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Lectures on Susy Phenomenology

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Abstract: I provide a pedagogical introduction to supersymmetry. The level is aimed at readers who have heard of the Standard Model and Quantum Field Theory, but who had no prior exposure to supersymmetry. Topics covered are: motivations for supersymmetry, some conventions for spinor notations, construction of supersymmetric Lagrangians including gauge interactions and soft supersymmetry breaking, the Minimal Supersymmetric Standard Model and its mass spectrum.

Contents

1 Introduction and Motivation

The first question any intelligent student would ask at the beginning of a series of lectures on supersymmetry (susy) is why speculate at all about going *beyond* the Standard Model (SM), unless (or until) an experiment forces us to? Yet, in the lectures on the SM by G. Bhattacharyya and M. Boonekamp, you were told that since its discovery, the SM has had tremendous experimental successes, like no other theory before. So if the SM is not broken, why try to fix it? In other words, why spend so much time† learning all this intricate susy formalism? A true theorist would certainly answer: "Why not? Just for the fun of it!". As a phenomenologist (who by definition is interested in phenomena), I would rather say "because the SM is not that perfect after all, in spite of what G. Bhattacharyya told you!". Indeed, although the SM (suitably extended to include neutrino masses) works remarkably well below the TeV scale, it has a weak point: the Higgs sector. In addition of playing hide-and-seek with experimentalists, theorists soon discovered that this sector was suffering from an extreme sensitivity to physics at energies above the TeV scale (the so-called "hierarchy problem"), questioning the validity of the SM above this scale. For this reason the SM is now believed to be only an effective theory at low energy and it is commonly agreed that some new physics should appear at or above the TeV scale. This is precisely the scale that the LHC will probe, so we better get ready to explain new phenomena that will undoubtedly appear as soon as it starts colliding protons (cf. lectures by M. Boonekamp on this subject)! As for today, susy is certainly the most studied candidate to extend the SM up to extremely high energy scales, such as the Planck scale $M_{\text{Planck}} = G_{\text{Newton}}^{-1/2} \simeq 1.2 \times 10^{19}$ GeV (where we know that gravity comes into play which cannot be accounted for by quantum field gauge theories). The main reason for this is that susy technically solves the hierarchy problem. There are other purely theoretical motivations for susy as well as quantitative indications on which we will come back in due time, but let us start by the hierarchy problem.

1.1 The Hierarchy Problem

Recall the potential for the neutral component of the SM Higgs SU(2) doublet:

$$V(H) = \mu^2 |H|^2 + \lambda |H|^4 \; . \tag{1.1}$$

If $\mu^2 < 0$ the electroweak symmetry is spontaneously broken and the Higgs field develops a vacuum expectation value (vev):

$$\langle H \rangle = \sqrt{\frac{-\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}} \; , \tag{1.2}$$

which in turn gives a mass to the Higgs field as well as to the weak gauge bosons (through gauge couplings g, g') and charged fermions (through Yukawa couplings λ_f):

$$m_H^2 = 2\lambda v^2 \; , \qquad m_W^2 = \frac{1}{4} g^2 v^2 \; , \qquad m_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2 \; , \qquad m_f^2 = \frac{1}{2}\lambda_f^2 v^2 \; . \tag{1.3}$$

† We will actually only spend 6 hours together, which is a very short time to learn susy phenomenology. The curious student will easily find very good reviews on the subject [1–4]. A few books have also recently been published [5–9]. Refs. [1, 5] are especially well suited for the beginners. All the necessary references on susy (shamelessly not cited in these lectures) will be found therein.

We know from experimental measurements of the weak interaction properties that $v \simeq$ 246 GeV (often called weak scale, $M_{\rm weak}$). If λ is to remain perturbative ($\lesssim 1$) up to a high energy cut-off Λ, this implies that $m_H \lesssim 1$ TeV depending on Λ. The same bound on the SM Higgs mass is obtained if one wants the $WW \to WW$ diffusion process to remain unitary at high energy.

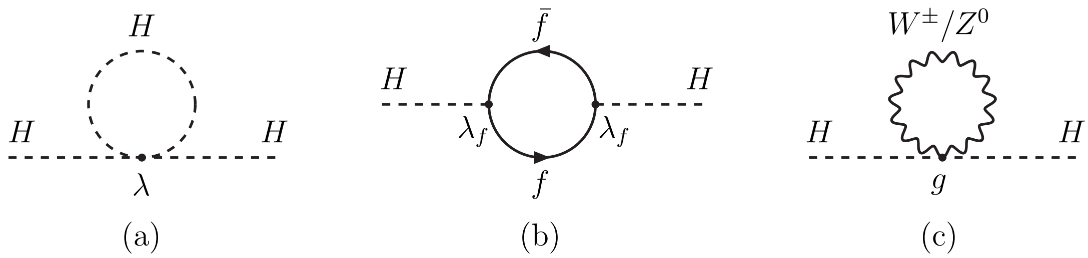


Figure 1.1: One loop quantum corrections to the Higgs squared mass parameter m_H^2, due to (a) the Higgs self interaction, (b) a Dirac fermion f and (c) a weak gauge boson.

However, the mass term for the Higgs boson is subject to potentially huge radiative corrections. Taking into account the one loop corrections due to the Higgs self interaction of fig. 1.1(a), one gets:

$$\delta m_H^2 = \lambda \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_H^2} = \frac{\lambda}{16\pi^2}\left(\Lambda^2 - 2m_H^2 \ln\left(\frac{\Lambda}{m_H}\right) + \ldots\right) . \tag{1.4}$$

If we take as our high energy cut-off $\Lambda = M_{\rm Planck}$, then $m_H \lesssim 1$ TeV requires to start with an equally huge value of the parameter μ^2 appearing in the potential (1.1), relying on a remarkable cancellation – or *fine-tuning* – between the tree level mass (1.3) and the quadratically divergent one loop corrections (1.4). Furthermore, these cancelations have to be repeated at each order in perturbations. Although technically possible, this is regarded as highly unnatural and is called the *hierarchy problem*: the presence of a fundamental scalar in the SM tends to destabilise the hierarchy between the scale $M_{\rm weak}$ and a high energy cut-off.

The attentive reader might object that this is an artefact of the regularization scheme used to renormalise m_H^2. Indeed, using dimensional regularization instead of a Pauli-Villars cut-off, we find no quadratic divergences from the diagram of fig. 1.1(a)! However, let suppose we have some new physics – say a grand unified theory – at a high energy scale $M_{\rm GUT} \sim 10^{16}$ GeV, containing massive states $\sim M_{\rm GUT}$. Then, these states will circulate in the loops of similar diagrams as in fig. 1.1 and the radiative corrections to the Higgs mass will diverge quadratically as a function of $M_{\rm GUT}$.

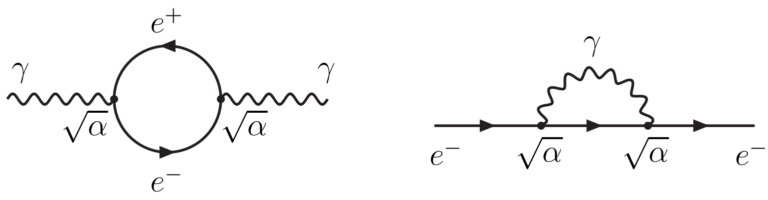
Why such divergences only seem to affect the scalar sector of the SM? For more simplicity, let us consider Quantum Electrodynamics (QED). The photon self-energy diagram of fig. 1.2 is apparently quadratically divergent. As in the scalar case, such a quadratic divergence would imply an enormous quantum correction to the photon mass (which has to remain zero):

$$\delta m_\gamma^2 \sim \alpha \int^\Lambda \frac{d^4k}{k^2} \sim \alpha\Lambda^2 . \tag{1.5}$$

But in fact this divergence is absent provided the theory is regularized in a gauge invariant way. In other words, gauge symmetry guarantees that no term of the form:

$$m_\gamma^2 A_\mu A^\mu \tag{1.6}$$

can be radiatively generated in an unbroken gauge theory: the photon remains massless. The diagram of fig. 1.2(a) is divergent, but only logarithmically. The divergence is absorbed in a field strength renormalization constant, and is ultimately associated with the running of the fine structure constant α. The same mechanism happens in the SM, although things are a bit more complicated (as the gauge symmetry of the SM is non-abelian and spontaneously broken).



(a) (b)

Figure 1.2: One loop self-energies in QED for (a) the photon, (b) the electron.

We may also consider the electron self energy diagram of fig. 1.2(b). This produces a correction to the electron mass which seems to vary linearly with the cut-off:

$$\delta m \sim \alpha \int^{\Lambda} \frac{\mathrm{d}^4 k}{\not{k} k^2} \sim \alpha\Lambda \ . \tag{1.7}$$

Though perhaps not so bad as a quadratic divergence, such a linear one would still lead to unacceptable fine-tuning in order to arrive at the physical electron mass. In fact, however, when the calculation is done in detail one finds:

$$\delta m \sim \alpha m \ln \Lambda \ , \tag{1.8}$$

so that even if $\Lambda \sim 10^{19}$ GeV, we have $\delta m \sim m$ and no unpleasant fine-tuning is necessary after all.

Why does it happen in this case that $\delta m \sim m$? It is because QED (and the SM for that matter) has an extra symmetry as the electron mass goes to zero, namely chiral symmetry. This is the symmetry under transformations on fermion fields of the form:

$$\psi \to \mathrm{e}^{i\alpha\gamma_5}\psi \tag{1.9}$$

in the U(1) case, or

$$\psi \to \mathrm{e}^{i\alpha_a T^a \gamma_5}\psi \tag{1.10}$$

in the general SU(N) case. This symmetry guarantees that all radiative corrections to m, computed in perturbation theory, will vanish as $m \to 0$. Hence δm must be proportional to m, and the dependence on Λ is therefore (from dimensional analysis) only logarithmic.

In these two examples from QED, we have seen how unbroken gauge and chiral symmetries protect gauge boson and fermion masses from potentially dangerous quadratic and linear divergences. If we could find a symmetry which grouped scalar particles with either fermions or gauge bosons, then the fundamental scalars would enjoy the same protection from unwanted divergences. susy is precisely such a symmetry: it groups together particles of different spins in supermultiplets.

To see how susy works, let us come back to the one loop corrections to the Higgs mass. Up to now we have only computed the contribution from the scalar loop of fig. 1.1(a) (Higgs self interaction). Let us now compute the fermion loop contribution of fig. 1.1(b):

$$\delta m_H^2 = -\lambda_f^2 \int^\Lambda \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{k^2 - m_f^2}\right] = \frac{\lambda_f^2}{16\pi^2}\left(-2\Lambda^2 + 6m_f^2 \ln\left(\frac{\Lambda}{m_f}\right) + \dots\right) . \quad (1.11)$$

Notice the sign difference compared with (1.4) due to fermion/boson different statistics. Hence, provided one has:

1) a relation between the fermionic and bosonic couplings $\lambda = \lambda_f^2$
2) an equal number of fermionic and bosonic degrees of freedom (1.12)
(a Dirac fermion has 4 degrees of freedom while a complex scalar has only 2)

the quadratic divergences cancel and we are left with:

$$\delta m_H^2 \sim \Delta m^2 \ln \Lambda , \quad (1.13)$$

where Δm^2 is the mass difference between fermions and scalars. Of course the two conditions (1.12) are not satisfied in the SM where we have just one fundamental scalar field (the Higgs boson), plenty of Dirac fermions (quarks and leptons) and no link between the Higgs self coupling and the fermion Yukawa couplings. We will see how the Minimal Susy Standard Model (MSSM) guarantees these two conditions by doubling the number of degrees of freedom of the SM, associating spin 1/2 Higgsinos and gauginos to the Higgs and gauge bosons and spin 0 squarks and sleptons to the quarks and leptons.

1.2 Quantitative Indications

Here we state briefly four quantitative results of the MSSM, which together with the technical solution to the hierarchy problem and the strong theoretical argument developed in the next section, have inclined many physicists to take the model seriously. We shall explore each in more detail in chapters 5 and 6.

- The precision fits to electroweak data show that m_H is less than about 200 GeV, at the 99% confidence level. The MSSM, which has three neutral Higgs states as we shall see, predicts that the lightest one should be no heavier than about 140 GeV. In the SM, one can certainly say, from (1.3), that if λ is not much greater than 1, so that perturbation theory has a hope of being applicable, then m_H can't be much greater than 1 TeV. This is not such a strong constraint however, at least not in quite the same sense of a mathematical bound.

- In any renormalizable theory, the parameters in the Lagrangian depend on the energy scale Q (they "run"). At one loop order, the inverse gauge couplings $\alpha_1^{-1}(Q)$, $\alpha_2^{-1}(Q)$, $\alpha_3^{-1}(Q)$ of the SM run linearly with $\ln Q$. Although α_1^{-1} decreases with $\ln Q$, and α_2^{-1} and α_3^{-1} increase, all three tending to meet at $M_{\mathrm{GUT}} \sim 10^{16}$ GeV, they do not in fact meet convincingly in the SM. On the other hand, in the MSSM they do, thus encouraging ideas of a Grand Unified Theory.

- The mass parameters also run, just as the gauge coupling parameters do. In the MSSM, the evolution of the Higgs squared mass parameter from a typical positive value at the scale $M_{\rm GUT}$, takes it to a negative value of the correct order of magnitude at the scale $M_{\rm weak}$, thus providing a possible explanation for the origin of electroweak symmetry breaking (EWSB). Actually, this happens because the Yukawa coupling of the top quark is large (being proportional to its mass), and this has a dominant effect on the evolution of the Higgs squared mass parameter.

- Assuming an extra symmetry, called R-parity, the MSSM also provides a candidate for Cold Dark Matter (see lectures by P. Gondolo and Sang Pyo Kim): the lightest susy particle (LSP), which is stable with a mass ~ 100 GeV and weak couplings to SM particles. The latest WMAP results on Dark Matter relic density actually put strong constraints on the MSSM parameters.

1.3 Theoretical Considerations

If one wants the first condition of (1.12) to hold at any order in perturbation theory, it must result from some symmetry. Such a symmetry linking bosons and fermions, as we already mentioned, is precisely what we call susy. But what are the normal (i.e. non-super) symmetries of nature? They are of two kinds:

- External symmetries (Poincaré group). They transform a point in spacetime as:

$$x'^\mu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu)x^\nu + a^\mu \ , \tag{1.14}$$

  where a^μ is the (constant 4-vector) parameter of a translation and ω^μ_ν the (constant antisymmetric tensor) parameter of an infinitesimal Lorentz transformation[‡]. They correspond to unitary transformations acting on the quantum fields:

$$U(a) = e^{ia_\mu P^\mu} \ , \qquad U(\omega) = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \ , \tag{1.15}$$

  where P^μ and $M^{\mu\nu}$ are the (hermitian) generators of translations and Lorentz transformations (resp.). We will come back to their algebra (ie their commutation relations) and representations in the next chapter.

