First, Second Quantization and Q-Deformed Harmonic Oscillator

First, Second Quantization and Q-Deformed Harmonic Oscillator

Man Van Ngu[1], Ngo Gia Vinh[2], Nguyen Tri Lan[3], Luu Thi Kim Thanh[4] and Nguyen Ai Viet[3]

[1]Hung Yen Industrial Colleage, Hung Yen Province, Vietnam
[2]Bac Ninh Department of Education and Training, Bac Ninh Province, Vietnam
[3]Institute of Physics, Vietnam Academy of Science and Technology, 10 Dao Tan street, Ba Dinh, Hanoi, Vietnam
[4]Hanoi Pedagogical University No.2, Vinh Phuc Province, Vietnam

E-mail: `cnttcs2@gmail.com`

Abstract. Relations between the first, the second quantized representations and deform algebra are investigated. In the case of harmonic oscillator, the axiom of first quantization (the commutation relation between coordinate and momentum operators) and the axiom of second quantization (the commutation relation between creation and annihilation operators) are equivalent. We shown that in the case of q-deformed harmonic oscillator, a violence of the axiom of second quantization leads to a violence of the axiom of first quantization, and inverse. Using the coordinate representation, we study fine structures of the vacuum state wave function depend in the deformation parameter q. A comparison with fine structures of Cooper pair of superconductivity in the coordinate representation is also performed.

1. Introduction

In the last decade, studies of quantum algebras (or groups) have a great attention in modern physics and mathematics. They motivated by the passage from classical physical systems to quantum systems. Despite concept of algebras was suggested long before the discovery of quantum mechanics in the beginning of the last century, and an important achievement was mad by Heisenberg, the Heisenberg deformed algebra still rare applications in comparison with wave representation (Schrodinger wave equation, or the first quantization), and particle representation (occupation numbers approach, or second quantization) of quantum mechanics. Recently, quantum group and deformed Heisenberg algebras with q-deformed harmonic oscillator have been a subject of intensive investigation. This approach is found some applications in various branches of physics and chemistry [1–7]. The method of q-deformed quantum mechanics was developed on the base of Heisenberg commutation relation (the Heisenberg algebra). The main parameter of this method is the deformation parameter q usually considered to variety in the range $0 < q < 1$, and the models have been constructed that the behavior of studying objects reduce to theirs standard counterparts as $q \to 1$. In this work we investigate relations between the first, the second quantized representations of quantum mechanics and deform algebra. In the case of harmonic oscillator, the axiom of first quantization (the commutation relation between coordinate and momentum operators) and the axiom of second quantization (the commutation relation between creation and annihilation operators) are equivalent, so harmonic

oscillator is a good object for investigation the link between the two representations of quantum mechanics. We shown that in the case of q-deformed harmonic oscillator, a violence of the axiom of second quantization leads to a violence of the axiom of first quantization, and inverse. Using the coordinate representation, we study fine structures of the vacuum state wave function depend on the deformation parameter q. A comparison with fine structures of Cooper pair of superconductivity in the coordinate representation also is performed. In this work we use the atomic unit system with $\hbar = c = k_B = 1$.

2. Relation between the first and the second quantized representations of quantum mechanics

One of most important behaviors of nature is the wave-particle dualism. In the first quantization, classical particles turned to some waves (or fields in general). Example electrons can be expressed by wave function $\Psi_e(\mathbf{r}, t)$, satisfied the wave equation Schrodinger. The regular habitants in first quantization are plane waves. Vacuum of the first quantization (the home land of wave habitants) is four-dimension flat Hilbert space. which is denoted by the ket-bra $|0\rangle = (\mathbf{r}, t)$. The axioms of the first quantization are the commutation relations between coordinate and momentum $[\hat{x}, \hat{p}] = -i$. In the second quantization, waves (or fields) become particle (quasi-particles in general). Example light wave become photon. The regular habitants in second quantization are particles (quasi-particles). Vacuum of the second quantization (the home land of particle habitants) is the Fock space (or occupation number space), which is denoted by the round-bras $|0\rangle$. The axioms of the second quantization are the commutation relations between annihilation and creation operators (commutation for bosons and anti-Commutation for fermions). Relations between the first and the second quantized representations of quantum mechanics are expressed in the table 1.

