Applications of scale invariant scattering formalism to critical systems

quenched disorder, local symmetry, coupled symmetries

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Based on:

1. Delfino, G., and **Lamsen, N.**, 2018. J. High Energy Phys. 77. https://doi.org/10.1007/JHEP04(2018)077
2. Delfino, G., and **Lamsen, N.**, 2019. J. Stat. Mech. 024001. https://doi.org/10.1088/1742-5468/aaf716
3. Delfino, G., and **Lamsen, N.**, 2019. J. Phys. A: Math. Theory 52 35LT02. https://doi.org/10.1088/1751-8121/ab3055
4. Delfino, G., and **Lamsen, N.**, 2019. Eur. Phys. J. B 92 278. https://doi.org/10.1140/epjb/e2019-100451-6
5. Delfino, G., Diouane, Y., **Lamsen, N.**, 2020. J. Phys. A: Math. Theory 54 03LT01. https://doi.org/10.1088/1751-8121/abd2fc
6. Diouane, Y., **Lamsen, N.**, and Delfino, G., 2021. J. Stat. Mech. 033214. https://doi.org/10.1088/1742-5468/abe6fc
7. Diouane, Y., **Lamsen, N.**, and Delfino, G., 2022. J. Stat. Mech. 023201. https://doi.org/10.1088/1742-5468/ac4983
8. **Lamsen, N.**, Diouane, Y., Delfino, G., 2023. J. Stat. Mech. 013203. https://doi.org/10.1088/1742-5468/aca901
9. Diouane, Y., **Lamsen, N.**, Delfino, G., 2023. J. Stat. Mech. 043204. https://doi.org/10.1088/1742-5468/acc8c9

Outline

Introduction
- Motivation
- Scale invariant scattering

Applications
- $O(N)$ vector model
- Quenched disordered $O(N)$ and Potts model
- Local symmetry: RP^{N-1} and CP^{N-1}
- $O(N) \times Z_2$ and $S_q \times S_r$

Summary

Motivation

- Critical phenomena
  - Universality
  - Renormalization group (RG) fixed points (FP's)
- Symmetry and dimensionality
  - Scale Invariance $\rightarrow$ Conformal invariance
  - Infinitely many generators in two dimensions

Lack of exact results for several relevant problems

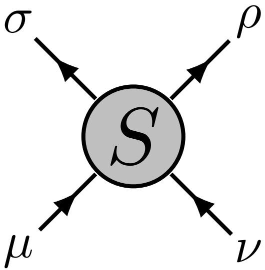
- Composite symmetry – Fully frustrated XY (FFXY) model
- Liquid crystals
- Quenched disorder with short ranged interactions

Scale invariant scattering (Delfino '13)

Main Idea

Conformal invariance applied on particles instead of fields

- Massless QFT in $d = (1+1)$
- Elastic scattering
- No momentum dependence

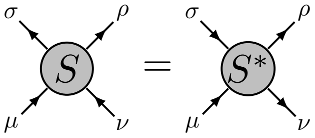


S-matrix, $S^{\rho,\sigma}_{\mu,\nu}$

Conditions for scattering amplitudes:

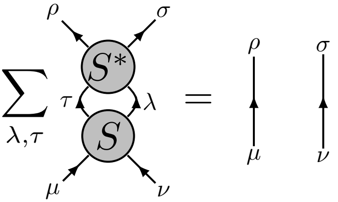
- Crossing symmetry

$$S^{\rho,\sigma}_{\mu,\nu} = \left[S^{\rho,\nu}_{\mu,\sigma}\right]^*$$



- Unitarity

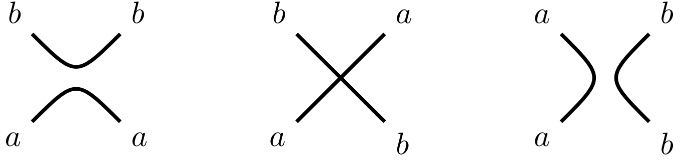
$$\sum_{\lambda,\tau} S^{\lambda,\tau}_{\mu,\nu} \left[S^{\rho,\sigma}_{\lambda,\tau}\right]^* = \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}$$



The same set of conditions applied to different scenarios.