- Internal symmetries (gauge groups). They act directly on the fields through unitary transformations:

$$U(\alpha) = e^{i\alpha_a(x)T^a} \ . \tag{1.16}$$

  The parameters $\alpha_a(x)$ of the transformation depend on the spacetime coordinate. The T^a's are the generators and they obey the commutation relations:

$$[T^a, T^b] = if_{abc}T^c \ , \tag{1.17}$$

  where f_{abc} are called the structure constants of the corresponding algebra.

[‡] We work in four dimension Minkowski spacetime with a flat metric $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

The Coleman-Mandula theorem states that a Lie algebra containing the Poincaré generators plus another Lie algebra must be the direct sum of the two. This implies that

$$[T^a, P_\mu] = [T^a, M_{\mu\nu}] = 0 \ , \tag{1.18}$$

i.e. any new symmetry can only connect particles with the same mass and spin: this is not what we want! Haag, Lopuszanski and Sohnius showed however that the Coleman-Mandula theorem can be evaded if the generators of the new symmetry satisfy *anticommutation* relations (this is called a graded algebra).

A susy transformation turns bosons into fermions and vice-versa:

$$Q|\text{boson}\rangle = |\text{fermion}\rangle \ , \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle \ . \tag{1.19}$$

The generator Q of a susy transformation must be of fermionic nature (i.e. an anticommuting spinor) and its hermitian conjugate $Q^\dagger$ (also noted $\bar{Q}$) is a distinct generator. It can therefore have non-trivial commutation relations with the Poincaré generators. For this reason, susy is regarded as the only non-trivial extension of the Poincaré group, i.e. the most general symmetry of spacetime.

Since this is a symmetry, it must commute with the Hamiltonian of the system and so does the anticommutator of two different components:

$$[Q, H] = [\{Q, \bar{Q}\}, H] = 0 \ . \tag{1.20}$$

We can guess that $\{Q, \bar{Q}\}$ transforms as a spin 1 object (because it is the combination of two spin 1/2 objects) which should be described by a 4-vector. Further, according to (1.20) this 4-vector is conserved. The Coleman-Mandula theorem implies that there is only one such operator, namely P_μ:

$$\{Q, \bar{Q}\} \sim P_\mu \ . \tag{1.21}$$

Clearly, this is sloppy (the Lorentz indices don't match!) but it captures the essence of susy: its generators are square roots of translation operators, or square roots of derivatives! It is worth pausing to take this in properly. Four-dimensional derivatives are firmly locked to our notions of a four-dimensional spacetime. In now entertaining the possibility that we can take square roots of them, we are effectively extending our concept of spacetime itself, just as, when the square root of -1 is introduced, we enlarge the real axis to the complex plane. That is to say, if we take seriously an algebra involving both P_μ and the Q's, we shall have to say that the spacetime coordinates are being extended to include further degrees of freedom, which are acted on by the Q's, and that these degrees of freedom are connected to the standard ones by means of transformations generated by the Q's. These new degrees of freedom are, in fact, fermionic. So we may say that susy invites us to contemplate "fermionic dimensions", and enlarge spacetime to "superspace". Unfortunately, we won't have time in these lectures to introduce superspace and superfields (but they usually are in any textbook on susy).

One final remark on motivations: taking local instead of global susy transformations (i.e. taking the parameter of a transformation dependent on the spacetime coordinate x as in gauge symmetries) you get general coordinate changes, i.e. gravitation. This theory, called supergravity (or sugra) is far beyond the scope of these lectures, although we will use some of its general results in the MSSM.

2 Representations of the Poincaré and Susy Algebras

2.1 Poincaré and Lorentz Algebras

The Poincaré group is the largest symmetry group of spacetime. It contains the Lorentz transformations with six (antisymmetric) generators $M_{\mu\nu} = -M_{\nu\mu}$ obeying:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho}) \ , \tag{2.1}$$

where $g_{\mu\nu}$ is the flat metric. This commutation relation has actually the general form of orthogonal transformations $O(N)$ in a space with metric $g_{\mu\nu}$. Because of the different signs between time-like and space-like coordinates of Minkowski space, the Lorentz group is noted $O(3,1)$. In addition we have translations, which four generators P^μ commute among themselves and transform as true 4-vectors:

$$[P_\mu, P_\nu] = 0 \ , \qquad [M_{\mu\nu}, P_\rho] = i(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu) \ . \tag{2.2}$$

The above commutation rules (2.1) and (2.2) can easily be derived by taking the differential operator form of the generators:

$$P_\mu = i\partial_\mu \ , \qquad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \ . \tag{2.3}$$

The Lorentz generators can be rewritten in the standard form of three rotations J_i and three boosts K_i defined by:

$$M_{0i} = K_i \quad \text{and} \quad M_{ij} = \epsilon_{ijk} J_k \ . \tag{2.4}$$

The commutation rule (2.1) then splits as:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ , \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k \ , \qquad [J_i, K_j] = i\epsilon_{ijk} K_j \ . \tag{2.5}$$

To identify the mathematical structure and to construct representations of this algebra one introduces the linear combinations:

$$J_j^\pm = \frac{1}{2}(J_j \pm iK_j) \ , \tag{2.6}$$

in terms of which the algebra separates into two commuting SU(2) algebras:

$$[J_i^\pm, J_j^\pm] = i\epsilon_{ijk} J_k^\pm \quad , \qquad [J_i^\pm, J_j^\mp] = 0 \ . \tag{2.7}$$

These generators are not hermitian however because of the i in (2.6), and we see that the Lorentz group is a complexified version of SU(2) × SU(2): this group is $\mathrm{Sl}(2, \mathbf{C})$. More precisely, $\mathrm{Sl}(2, \mathbf{C})$ is the universal cover of the Lorentz group, just as SU(2) is the universal cover of SO(3). To see that this group is really $\mathrm{Sl}(2, \mathbf{C})$ is easy: introduce the four 2×2 matrices σ^μ where σ^0 is the identity matrix and σ^i, $i = 1..3$ are the three Pauli matrices. Note that we always write the Pauli matrices with an upper index i, while $\sigma_0 = \sigma^0$ and $\sigma_i = -\sigma^i$. Then for every 4-vector x_μ the 2×2 matrix $x_\mu \sigma^\mu$ is hermitian and has determinant equal to $x^\mu x_\mu$ which is a Lorentz invariant. Hence a Lorentz transformation preserves the determinant and the hermiticity of this matrix, and thus must act as $x^\mu \sigma_\mu \to A x^\mu \sigma_\mu A^\dagger$ with $|\det A| = 1$. We see that up to an irrelevant phase, A is a complex 2×2 matrix of unit determinant, i.e. an element of $\mathrm{Sl}(2, \mathbf{C})$. This establishes the mapping between an element of the Lorentz group and the group $\mathrm{Sl}(2, \mathbf{C})$.

2.2 Weyl, Dirac and Majorana Spinors

Spinors are the elementary non-trivial representations of the Lorentz group (trivial representations being Lorentz invariant scalar fields). There are two non-equivalent fundamental representations of $\mathrm{Sl}(2, \mathbf{C})$ which decomposes into $\mathrm{SU}(2) \times \mathrm{SU}(2)$ called left and right chiralities. The corresponding (two components) spinor fields are called Weyl spinors, noted with dotted and undotted indices and labelled according to their eigenvalues for (J^+, J^-):

$$\xi^{\alpha=1,2} \in \left(\frac{1}{2}, 0\right) \equiv 2_L \ , \qquad \bar{\xi}^{\dot{\alpha}=1,2} \in \left(0, \frac{1}{2}\right) \equiv 2_R \ . \tag{2.8}$$

The bar on the 2_R spinor is not necessary as the dot on the index allows to distinguish between the two types of spinors (note that this bar has nothing to do with the particle/antiparticle conjugation or the bar used in 4 component Dirac spinors!). However, in susy indices are usually not written, so that the bar notation allows to tell in which representation the spinor is.

In this representation $\mathrm{Sl}(2, \mathbf{C})$ matrices are represented by:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \ a, b, c, d \in \mathbb{C} \, / \, \det M = 1 \ . \tag{2.9}$$

For a rotation of parameter θ_i and a boost of parameter η_i, we can explicitly display the $\mathrm{Sl}(2, \mathbf{C})$ generators as the spin $\frac{1}{2}$ representations of the complexified $\mathrm{SU}(2) \times \mathrm{SU}(2)$, in accordance with (2.6):

$$M = e^{\eta_i \sigma^i + i\theta_i \sigma^i} \qquad M^* = e^{\eta_i \sigma^{*i} - i\theta_i \sigma^{*i}} \ . \tag{2.10}$$

Under a Lorentz transformation, we get:

$$\xi^\alpha \to \xi'^\alpha = M^\alpha_{\ \beta} \, \xi^\beta \ , \qquad \bar{\xi}^{\dot{\alpha}} \to \bar{\xi}'^{\dot{\alpha}} = M^{*\dot{\alpha}}_{\ \ \dot{\beta}} \, \xi^{\dot{\beta}} \ . \tag{2.11}$$

One can then identify $(\xi^\alpha)^* \equiv \bar{\xi}^{\dot{\alpha}}$ and $(\bar{\xi}^{\dot{\alpha}})^* \equiv \xi^\alpha$. It proves useful to introduce the antisymmetric two index tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$:

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.12}$$

which are used to raise and lower indices as follows:

$$\xi_\alpha = \epsilon_{\alpha\beta} \xi^\beta \ , \quad \xi^\alpha = \epsilon^{\alpha\beta} \xi_\beta \ , \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\xi}^{\dot{\beta}} \ , \quad \bar{\xi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}} \ . \tag{2.13}$$

It is a Lorentz invariant: $M^\alpha_{\ \gamma} M^\beta_{\ \delta} \epsilon_{\alpha\beta} = \epsilon_{\gamma\delta} \det M = \epsilon_{\gamma\delta}$. One can then easily check:

$$\xi'_\alpha = \epsilon_{\alpha\beta} \xi'^\beta = \epsilon_{\alpha\beta} M^\beta_{\ \delta} \xi^\delta = \epsilon_{\gamma\delta} (M^{-1})^\gamma_{\ \alpha} \xi^\delta = (M^{-1T})_\alpha^{\ \gamma} \xi_\gamma \ , \tag{2.14}$$

which means that ξ_α transforms with M^{-1T}. Similarly, $\bar{\xi}_{\dot{\alpha}}$ transforms with $M^{-1\dagger}$.
One can then form Lorentz invariant products:

$$\xi^1 \chi_1 + \xi^2 \chi_2 = \xi^\alpha \chi_\alpha \equiv \xi\chi \ , \qquad \bar{\xi}_{\dot{1}} \bar{\chi}^{\dot{1}} + \bar{\xi}_{\dot{2}} \bar{\chi}^{\dot{2}} = \bar{\xi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \equiv \bar{\xi}\bar{\chi} \ . \tag{2.15}$$

As a convention, when indices are not written, undotted indices are descending (${}^{\alpha}{}_{\alpha}$) while dotted indices are ascending (${}_{\dot\alpha}{}^{\dot\alpha}$). Another convention is to reverse the order of spinors when performing complex conjugation:

$$(\xi\chi)^\dagger = (\xi^\alpha\chi_\alpha)^\dagger = (\chi_\alpha)^*(\xi^\alpha)^* = \bar\chi_{\dot\alpha}\bar\xi^{\dot\alpha} = \bar\chi\bar\xi \ . \tag{2.16}$$

Recall that fermionic degrees of freedom always anticommute (i.e. $\xi_\alpha\chi_\beta = -\chi_\beta\xi_a$, $\bar\xi^{\dot\alpha}\bar\chi^{\dot\beta} = -\bar\chi^{\dot\beta}\bar\xi^{\dot\alpha}$, etc...). With this in mind, one easily shows:

$$\chi\xi = \xi\chi \quad \text{and} \quad \bar\chi\bar\xi = \bar\xi\bar\chi \ . \tag{2.17}$$

We have seen how to form a Lorentz invariant (i.e. spin 0) object out of two spinors. One can also obtain a spin 1 object using σ^μ matrices:

$$V^\mu = \xi^\alpha\sigma^\mu{}_{\alpha\dot\alpha}\bar\xi^{\dot\alpha} \ \in \ \left(\frac{1}{2},\frac{1}{2}\right) \ . \tag{2.18}$$

the Pauli matrices, being hermitian, satisfy $(\sigma^\mu{}_{\alpha\dot\beta})^* = (\sigma^{\mu *})_{\dot\alpha\beta} = (\sigma^{\mu\dagger})_{\beta\dot\alpha} = \sigma^\mu{}_{\beta\dot\alpha}$. One introduces the $\bar\sigma^\mu$ defined by: $\bar\sigma^{\mu\ \dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu{}_{\beta\dot\beta}$ ($\bar\sigma^0 \sim \sigma^0$ and $\bar\sigma^i \sim -\sigma^i$, but this is not an equality: indices are not the same!). They have the following properties:

$$\begin{aligned} &\sigma^\mu_{\alpha\dot\alpha}\bar\sigma_\mu^{\dot\beta\beta} = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} \ , \quad \sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} \ , \quad \bar\sigma^{\mu\alpha\dot\alpha}\bar\sigma_\mu^{\dot\beta\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} \ , \\ &\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} + \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta} = g^{\mu\nu}\delta^\beta_\alpha \ , \qquad \bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} + \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta} \ , \end{aligned} \tag{2.19}$$

from which we deduce $\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = 2g^{\mu\nu}$ and $V^\mu V_\mu = 2\xi^2\bar\xi^2$. Hence V^μ is a true 4-vector under Lorentz transformations. Other useful identities are:

$$\begin{aligned} &\chi\sigma^\mu\bar\xi = -\bar\xi\bar\sigma^\mu\chi \ , \qquad \chi\sigma^\mu\bar\sigma^\nu\xi = \xi\sigma^\nu\bar\sigma^\mu\chi \ , \\ &(\chi\sigma^\mu\bar\xi)^\dagger = \xi\sigma^\mu\bar\chi \ , \qquad (\chi\sigma^\mu\bar\sigma^\nu\xi)^\dagger = \bar\xi\bar\sigma^\nu\sigma^\mu\bar\chi \ , \\ &(\chi_1\sigma^\mu\bar\xi_1)(\chi_2\sigma_\mu\bar\xi_2) = 2(\xi_1\xi_2)(\chi_1\chi_2) \quad \text{(Fierz)} \ . \end{aligned} \tag{2.20}$$

(The indices 1, 2 are not spinorial – they only denote different spinors). For a massive Weyl spinor ξ, one can then form the Lorentz invariant Lagrangian:

$$\mathcal{L} = i\bar\xi\bar\sigma^\mu\partial_\mu\xi - \frac{m}{2}(\xi\xi + \bar\xi\bar\xi) \ , \tag{2.21}$$

and the Lorentz generators for the 2_L, 2_R representations can be written:

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}) \ , \quad \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} = \frac{i}{4}(\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}) \quad \text{(resp.)}. \tag{2.22}$$

(Note that $\sigma^{12} = \bar\sigma^{12} = \frac{1}{2}\sigma^3$ which is the expected rotation generator).