Name	First quantization	Second quantization	Connection
Vacuum	Hilbert space $|0\rangle$	Fock space $|0)$	
Habitant	Wave	Particle	Fourier transform
Main variable	Coordinate x Momentum p	Level number N	
Axiom	Commutation relations of coordinate and momentum $[\hat{x}, \hat{p}] = i$	Relations for annihilation and creation operators Commutation (boson)$[a, a^+] = 1$ Anti-Commutation (fermions) $\{b, b^+\} = 1$	
Evolution variable	Time t	Temperature T	Conjugate
Physical constant	Planck constant $\hbar$	Boltzmann constant k_B	
Main uncertainty or chaos	Uncertainty principle $\triangle x \triangle p = \hbar$, $\triangle t \triangle E = \hbar$	Zero-level (vacuum) fluctuation Zero temperature fluctuation	
Main state	Steady	Equilibrium	
Behavior	Spreading	Rotation (spin)	
Distribution& Statistics	Uniform (free) Wave packet (bound)	Fermi-Dirac (fermion) Bose-Einstein (boson)	
System main principle	Minimal energy	Maximal entropy	
Main presentation	Schrodinger equation	Hamiltonian operator	Harmonic oscillator

Table 1. Relations between the first and the second quantized representations of quantum mechanics.

Note here harmonic oscillator is a good bridge between the first and the second quantized representations of quantum mechanics.

3. Harmonic oscillator ($q = 1$)

For simplicity we put the mass of oscillator is $m = 1$, and take the case of one dimensional harmonic oscillator. In the first quantization representation the Hamiltonian of harmonic oscillator is

$$H_0 = \frac{\omega}{2} \left(\hat{p}^2 + \hat{x}^2 \right),\tag{1}$$

where ω is the oscillation frequency, $\hat{x}$ and $\hat{p} = -i\frac{\mathrm{d}}{\mathrm{d}x}$ are the coordinate and momentum operators, which satisfied the commutative relation $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i$. In the second quantization representation the Hamiltonian operator of harmonic oscillator is

$$H_0 = \frac{\omega}{2} \left(a_0 a_0^+ + a_0^+ a_0 \right),\tag{2}$$

where a_0^+, a_0 are creation and annihilation operators, which satisfied the commutation relation $\left[a_0, a_0^+ \right] = a_0 a_0^+ - a_0^+ a_0 = 1$. Energy spectrum of harmonic oscillator has the form

$$E_{0n} = \frac{\omega}{2} \left(2n + 1 \right),\tag{3}$$

where $n = 0, 1, 2, 3, \ldots$ are the integers. The creation and annihilation operators of harmonic oscillator can be expressed in terms of coordinate and momentum operators as

$$a_0 = \frac{(\hat{x} + i\hat{p}_x)}{\sqrt{2}}, \tag{4}$$

$$a_0^+ = \frac{(\hat{x} - i\hat{p}_x)}{\sqrt{2}}. \tag{5}$$

From the definition of the annihilation operator $a_0|0\rangle$, the ground state wave function of harmonic oscillator Ψ_0 satisfies the equation

$$a_0|0\rangle = 0 = \left[\frac{(\hat{x} + i\hat{p}_x)}{\sqrt{2}}\right]\Psi_0 = \frac{1}{\sqrt{2}}(\hat{x} + \frac{\mathrm{d}}{\mathrm{d}x})\Psi_0. \tag{6}$$

The solution of this equation gives us the ground state wave function of harmonic oscillator Ψ_0 in a Gaussian form

$$\Psi_0(x) = Ce^{-\frac{x^2}{2}}, \tag{7}$$

where C is the normalization constant. The ground state wave function of harmonic oscillator $C\Psi_0$ with is plotted in the figure 1.
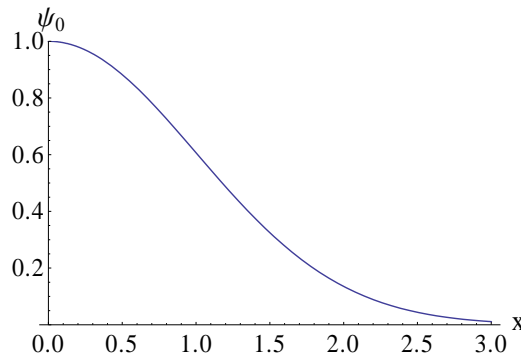

Figure 1. The ground state wave function of harmonic oscillator Ψ_0 has a Gaussian form.

