Study Chaotic Behavior of a 3 Body Systems: Simple Application to Earth-Sun-Moon like System

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Abstract. It is well known that three-body systems, in general will exhibit chaotic behavior. In this work, we study the case of the simple restricted planar three-body problem, and its application to the Earth-Sun-Moon like system. Earth-like and Moon-like have a small mass relative to the Sun-like object. The Moon-like is affected both centrifugal and Coriolis forces, and it would not be able to escape from the Hill disc. The trajectories of the Moon-like was viewed in a rotating frame which fixes the two more massive bodies Sun-like and Earth-like, so able reduced to a simple two-degrees of freedom system. It is possible to construct a more generalized model for investigation the chaotic behavior of tide forces on Earth also discussed.

1. Introduction
Changes in the sea level are very well known as tides but the investigation it accurately is not easy because the Earth-Sun-Moon like system is a three-body problem, in general will exhibit chaotic behavior. The motion of the Moon-like around the Earth-like can be considered in first approximation as a Keplerian motion perturbed by the action of the distant Sun. The perturbation here is more important than in the planetary case and the problem was the object of major works since Newton himself and then the others later. In this first approximation the mass of the Moon-like is supposed to be zero, Poincarï£¡, followed by Birkhoff , developed the so-called restricted problem, where the motion of the two massive planets, being unperturbed by the zero mass body, is Keplerian. In this work in the frame of restricted problem we investigate the chaotic behavior of Moon-like object. This will be useful in the further understanding and analyzing our collected modern moon tide data of Hon-dau (Batshaw in the Newton' book Principia) tide observation station, where Newton took as an example to develop his famous gravity law.

2. The model
We treat the Sun-Earth-Moon like system as a three-body restricted problem in plane. In this case when the motion is circular and the small mass body (Moon-like) like body (M) in the same plane of Sun-like (S) and Earth-like (E) bodies. The problem can be studied in a rotating frame which fixes the two massive bodies (Sun-like and Earth-like). Assume the Earth-like object is fixed in the center 0, Sun-like object is located in the 0y direction of the rotating coordinate system (x0y), and Moon-like object is moving around the Earth-like body (see Fig.1).
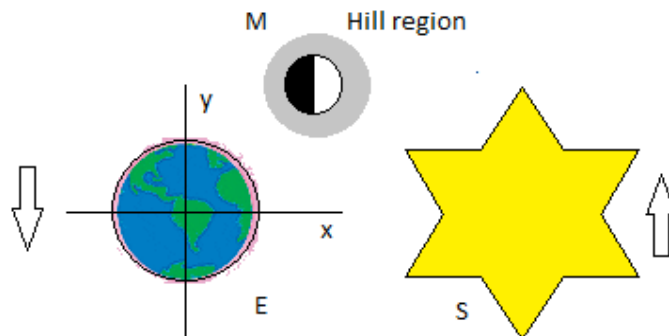
Figure 1. The Sun-Earth-Moon like system in a rotating frame.

In this rotating frame, the trajectory of the Moon-like object is defined by an autonomous Hamiltonian with two degrees of freedom, and the Jacobi constants. We have

$$
\begin{aligned}
x''(t) &= 2y'(t) + x(t) - \frac{\mu_1(\mu_2 + x(t))}{\left[(\mu_2 + x(t))^2 + y(t)^2\right]^{3/2}} \\
&\quad - \frac{\mu_2(-\mu_1 + x(t))}{\left[(-\mu_1 + x(t))^2 + y(t)^2\right]^{3/2}}, \\
y''(t) &= 2y(t) - 2x'(t) - \frac{\mu_1 y(t)}{\left[(\mu_2 + x(t))^2 + y(t)^2\right]^{3/2}} \\
&\quad - \frac{\mu_2 y(t)}{\left[(-\mu_1 + x(t))^2 + y(t)^2\right]^{3/2}}.
\end{aligned}
\tag{1}
$$

where μ_i are the reduced masses of the Earth-like and Moon-like with Sun-like object, while x and y are the coordinates in the rotating system, the prime sign means derivative on time $\partial/\partial t$. The small body is subjected to both centrifugal and Coriolis forces. The masses are in the unit of the Sun-like object M_S, and the coordinates are taken in the units of distance between the Sun-like and Earth-like objects $R = R_E - R_S$, and the time unit is the period (year) of Earth-like body circling around the Sun-like object. The initial conditions are:

$$
\begin{aligned}
x(t = 0) &= x_0, \\
y(t = 0) &= y_0.
\end{aligned}
\tag{2}
$$

for coordinates, and

$$
\begin{aligned}
x'(t = 0) &= \nu_{x0}, \\
y'(t = 0) &= \nu_{y0}.
\end{aligned}
\tag{3}
$$

for initial velocities, correspondingly.

Actually the mass of Moon-like object is very small, so it is supposed to be zero $\mu_2 = \mu_M = 0$. The mass of the Earth-like body is also small in comparing with the Sun-like object $\mu = \mu_1 = \mu_E \approx M_E/M_S \lll 1$. We plot the trajectories of the Moon-like object with several sets of the initial conditions and $\mu = 3.10^{-6}$ like in the Sun-Earth system in the Fig. 2. The chaos behavior of trajectories of Moon-like object can be seen quite clearly.