A 4-componant Dirac spinor (the one you are used to) is formed of two Weyl spinors:

$$\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix} \ \in \ \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right) \tag{2.23}$$

One may introduce the following basis for gamma matrices, called the Weyl (or chiral) basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \bar\sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix} \ , \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{2.24}$$

for which we can check the usual relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ , \qquad \{\gamma^\mu, \gamma^5\} = 0 \ , \qquad (\gamma^5)^2 = 1 \ , \ ... \tag{2.25}$$

One defines the chiral projectors $P_L = \dfrac{1-\gamma^5}{2}$ and $P_R = \dfrac{1+\gamma^5}{2}$. It is easy to check that $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = 0$ and $P_L + P_R = 1$. Hence, any Dirac spinor can be written as the some of two chiral eigenstates:

$$\psi_D = \psi_L + \psi_R \ , \ \text{ where } \psi_L = P_L\psi_D = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix} \text{ and } \psi_R = P_R\psi_D = \begin{pmatrix} 0 \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \ , \tag{2.26}$$

eigenstates of γ^5 with eigenvalues -1 and $+1$ (resp.). The conjugate (or bar) spinor of ψ_D is:

$$\bar{\psi}_D = \psi_D^\dagger \begin{pmatrix} 0 & \delta^{\dot{\alpha}}_{\dot{\beta}} \\ \delta^{\beta}_{\alpha} & 0 \end{pmatrix} = (\xi^\beta \bar{\chi}_{\dot{\beta}}) \ . \tag{2.27}$$

(The matrix in (2.27) is similar to γ^0 but the two component indices do not match). One can then form the following Lorentz invariant bilinear products of two Dirac spinors ψ_1, ψ_2 made out of Weyl spinors ξ_1, χ_1 and ξ_1, χ_1 (resp.) following the same notation as in (2.23):

$$\begin{aligned}
&\bar{\psi}_1\psi_2 = \xi_1\chi_2 + \bar{\chi}_1\bar{\xi}_2 = (\bar{\psi}_2\psi_1)^\dagger \ , \\
&\bar{\psi}_1\gamma^\mu\psi_2 = \xi_1\sigma^\mu\bar{\xi}_2 + \bar{\chi}_1\bar{\sigma}^\mu\chi_2 = (\bar{\psi}_2\gamma^\mu\psi_1)^\dagger \ , \\
&\bar{\psi}_1\gamma^5\psi_2 = -\xi_1\chi_2 + \bar{\chi}_1\bar{\xi}_2 = -(\bar{\psi}_2\gamma^5\psi_1)^\dagger \ , \\
&\bar{\psi}_1\gamma^\mu\gamma^5\psi_2 = \xi_1\sigma^\mu\bar{\xi}_2 - \bar{\chi}_1\bar{\sigma}^\mu\chi_2 = (\bar{\psi}_2\gamma^\mu\gamma^5\psi_1)^\dagger
\end{aligned} \tag{2.28}$$

and the Lorentz invariant Lagrangian for a Dirac spinor ψ_D defined as in (2.23) is:

$$\mathcal{L} = i\bar{\psi}_D\gamma^\mu\partial_\mu\psi_D - m\bar{\psi}_D\psi_D = i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi + i\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi - m(\xi\chi + \bar{\xi}\bar{\chi}) \tag{2.29}$$

(note that the Weyl spinors propagate separately while the mass term mixes them).

Finally, the charge conjugation operator in this representation reads:

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \ , \quad C^{-1} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \implies C\gamma_\mu^T C^{-1} = -\gamma_\mu \ . \tag{2.30}$$

For a Dirac spinor ψ_D, one defines the charge conjugate spinor:

$$\psi_D^c = C\bar{\psi}_D^T = \begin{pmatrix} \xi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{2.31}$$

The Majorana condition $\psi_M^c = \psi_M$ is therefore equivalent to $\chi_\alpha = \xi_\alpha$ and a Majorana spinor can be constructed with just one Weyl spinor:

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{2.32}$$

For two Majorana spinors ψ_M, ϕ_M, made out of Weyl spinors ψ, ϕ (resp.), one can construct the following Lorentz invariant products:

$$\begin{aligned}
&\bar{\psi}_M\phi_M = \bar{\psi}\bar{\phi} + \psi\phi = \bar{\phi}_M\psi_M = (\bar{\psi}_M\phi_M)^\dagger \\
&\bar{\psi}_M\gamma^\mu\phi_M = \psi\sigma^\mu\bar{\phi} + \bar{\psi}\bar{\sigma}^\mu\phi = -\bar{\phi}_M\gamma^\mu\psi_M = -(\bar{\psi}_M\gamma^\mu\phi_M)^\dagger \\
&\bar{\psi}_M\gamma^5\phi_M = \bar{\psi}\bar{\phi} - \psi\phi = \bar{\phi}_M\gamma^5\psi_M = -(\bar{\psi}_M\gamma^5\phi_M)^\dagger \\
&\bar{\psi}_M\gamma^\mu\gamma^5\phi_M = \psi\sigma^\mu\bar{\phi} - \bar{\psi}\bar{\sigma}^\mu\phi = \bar{\phi}_M\gamma^\mu\gamma^5\psi_M = (\bar{\psi}_M\gamma^\mu\gamma^5\phi_M)^\dagger
\end{aligned} \tag{2.33}$$

And the Lorentz invariant Lagrangian for a Majorana spinor ψ_M reads:

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_M\gamma^\mu\partial_\mu\psi_M - \frac{m}{2}\bar{\psi}_M\psi_M = i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \frac{m}{2}(\psi\psi + \bar{\psi}\bar{\psi}) \ . \tag{2.34}$$

Before moving on to the susy algebra and representations, let us say a word about representations of the Poincaré group. As we have seen, it is obtained by adding to the Lorentz generators $M_{\mu\nu}$ the translations P_μ. It can be shown that it is a group of rank 2 and that the only Casimirs (i.e. the equivalents of J^2 for SU(2)) are P^2 and W^2, W^μ being the Pauli-Lubanski operator:

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma} \ , \tag{2.35}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the fully antisymmetric tensor with the convention $\epsilon^{0123} = 1 = -\epsilon_{0123}$. Irreducible representations are then characterised by their mass m with $P^2 = m^2$. For massive particles, the other quantum number is the spin s with $W^2 = -m^2 s(s+1)$. For massless particles, $P^2 = W^2 = 0$ and it can be shown that $W_\mu = \lambda P_\mu$ where λ is the helicity which can take only two integral or half-integral values $\pm\lambda$.

2.3 Susy Algebra and Supermultiplets

We are now ready to display the susy algebra. The extra generators transform either as undotted spinors Q_α or dotted spinors $\bar{Q}_{\dot\alpha}$. The anticommutator of Q_α and $\bar{Q}_{\dot\alpha}$ must transform as a 4-vector, the only candidate being P_μ. Finally, it can be checked that the susy generators commute with translations. So, in addition to (2.1), the graded algebra of the susy and Poincaré generators is (up to multiplicative constants):

$$\begin{aligned}
&[Q_\alpha, M_{\mu\nu}] = \sigma_{\mu\nu\alpha}{}^{\beta} Q_\beta \\
&[\bar{Q}^{\dot\alpha}, M_{\mu\nu}] = \bar{\sigma}_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta} Q^{\dot\beta} \\
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \\
&\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \\
&[Q_\alpha, P_\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = 0 \ .
\end{aligned} \tag{2.36}$$

(These relation can easily be obtained if one defines Q_α and $\bar{Q}_{\dot\alpha}$ as differential operators acting on the fermionic components of the superspace). In particular, $M_{12} \equiv J_3$ thus $[J_3, Q_1] = -\frac{1}{2}Q_1$ and $[J_3, Q_2] = \frac{1}{2}Q_2$. Since $\bar{Q}^{\dot 1} = -Q_2^\dagger$ and $\bar{Q}^{\dot 2} = Q_1^\dagger$ one similarly has $[J_3, Q_1^\dagger] = \frac{1}{2}Q_1^\dagger$ and $[J_3, Q_2^\dagger] = -\frac{1}{2}Q_2^\dagger$. We conclude that Q_1, $Q_2^\dagger$ lower the z-component of the spin (or helicity) by 1/2, while Q_2, $Q_1^\dagger$ rise it by 1/2.

Since the susy algebra contains the Poincaré algebra as a subalgebra, any representation of the former also gives a representation of the latter, although in general a reducible one. Since each irreducible representation of the Poincaré algebra corresponds to a massive or massless particle, an irreducible representation of the susy algebra in general corresponds to several particles. The corresponding states are related to each other by the Q_α and $\bar{Q}_{\dot\alpha}$ and thus have spins differing by 1/2. They are called superpartners and form a supermultiplet.

From (2.36) it is easy to show that P^2 commutes with all generators of the susy algebra, i.e. it is a Casimir operator. This implies that all superpartners in a supermultiplet must have the same mass. In addition, if T_a's are the generators of an internal (gauge) symmetry, in addition to (1.18) we have the following commutation relations:

$$[Q_\alpha, T_a] = [\bar{Q}_{\dot{\alpha}}, T_a] = 0 \ . \tag{2.37}$$

This means that superpartners in a supermultiplet must be in the same representation of the gauge groups, i.e. they have the same gauge quantum numbers (charge, colour, etc...). Finally, a supermultiplet contains an equal number of bosonic and fermionic degrees of freedom, i.e. physical on-shell states (e.g. a photon has two degrees of freedom corresponding to the two helicities $+1$ and -1). Let the fermion number be N_F equal 1 on a fermionic state and 0 on a bosonic one. Equivalently $(-1)^{N_F}$ is $+1$ on bosons and -1 on fermions. We want to show that:

$$\mathrm{Tr}\ (-1)^{N_F} = 0 \ , \tag{2.38}$$

if the trace is taken over a supermultiplet. Note that $(-1)^{N_F}$ anticommutes with Q. Using the cyclicity of the trace, one has:

$$\begin{aligned} 0 &= \mathrm{Tr}\left(-Q_\alpha(-1)^{N_F}\bar{Q}_{\dot{\beta}} + (-1)^{N_F}\bar{Q}_{\dot{\beta}}Q_\alpha\right) \\ &= \mathrm{Tr}\left((-1)^{N_F}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}\right) = 2\sigma^\mu_{\alpha\dot{\beta}}\mathrm{Tr}\left((-1)^{N_F}P_\mu\right) \ . \end{aligned} \tag{2.39}$$

Choosing any non-vanishing momentum P_μ gives the desired result.

Let us see now what are the possible massless representations. Since in this case $P^2 = 0$ we can choose a reference frame where $P_\mu = (E, 0, 0, E)$, so that

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \ . \tag{2.40}$$

In particular $\{Q_2, \bar{Q}_{\dot{2}}\} = 0$. Let $|\Phi\rangle$ be any state. The Hilbert space of states being definite positive, one has:

$$0 = \langle\Phi|\{Q_2, \bar{Q}_{\dot{2}}\}|\Phi\rangle = ||Q_2|\Phi\rangle||^2 + ||\bar{Q}_{\dot{2}}|\Phi\rangle||^2 \implies Q_2 = \bar{Q}_{\dot{2}} = 0 \ . \tag{2.41}$$

Thus we are left with 2 fermionic generators: Q_1, $\bar{Q}_{\dot{1}}$. If we define:

$$a = Q_1/\sqrt{4E} \ , \qquad a^\dagger = \bar{Q}_{\dot{1}}/\sqrt{4E} \ , \tag{2.42}$$

we can deduce from (2.36) that a, $a^\dagger$ are anticommuting annihilation/creation operators:

$$\{a, a^\dagger\} = 1 \ , \qquad \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \ , \tag{2.43}$$

(note that $a^\dagger$ is in the 2_R representation and carries helicity $\lambda = +1/2$ while a is in the 2_L representation with helicity $\lambda = -1/2$). One then chooses a "vacuum state" annihilated by a. Such a state will carry some irreducible representation of the Poincaré algebra, i.e. in addition to its zero mass it is characterised by some helicity λ_0. We denote this state as $|\lambda_0\rangle$. From the commutators of Q_1 and $\bar{Q}_{\dot{1}}$ with the helicity operator which in the present frame is $J_3 = M_{12}$ one sees that Q_1 lowers the helicity by one half and $\bar{Q}_{\dot{1}}$ rises it by one half. The supermultiplet then contains only two states:

$$|\lambda_0\rangle \ , \qquad a^\dagger|\lambda_0\rangle = |\lambda_0 + \frac{1}{2}\rangle \ . \tag{2.44}$$

We denote this supermultiplet by $\left(\lambda_0, \lambda_0 + \frac{1}{2}\right)$. In such a supermultiplet however helicities are not distributed symmetrically about 0. Hence it is not invariant under CPT, since CPT flips the sign of the helicity. To satisfy CPT one then needs to double these supermultiplets by adding their CPT conjugate with opposite helicities and opposite quantum numbers. Thus one arrives at the following representations:

- The massless chiral supermultiplet ($\lambda_0 = 0$) consists of $\left(0, \frac{1}{2}\right)$ and its CPT conjugate $\left(-\frac{1}{2}, 0\right)$, corresponding to a Weyl spinor and a complex scalar.

- The massless vector supermultiplet ($\lambda_0 = \frac{1}{2}$) consists of $\left(\frac{1}{2}, 1\right)$ plus $\left(-1, -\frac{1}{2}\right)$, corresponding to a gauge boson and a Weyl spinor, both in the adjoint representation of the gauge group.

Higher helicity supermultiplets – including the graviton (spin 2) and its superpartner the gravitino (spin 3/2) – are possible, but we won't study them in detail in these lectures.

In the case of massive supermultiplets, one chooses the rest frame with $P_\mu = (m, 0, 0, 0)$ and $W^2 = -m^2 J^2$, where the Lorentz generators $J_i = (M_{23}, M_{31}, M_{12})$ form a SU(2) algebra. The susy algebra becomes:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}} \ , \tag{2.45}$$

which defines two annihilation and creation operators

$$a_\alpha = Q_\alpha/\sqrt{2m} \ , \qquad a^\dagger_{\dot{\beta}} = \bar{Q}_{\dot{\beta}}/\sqrt{2m} \tag{2.46}$$

satisfying

$$\{a_\alpha, a^\dagger_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}} \ . \tag{2.47}$$

Starting from a state $|\lambda_0\rangle$ of helicity λ_0 annihilated by a_1, a_2, one finds 4 different states:

$$|\lambda_0\rangle \ , \quad a^\dagger_{\dot{1}}|\lambda_0\rangle = |\lambda_0 + \frac{1}{2}\rangle \ , \quad a^\dagger_{\dot{2}}|\lambda_0\rangle = |\lambda_0 - \frac{1}{2}\rangle \ , \quad a^\dagger_{\dot{\alpha}} a^\dagger_{\dot{\beta}}|\lambda_0\rangle = |\lambda'_0\rangle \ . \tag{2.48}$$

To establish that the last state is of helicity λ_0, one can use the following equality: $a^\dagger_{\dot{\alpha}} a^\dagger_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\ a^\dagger_{\dot{\gamma}} a^{\dagger\dot{\gamma}}$, the last term being a Lorentz scalar (spin 0) made out of two spin $\frac{1}{2}$ objects. One then gets the following representations, again labelled by their helicities:

- The massive chiral supermultiplet ($\lambda_0 = 0$) consists of $\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ and is the same as the massless chiral supermultiplet.