$O(N)$-vector model

- Particles: $O(N)$-vector multiplet $a = 1, \ldots, N$
- Scattering amplitudes:

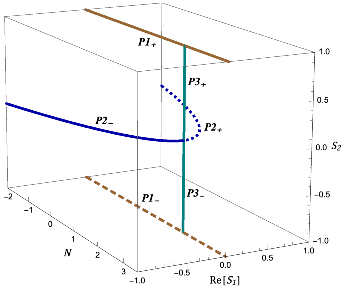


$S_1 = \rho_1 e^{i\phi}$ $\quad S_2 = \rho_2$ $\quad S_3 = \rho_1 e^{-i\phi}$

- Unitarity:

$$
\begin{aligned}
1 &= \rho_1^2 + \rho_2^2 \\
0 &= \rho_1 \rho_2 \cos\phi \\
0 &= N\rho_1^2 + 2\rho_1^2 \cos 2\phi
\end{aligned}
$$

Solution	N	ρ_1	ρ_2	$\cos\phi$
$P1_\pm$	$(-\infty,\infty)$	0	± 1	-
$P2_\pm$	$[-2,2]$	1	0	$\pm\frac{1}{2}\sqrt{2-N}$
$P3_\pm$	2	$[0,1]$	$\pm\sqrt{1-\rho_1^2}$	0



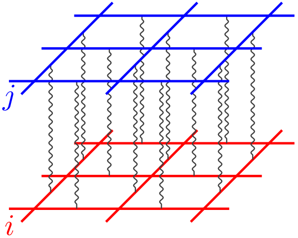
- $P1$ – Free, $T = 0$
- $P2$ – Dilute and dense loops
- $P3$ – Berezinskii-Kosterlitz-Thouless (BKT)

All FP's from a single set of equations

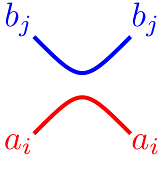
Disordered $O(N)$ model (Delfino, Lamsen '18-'19)

Replica trick

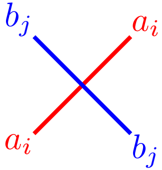
$$\overline{\ln Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}$$



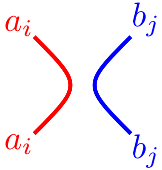
- Particles with replica index: a_i, $i = 1, \ldots, n$



$S_4 = \rho_4 e^{i\theta}$



$S_5 = \rho_5$



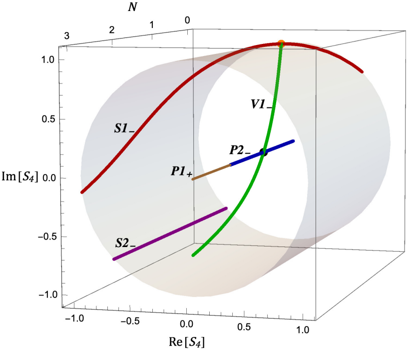
$S_6 = \rho_4 e^{-i\theta}$

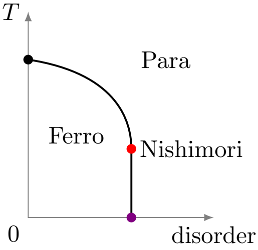
- Unitarity

$$1 = \rho_1^2 + \rho_2^2\,,$$
$$0 = \rho_1\rho_2\cos\phi\,,$$
$$0 = N\rho_1^2 + N(n-1)\rho_4^2 + 2\rho_1^2\cos 2\phi\,,$$
$$1 = \rho_4^2 + \rho_5^2\,,$$
$$0 = \rho_4\rho_5\cos\theta\,,$$
$$0 = 2N\rho_1\rho_4\cos(\phi-\theta) + N(n-2)\rho_4^2 + 2\rho_2\rho_4\cos\theta + 2\rho_1\rho_4\cos(\phi+\theta)\,.$$

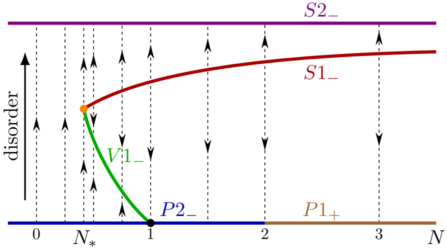
- Disorder strength $\sim \rho_4$:

$$\rho_4 \to 0 \Longrightarrow \text{Pure case}$$





random $\pm J$ Ising



$\Leftarrow$ Schematic RG flow

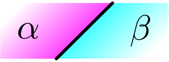
Nishimori-like end point:

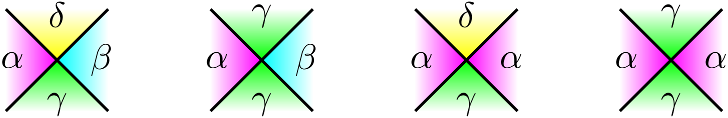
$$N_* = \sqrt{2} - 1 = 0.414\ldots$$

$$N_*^{\text{num.}} \approx 0.5$$ (Shimada et al. '14)

Potts model

- Permutation symmetry, $\mathbb{S}_q$
- Particles $A_{\alpha,\beta}$, $\alpha \neq \beta = 1, \ldots, q$





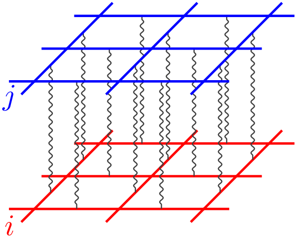
$S_0 = \rho_0$ $\qquad S_1 = \rho e^{i\varphi}$ $\qquad S_2 = \rho e^{-i\varphi}$ $\qquad S_3 = \rho_3$

Solution	Range	ρ_0	ρ	$2\cos\varphi$	ρ_3
I	$q = 3$	$0,\ 2\cos\varphi$	1	$\in [-2, 2]$	0
$\text{II}_\pm$	$q \in [-1, 3]$	0	1	$\pm\sqrt{3-q}$	$\pm\sqrt{3-q}$
$\text{III}_\pm$	$q \in [0, 4]$	± 1	$\sqrt{4-q}$	$\pm\sqrt{4-q}$	$\pm(3-q)$
$\text{IV}_\pm$	$q \in [\frac{7-\sqrt{17}}{2}, 3]$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$	$\sqrt{\frac{q-4}{q^2-5q+5}}$	$\pm\sqrt{(3-q)(4-q)}$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$
$\text{V}_\pm$	$q \in [4, \frac{7+\sqrt{17}}{2}]$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$	$\sqrt{\frac{q-4}{q^2-5q+5}}$	$\mp\sqrt{(3-q)(4-q)}$	$\pm\sqrt{\frac{q-3}{q^2-5q+5}}$

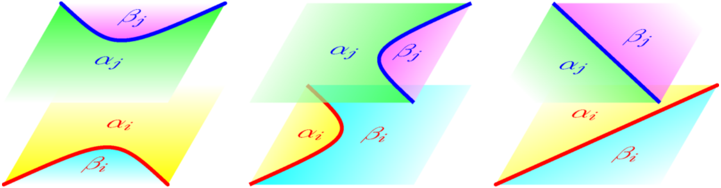
(Delfino & Tartaglia '17)

Replica trick

$$\overline{\ln Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}$$



- Particles with replica index A_{α_i,β_i}, $i = 1, \ldots, n$



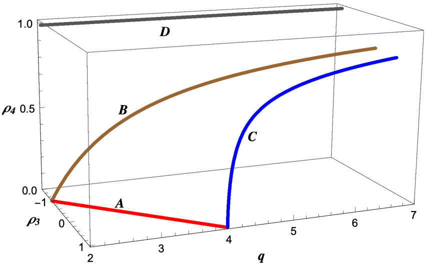
$S_4 = \rho_4 e^{i\theta}$ $\quad$ $S_5 = \rho_4 e^{-i\theta}$ $\quad$ $S_6 = \rho_6$

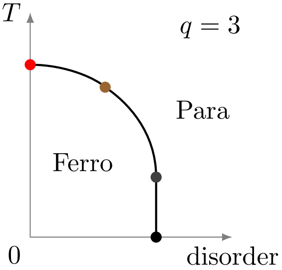
(Delfino '17)

- Disorder strength $\sim \rho_4$

Disordered Potts RG flow (Delfino, Lamsen '19)

- Disorder strength $\sim \rho_4$





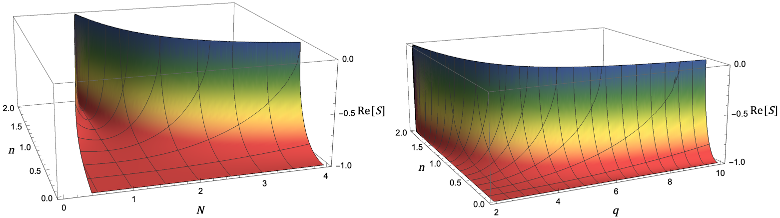
(A) Pure ferromag. (B) Stable "random" crit. pt.
(C) Unstable line (D) Nishimori-like & $T = 0$

- (B) extends for $q > 2$ (Ludwig '90, Dotsenko et al. '95, ...) up to $q \to \infty$
- (C) provides RG flow source towards (B) for $q > 4$