Easily to check that

$$\left(0|a_0^+ a_0|0\right) = 0, \tag{8}$$

and

$$\langle 0|a_0^+ a_0|0\rangle = \langle 0|\Psi_0^+ \Psi_0|0\rangle = \langle 0|C^2 e^{-x^2}|0\rangle = C^2 \int_{-\infty}^{\infty} dx e^{-x^2} = C^2 \Pi^{\frac{1}{2}} \neq 0, \tag{9}$$

therefore, but $\langle 0|a^+ a|0\rangle \neq 0$ so $|0) \neq |0\rangle$. We note here the vacuums of the first and the second quantized representations of quantum mechanics are not the same. Introducing density operator $\rho_0 = \Psi_0^+ \Psi_0$ and consider that the density fluctuation of vacuum in first quantization (Hilbert space) is corresponding the zero level oscillation $\left(n = \frac{1}{2}\right)$ of harmonic oscillator in the second quantization Fock space (see Casimir effect)

$$\langle 0|\rho_0|0\rangle = \frac{1}{2}, \tag{10}$$

from that we can calculate the value of constant C

$$C = \left(2^{\frac{1}{2}} \Pi^{\frac{1}{4}}\right)^{-1}. \tag{11}$$

4. q-deformed harmonic oscillator

Creation and annihilation operators of q-deformed harmonic oscillator satisfied the commutation relation

$$\left[a, a^+\right]_q = aa^+ - qa^+a = 1, \tag{12}$$

where q is deformation parameter taking values in $[0, 1]$. Introducing the new deformation parameter α taking values in $[0, \infty]$

$$\alpha = \sqrt{-\frac{\ln q}{2}}. \tag{13}$$

In the second quantized representation, the Hamiltonian operator of q-deformed harmonic oscillator is

$$H = \frac{\omega}{2}\left(aa^+ + a^+a\right). \tag{14}$$

Energy spectrum of q-deformed harmonic oscillator has the form

$$E_n = \frac{\omega}{2}\left([n] + [n+1]\right), \tag{15}$$

where $[n] = \frac{1-q^n}{1-q}$ are the d-integer, they differ from natural numbers. The values of ratio $\frac{[n]}{n}$ depend on deformation parameter q are presented in the figure 2.
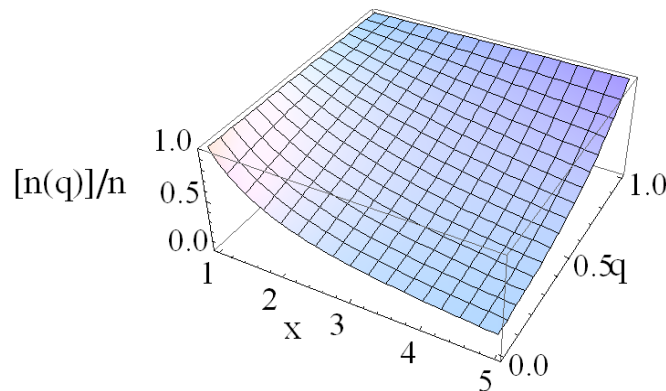

Figure 2. The values of ratio $\frac{[n]}{n}$ depend on deformation parameter q.