- The massive vector supermultiplet ($\lambda_0 = \frac{1}{2}$) consists of $\left(0, \frac{1}{2}, \frac{1}{2}, 1\right)$ to which we must add the CPT conjugate $\left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right)$. In total we have the same states as in a massless vector plus a massless chiral supermultiplets and the massive vector supermultiplet can be obtained from them via a Higgs mechanism. In terms of massive representations, this is a vector (3 degrees of freedom) a Dirac spinor (4 degrees of freedom) and a single real scalar (1 degrees of freedom).

3 Chiral Supermultiplets and their Interactions

3.1 The Free Chiral Supermultiplet

The minimum fermion content of a field theory in four dimensions consists of a single left-handed two component Weyl spinor ψ_α. Since this is an intrinsically complex object, it seems sensible to choose as its superpartner a complex scalar field ϕ. The simplest action we can write down for these fields just consists of kinetic energy terms for each:

$$S = \int d^4x \ (\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}}) \ , \quad \mathcal{L}_{\text{scalar}} = \partial^\mu \phi^\dagger \partial_\mu \phi \ , \quad \mathcal{L}_{\text{fermion}} = i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi \ . \tag{3.1}$$

This is called the *Wess-Zumino model*, and it corresponds to a massles chiral supermultiplet as discussed in the previous chapter.

A susy transformation should turn the boson field ϕ into something involving the fermion field ψ_α. The simplest possibility for the transformation of the scalar field is:

$$\delta_\epsilon \phi = \epsilon\psi \ , \qquad \delta_\epsilon \phi^\dagger = \bar{\epsilon}\bar{\psi} \ , \tag{3.2}$$

where ϵ^α is an infinitesimal, anticommuting, two component Weyl spinor parameterizing the susy transformation. In these lectures, we will be discussing global susy only, which means that ϵ^α is constant, satisfying $\partial_\mu \epsilon^\alpha = 0$. Since ψ has dimensions of $[\text{mass}]^{3/2}$ and ϕ has dimensions of [mass], ϵ has dimensions of $[\text{mass}]^{-1/2}$. Using (3.2), we find that the scalar part of the Lagrangian transforms as:

$$\delta_\epsilon \mathcal{L}_{\text{scalar}} = \epsilon \partial^\mu \psi \partial_\mu \phi^\dagger + \bar{\epsilon}\partial^\mu \bar{\psi}\partial_\mu \phi \ . \tag{3.3}$$

We would like this to be canceled by $\delta_\epsilon \mathcal{L}_{\text{fermion}}$, at least up to a total derivative, so that the action is invariant under the susy transformation. Comparing (3.3) with $\mathcal{L}_{\text{fermion}}$, we see that for this to happen, $\delta_\epsilon \psi$ should be linear in $\bar{\epsilon}$ and in ϕ, and should contain one spacetime derivative. Up to a multiplicative constant, there is only one possibility:

$$\delta_\epsilon \psi_\alpha = -i\sigma^\mu_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}} \partial_\mu \phi \ , \qquad \delta_\epsilon \bar{\psi}^{\dot{\alpha}} = i\bar{\sigma}^{\mu\dot{\alpha}\beta} \epsilon_\beta \partial_\mu \phi^\dagger \ . \tag{3.4}$$

With this guess, one immediately obtains:

$$\delta_\epsilon \mathcal{L}_{\text{fermion}} = -\epsilon\sigma^\mu \bar{\sigma}^\nu \partial_\nu \psi \partial_\mu \phi^\dagger + \bar{\psi}\bar{\sigma}^\nu \sigma^\mu \bar{\epsilon}\partial_\mu \partial_\nu \phi \ . \tag{3.5}$$

This can be put in a slightly more useful form by employing the identities (2.19) and using the fact that partial derivatives commute ($\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$):

$$\delta_\epsilon \mathcal{L}_{\text{fermion}} = -\epsilon\partial^\mu \psi \partial_\mu \phi^\dagger - \bar{\epsilon}\partial^\mu \bar{\psi}\partial_\mu \phi - \partial_\mu \left(\epsilon\sigma^\nu \bar{\sigma}^\mu \psi \partial_\nu \phi^\dagger - \epsilon\psi \partial^\mu \phi^\dagger - \bar{\epsilon}\bar{\psi}\partial^\mu \phi\right) \ . \tag{3.6}$$

The first two terms here just cancel against $\delta_\epsilon \mathcal{L}_{\text{scalar}}$, while the remaining contribution is a total derivative. So we arrive at:

$$\delta_\epsilon S = \int d^4x \ (\delta_\epsilon \mathcal{L}_{\text{scalar}} + \delta_\epsilon \mathcal{L}_{\text{fermion}}) = 0 \ , \tag{3.7}$$

justifying our guess of the numerical multiplicative factor made in (3.4).

We are not quite finished in showing that the theory described by (3.1) is supersymmetric. We must also show that the susy algebra closes, in other words, that the commutator of two susy transformations parameterized by two different spinors ϵ_1 and ϵ_2 is another symmetry of the theory. Using (3.4) in (3.2), one finds for the scalar field:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi = \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) = i(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)\partial_\mu\phi \ . \tag{3.8}$$

Hence, the commutator of two susy transformations gives us back the derivative of the original field. Since $i\partial_\mu$ corresponds to the generator of spacetime translations P_μ, (3.8) implies the form of the susy algebra that was given in (2.36).

For the spinor ψ_α, using (3.2) in (3.4), we get:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha = -i\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}_1^{\dot{\beta}}\epsilon_2^\beta\partial_\mu\psi_\beta + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}_2^{\dot{\beta}}\epsilon_1^\beta\partial_\mu\psi_\beta \ . \tag{3.9}$$

This can be put into a more useful form by applying the following identity:

$$\chi_\alpha(\xi\eta) + \xi_\alpha(\eta\chi) + \eta_\alpha(\chi\xi) = 0 \ , \tag{3.10}$$

with $\chi = \sigma^\mu\bar{\epsilon}_1$, $\xi = \epsilon_2$, $\eta = \partial_\mu\psi$, and again with $\chi = \sigma^\mu\bar{\epsilon}_2$, $\xi = \epsilon_1$, $\eta = \partial_\mu\psi$, followed by an application of the identities (2.20). The result is:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha = i(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)\partial_\mu\psi_\alpha + i\epsilon_{1\alpha}\bar{\epsilon}_2\bar{\sigma}^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\bar{\epsilon}_1\bar{\sigma}^\mu\partial_\mu\psi \ . \tag{3.11}$$

The last two terms in (3.11) vanish on-shell, that is, if the equation of motion $\bar{\sigma}^\mu\partial_\mu\psi = 0$ following from the action is enforced. The remaining piece is exactly the same spacetime translation that we found for the scalar field.

The fact that the susy algebra only closes on-shell (when the classical equations of motion are satisfied) is somewhat worrisome, since we would like susy to hold at the quantum level. This can be fixed by introducing a new complex scalar field F, which does not have a kinetic term. Such fields are called *auxiliary*, and they are really just book-keeping devices that allow the susy algebra to close off-shell. The Lagrangian for F and its complex conjugate is simply:

$$\mathcal{L}_{\text{auxiliary}} = F^\dagger F \ . \tag{3.12}$$

The dimensions of F are $[\text{mass}]^2$, unlike an ordinary scalar field, which has dimensions of [mass]. The (not-very-exciting) equations of motion are $F = F^\dagger = 0$. However, we can use the auxiliary fields to our advantage by including them in the susy transformations. In view of (3.11), a plausible thing to do is to make F transform into a multiple of the equation of motion for ψ:

$$\delta_\epsilon F = -i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \ , \qquad \delta_\epsilon F^\dagger = i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\epsilon \ . \tag{3.13}$$

Once again we have chosen the overall factor to get things right. Now the auxiliary part of the Lagrangian transforms as:

$$\delta\mathcal{L}_{\text{auxiliary}} = -i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi F^\dagger + i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\epsilon F \ , \tag{3.14}$$

which vanishes on-shell, but not for arbitrary off-shell field configurations. Now, by adding an extra term to the transformation rule for ψ and $\bar{\psi}$:

$$\delta_\epsilon\psi_\alpha = -i\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_\mu\phi + \epsilon_\alpha F \ , \qquad \delta_\epsilon\bar{\psi}^{\dot{\alpha}} = i\bar{\sigma}^{\mu\dot{\alpha}\beta}\epsilon_\beta\partial_\mu\phi^\dagger + \bar{\epsilon}^{\dot{\alpha}}F^\dagger \ , \tag{3.15}$$

one obtains an additional contribution to $\delta_\epsilon\mathcal{L}_{\text{fermion}}$, which just cancels with $\delta_\epsilon\mathcal{L}_{\text{auxiliary}}$, up to a total derivative term. So our "modified" theory with $\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{auxiliary}}$ is invariant under susy transformations. Proceeding as before, one now obtains for each of the fields $X = \phi, \phi^\dagger, \psi, \bar{\psi}, F, F^\dagger$:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = i(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)\partial_\mu X \ , \tag{3.16}$$

using (3.2), (3.13), and (3.15), but now without resorting to any of the equations of motion. So we have succeeded in showing that susy is a valid symmetry of the Lagrangian off-shell.

In retrospect, one can see why we needed to introduce the auxiliary field F in order to get the susy algebra to work off-shell. On-shell, the complex scalar field ψ has two real propagating degrees of freedom, matching the two spin polarization states of ψ. Off-shell, however, the Weyl spinor ψ is a complex two component object, so it has four real degrees of freedom. Going on-shell eliminates half of the propagating degrees of freedom for ψ, because the Lagrangian is linear in time derivatives, so that the canonical momentum can be reexpressed in terms of the field without time derivatives and are not independent phase space coordinates:

$$\pi_\psi = \frac{\partial\mathcal{L}_{\text{fermion}}}{\partial(\partial_0\psi)} = i\bar{\psi}\bar{\sigma}^0 \ . \tag{3.17}$$

Another way to see this is to consider the frame in which the fermion momentum is $P^\mu = (E, 0, 0, E)$ so that the equation of motion reads:

$$\bar{\sigma}^\mu P_\mu \psi = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ , \tag{3.18}$$

which simply projects out half of the fermionic degrees of freedom. To make the numbers of bosonic and fermionic degrees of freedom match off-shell as well as on-shell, we had to introduce two more real scalar degrees of freedom in the complex field F, which are eliminated when one goes on-shell. The auxiliary fields will also play an important role when we add interactions to the theory and in gaining a simple understanding of susy breaking.

3.2 Recovering the Susy Algebra

We are now going to show explicitly how to construct susy generators satisfying the algebra (2.36). Invariance of the action under a susy transformation implies the existence of a conserved *supercurrent* J^μ_α, which is an anticommuting 4-vector carrying a spinor index. By the usual Noether procedure, one finds for the supercurrent (and its hermitian conjugate) in terms of the variations of the fields $X = \phi, \phi^\dagger, \psi, \bar{\psi}, F, F^\dagger$:

$$\epsilon J^\mu + \bar{\epsilon}\bar{J}^\mu \equiv \sum_X \frac{\delta\mathcal{L}}{\delta(\partial_\mu X)}\ \delta X - K^\mu, \tag{3.19}$$

where K^μ satisfies $\delta\mathcal{L} = \partial_\mu K^\mu$. Note that K^μ is not unique: one can always replace K^μ by $K^\mu + k^\mu$, where k^μ is any vector satisfying $\partial_\mu k^\mu = 0$, for example $k^\mu = \partial^\mu\partial_\nu a^\nu - \partial_\nu\partial^\nu a^\mu$. Up to this ambiguity, one gets:

$$J^\mu_\alpha = (\sigma^\nu\bar{\sigma}^\mu\psi)_\alpha\partial_\nu\phi^\dagger \ , \qquad \bar{J}^\mu_{\dot{\alpha}} = (\bar{\psi}\bar{\sigma}^\mu\sigma^\nu)_{\dot{\alpha}}\partial_\nu\phi \ . \tag{3.20}$$

The supercurrent and its hermitian conjugate are separately conserved:

$$\partial_\mu J^\mu_\alpha = 0 \ , \qquad \partial_\mu \bar{J}^\mu_{\dot\alpha} = 0, \tag{3.21}$$

as can be verified by use of the equations of motion. From these currents one constructs the conserved charges:

$$Q_\alpha = \sqrt{2}\int d^3\vec{x}\ J^0_\alpha \ , \qquad \bar{Q}_{\dot\alpha} = \sqrt{2}\int d^3\vec{x}\ \bar{J}^0_{\dot\alpha} \ , \tag{3.22}$$

which are the generators of susy transformations. As quantum mechanical operators, they satisfy:

$$\left[\epsilon Q + \bar\epsilon\bar{Q}, X\right] = -i\sqrt{2}\ \delta X \tag{3.23}$$

for any field X, up to terms that vanish on-shell. This can be verified explicitly by using the canonical equal-time commutation and anticommutation relations:

$$[\phi(\vec{x}),\pi(\vec{y})] \ = \ [\phi^\dagger(\vec{x}),\pi^\dagger(\vec{y})] \ = \ i\delta^{(3)}(\vec{x}-\vec{y}) \ , \quad \{\psi_\alpha(\vec{x}),\bar\psi_{\dot\alpha}(\vec{y})\} \ = \ \sigma^0_{\alpha\dot\alpha}\,\delta^{(3)}(\vec{x}-\vec{y}) \ , \tag{3.24}$$

where $\pi = \partial_0\phi^\dagger$ and $\pi^\dagger = \partial_0\phi$ are the momenta conjugate of ϕ and $\phi^\dagger$ (resp.).