Superuniversality

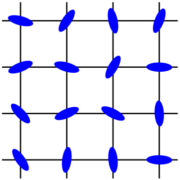
- Theory sectors independent of symmetry parameters N or q
- Arises exactly at $n \to 0$ replica limit (first analytic evidence)

$$S = \begin{cases} S_3 + (q-2)S_2 + (n-1)(q-1)S_4\,, & q\text{-Potts} \\ NS_1 + S_2 + S_3 + (n-1)NS_4\,, & O(N) \end{cases}$$



- Mechanism for superuniversal crit. exponents, explains puzzling findings since '90s (Chen et al. '93 & '95, Wisemann & Domany '95, ...)

Liquid crystals (Delfino, Diouane, Lamsen '20-'23)



Lebwohl-Lasher model

- Nearest neighbor interaction: $(\mathbf{s}_i \cdot \mathbf{s}_j)^2$
- Nematic-isotropic topological transition?

- $O(N)$ + "head-tail" symmetry $\Longrightarrow RP^{N-1}$ model
- Order parameter: $Q_i^{a,b} = s_i^a s_i^b - \frac{1}{N}\delta_{a,b}$ (de Gennes '70s)
- Scattering of particles with two indices

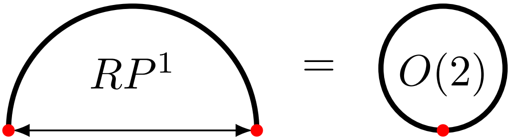
$S_1 = \rho_1 e^{i\phi}$ $S_2 = \rho_2$ $S_3 = \rho_1 e^{-i\phi}$ $S_4 = \rho_4 e^{i\theta}$ $S_5 = \rho_5$ $S_6 = \rho_4 e^{-i\theta}$

$S_7 = \rho_7$ $S_8 = \rho_8 e^{i\psi}$ $S_9 = \rho_9$ $S_{10} = \rho_{10}$ $S_{11} = \rho_8 e^{-i\psi}$

- Traceless sector → reduces to amplitudes $S_{1\leq k\leq 6}$ only
- Unitarity ($M_N = \frac{1}{2}N(N+1) - 1$)

$$
\begin{aligned}
1 &= \rho_1^2 + \rho_2^2 + 4\rho_4^2\,,\\
0 &= 2\rho_1\rho_2\cos\phi + 4\rho_4^2\,,\\
0 &= M_N\rho_1^2 + 2\rho_1^2\cos 2\phi + 2\rho_1\rho_2\cos\phi + 4\left(1 - \frac{2}{N} + N\right)\rho_1\rho_4\cos(\phi-\theta)\\
&\quad + 4\left(1 - \frac{2}{N}\right)\rho_1\rho_4\cos(\phi+\theta) + \frac{32}{N^2}\rho_4^2\cos 2\theta + 4\left(1 - \frac{2}{N} + N\right)\rho_1\rho_5\cos\phi\\
&\quad + 8\left(1 + \frac{8}{N^2}\right)\rho_4\rho_5\cos\theta + 4\left(1 + \frac{8}{N^2}\right)\rho_4^2 + 4\left(1 + \frac{4}{N^2}\right)\rho_5^2\,,\\
0 &= 2\rho_2\rho_5 + 2\rho_1\rho_4\cos(\phi+\theta) - \frac{8}{N}\rho_4^2 + 2\left(1 - \frac{4}{N}\right)\rho_4^2\cos 2\theta\\
&\quad + 2\left(3 - \frac{8}{N} + N\right)\rho_4\rho_5\cos\theta - \frac{4}{N}\rho_5^2\,,\\
0 &= 2\rho_2\rho_4\cos\theta + \left(2 - \frac{8}{N} + N\right)\rho_4^2 + 2\left(1 - \frac{4}{N}\right)\rho_4^2\cos 2\theta + 2\rho_1\rho_5\cos\phi\\
&\quad + 2\left(1 - \frac{8}{N}\right)\rho_4\rho_5\cos\theta + \left(2 - \frac{4}{N} + N\right)\rho_5^2\,,\\
0 &= 2\rho_1\rho_4\cos(\phi-\theta) + 2\rho_2\rho_4\cos\theta + 2\rho_4^2\,.
\end{aligned}
$$

- Contains $O(M_N)$ FP's as a particular case
- Line of FP's at $N = 2$: $M_2 = 2 \Longrightarrow O(2) \sim RP^1$