This ratio tends to zero $\frac{[n]}{n} \to 0$ when $q \to 0$. The maximum value of $[n]_{\max} \to \infty$ when $q \to 1$ and we return to case of standard harmonic oscillator. The creation and annihilation operators of q-deformed harmonic oscillator can be expressed in terms of coordinate and momentum operators as

$$a = \frac{\exp\left(-2i\alpha x\right) - \exp\left(i\alpha\frac{\mathrm{d}}{\mathrm{d}x}\right)\exp\left(-i\alpha x\right)}{\sqrt[-i]{1 - \exp\left(-2\alpha^2\right)}}, \tag{16}$$

$$a^+ = \frac{\exp\left(2i\alpha x\right) - \exp\left(i\alpha x\right)\exp\left(i\alpha\frac{\mathrm{d}}{\mathrm{d}x}\right)}{\sqrt[i]{1 - exp\left(-2\alpha^2\right)}}. \tag{17}$$

Denote $\varepsilon = 1 - q$, the energy spectrum of q-deformed harmonic oscillator becomes quadratic

$$E_n = \omega\left(n + \frac{1}{2} - \frac{n^2}{2}\varepsilon + O\left(\varepsilon^2\right)\right). \tag{18}$$

As above, the solution of the equation $a|0) = 0$

$$a|0) = \frac{\exp\left(-2i\alpha x\right) - \exp\left(i\alpha \frac{\mathrm{d}}{\mathrm{d}x}\right)\exp\left(-i\alpha x\right)}{\sqrt[i]{1 - \exp\left(-2\alpha^2\right)}}, \tag{19}$$

gives us the ground state wave function Ψ_0 of q-deformed harmonic oscillator in the form

$$\Psi_0\left(x\right) = C \exp\left(-\frac{x^2}{2} + \frac{3}{2}i\alpha x\right), \tag{20}$$

In the case of $q \to 1$, $\alpha \to 0$ we return to case of standard harmonic oscillator. Introduce density operator $\rho = \Psi_0^+ \Psi_0$ of q-deformed harmonic oscillator, because $\Psi_0^+\Psi_0 = \psi_0^+\psi_0$ therefore $\rho = \rho_0$, so that $\langle 0|\rho|0\rangle = \langle 0|\rho_0|0\rangle = \frac{1}{2}$. Values of density vacuum fluctuations are the same for both harmonic oscillator and q-deformed harmonic oscillator. Real part ReΨ_0 and imaginary part Im Ψ_0 of ground state wave function Ψ_0 are presented in figure 3.
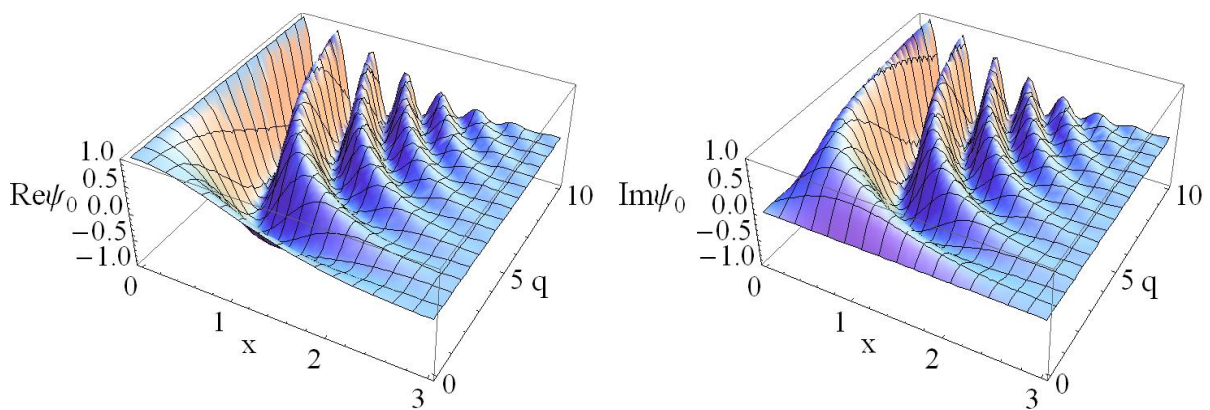

Figure 3. Real part ReΨ_0 and imaginary part ImΨ_0 of ground state wave function Ψ_0 are presented in figure 3.