Using (3.23), the content of (3.16) can be expressed in terms of canonical commutators:

$$\Big[\epsilon_2 Q + \bar\epsilon_2\bar{Q},\ \big[\epsilon_1 Q + \bar\epsilon_1\bar{Q},\ X\big]\Big] - \Big[\epsilon_1 Q + \bar\epsilon_1\bar{Q},\ \big[\epsilon_2 Q + \bar\epsilon_2\bar{Q},\ X\big]\Big] = \\ 2(\epsilon_1\sigma^\mu\bar\epsilon_2 - \epsilon_2\sigma^\mu\bar\epsilon_1)\ i\partial_\mu X \ , \tag{3.25}$$

up to terms that vanish on-shell. The spacetime momentum operator is $P^\mu = (H,\vec{P})$, where H is the Hamiltonian and $\vec{P}$ is the three-momentum operator, given in terms of the canonical fields by:

$$\begin{aligned} H &= \int d^3\vec{x}\left[\pi^\dagger\pi + (\vec\nabla\phi^\dagger)\cdot(\vec\nabla\phi) + i\bar\psi\vec\sigma\cdot\vec\nabla\psi\right] \ , \\ \vec{P} &= -\int d^3\vec{x}\left[\pi\vec\nabla\phi + \pi^\dagger\vec\nabla\phi^\dagger + i\bar\psi\bar\sigma^0\vec\nabla\psi\right] \ . \end{aligned} \tag{3.26}$$

It generates spacetime translations on the fields X according to:

$$[P^\mu, X] = -i\partial^\mu X \ . \tag{3.27}$$

Rearranging the terms in (3.25) using the Jacobi identity, we therefore have:

$$\Big[\big[\epsilon_2 Q + \bar\epsilon_2\bar{Q},\ \epsilon_1 Q + \bar\epsilon_1\bar{Q}\big],\ X\Big] = 2(-\epsilon_1\sigma_\mu\bar\epsilon_2 + \epsilon_2\sigma_\mu\bar\epsilon_1)\ [P^\mu, X] \ , \tag{3.28}$$

for any X, up to terms that vanish on-shell, so:

$$\big[\epsilon_2 Q + \bar\epsilon_2\bar{Q},\ e_1 Q + \bar\epsilon_1\bar{Q}\big] = 2(-\epsilon_1\sigma_\mu\bar\epsilon_2 + \epsilon_2\sigma_\mu\bar\epsilon_1)\ P^\mu \ . \tag{3.29}$$

Now by expanding (3.29), one obtains the precise form of the susy algebra:

$$\begin{aligned} &\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu \ , \\ &\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \ , \end{aligned} \tag{3.30}$$

as in section 2.3. The result:

$$[Q_\alpha, P^\mu] = [\bar{Q}_{\dot\alpha}, P^\mu] = 0 \tag{3.31}$$

follows immediately from (3.27) and the fact that the susy transformations are global (independent of spacetime coordinates). This demonstration of the susy algebra in terms of the canonical generators Q and $\bar{Q}$ requires the use of the Hamiltonian equations of motion, but the symmetry itself is valid off-shell at the level of the Lagrangian, as we have already shown.

3.3 Interactions of Chiral Supermultiplets

We are now going to construct the most general theory of (non-gauge) interactions for particles that live in chiral supermultiplets (gauge interactions will be dealt with in the next chapter). We start with the Lagrangian for a collection of free chiral supermultiplets labeled by an index i. Since we want the susy algebra to close off-shell, each supermultiplet contains a complex scalar ϕ_i and a left-handed Weyl spinor ψ_i as physical degrees of freedom, plus a complex auxiliary field F_i which does not propagate. The Lagrangian is:

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i + i\bar{\psi}^i \bar{\sigma}^\mu \partial_\mu \psi_i + F^{\dagger i} F_i, \tag{3.32}$$

where we sum over repeated indices i (not to be confused with the hidden spinor indices), with the convention that fields ϕ_i and ψ_i always carry lowered indices, while their conjugates always carry raised indices. The free Lagrangian is invariant under the susy transformations:

$$\begin{array}{ll} \delta_\epsilon \phi_i = \epsilon \psi_i \ , & \delta \phi^{\dagger i} = \bar{\epsilon} \bar{\psi}^i \ , \\ \delta_\epsilon (\psi_i)_\alpha = -i(\sigma^\mu \bar{\epsilon})_\alpha \partial_\mu \phi_i + \epsilon_\alpha F_i \ , & \delta_\epsilon (\bar{\psi}^i)_{\dot{\alpha}} = i(\epsilon \sigma^\mu)_{\dot{\alpha}} \partial_\mu \phi^{\dagger i} + \bar{\epsilon}_{\dot{\alpha}} F^{\dagger i} \ , \\ \delta_\epsilon F_i = -i\bar{\epsilon} \bar{\sigma}^\mu \partial_\mu \psi_i \ , & \delta_\epsilon F^{\dagger i} = i\partial_\mu \bar{\psi}^i \bar{\sigma}^\mu \epsilon \ . \end{array} \tag{3.33}$$

We will now find the most general set of renormalizable interactions for these fields consistent with susy. We do this before integrating out the auxiliary fields. In order to be renormalizable by power counting, each term must have a total mass dimension ≤ 4. So, the only possible terms are:

$$\mathcal{L}_{\text{int}} = \left(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + x^{ij} F_i F_j \right) + \text{c.c.} - U \ , \tag{3.34}$$

where W^{ij}, W^i, x^{ij}, and U are polynomials in the scalar fields $\phi_i, \phi^{\dagger i}$ with degrees 1, 2, 0, and 4 (resp.). Terms of the form $F^{\dagger i} F_j$ are already included in (3.32), with coefficients fixed by the transformation rules (3.33).

We must now require that $\mathcal{L}_{\text{int}}$ is invariant under susy transformations, since $\mathcal{L}_{\text{free}}$ is already invariant by itself. This implies that the candidate term $U(\phi_i, \phi^{\dagger i})$ must vanish. If there were such a term, then under a susy transformation (3.33) it would transform into another function of the scalar fields multiplied by $\epsilon\psi_i$ or $\bar{\epsilon}\bar{\psi}^i$, and no spacetime derivatives or auxiliary fields F_i, $F^{\dagger i}$. It is easy to see from (3.33) and (3.34) that nothing of this form can possibly be canceled by the susy transformation of any other term in the Lagrangian. Similarly, the dimensionless couplings x^{ij} must be zero, because their susy transformation likewise cannot possibly be canceled by any other term. So, we are left with:

$$\mathcal{L}_{\text{int}} = \left(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i \right) + \text{c.c.} \tag{3.35}$$

as the only possibilities. At this point, we are not assuming that W^{ij} and W^i are related to each other. Soon we will find out that they *are*, which is why we have chosen the same letter for them. Note that (2.17) implies that W^{ij} is symmetric under $i \leftrightarrow j$.

It is easier to divide the variation of $\mathcal{L}_{\text{int}}$ into parts which must cancel separately. First, we consider the part that contains four spinors:

$$\delta \mathcal{L}_{\text{int}}|_{4-\text{spinor}} = \left[-\frac{1}{2} \frac{\delta W^{ij}}{\delta \phi_k} (\epsilon \psi_k)(\psi_i \psi_j) - \frac{1}{2} \frac{\delta W^{ij}}{\delta \phi^{\dagger k}} (\bar{\epsilon} \bar{\psi}^k)(\psi_i \psi_j) \right] + \text{c.c.} \ . \tag{3.36}$$

The term proportional to $(\epsilon\psi_k)(\psi_i\psi_j)$ cannot cancel against any other term. Fortunately, however, the identity (3.10) implies:

$$(\epsilon\psi_i)(\psi_j\psi_k) + (\epsilon\psi_j)(\psi_k\psi_i) + (\epsilon\psi_k)(\psi_i\psi_j) = 0 \ , \tag{3.37}$$

so this contribution to $\delta\mathcal{L}_{\rm int}$ vanishes identically if $\delta W^{ij}/\delta\phi_k$ is totally symmetric under interchange of i, j, k. There is no such identity available for the term proportional to $(\bar\epsilon\bar\psi^k)(\psi_i\psi_j)$. Since that term cannot cancel with any other, requiring it to be absent just tells us that W^{ij} cannot contain $\phi^{\dagger k}$. In other words, W^{ij} is analytic (or holomorphic) in the complex fields ϕ_k.

Combining what we have learned so far, we can write:

$$W^{ij} = M^{ij} + y^{ijk}\phi_k \ , \tag{3.38}$$

where M^{ij} is a symmetric mass matrix for the fermion fields, and y^{ijk} is a Yukawa coupling of a scalar ϕ_k and two fermions $\psi_i\psi_j$ that must be totally symmetric under interchange of i, j, k. It is therefore possible, and it turns out to be convenient, to write:

$$W^{ij} = \frac{\delta^2 W}{\delta\phi_i\delta\phi_j} \ , \tag{3.39}$$

where we have introduced a useful object:

$$W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \ , \tag{3.40}$$

called the *superpotential*. This is not a scalar potential in the ordinary sense; in fact, it is not even real. It is instead an analytic function of the scalar fields ϕ_i treated as complex variables.

Continuing on our quest, we next consider the parts of $\delta\mathcal{L}_{\rm int}$ that contain a spacetime derivative:

$$\delta\mathcal{L}_{\rm int}|_\partial = \left(iW^{ij}\partial_\mu\phi_j\psi_i\sigma^\mu\bar\epsilon + iW^i\partial_\mu\psi_i\sigma^\mu\bar\epsilon\right) + {\rm c.c.} \tag{3.41}$$

Here we have used the identity (2.20) on the second term, which came from $(\delta F_i)W^i$. Now we can use (3.39) to observe that:

$$W^{ij}\partial_\mu\phi_j = \partial_\mu\left(\frac{\delta W}{\delta\phi_i}\right). \tag{3.42}$$

Therefore (3.41) is a total derivative if

$$W^i = \frac{\delta W}{\delta\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k, \tag{3.43}$$

which explains why we chose the same name. The remaining terms in $\delta\mathcal{L}_{\rm int}$ are all linear in F_i or $F^{\dagger i}$, and it is easy to show that they cancel, given the results for W^i and W^{ij} that we have already found.

Actually, we can include a linear term in the superpotential without disturbing the validity of the previous discussion at all:

$$W = L^i\phi_i + \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \ . \tag{3.44}$$

Here L^i are parameters with dimensions of [mass]^2, which affect only the scalar potential part of the Lagrangian. Such linear terms are only allowed when ϕ_i is a gauge singlet, and there are no such gauge singlet chiral supermultiplets in the MSSM with minimal field content. We will therefore omit these terms from the remaining discussion of this section.

To recap, we have found that the most general non-gauge interactions for chiral supermultiplets are determined by a single analytic function of the complex scalar fields, the superpotential W. The auxiliary fields F_i and $F^{\dagger i}$ can be eliminated using their classical equations of motion. The part of $\mathcal{L}_{\rm free} + \mathcal{L}_{\rm int}$ that contains the auxiliary fields is $F_i F^{\dagger i} + W^i F_i + W_i^\dagger F^{\dagger i}$, leading to the equations of motion:

$$F_i = -W_i^\dagger \ , \qquad F^{\dagger i} = -W^i. \tag{3.45}$$

Thus the auxiliary fields are algebraic expressions in terms of the scalar fields (without any derivatives). After making the replacement (3.45) in $\mathcal{L}_{\rm free}+\mathcal{L}_{\rm int}$, we obtain the Lagrangian:

$$\mathcal{L}_{\rm chiral} = \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i + i\bar{\psi}^i \bar{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^\dagger_{ij}\bar{\psi}^i\bar{\psi}j \right) - W^i W_i^\dagger \ . \tag{3.46}$$

Now that the non-propagating fields F_i, $F^{\dagger i}$ have been eliminated, it follows from (3.46) that the scalar potential for the theory is just given in terms of the superpotential by:

$$\begin{aligned} V(\phi, \phi^\dagger) &= W^k W_k^\dagger \ = \ F^{\dagger k} F_k \\ &= M^*_{ik} M^{kj} \phi^{\dagger i}\phi_j + \frac{1}{4} y^{ijn} y^*_{kln} \phi_i \phi_j \phi^{\dagger k}\phi^{\dagger l} \\ &\quad + \frac{1}{2} M^{in} y^*_{jkn} \phi_i \phi^{\dagger j}\phi^{\dagger k} + \frac{1}{2} M^*_{in} y^{jkn} \phi^{\dagger i}\phi_j\phi_k \ . \end{aligned} \tag{3.47}$$

This scalar potential is automatically bounded from below; in fact, since it is a sum of squares it is always ≥ 0. If we substitute the general form for the superpotential (3.40) into (3.46), we obtain for the full Lagrangian:

$$\begin{aligned} \mathcal{L}_{\rm chiral} &= \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i + i\bar{\psi}^i \bar{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2} M^{ij}\psi_i\psi_j - \frac{1}{2} M^*_{ij}\bar{\psi}^i\bar{\psi}^j \\ &\quad - \frac{1}{2} y^{ijk}\phi_i\psi_j\psi_k - \frac{1}{2} y^*_{ijk}\phi^{\dagger i}\bar{\psi}^j\bar{\psi}^k - V(\phi,\phi^\dagger) \ . \end{aligned} \tag{3.48}$$

Now we can compare the masses of the fermions and scalars by looking at the equations of motion:

$$\partial^\mu\partial_\mu \phi_i = -M^*_{ik}M^{kj}\phi_j + \dots \ , \quad i\bar{\sigma}^\mu \partial_\mu \psi_i = M^*_{ij}\bar{\psi}^j + \dots \ , \quad i\sigma^\mu\partial_\mu\bar{\psi}^i = M^{ij}\psi_j + \dots \ . \tag{3.49}$$

One can eliminate ψ in terms of $\bar{\psi}$ and vice versa in (3.49), obtaining after use of the identities (2.19):

$$\partial^\mu\partial_\mu\psi_i = -M^*_{ik}M^{kj}\psi_j + \dots \ , \qquad \partial^\mu\partial_\mu\bar{\psi}^j = -\bar{\psi}^i M^*_{ik} M^{kj} + \dots \ . \tag{3.50}$$

Therefore, the fermions and the bosons satisfy the same wave equation with exactly the same squared mass matrix with real positive eigenvalues, namely $(M^2)_i^{\ j} = M^*_{ik}M^{kj}$. It follows that diagonalizing this matrix by redefining the fields with a unitary matrix gives a collection of chiral supermultiplets, each of which contains a mass-degenerate complex scalar and Weyl spinor, in agreement with the general argument in section 2.3.

Finally, note that the quartic scalar coupling in (3.47) is equal to the fermion Yukawa coupling in (3.48). This is essential in proving that susy invariant theories are free of quadratic divergences.