- No line of FP's for $N > 2$: No quasi-long-range ordering (QLRO)
- Correlation length suppression as $T \to 0$ for $N > 2$

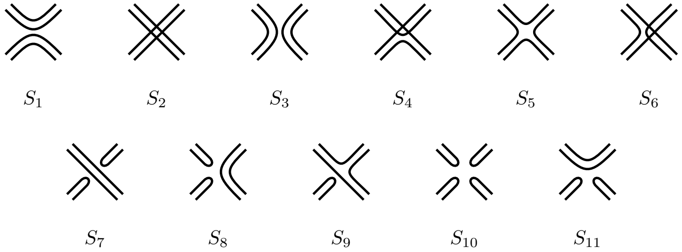
$$\xi_M \propto T^{1/(M-2)} e^{A/[(M-2)T]} \Longrightarrow \xi_{M_N} \ll \xi_N$$

explains puzzling results of simulations (Sinclair '82, Carraciolo et al. '93, ...)

CP^{N-1} model (Diouane, Lamsen, Delfino '22-'23)

Consider continuous local $U(1)$ symmetry (c.f. Z_2)

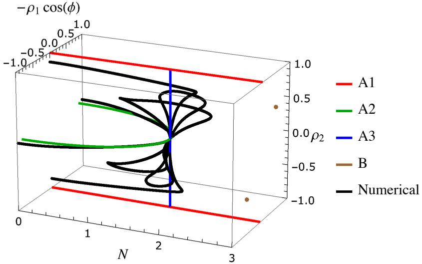
- $U(N)$ global + $U(1)$ local symmetry $\Longrightarrow CP^{N-1}$ model
- Order parameter: $Q_i^{a,b} = z_i^a (z_i^b)^* - \frac{1}{N}\delta_{a,b}$
- Scattering of particles matching appropriate indices

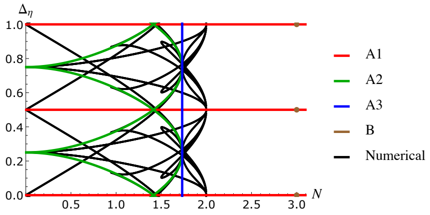


- Traceless sector $\to$ reduces to amplitudes $S_{1\leq k\leq 6}$ only
- Unitarity ($M_N = N^2 - 1$)

$$
\begin{aligned}
1 &= \rho_1^2 + \rho_2^2 + 2\rho_4^2 \\
0 &= 2\rho_1\rho_2\cos\phi + 2\rho_4^2 \\
0 &= M_N\rho_1^2 + 2\rho_1^2\cos 2\phi + 2\rho_1\rho_2\cos\phi + 4\left(N - \tfrac{1}{N}\right)\rho_1\left(\rho_4\cos(\theta-\phi) + \rho_5\cos\phi\right) \\
&\quad - \tfrac{4}{N}\rho_1\rho_4\cos(\theta+\phi) + \tfrac{8}{N^2}\rho_4^2\cos 2\theta + 2\left(1 + \tfrac{4}{N^2}\right)\rho_4\left(\rho_4 + 2\rho_5\cos\theta\right) \\
&\quad + 2\left(1 + \tfrac{2}{N^2}\right)\rho_5^2 \\
0 &= 2\rho_1\rho_5\cos\phi + 2\rho_2\rho_4\cos\theta - \tfrac{4}{N}\rho_4^2\cos 2\theta + \left(N - \tfrac{4}{N}\right)\rho_4^2 - \tfrac{8}{N}\rho_4\rho_5\cos\theta \\
&\quad + \left(N - \tfrac{2}{N}\right)\rho_5^2 \\
0 &= 2\rho_1\rho_4\cos(\theta+\phi) + 2\rho_2\rho_5 - \tfrac{4}{N}\rho_4^2\cos 2\theta - \tfrac{4}{N}\rho_4^2 + 2\left(N - \tfrac{4}{N}\right)\rho_4\rho_5\cos\theta - \tfrac{2}{N}\rho_5^2 \\
0 &= 2\rho_1\rho_4\cos(\theta-\phi) + 2\rho_2\rho_4\cos(\theta)\,.
\end{aligned}
$$

- Contains $O(M_N)$ FP's as a particular case
- Line of FP's at $N = \sqrt{3}$: $M_{\sqrt{3}} = 2$
- No line of FP's for $N > 2$: No quasi-long-range ordering (QLRO)
- Branches of FP's for $N < 2 \longrightarrow$ loop gas w/ intersections