2.1. The Hill region

The projections on the configuration plane of the constant energy surfaces define regions of possible motions, called the Hill regions. It occurs when, the Jacobi constant being negative and big enough. For simplicity, we consider the Moon-like object as a zero body mass turning around the more massive body (the Earth-like), moves in a component of the Hill region which
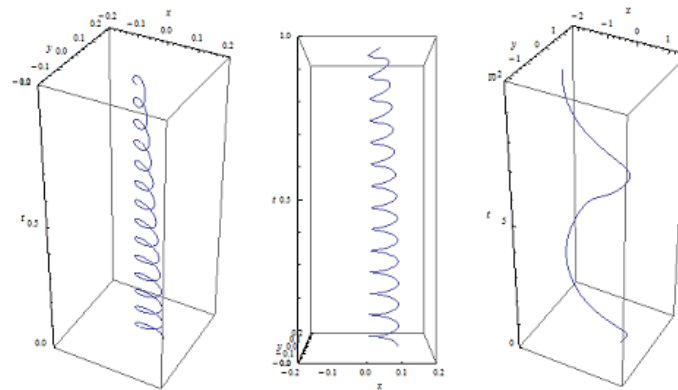
Figure 2. Trajectories of Moon-like object with chaos behavior.

is expressed by a disc with gray color in the Fig. 3. This fact already implies Hill's rigorous stability result: for all times such a Moon-like body would not be able to escape from this disc.
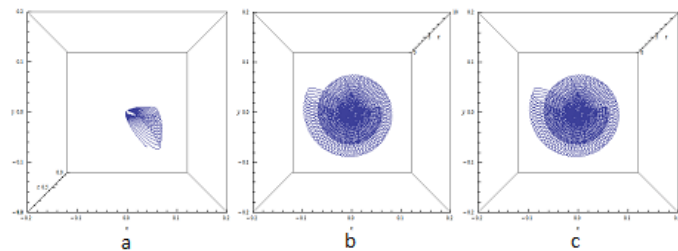

Figure 3. Hill regions of the Moon-like object.

2.2. Lagrange points
In some case, the Moon-like object appears to orbit around points other than the Earth-like and Sun-like objects. These positions are called "Lagrange points", are the equilibrium states in the co-rotating system. An example of this point with its chaotic behavior is shown in the

2.3. Tide and tide potential and force
Changes in the see level on the Earth are very well known as tides but the explanation is not easy as the simple statement that they are caused by gravity of the Moon. Beside that the Sun also gives an important contribution to the tides.
The magnitude of the potential and force attracting two bodies with masses m and M each other in the distance is given by the Newton's law of gravity with the gravitational constant G is given by $V(r) = GmM/r$ and $F(r) = GmM/r^2$, so the gravitational potential (in the unit mass) from the Moon-like body in the center O of the Earth-like object is

$$V_M(0) = \frac{GM_M}{r^2_M}, \qquad (4)$$

where r_M is the distance between Earth-like and Moon-like objects.
Assume the distances between Sun or Earth like objects and the Moon-like object are much large than the Moon-Earth-like separation , the Moon-tide force in the surface of Earth-like object in lowest order of approximation can be considered its value at its center
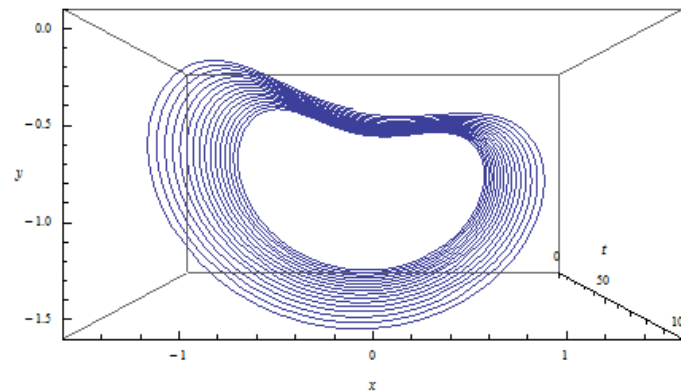
Figure 4. Example of Lagrange points.

$$F_M(0) = \frac{GM_M}{r_M^2}.$$

(5)

Because the distance between Moon and Earth like objects exhibit chaotic behavior so the values of tide potential and force also are chaos.

The Earth's axial tilt is 23.4o and the Moon's inclination to elliptic is 5.1o, but for simplicity we assume both are zero for the system of Earth-like and Moon-like, that the Earth-like's rotational axis is orthogonal to the orbital plane and the Moon-like orbits around the Earth-like above the Earth-like's equator, also assume that the hole Earth-like is covered with water.
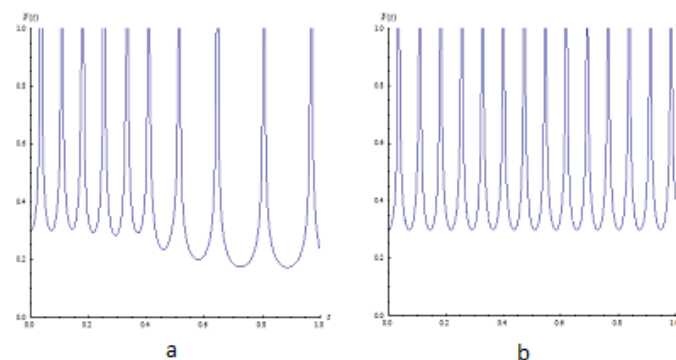

Figure 5. Chaotic behavior of the tide potential in the Earth like body from Moon and Sun like objects.

3. Discussions

In this work, chaotic behavior of a three body problem: the Earth-Sun-Moon like system was investigated in the frame work of restricted problem. Taken the mass ratios as those of the real Sun-Earth-Moon system we investigated the trajectories and chaotic properties of the Moon-like object in rotating frame. We show that because the distance between Moon and Earth like objects exhibit chaotic behavior so the values of tide potential and force in the Earth like body also are chaos.

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