4 Susy Gauge Interactions and Susy Breaking

4.1 The Free Vector Supermultiplet

The propagating degrees of freedom in a vector supermultiplet are a massless gauge boson A^a_μ and a two component Weyl spinor λ^a_α, called gaugino. The index a runs over the adjoint representation of the gauge group ($a = 1..8$ for $SU(3)_C$ gluons and gluinos; $a = 1..3$ for $SU(2)_L$ W bosons and winos; $a = 1$ for $U(1)_Y$ B boson and bino). The gauge transformations of the vector supermultiplet fields are:

$$\delta_\Lambda A^a_\mu = -\partial_\mu \Lambda^a - g f^{abc} \Lambda^b A^c_\mu \ , \qquad \delta_\Lambda \lambda^a_\alpha = -g f^{abc} \Lambda^b \lambda^c_\alpha \ , \tag{4.1}$$

where Λ^a is an infinitesimal gauge transformation parameter, g is the gauge coupling, and f^{abc} are the antisymmetric structure constants that define the gauge group. They satisfy:

$$[T^a_r, T^b_r] = i f^{abc} T^c_r \ , \tag{4.2}$$

for the generators T^a_r of any representation r. In the adjoint representation, the generator matrices are given by the structure constants themselves:

$$(T^a_{\rm Ad})_{bc} = -i f^{abc} \ . \tag{4.3}$$

The special case of an abelian U(1) group is obtained by just setting $a = 1$ and $f^{abc} = 0$ (the corresponding gaugino is a gauge singlet in that case).

The on-shell degrees of freedom for A^a_μ and λ^a_α amount to two bosonic and two fermionic helicity states (for each a), as required by the susy algebra. However, off-shell λ^a_α consists of two complex, i.e. four real fermionic degrees of freedom, while A^a_μ only has three real bosonic degrees of freedom. Indeed, A^a_μ has 4 real components, one of which can be removed by the gauge transformation (4.1), e.g. one can choose Λ such that the Lorentz condition $\partial_\mu A^\mu = 0$ is verified. So, we will need one real bosonic auxiliary field D^a in order for the susy algebra to be closed off-shell. This field is also in the adjoint representation of the gauge group, i.e. it transforms like λ^a_α in (4.1), and satisfies $(D^a)^* = D^a$. Like the chiral auxiliary fields F_i, the gauge auxiliary field D^a has dimensions of $[\text{mass}]^2$ and no kinetic term, so it can be eliminated on-shell using its algebraic equation of motion.

The Lagrangian for a vector supermultiplet can then be written as:

$$\mathcal{L}_{\rm vector} = i \bar\lambda^a \bar\sigma^\mu D_\mu \lambda^a - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} D^a D^a \ , \tag{4.4}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{4.5}$$

is the usual Yang-Mills field strength, and

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f^{abc} A^b_\mu \lambda^c \tag{4.6}$$

is the covariant derivative of the gaugino field. To check that (4.4) is really supersymmetric, one must specify the susy transformations of the fields. The forms of these follow from the requirements that they should be linear in the infinitesimal parameters $\epsilon, \bar\epsilon$ with dimensions of $[\text{mass}]^{-1/2}$, that δA^a_μ is real, and that δD^a should be real and proportional to the field equations for the gaugino, in analogy with the role of the auxiliary field F in the chiral supermultiplet case. Thus one can guess, up to multiplicative factors, that:

$$\begin{aligned} \delta_\epsilon A^a_\mu &= -\frac{1}{\sqrt{2}}\left(\bar{\epsilon}\bar{\sigma}_\mu\lambda^a + \bar{\lambda}^a\bar{\sigma}_\mu\epsilon\right) , & \delta_\epsilon D^a &= \frac{i}{\sqrt{2}}\left(-\bar{\epsilon}\bar{\sigma}^\mu D_\mu\lambda^a + D_\mu\bar{\lambda}^a\bar{\sigma}^\mu\epsilon\right) , \\ \delta_\epsilon\lambda^a_\alpha &= -\frac{i}{2\sqrt{2}}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon_\alpha D^a , & \delta_\epsilon\bar{\lambda}^a_{\dot{\alpha}} &= \frac{i}{2\sqrt{2}}(\bar{\epsilon}\bar{\sigma}^\nu\sigma^\mu)_{\dot{\alpha}} F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\bar{\epsilon}_{\dot{\alpha}} D^a .\end{aligned} \tag{4.7}$$

It is now a little bit tedious, but straightforward, to also check that

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = i(-\epsilon_1\sigma^\mu\bar{\epsilon}_2 + \epsilon_2\sigma^\mu\bar{\epsilon}_1)D_\mu X \tag{4.8}$$

for X equal to any of the gauge-covariant fields $F^a_{\mu\nu}$, λ^a, $\bar{\lambda}^a$, D^a, as well as for arbitrary covariant derivatives acting on them. This ensures that the susy algebra (2.36) is realized on gauge invariant combinations of fields in vector supermultiplets, as they are for the chiral supermultiplets in (3.16). This check requires the use of identities (2.19) and (2.20) as well as:

$$\begin{aligned} \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho &= g^{\mu\nu}\bar{\sigma}^\rho + g^{\nu\rho}\bar{\sigma}^\mu - g^{\mu\rho}\bar{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\bar{\sigma}_\kappa , \\ \sigma^\mu\bar{\sigma}^\nu\sigma^\rho &= g^{\mu\nu}\sigma^\rho + g^{\nu\rho}\sigma^\mu - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa .\end{aligned} \tag{4.9}$$

If we had not included the auxiliary field D^a, then the susy algebra (4.8) would hold only after using the equations of motion for λ^a and $\bar{\lambda}^a$. The equation of motion for the field strength is $\partial^\mu F^a_{\mu\nu} = 0$ while auxiliary fields satisfies a trivial equation of motion $D^a = 0$. However, the latter is modified if one couples the gauge supermultiplets to chiral supermultiplets, as we do in the next section.

4.2 Susy Gauge Interactions

Now we are ready to consider a general Lagrangian for a susy theory with both chiral and vector supermultiplets. Suppose that the chiral supermultiplets transform under the gauge group in a representation with hermitian matrices $(T^a)_i{}^j$ satisfying (4.2). Since susy and gauge transformations commute, the scalar, fermion, and auxiliary fields must be in the same representation of the gauge group, so:

$$\delta_\Lambda X_i = ig\Lambda^a(T^a X)_i \tag{4.10}$$

for $X_i = \phi_i, \psi_i, F_i$. To have a gauge invariant Lagrangian, we now need to replace the ordinary derivatives in (3.32) with covariant derivatives:

$$\begin{aligned} \partial_\mu\phi_i &\rightarrow D_\mu\phi_i = \partial_\mu\phi_i + igA^a_\mu(T^a\phi)_i , \\ \partial_\mu\phi^{\dagger i} &\rightarrow D_\mu\phi^{\dagger i} = \partial_\mu\phi^{\dagger i} - igA^a_\mu(\phi^\dagger T^a)^i , \\ \partial_\mu\psi_i &\rightarrow D_\mu\psi_i = \partial_\mu\psi_i + igA^a_\mu(T^a\psi)_i .\end{aligned} \tag{4.11}$$

Naively, this simple procedure achieves the goal of coupling the gauge bosons in the vector supermultiplet to the scalars and fermions in the chiral supermultiplets. However, we also have to consider whether there are any other interactions allowed by gauge invariance and involving the gaugino and D^a fields, which might have to be included to make a susy invariant Lagrangian. Since A^a_μ couples to ϕ_i and ψ_i, it makes sense that λ^a and D^a should as well.

There are three such possible terms that are renormalizable (of mass dimension ≤ 4):

$$(\phi^\dagger T^a \psi)\lambda^a \ , \qquad \bar{\lambda}^a(\bar{\psi} T^a \phi) \ , \qquad \text{and} \qquad (\phi^\dagger T^a \phi) D^a \ . \tag{4.12}$$

One can add them with unknown dimensionless coupling coefficients to the Lagrangians for the chiral and vector supermultiplets and demand that the whole mess be real and invariant under susy, up to a total derivative. Not surprisingly, this is possible only if the susy transformations for the chiral fields are modified to include gauge-covariant rather than ordinary derivatives. Also, it is necessary to include one extra term in $\delta_\epsilon F_i$:

$$\begin{aligned} \delta_\epsilon \phi_i &= \epsilon \psi_i \ , \\ \delta_\epsilon \psi_{i\alpha} &= -i(\sigma^\mu \bar{\epsilon})_\alpha D_\mu \phi_i + \epsilon_\alpha F_i \ , \\ \delta_\epsilon F_i &= -i\bar{\epsilon}\bar{\sigma}^\mu D_\mu \psi_i + \sqrt{2} g (T^a \phi)_i \bar{\epsilon}\bar{\lambda}^a \ . \end{aligned} \tag{4.13}$$

After some algebra one can now fix the coefficients for the terms in (4.12), with the result that the full Lagrangian for a renormalizable supersymmetric theory is:

$$\mathcal{L} = \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{vector}} - \sqrt{2} g (\phi^\dagger T^a \psi)\lambda^a - \sqrt{2} g \bar{\lambda}^a (\bar{\psi} T^a \phi) + g(\phi^\dagger T^a \phi) D^a \ . \tag{4.14}$$

Here $\mathcal{L}_{\text{chiral}}$ is the chiral supermultiplet Lagrangian (3.46) or (3.48), but with ordinary derivatives replaced everywhere by gauge-covariant derivatives, and $\mathcal{L}_{\text{vector}}$ is given in (4.4). To prove that the full Lagrangian (4.14) is invariant under the susy transformations, one must use the identity

$$W^i (T^a \phi)_i = 0 \ . \tag{4.15}$$

This is precisely the condition that must be satisfied anyway in order for the superpotential, and thus $\mathcal{L}_{\text{chiral}}$, to be gauge invariant, since the left side is proportional to $\delta_\Lambda W$.

The new terms in (4.14) consists of interactions which strengths are fixed to be gauge couplings by the requirements of susy, even though they are not gauge interactions from the point of view of an ordinary field theory. The first two terms are a direct coupling of gauginos to matter fields; this can be thought of as the "supersymmetrization" of the usual gauge boson couplings to matter fields. The last term combines with the $\frac{1}{2}D^a D^a$ term in $\mathcal{L}_{\text{vector}}$ to provide an equation of motion for the vector auxiliary field;

$$D^a = -g(\phi^\dagger T^a \phi) \ . \tag{4.16}$$

Thus, like the auxiliary fields F_i and $F^{\dagger i}$, the D^a are expressible purely algebraically in terms of the scalar fields. Replacing the auxiliary fields in (4.14) using (4.16), one finds that the complete scalar potential is (recall that $\mathcal{L}$ contains $-V$):

$$V(\phi, \phi^\dagger) = F^{\dagger i} F_i + \frac{1}{2} \sum_a D^a D^a = W_i^\dagger W^i + \frac{1}{2} \sum_a g_a^2 (\phi^\dagger T^a \phi)^2 \ . \tag{4.17}$$

The two types of terms in this expression are called "F term" and "D term" contributions, respectively. In the second term of (4.17), we have now written an explicit sum to cover the case where the gauge group has several distinct factors with different gauge couplings g_a. Since $V(\phi, \phi^\dagger)$ is a sum of squares, it is always ≥ 0 for every field configurations. It is an interesting and unique feature of supersymmetric theories that the scalar potential is completely determined by the other interactions of the theory. The F terms are fixed by Yukawa couplings and fermion mass terms, and the D terms are fixed by the gauge interactions. Here again, note the relation between scalar quartic couplings and gauge couplings, which is essential in proving the absence of quadratic divergences in susy theories.

4.3 Spontaneous vs. Soft Susy Breaking

As we have seen, susy models are extremely constrained: the masses and interactions of the various fields are described by a handful of parameters: namely the masses and Yukawa couplings in the superpotential and the gauge interactions. In particular, all particles within a supermultiplet must have the same mass. This is a direct consequence of the susy algebra (2.36) in which P^2 is a Casimir operator. Clearly, this implies that susy cannot be an exact symmetry of particle physics. Otherwise, there would be scalar partners of the electron (selectrons) with masses exactly equal to $m_e = 511$ keV, as well as massless fermionic partners of the photon (photino) and gluons (gluinos). Such particles would have been extraordinarily easy to detect long ago, yet none has been reported yet. We must therefore find a way to give susy breaking masses to the superpartners of the SM particles without reintroducing quadratic divergences in the theory (which was motivation number one for the introduction of susy). There are two approaches: The first (idealistic) one is to devise a mechanism of spontaneous susy breaking (in this way, susy would be hidden at low energies in a similar way as the electroweak symmetry is in the SM); The second (pragmatic) approach is to introduce "by hand" the desired susy breaking terms.

Let us start with the first approach. If susy is broken spontaneously, the full Lagrangian of the theory is susy invariant, but the the vacuum state $|0\rangle$ is not:

$$Q_\alpha|0\rangle \neq 0 \ , \qquad \bar{Q}_{\dot{\alpha}}|0\rangle \neq 0 \ . \tag{4.18}$$

From the susy algebra (2.36), we can derive a relation between the the susy generators and the Hamiltonian:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \quad \Longrightarrow \quad H = P^0 = \frac{1}{4}\left(Q_1\bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}}Q_1 + Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}}Q_2\right) \ . \tag{4.19}$$

This implies that the energy of any state $|\Phi\rangle$ is positive definite:

$$\langle\Phi|H|\Phi\rangle = \frac{1}{4}\left(||Q_1|\Phi\rangle||^2 + ||\bar{Q}_{\dot{1}}|\Phi\rangle||^2 + ||Q_2|\Phi\rangle||^2 + ||\bar{Q}_{\dot{2}}|\Phi\rangle||^2\right) \ \geq \ 0 \ . \tag{4.20}$$

In particular, it means that susy is spontaneously broken if and only if:

$$\langle 0|H|0\rangle \ = \ \langle 0|V|0\rangle \ > \ 0 \ , \tag{4.21}$$

where V is the scalar potential (4.17) which consists of a sum of squares of auxiliary fields F_i and D^a. Therefore, susy will be spontaneously broken if the vev of some F_i or D^a does not vanish. One way to guarantee spontaneous susy breaking is then to look for models in which the equations $F_i = 0$ and $D^a = 0$ cannot be simultaneously fulfilled for *any* field configuration (as soon as it is for one field configuration, the global minimum will be for $\langle V\rangle = 0$, i.e. susy invariant).

Susy breaking with a non-zero D term vev can occur through the Fayet-Iliopoulos mechanism: if the gauge symmetry includes a U(1) factor, then one can introduce a term linear in the corresponding auxiliary field of the vector supermultiplet:

$$\mathcal{L}_{\rm FI} = -\kappa D \ , \tag{4.22}$$

where κ is a constant with dimensions of [mass]^2. For a U(1) gauge symmetry, the susy transformation $\delta_\epsilon D$ in (4.7) is a total derivative and D^a is a gauge singlet. Hence, (4.22) is both susy and gauge invariant (this would not be the case for a non-abelian gauge group). If we include this Fayet-Iliopoulos term in the Lagrangian, then D may be forced to get a non-zero vev. To see this, consider the relevant part of the scalar potential from (4.4) and (4.14):

$$V = \kappa D - \frac{1}{2}D^2 - gD\sum_i q_i|\phi_i|^2 \ . \tag{4.23}$$

Here the q_i are the charges of the scalar fields ϕ_i under the U(1) gauge group in question. The presence of the Fayet-Iliopoulos term modifies the equation of motion (4.16) for D:

$$D = \kappa - g\sum_i q_i|\phi_i|^2 \ . \tag{4.24}$$

Suppose that the scalar fields ϕ_i charged under U(1) have superpotential masses m_i. Then the potential will have the form:

$$V = \sum_i |m_i|^2|\phi_i|^2 + \frac{1}{2}(\kappa - g\sum_i q_i|\phi_i|^2)^2. \tag{4.25}$$

Since this cannot vanish, susy must be broken. For the simplest case in which $|m_i|^2 > gq_i\kappa$, the minimum is realized for $\phi_i = 0$ and $D = \kappa$, with the U(1) gauge symmetry unbroken. As further evidence that susy has indeed been spontaneously broken, note that the scalars have squared masses $|m_i|^2 - gq_i\kappa$, while their fermion partners have squared masses $|m_i|^2$. The gaugino however remains massless.

