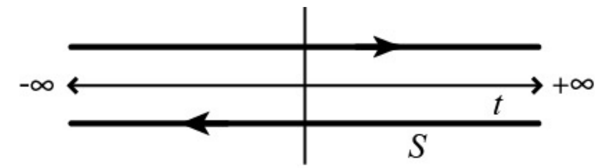
$$+ \int_0^t dt'\, G_{11,0}^{\mathrm{CC},<}(t_1,t')\, U(t')\, G_{11}^{\mathrm{CC},a}(t',t_2)$$

$$+ \int_0^t dt' \int_0^t dt''\, G_{11}^{\mathrm{CC},r}(t_1,t')\, U(t')\, G_{11,0}^{\mathrm{CC},<}(t',t'')\, U(t'')\, G_{11}^{\mathrm{CC},a}(t'',t_2)$$

Steady-State Green's Functions

Adiabatic switch-on of the leads-channel coupling:





Steady-State Contour-Ordered Green's Function:

$$G^{CC}_{11,0}(\tau_1,\tau_2) = -\frac{i}{\hbar}\left\langle \mathrm{T}_c\, c_1(\tau_1) c_1^\dagger(\tau_2)\right\rangle_0 = -\frac{i}{\hbar}\left\langle \mathrm{T}_c\, e^{-\frac{i}{\hbar}\int_c d\tau'\left(H^{LC}(\tau')+H^{RC}(\tau')\right)}\, c_1(\tau_1) c_1^\dagger(\tau_2)\right\rangle_\infty$$

$$= g^{C}_{11}(\tau_1,\tau_2) + \int_c d\tau' \int_c d\tau''\, g^{C}_{11}(\tau_1,\tau')\, \Sigma^{C}_{11}(\tau',\tau'')\, G^{CC}_{11,0}(\tau'',\tau_2)$$

Self-Energy: $\Sigma^{C}_{11}(\tau',\tau'') = v^{CL}_{10}\, g^{L}_{00}(\tau',\tau'')\, v^{LC}_{01} + v^{CR}_{12}\, g^{R}_{22}(\tau',\tau'')\, v^{RC}_{21}$

Analytic continuation and Langreth's theorem + Fourier transform:

Retarded: $G^{CC,r}_{11,0}(E) = \left[(E+i\eta) - \varepsilon^{C}_1 - \Sigma^{C,r}_{11}(E)\right]^{-1}$

Advanced: $G^{CC,a}_{11,0}(E) = \left[G^{CC,r}_{11,0}(E)\right]^*$

Lesser: $G^{CC,<}_{11,0}(E) = G^{CC,r}_{11,0}(E)\, \Sigma^{C,<}_{11}(E)\, G^{CC,a}_{11,0}(E)$

Haug and Jauho, Quantum Kinetics in Transport and Optics of Semiconductors

CL Steady-State Green's Functions:

CL Steady-State Green's Functions:

Retarded: $$G^{\mathrm{CL},r}_{10,0}(E) = G^{\mathrm{CC},r}_{11,0}(E)\, v^{\mathrm{CL}}_{10}\, g^{\mathrm{L},r}_{00}(E)$$

Advanced: $$G^{\mathrm{CL},a}_{10,0}(E) = \left[G^{\mathrm{CL},r}_{10,0}(E)\right]^*$$

Lesser: $$G^{\mathrm{CL},<}_{10,0}(E) = G^{\mathrm{CC},r}_{11,0}(E)\, v^{\mathrm{CL}}_{10}\, g^{\mathrm{L},<}_{00}(E) + G^{\mathrm{CC},r}_{11,0}(E)\, \Sigma^{\mathrm{C},<}_{11}(E)\, G^{\mathrm{CL},a}_{10,0}(E)$$

Equilibrium Green's Functions of the Leads:

Can be determined from the Equation of Motion of a Free Lead:

Retarded Equilibrium Green's Function:

$$g^r(E) = \frac{2}{v^2}\left((E+i\eta)-\varepsilon\right) \pm i\frac{2}{v^2}\sqrt{v^2-(\varepsilon-E)^2} \qquad -v \le \varepsilon - E \le v$$

Advanced Equilibrium Green's Function:

$$g^a(E) = \left(g^r(E)\right)^*$$

Lesser Equilibrium Green's Function:

$$g^<(E) = -f_{FD}(E)\left(g^r(E) - g^a(E)\right) \qquad f_{FD}(E) = \left(e^{(E-\mu)/k_BT}+1\right)^{-1}$$

Summary: How to calculate the time-dependent current

1. Equilibrium Green's Functions of the Free Leads in E-Space:

$$g_{00}^{\mathrm{L},r}, g_{00}^{\mathrm{L},a}, g_{00}^{\mathrm{L},<} \qquad g_{22}^{\mathrm{R},r}, g_{22}^{\mathrm{R},a}, g_{22}^{\mathrm{R},<}$$

2. Steady-State Green's Functions in E-Space:

$$G_{11,0}^{\mathrm{CC},r}, G_{11,0}^{\mathrm{CC},a}, G_{11,0}^{\mathrm{CC},<} \qquad G_{10,0}^{\mathrm{CL},r}, G_{10,0}^{\mathrm{CL},a}, G_{10,0}^{\mathrm{CL},<} \qquad G_{12,0}^{\mathrm{CR},r}, G_{12,0}^{\mathrm{CR},a}, G_{12,0}^{\mathrm{CR},<}$$

3. Fourier transforms of the Steady-State Green's Functions into t-Space

4. Nonequilibrium Green's Functions in t-Space:

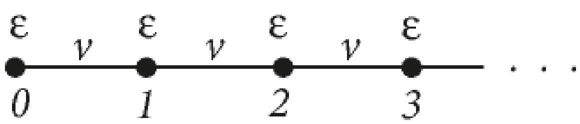
$$G_{11}^{\mathrm{CC},r}, G_{11}^{\mathrm{CC},a}, G_{11}^{\mathrm{CC},<} \qquad G_{10}^{\mathrm{CL},r}, G_{10}^{\mathrm{CL},a}, G_{10}^{\mathrm{CL},<} \qquad G_{12}^{\mathrm{CR},r}, G_{12}^{\mathrm{CR},a}, G_{12}^{\mathrm{CR},<}$$

5. Left and Right current:

$$I^{\mathrm{L}}(t) = 2q\,\mathrm{Re}\Big[v_{01}^{\mathrm{LC}}\, G_{10}^{\mathrm{CL},<}(t,t)\Big] \qquad I^{\mathrm{R}}(t) = -2q\,\mathrm{Re}\Big[v_{21}^{\mathrm{RC}}\, G_{12}^{\mathrm{CR},<}(t,t)\Big]$$

Thank you for your attention!

Calculating the equilibrium Green's function of a free lead:



$$H = \sum_k \varepsilon_k \, a_k^\dagger a_k + \sum_{kj} v_{kj} \left(a_k^\dagger a_j + a_j^\dagger a_k \right)$$

Equation of motion of electrons in the lead:

$$(H - E)\, g = -I$$

$$\begin{pmatrix} \varepsilon - E & v & 0 & 0 & \cdots \\ v & \varepsilon - E & v & 0 & \\ 0 & v & \varepsilon - E & v & \\ 0 & 0 & v & \varepsilon - E & \\ \vdots & & & & \ddots \end{pmatrix} \begin{pmatrix} g_{00} & g_{01} & g_{02} & \cdots \\ g_{10} & g_{11} & g_{12} & \\ g_{20} & g_{21} & g_{22} & \\ g_{30} & g_{31} & g_{32} & \\ \vdots & & & \ddots \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$(\varepsilon - E)\, g_{00} + v\, g_{10} = -1$$

$$v\, g_{00} + (\varepsilon - E)\, g_{10} + v\, g_{20} = 0$$

Ansatz: $g_{mn} = c\lambda^{|m-n|}$

$$\lambda = e^{\pm iq} \qquad \cos q = \frac{\varepsilon - E}{v} \qquad \sin q = \frac{\sqrt{v^2 - (\varepsilon - E)^2}}{v} \qquad c = -\frac{2}{v} e^{\pm iq}$$

we get: $g_{mn} = -\frac{2}{v} e^{\pm iq(|m-n|-1)}$

$$g_{00} = \frac{2}{v^2}(E - \varepsilon) \pm \frac{2i}{v^2}\sqrt{v^2 - (\varepsilon - E)^2}$$