At $q = 1$, back to harmonic oscillator case with ReΨ_0 has a Gaussian form, and Im$\Psi_0 = 0$.

5. Comparison with Cooper pairs in superconductivity

From condensed matter physics we have learn that the real part of wave function plays important role. In this part we try to explore the meaning of real part ReΨ_0 of ground state wave function Ψ_0 of q-deformed harmonic oscillator. For comparison we take the well known case of Cooper pairs in superconductivity. In a long history of the BCS theory, the Cooper pair usually analyzed in the momentum-space. The first investigation Cooper pair in coordinate-space was done in the work [8], where was shown that this leads to a spherically symmetrical quasi-atomic wave function, with an identical onion-like layered structure for each of the electrons constituting the Cooper pair. The internal structure of Cooper pair wave function Ψ_C (also called the singlet pair function or the Gorkov function) is given by

$$\Psi(r) \propto \int \frac{cos\left(k_F r + e'r'\right)d\varepsilon'}{\sqrt{\varepsilon'^2 + 1}} \approx cos\left(k_F r\right) \int \frac{cos\left(\varepsilon'r'\right)d\varepsilon'}{\sqrt{\varepsilon'^2 + 1}} = cos\left(k_F r\right) K_0\left(\frac{r}{\pi \xi_0}\right), \tag{21}$$

where k_F is the Fermi wave vector at the top of the Fermi sea, K_0 is the zero-order modified Bessel function with an asymptotic form that is similar to an exponential for large x. The wave function Ψ_C of Cooper pair is plotted in the figure 4.
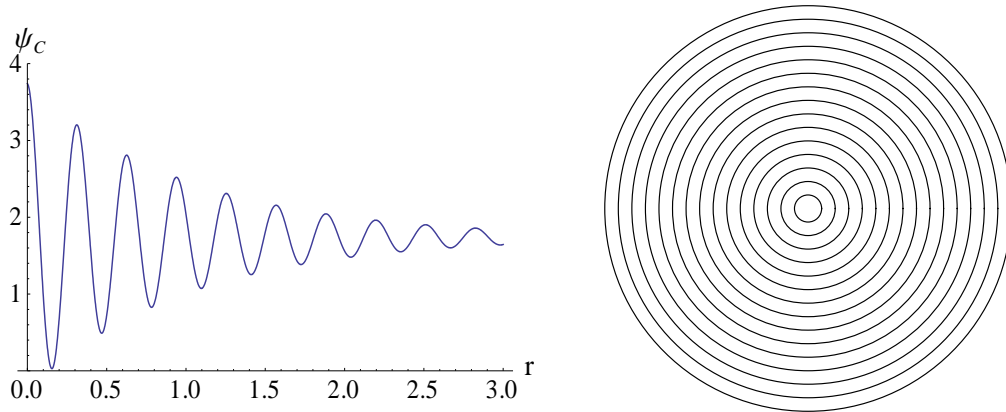

Figure 4. The wave function Ψ_C of Cooper pairs in superconductivity.

We can realize that the real part $\mathrm{Re}\Psi_0$ of ground state wave function Ψ_0 of q-deformed harmonic oscillator (see figure 5) is very similar the wave function Ψ_C of Cooper pair.
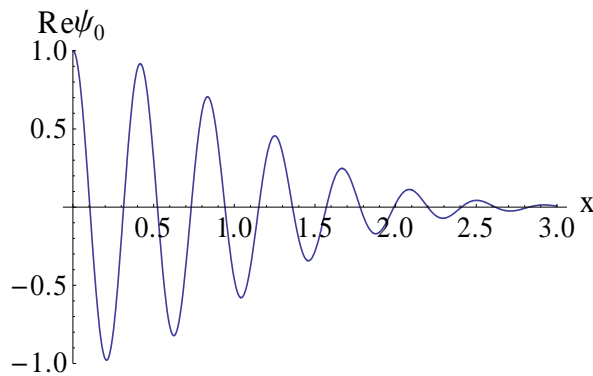

Figure 5. The real part $\mathrm{Re}\Psi_0$ of ground state wave function Ψ_0 of q-deformed harmonic oscillator $\alpha = 10$.