In models where spontaneous susy breaking is due to a non-zero F term vev, called O'Raifeartaigh models, the idea is to pick a set of chiral supermultiplets (ϕ_i, ψ_i, F_i) and a superpotential W in such a way that the equations $F_i = -\delta W^\dagger/\delta\phi^{\dagger i} = 0$ have no simultaneous solution. Then $V = \sum_i |F_i|^2$ will have to be positive at its minimum, ensuring that susy is broken. The simplest example that does the trick has three chiral supermultiplets with:

$$W = -k\phi_1 + m\phi_2\phi_3 + \frac{y}{2}\phi_1\phi_3^2 \ . \tag{4.26}$$

Note that W contains a linear term, with k having dimensions of [mass]^2. Such a term is allowed if the corresponding chiral supermultiplet is a gauge singlet. In fact, a linear term is necessary to achieve F term breaking at tree level in renormalizable theories, since otherwise setting all $\phi_i = 0$ will always give a supersymmetric global minimum with all $F_i = 0$. Without loss of generality, we can choose k, m, and y to be real and positive (by a phase rotation of the fields). The scalar potential then reads:

$$V = |F_1|^2 + |F_2|^2 + |F_3|^2 \ , \quad , \quad F_1 = k - \frac{y}{2}\phi_3^{\dagger 2} \ , \quad F_2 = -m\phi_3^\dagger \ , \quad F_3 = -m\phi_2^\dagger - y\phi_1^\dagger\phi_3^\dagger \ . \tag{4.27}$$

Clearly, $F_1 = 0$ and $F_2 = 0$ are not compatible, so susy must indeed be broken. If $m^2 > yk$, the absolute minimum of the potential is at $\phi_2 = \phi_3 = 0$ with ϕ_1 undetermined, so $F_1 = k$ and $V = k^2$ at the minimum. The fact that ϕ_1 is undetermined is an example of a "flat direction" in the scalar potential; this is a common feature of susy models. The mass spectrum of the theory consists of six real scalars with tree level squared masses: 0, 0, m^2, m^2, $m^2 - yk$, $m^2 + yk$. Meanwhile, there are three Weyl fermions with squared masses: 0, m^2, m^2. The non-degeneracy of scalars and fermions is a clear sign that susy has been spontaneously broken. The 0 eigenvalues correspond to the complex scalar ϕ_1 and its fermionic partner ψ_1.

In both D term and F term spontaneous susy breaking, one then encounters light states which are not welcome in a susy extension of the SM. This could have been foreseen, as the spontaneous breaking of a global symmetry always yields a massless Goldstone mode with the same quantum numbers as the broken symmetry generator. In the case of global susy, the broken generator is the fermionic charge Q_α, so the Goldstone mode is a massless neutral Weyl fermion, called the *goldstino*. To prove it, consider a generic susy model with both chiral and vector supermultiplets as in the previous section. The fermionic degrees of freedom consist of gauginos (λ^a) and chiral fermions (ψ_i). After some of the scalar fields in the theory get vevs, the fermion mass matrix in the (λ^a, ψ_i) basis has the form:

$$\mathbf{m}_\mathrm{F} = \begin{pmatrix} 0 & \sqrt{2}g_b(\langle\phi^\dagger\rangle T^b)^i \\ \sqrt{2}g_a(\langle\phi^\dagger\rangle T^a)^j & \langle W^{ji}\rangle \end{pmatrix} . \tag{4.28}$$

Now $\mathbf{m}_\mathrm{F}$ annihilates the vector:

$$\widetilde{G} = \begin{pmatrix} \langle D^a\rangle/\sqrt{2} \\ \langle F_i\rangle \end{pmatrix} . \tag{4.29}$$

The first row of $\mathbf{m}_\mathrm{F}$ annihilates $\widetilde{G}$ by virtue of the requirement (4.15) that the superpotential is gauge invariant, and the second row does so because of the condition $\langle\partial V/\partial\phi_i\rangle = 0$, which must be satisfied at a local minimum of the scalar potential. Equation (4.29) is therefore proportional to the goldstino wavefunction; it is non-trivial if and only if at least one of the auxiliary fields has a vev, breaking susy. So if global susy is spontaneously broken, there must be a massless goldstino, and its components among the various fermions in the theory are just proportional to the corresponding auxiliary field vevs.

However, if susy is promoted to be a local (gauge) symmetry, i.e. if the susy transformation parameter ϵ^α depends on the spacetime coordinate, the goldstino will no longer be massless. Such theories are called supergravity (sugra) as they include a description of gravitation (cf. remark at the end of section 1.3). In sugra, the spin 2 graviton has a spin 3/2 fermion superpartner Ψ^α_μ, the *gravitino*. It carries both a vector index (μ) and a spinor index (α), and transforms inhomogeneously under local susy transformations:

$$\delta\Psi^\alpha_\mu = \partial_\mu\epsilon^\alpha + \ldots . \tag{4.30}$$

Thus the gravitino should be thought of as the "gauge" field of local susy transformations (compare (4.30) with (4.1)). As long as susy is unbroken, the graviton and the gravitino are massless, each with two spin helicity states. Once susy is spontaneously broken, the gravitino acquires a mass by absorbing ("eating") the goldstino, which becomes its longitudinal (helicity $\pm 1/2$) components. This is called the *super-Higgs* mechanism, and it is analogous to the SM Higgs mechanism by which the $W^\pm$ and Z^0 gauge bosons gain mass by absorbing the Goldstone bosons associated with the spontaneously broken electroweak gauge symmetry. The massive spin 3/2 gravitino now has four helicity states, of which two were originally assigned to the would-be goldstino. The gravitino mass is traditionally called $m_{3/2}$, and in the case of F term breaking it can be estimated as

$$m_{3/2} \sim \frac{\langle F\rangle}{M_\mathrm{Planck}}, \tag{4.31}$$

This follows simply from dimensional analysis, since $m_{3/2}$ must vanish in the limits that susy is restored ($\langle F\rangle \to 0$) and that gravity is turned off ($M_\mathrm{Planck} \to \infty$).

It is usually assumed that susy (or sugra) is spontaneously broken at high energy in some hidden sector and the breaking transmitted to low energy fields by some interaction (e.g. gravitation). Many models of susy breaking have been proposed and there is no consensus on exactly how this should be done. From a practical point of view, it is useful to simply parameterize our ignorance of these issues by just introducing extra terms that break susy explicitly in the effective low energy Lagrangian. These terms however should not reintroduce quadratic divergences in the theory, i.e. they should be "soft". The cancellation of quadratic divergences in susy models is due to the relation between the dimensionless couplings of the theory (namely the scalar quartic couplings, the fermion Yukawa couplings and the gauge couplings). Soft susy breaking terms should therefore be of positive mass dimension. The most general soft susy breaking Lagrangian is:

$$\mathcal{L}_{\text{soft}} = -\left(\frac{1}{2}M_a\,\lambda^a\lambda^a + \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b^{ij}\phi_i\phi_j + t^i\phi_i\right) + \text{c.c.} - (m^2)^i_j\phi^{\dagger j}\phi_i\ , \tag{4.32}$$

It containst gaugino masses M_a for each gauge group, scalar squared mass terms $(m^2)^j_i$, and scalar couplings a^{ijk}, b^{ij} and t^i. The last of these can only occur if ϕ_i is a gauge singlet. One might wonder why we have not included possible soft mass terms for the chiral supermultiplet fermions, like $\mathcal{L} = -\frac{1}{2}m^{ij}\psi_i\psi_j + \text{c.c.}$. Such terms however can always be absorbed into a redefinition of the superpotential and the terms $(m^2)^i_j$. In the special case of a theory withs chiral supermultiplets that are singlets or in the adjoint representation of the gauge group, there are also possible soft susy breaking Dirac mass terms between the corresponding fermions ψ_a and the gauginos:

$$\mathcal{L}'_{\text{soft}} \;=\; -M^a_{\text{Dirac}}\lambda^a\psi_a + \text{c.c.} \tag{4.33}$$

This is not relevant for the MSSM with minimal field content, which does not have adjoint representation nor singlet chiral supermultiplets.

It has been shown rigorously that a softly broken supersymmetric theory with $\mathcal{L}_{\text{soft}}$ as given by (4.32) is indeed free of quadratic divergences in quantum corrections to scalar masses, to all orders in perturbation theory. The terms in $\mathcal{L}_{\text{soft}}$ clearly do break susy, because they involve only scalars and gauginos and not their respective superpartners. In fact, the soft terms in $\mathcal{L}_{\text{soft}}$ are capable of giving masses to all the scalars and gauginos, even if the gauge bosons and fermions in chiral supermultiplets are massless (or relatively light). The gaugino masses M_a are always allowed by gauge symmetry. The $(m^2)^i_j$ terms are allowed for i, j such that ϕ_i, $\phi^{\dagger j}$ transform in complex conjugate representations of each other under all gauge symmetries; in particular this is true of course when $i = j$, so every scalar is eligible to get a mass in this way if susy is broken. The remaining soft terms may or may not be allowed by the symmetries. The a^{ijk}, b^{ij}, and t^i terms have the same form as the y^{ijk}, M^{ij}, and L^i terms in the superpotential (3.44), so they will each be allowed by gauge invariance if and only if a corresponding superpotential term is allowed. For the superpartners to have escaped detection at LEP and Tevatron, the soft susy breaking masses must be at least of the order of 100 GeV. On the other hand, if susy is broken softly, according to (1.13) the radiative corrections to the Higgs mass are:

$$\delta m_H^2 \sim m^2_{\text{soft}} \ln\Lambda\ , \tag{4.34}$$

where Λ is the ultraviolet cut-off. To avoid an excessive fine tuning, the soft masses should not exceed the TeV scale. Hence, susy should be just around the corner, within reach of the LHC!

5 The Minimal Susy Standard Model (MSSM)

5.1 The Minimal Field Content

We now have all the ingredients to construct a realistic susy extension of the SM. The minimal susy extension, the MSSM, has a minimal field content. First note that all the SM fermions (quarks and leptons) have their left-handed parts transforming differently under the gauge group than their right-handed parts. Only chiral supermultiplets can contain fermions with this property, so they must be members of chiral supermultiplets. The spin 0 partners of the quarks and leptons are called *squarks* and *sleptons* (short for "scalar quark" and "scalar lepton"), or *sfermions*. The left-handed and right-handed pieces of the quarks and leptons are separate two component Weyl fermions with different gauge transformation properties, so each must have its own complex scalar partner. The symbols for the squarks and sleptons are the same as for the corresponding fermion, but with a tilde. For example, the superpartners of the left- and right-handed parts of the electron field are called left- and right-handed selectrons, and are denoted $\widetilde{e}_L$ and $\widetilde{e}_R$. It is important to keep in mind that the "handedness" here does not refer to the helicity of the selectrons (they are spin 0 particles) but to that of their superpartners. A similar nomenclature applies for smuons and staus: $\widetilde{\mu}_L$, $\widetilde{\mu}_R$, $\widetilde{\tau}_L$, $\widetilde{\tau}_R$. The neutrinos (neglecting their very small masses) are always left-handed, so the sneutrinos are denoted generically by $\widetilde{\nu}$, with a possible subscript indicating which lepton flavor they carry: $\widetilde{\nu}_e$, $\widetilde{\nu}_\mu$, $\widetilde{\nu}_\tau$. Finally, a complete list of the squarks is $\widetilde{q}_L$, $\widetilde{q}_R$ with $q = u, d, s, c, b, t$.

It seems clear that the Higgs boson must reside in a chiral supermultiplet, since it has spin 0. Actually, it turns out that just one chiral supermultiplet is not enough. One reason is that if there were only one Higgs chiral supermultiplet, the electroweak gauge symmetry would suffer a gauge anomaly, and would be inconsistent as a quantum theory. This is because the conditions for cancellation of gauge anomalies include $\mathrm{Tr}[T_3^2 Y] = \mathrm{Tr}[Y^3] = 0$, where T_3 and Y are the third component of weak isospin and the hypercharge (resp.) in a normalization where the ordinary electric charge is $Q = T_3 + Y$. The traces run over all of the left-handed Weyl fermionic degrees of freedom in the theory. In the SM, these conditions are already satisfied, by the known quarks and leptons. Now, a fermionic partner of a Higgs chiral supermultiplet must be a $\mathrm{SU}(2)_L$ doublet with hypercharge $Y = 1/2$ or $Y = -1/2$. In either case alone, such a fermion will make a non-zero contribution to the traces and spoil the anomaly cancellation. This can be avoided if there are two Higgs doublets, with $Y = \pm 1/2$, so that the total contribution to the anomaly traces from the two fermionic members of the Higgs chiral supermultiplets vanishes. As we will see in the next section, only a $Y = 1/2$ Higgs chiral supermultiplet can have the Yukawa couplings necessary to give masses to up-type quarks (u, c, t), while only a $Y = -1/2$ Higgs can have the Yukawa couplings necessary to give masses to down-type quarks (d, s, b) and charged leptons (e, μ, τ). We call the $\mathrm{SU}(2)_L$ doublet complex scalar fields with $Y = \pm 1/2$ H_u and H_d (resp.).The components of H_u with $T_3 = (1/2, -1/2)$ have electric charges $+1$, 0 (resp.) and are denoted $(H_u^+,\ H_u^0)$. Similarly, the $\mathrm{SU}(2)_L$ doublet complex scalar H_d has $T_3 = (1/2, -1/2)$ components $(H_d^0,\ H_d^-)$. The generic nomenclature for a spin 1/2 superpartner is to append "-ino" to the name of the SM particle, so the fermionic partners of the Higgs scalars are called higgsinos. They are denoted by $\widetilde{H}_u$, $\widetilde{H}_d$ for the $\mathrm{SU}(2)_L$ doublet left-handed Weyl spinor fields, with weak isospin components $(\widetilde{H}_u^+,\ \widetilde{H}_u^0)$ and $(\widetilde{H}_d^0,\ \widetilde{H}_d^-)$.