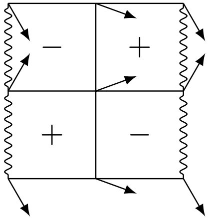




Vector-Ising model (Delfino, Lamsen '19)

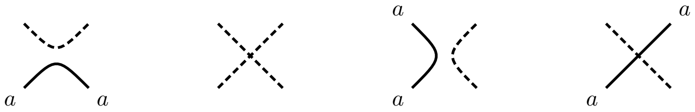
Fully frustrated XY model

- Josephson junction array in magnetic field (Teitel & Jayaprakash '83)
- Non-perturbative (BKT)

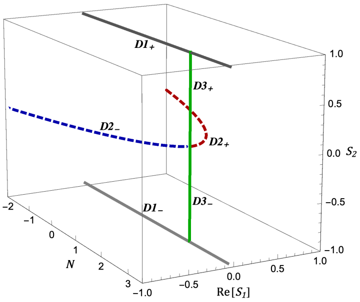


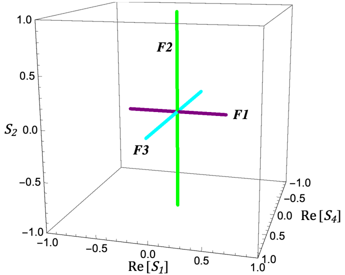
- Vector-Ising coupling, B,

$$\mathcal{H} = -\sum_{\langle i,j\rangle} \left[(A + B\sigma_i\sigma_j)\mathbf{s}_i \cdot \mathbf{s}_j + C\sigma_i\sigma_j \right],$$



$S_4 = \rho_4 e^{i\theta}$ $S_5 = \rho_5$ $S_6 = \rho_4 e^{-i\theta}$ $S_7 = \rho_7$





Solution set contains decoupled case

- $N = 2$: Decoupled solutions only

No new universality class at $N = 2$

including FFXY model

Lines of FP's at $N = 1$

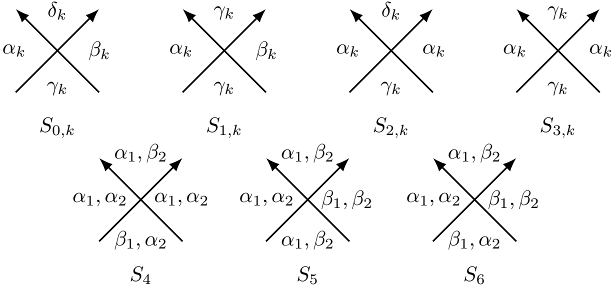
first exact determination

Correlated percolation (Lamsen, Diouane, Delfino '23)

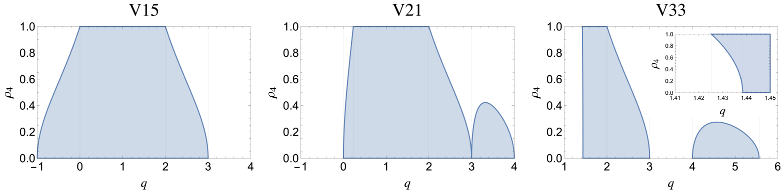
- Coupled q-state Potts with r-state Potts

$$\mathcal{H}_{q,r} = -J_1 \sum_{\langle i,j \rangle} \delta_{s_{i,1},s_{j,1}} - J_2 \sum_{\langle i,j \rangle} \delta_{s_{i,2},s_{j,2}} - J \sum_{\langle i,j \rangle} \delta_{s_{i,1}s_{j,1}} \delta_{s_{i,2},s_{j,2}}$$

- Correlated percolation at $r \to 1$
- Scattering amplitudes



- For integer $q, r > 1$, only $q = r = 2$ coupled criticality is possible – Ashkin-Teller criticality
- At $r \to 1$, no critical line continuously defined in $q \in [2, 4]$ – Potts spin clusters as analytic continuation of FK clusters cannot hold for all cluster properties
- Appearance of surfaces of FP at $q = r$



corresponding to results from coupled square lattice antiferromagnets (Fendley & Jacobsen '08; Vernier, Jacobsen, Saleur '14)

Summary

- RG FP's of two-dimensional statistical models obtained exactly from scale invariant scattering.

- Non-perturbative insights on critical behavior of various models in two dimensions.

    - Phase diagram
    - RG flows
    - Behavior of critical exponents

–End–

Thank you

Based on:

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