Solving the iterative equation for the retarded NEGF:

Retarded Nonequilibrium Green's Function:

$$G_{10}^{\mathrm{RL},r}(t_1,t_2) = G_{10,0}^{\mathrm{RL},r}(t_1,t_2) + \int_0^t dt'\, G_{10,0}^{\mathrm{RL},r}(t_1,t')\, v_{01}^{\mathrm{LR}}(t')\, G_{10}^{\mathrm{RL},r}(t',t_2)$$

In the form:

$$f(t_a,t_b) = f_0(t_a,t_b) + \int_{t_1}^{t_N} f_0(t_a,t')v(t')f(t',t_b)dt'$$

Use numerical integration:

$$f(t_a,t_b) = f_0(t_a,t_b) + \Delta t \sum_{i=1}^{N} c_i\, f_0(t_a,t_i)v(t_i)f(t_i,t_b)$$

$c_i =$ numerical integration coefficient

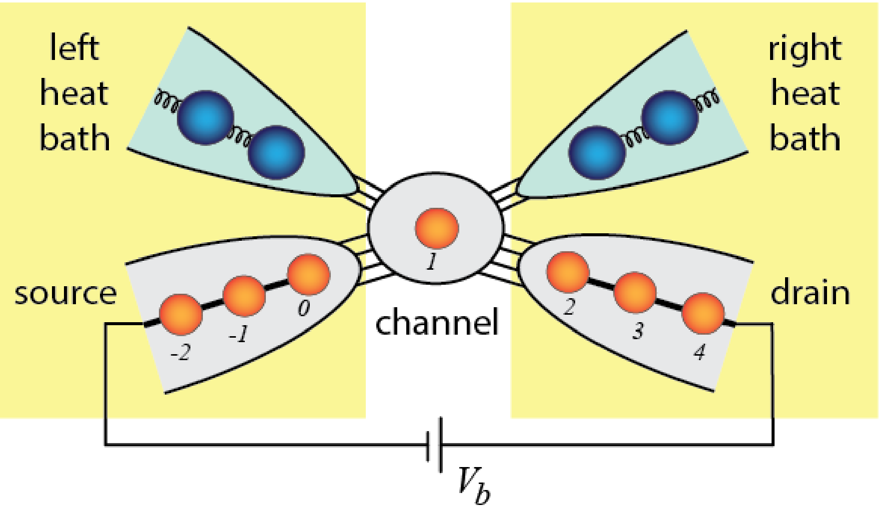
$$f(t_a,t_b) - \Delta t \sum_{i=1}^{N} c_i\, f_0(t_a,t_i)v(t_i)f(t_i,t_b) = f_0(t_a,t_b)$$

Construct the equivalent matrix equation: $Ax = b$

$$\begin{pmatrix} 1-\Delta t\, c_1 f_0(t_1,t_1)v(t_1) & -\Delta t\, c_2 f_0(t_1,t_2)v(t_2) & \cdots \\ -\Delta t\, c_1 f_0(t_2,t_1)v(t_1) & 1-\Delta t\, c_2 f_0(t_2,t_2)v(t_2) & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} f(t_1,t_b) \\ f(t_2,t_b) \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0(t_1,t_b) \\ f_0(t_2,t_b) \\ \vdots \end{pmatrix}$$

which can be solved by taking the inverse: $x = A^{-1}b$

With electron-phonon interactions:



Electron part: $H_e = H_e^{\rm L} + H_e^{\rm R} + H_{e,0}^{\rm C} + H_e^{\rm LC} + H_e^{\rm RC}$ $H_{e,0}^{\rm C} = \varepsilon_{e,1}^{\rm C} c_1^\dagger c_1$

Phonon part:

Heat baths:

$$H_p^{\rm L} = \sum_k \varepsilon_{p,k}^{\rm L}\, q_k^\dagger q_k + \sum_{jk} v_{p,jk}^{\rm L}\, q_j^\dagger q_k$$

$$H_p^{\rm R} = \sum_k \varepsilon_{p,k}^{\rm R}\, r_k^\dagger r_k + \sum_{jk} v_{p,jk}^{\rm R}\, r_j^\dagger r_k$$

Couplings:

$$H_p^{\rm LC} = v_{p,01}^{\rm LC} \left(q_0^\dagger p_1 + p_1^\dagger q_0 \right)$$

$$H_p^{\rm RC} = v_{p,21}^{\rm RC} \left(r_2^\dagger p_1 + p_1^\dagger r_0 \right)$$

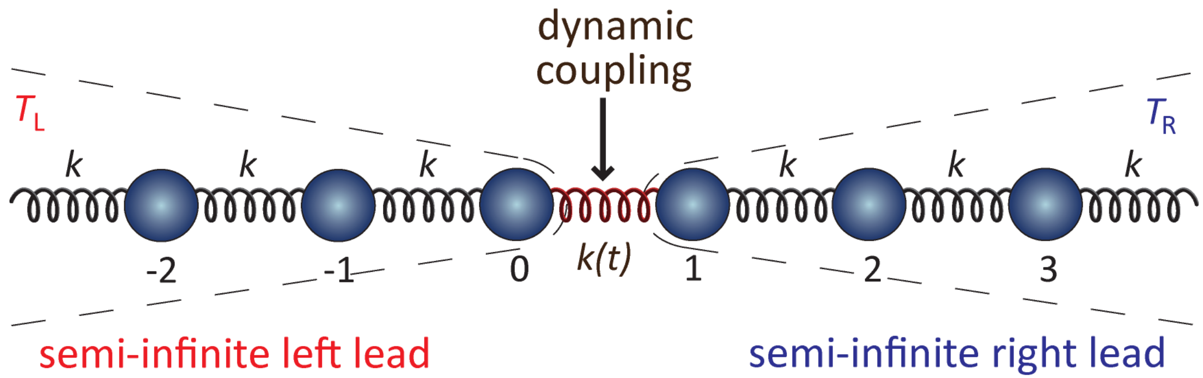
Channel: $H_p^{\rm C} = \varepsilon_{p,1}^{\rm C}\, p_1^\dagger p_1$

Phonon Hamiltonian: $H_p = H_p^{\rm L} + H_p^{\rm R} + H_p^{\rm C} + H_p^{\rm LC} + H_p^{\rm RC}$

Electron-Phonon Interaction: $H_{ep}^{\rm C} = M_{ep}(t) \left(p_1^\dagger + p_1 \right) c_1^\dagger c_1$

Total Hamiltonian: $H = H_e + H_p + H_{ep}^{\rm C}$

Thermal switch:



Hamiltonian for the left and right leads:

$$H^\alpha = \frac{1}{2}\sum_i \dot{u}_i^\alpha\,\dot{u}_i^\alpha + \frac{1}{2}\sum_{ij} u_i^\alpha\,K_{ij}^\alpha\,u_j^\alpha \qquad \alpha = \mathrm{L},\mathrm{R} \qquad u_i = \sqrt{m_i}\,x_i$$

Coupling between the leads:

$$H^{\mathrm{LR}}(t) = \sum_{ij} u_i^{\mathrm{L}}\,V_{ij}^{\mathrm{LR}}\,u_j^{\mathrm{R}}$$

Total Hamiltonian of the system: $H = H^{\mathrm{L}} + H^{\mathrm{R}} + H^{\mathrm{LR}}(t)$