The similarity between $\mathrm{Re}\Psi_0$ of q-deformed harmonic oscillator and the wave function Ψ_C of Cooper pair might be not accidental and will be a subject for further investigation.

6. Uncertainty relation for q-deformed harmonic oscillator

The coordinate-momentum uncertainty relation is an important character of quantum world. This uncertainty for q-deformed harmonic oscillator is

$$\triangle x \triangle p = \frac{E}{\omega} = \frac{([n] + [n+1])}{2}.$$

For the case of harmonic oscillator $(\triangle x \triangle p)_0 = \frac{(2n+1)}{2}$. The ratio of $K = \frac{\triangle x \triangle p}{(\triangle x \triangle p)_0}$ of uncertainty for q-deformed harmonic oscillator and uncertainty for harmonic oscillator is plotted in the figure 6.
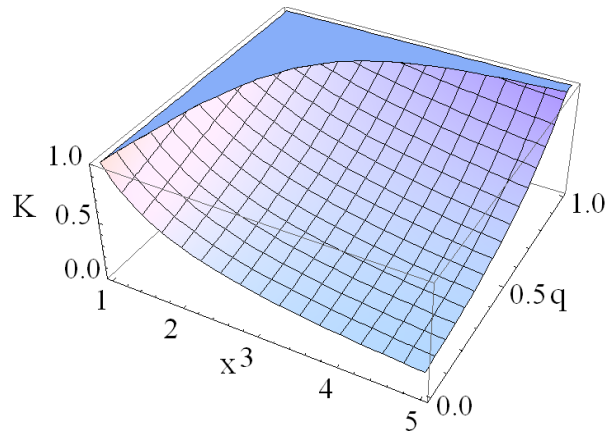

Figure 6. The ratio of $K = \frac{\triangle x \triangle p}{(\triangle x \triangle p)_0}$ of uncertainty for q-deformed harmonic oscillator and uncertainty for harmonic oscillator.

We note here uncertainty for q-deformed harmonic oscillator is less than the analogous value for standard harmonic oscillator for all n, except $n = 0$. At $n = 0$, the coordinate-momentum uncertainty is minimum and given by $1/2$.

7. Discussion

The main results of this work are the table1 showing the relations between the first and the second quantized representations of quantum mechanics with an attention that harmonic oscillator is a good bridge between them, also the investigation deformation parameter q depending on the most important physical characters of q-deformed harmonic oscillator in explicit coordinate representations, such as annihilation and creation operators, q-deformed integer, the ground wave function, coordinate-momentum uncertainty relation. Introducing density operators, we explored the physical meaning of wave function of ground state of the two types oscillators with and absence deformation, and shown that the vacuums of first and second quantization are not the same but the values of vacuum density fluctuation or zero level oscillation are equal for both. We shown that the real part $\text{Re}\Psi_0$ of ground state wave function Ψ_0 of q-deformed harmonic oscillator is very similar the wave function Ψ_C of Cooper pair in superconductivity. This similarity might be not accidental and will be a subject for further investigation. We noted that uncertainty for q-deformed harmonic oscillator is less than the analogous value for standard harmonic oscillator for all n, except $n = 0$, which in contrast to standard quantum mechanics that uncertainty of harmonic oscillator is minimal value. The physical meaning of this also will be a subject for further investigation.

References

[1] V. V. Eremin, A. A. Meldianov 2008 *arXiv:0810.1967v1*.
[2] S. Abe, C. Beck, E. G. D. Cohen 2007 *Phys. Rev.* E **76** 031102.
[3] S. Abe 2009 *Phys. Rev.* E **79** 041116.
[4] A. Lavagno, P. P. Swamy 2010 *Physica.* A **389** 933.

[5] A. Algin, M. Senay 2012 *Phys. Rev.* E **85** 041123.
[6] F. M. Andrade, E. O. Silva 2013 *Phys. Lett.* B **719** 467 .
[7] V. M. Tkachuck 2013 *Phys. Rev.* A **86** 062112 .
[8] A. M Kadin 2007 *JSNM* **20** 285.