Names		spin 0	spin 1/2	$SU(3)_C$, $SU(2)_L$, $U(1)_Y$
squarks, quarks	Q	$(\widetilde{u}_L \ \ \widetilde{d}_L)$	$(u_L \ \ d_L)$	$(\,\mathbf{3},\ \mathbf{2}\,,\ \frac{1}{6})$
(×3 families)	U^c	$\widetilde{u}^c_R$	u^c_R	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ -\frac{2}{3})$
	D^c	$\widetilde{d}^c_R$	d^c_R	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ \frac{1}{3})$
sleptons, leptons	L	$(\widetilde{\nu}\ \widetilde{e}_L)$	$(\nu \ \ e_L)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$
(×3 families)	E^c	$\widetilde{e}^c_R$	e^c_R	$(\,\mathbf{1},\ \mathbf{1},\ 1)$
Higgs, higgsinos	H_u	$(H^+_u \ \ H^0_u)$	$(\widetilde{H}^+_u \ \ \widetilde{H}^0_u)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ +\frac{1}{2})$
	H_d	$(H^0_d \ \ H^-_d)$	$(\widetilde{H}^0_d \ \ \widetilde{H}^-_d)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$

Table 5.1: Chiral supermultiplets in the MSSM. The spin 0 fields are complex scalars, and the spin 1/2 fields are left-handed two component Weyl fermions.

We now have all the chiral supermultiplets of the MSSM, summarized in table 5.1. Fields are classified according to their transformation under $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ and (u_L, d_L), (ν, e_L) are combined into $\mathrm{SU}(2)_L$ doublets. All chiral supermultiplets are defined in terms of left-handed Weyl spinors, so that the *charge conjugates* of the right-handed quarks and charged leptons (and their scalar partners) appear in table 5.1. The $\mathrm{SU}(2)_L$ singlet supermultiplets then contain left-handed antifermions and their scalar partners with charge +1 for $\widetilde{e}^c_R$, −2/3 for $\widetilde{u}^c_R$, and +1/3 for $\widetilde{d}^c_R$. It is also useful to have a symbol for each of the chiral supermultiplets as a whole; these are indicated in the second column of table 5.1. There are three families for each of the quark and lepton supermultiplets, for which a family index $i = 1..3$ can be affixed to the chiral supermultiplet names.

The gauge bosons of the SM clearly must reside in vector supermultiplets. Their fermionic superpartners are called gauginos. The $\mathrm{SU}(3)_C$ color interaction is mediated by the gluon g, whose spin 1/2 superpartner is the gluino $\widetilde{g}$. The electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ is associated with spin 1 gauge bosons W^+, W^0, W^- and B^0, whose spin 1/2 superpartners $\widetilde{W}^+, \widetilde{W}^0, \widetilde{W}^-$ and $\widetilde{B}^0$ are called *winos* and *bino*. After EWSB, the W^0, B^0 gauge eigenstates mix to give mass eigenstates Z^0 and γ. The corresponding gaugino mixtures of $\widetilde{W}^0$ and $\widetilde{B}^0$ are called zino ($\widetilde{Z}^0$) and photino ($\widetilde{\gamma}$). If susy were unbroken, they would be mass eigenstates with masses m_Z and 0. Table 5.2 summarizes the vector supermultiplets of the MSSM.

Names	spin 1/2	spin 1	$SU(3)_C$, $SU(2)_L$, $U(1)_Y$
gluino, gluon	$\widetilde{g}$	g	$(\,\mathbf{8},\ \mathbf{1}\,,\ 0)$
winos, W bosons	$\widetilde{W}^\pm \ \ \widetilde{W}^0$	$W^\pm \ \ W^0$	$(\,\mathbf{1},\ \mathbf{3}\,,\ 0)$
bino, B boson	$\widetilde{B}^0$	B^0	$(\,\mathbf{1},\ \mathbf{1}\,,\ 0)$

Table 5.2: Gauge supermultiplets in the MSSM.

5.2 The Superpotential and R-parity

The interaction content of the MSSM is also minimal. As we have seen in section 3.3, all the non-gauge interactions in susy theories are encoded in the superpotential, an analytic function of order 3 in the chiral complex fields. The superpotential for the MSSM is:

$$W_{\rm MSSM} = U^c \mathbf{y_u} Q H_u - D^c \mathbf{y_d} Q H_d - E^c \mathbf{y_e} L H_d + \mu H_u H_d \ , \qquad (5.1)$$

where tildes on the scalar fields of the chiral supermultiplets are omitted. The dimensionless Yukawa coupling parameters $\mathbf{y_u}, \mathbf{y_d}, \mathbf{y_e}$ are 3×3 matrices in family space. The gauge and family indices in (5.1) are suppressed. The "μ term" can be written as $\mu (H_u)_\alpha (H_d)_\beta \epsilon^{\alpha\beta}$, where $\epsilon^{\alpha\beta}$ is used to sum over SU(2)$_L$ indices $\alpha, \beta = 1, 2$ in a gauge invariant way (watch out: these are *not* spinor indices!). Likewise, the term $U^c \mathbf{y_u} Q H_u$ can be written as $(U^c)^{ia} \, (\mathbf{y_u})_i{}^j \, (Q)_{j\alpha a} \, (H_u)_\beta \epsilon^{\alpha\beta}$, where $i = 1..3$ is the family index, and $a = 1..3$ is the color index which is lowered (raised) in the $\mathbf{3}$ ($\overline{\mathbf{3}}$) representation of SU(3)$_C$.

The μ term in (5.1) is the susy version of the SM Higgs boson mass. It is unique, because terms like $H_u^\dagger H_u$ or $H_d^\dagger H_d$ are forbidden in the superpotential, which must be analytic in the scalar fields. We can also see from (5.1) why two Higgs doublets are needed: Since the superpotential must be analytic, the $U^c Q H_u$ term cannot be replaced by $U^c Q H_d^\dagger$. Similarly, the $D^c Q H_d$ and $E^c L H_d$ terms cannot be replaced by $D^c Q H_u^\dagger$ and $E^c L H_u^\dagger$. So we need both H_u and H_d in order to give Yukawa couplings, and thus masses, to all quarks and charged leptons, even without invoking the argument based on anomaly cancellation mentioned in the previous section.

The Yukawa matrices determine the current masses and CKM mixing angles of the ordinary quarks and leptons, after the neutral scalar components of H_u and H_d get vevs. Since the top quark, bottom quark and tau lepton are the heaviest fermions in the SM, it is often useful to make an approximation that only the $(3, 3)$ family components of $\mathbf{y_u}$, $\mathbf{y_d}$ and $\mathbf{y_e}$ are important:

$$\mathbf{y_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix} , \qquad \mathbf{y_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b \end{pmatrix} , \qquad \mathbf{y_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau \end{pmatrix} . \qquad (5.2)$$

In this limit, only the third family and Higgs fields contribute to the MSSM superpotential. It is instructive to rewrite it in terms of the SU(2)$_L$ components $Q_3 = (t_L \ , b_L)$, $L_3 = (\nu_\tau \ , \tau_L)$, $H_u = (H_u^+ \ , H_u^0)$, $H_d = (H_d^0 \ , H_d^-)$, $U_3^c = t_R^c$, $D_3^c = b_R^c$ and $E_3^c = \tau_R^c$:

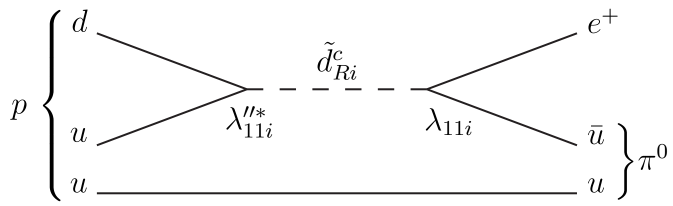
$$\begin{aligned} W_{\rm MSSM} \ \approx \ & y_t (t_R^c t_L H_u^0 - b_R^c b_L H_u^+) - y_b (b_R^c t_L H_d^- - b_R^c b_L H_d^0) \\ & - y_\tau (\tau_R^c \nu_\tau H_d^- - \tau_R^c \tau_L H_d^0) + \mu (H_u^+ H_d^- - H_u^0 H_d^0) \ . \end{aligned} \qquad (5.3)$$

The minus signs inside the parentheses appear because of the antisymmetry of the $\epsilon^{\alpha\beta}$ tensor used to tie up the SU(2)$_L$ indices. The other minus signs in (5.1) were chosen so that the terms $y_t t_R^c t_L H_u^0$, $y_b b_R^c b_L H_d^0$, and $y_\tau \tau_R^c \tau_L H_d^0$, which become the top, bottom and tau masses when H_u^0 and H_d^0 get vevs, have positive signs in (5.3).

The MSSM superpotential (5.1) is minimal in the sense that it is sufficient to produce a phenomenologically viable model. There are, however, other gauge invariant and renormalizable terms that are analytic and of order 3 in the chiral scalar fields:

$$W_{\not R_p} = \lambda_{ijk} L_i L_j E_k^c + \lambda'_{ijk} L_i Q_j D_k^c + \mu'_i L_i H_u + \lambda''_{ijk} U_i^c D_j^c D_k^c \ , \qquad (5.4)$$

Figure 5.1: Proton decay to $e^+\pi^0$ mediated by a down-type squark. The arrows on propagators are omitted for simplicity, and external fermion labels refer to physical states rather than two component spinors.



where family indices $i, j, k = 1..3$ have been restored. The chiral supermultiplets Q_i carry baryon number B $= +1/3$ while U_i^c, D_i^c carry B $= -1/3$ and all others have B $= 0$. Similarly, the total lepton number is L $= +1$ for L_i, L $= -1$ for E_i^c and L $= 0$ for all others. Hence, the three first terms in (5.4) violate L by one unit (as well as the individual lepton flavors) while the last term violates B by one unit.

The possible existence of such terms is rather disturbing, since processes violating B or L have never been observed experimentally. The most obvious experimental constraint comes from the non-observation of proton decay, which would violate B and L by one unit. If both λ' and λ'' couplings were present and unsuppressed, then the lifetime of the proton would be extremely short. For example, Feynman diagrams like the one in fig. 5.1 would lead to $p^+ \to e^+\pi^0$. As a rough estimate based on dimensional analysis, the partial width for the proton decay of fig. 5.1 is:

$$\Gamma(p \to e^+\pi^0) \sim m_p^5 \sum_i \frac{|\lambda'_{11i}\lambda''_{11i}|^2}{m^4_{\tilde{d}^c_{Ri}}} \ , \tag{5.5}$$

which would be substantial if the couplings were of order unity and the squarks masses of order 1 TeV. In contrast, the decay time of the proton is known experimentally to be in excess of 10^{32} years. Therefore, λ'_{11i} or λ''_{11i} must be extremely small. Many other processes also give strong constraints on the violation of lepton and baryon numbers.

One could take B and L conservation as a postulate in the MSSM. However, this is clearly a step backward from the SM, where the conservation of these quantum numbers is *not* assumed, but is rather a pleasantly accidental consequence of the fact that there are no possible renormalizable Lagrangian terms that violate B or L. Furthermore, there is a problem in treating B and L as fundamental symmetries, since they are known to be necessarily violated by non-perturbative electroweak effects, which are negligible for experiments at ordinary energies but might be relevant in the early universe. Therefore, in the MSSM one adds a new symmetry, which eliminates the possibility of B and L violating terms in the renormalizable superpotential, while allowing the terms in (5.1). This new symmetry, called R-parity, is multiplicatively conserved and defined by:

$$R_p = (-1)^{3(B-L)+2s} \ , \tag{5.6}$$

where s is the spin of the particle. One quickly finds that R_p is $+1$ for all SM particles and -1 for their superpartners (also called sparticles). Hence, it is clear that the terms in (5.4) do not conserve R_p, while the terms in (5.1) do. The conservation of R_p implies that SM particles cannot mix with their superpartners. In addition, it can be shown that every interaction vertex resulting from (5.1) contains an even number of sparticles (i.e. 0 or 2). This is also true for gauge interactions and has three extremely important phenomenological consequences:

- The lightest sparticle, called the "lightest susy particle" or LSP, must be absolutely stable. If the LSP is electrically neutral, it interacts only weakly with ordinary matter, and so make an attractive candidate for the non-baryonic dark matter that is required by cosmology.

- Each sparticle other than the LSP must eventually decay into a state that contains an odd number of LSP's (usually just one).

- In collider experiments, sparticles can only be produced in even numbers (usually in pair).

The LSP must lack electromagnetic and strong interactions, otherwise, LSP's surviving from the Big Bang era would have bound to nuclei forming objects with highly unusual charge to mass ratios. Searches for such exotics have excluded all models with stable charged or strongly interacting particles unless their mass exceeds several TeV, which is unacceptably high for the LSP if susy is supposed to solve the hierarchy problem. An important implication is that in collider experiments LSP's will carry away energy and momentum while escaping detection. Since all sparticles are pair-produced and each decays into at least one LSP (plus SM particles), it follows that at least twice the mass of the LSP will turn up as missing energy in every susy events. In e^+e^- machines (like LEP or ILC), the total visible energy and momentum can be well measured, and the beams have very small spread, so that the missing energy and momentum can be well correlated with the energy and momentum of the LSP's. In hadron colliders (as the LHC), the distribution of energy and longitudinal momentum of the partons (i.e. quarks and gluons) is very broad, so in practice only the missing transverse momentum (or missing transverse energy $\not{E}_{\rm T}$) is useful.

The conservation of R_p is *imposed* in the MSSM. While this decision seems to be well-motivated phenomenologically by proton decay constraints and the fact that the LSP provides a good dark matter candidate, it might appear somewhat artificial from a theoretical point of view. After all, the MSSM would not suffer any internal inconsistency if we did not impose R_p conservation. Furthermore, it is fair to ask why R_p should be exactly conserved, given that the discrete symmetries in the SM are known to be broken (namely the ordinary parity P, the charge conjugation C and the time reversal T which are individually broken although their product CPT is always conserved). Fortunately, it *is* sensible to formulate R_p as a discrete symmetry that is exactly conserved. In general, exactly conserved discrete gauge symmetries can exist provided that they satisfy certain anomaly cancellation conditions (much like continuous gauge symmetries). One particularly attractive way this could occur is if B−L is a continuous gauge symmetry that is spontaneously broken at some very high energy scale. A continuous $\rm U(1)_{B-L}$ forbids the renormalizable terms that violate B and L, but this gauge symmetry must be spontaneously broken, since there is no corresponding massless vector boson. However, if $\rm U(1)_{B-L}$ is broken by a scalar vev that carries an even integer value of $3(\rm B-L)$, then R_p will automatically survive as an exactly conserved discrete remnant subgroup. It may also be possible to have discrete gauge symmetries that do not owe their exact conservation to an underlying continuous gauge symmetry, but rather to some other structure such as can occur in string theory. Finally, it is also possible that R_p is broken after all, with strongs constraints however on R_p violating terrms in (5.4), or replaced by some alternative discrete symmetry.

5.3 Soft Susy Breaking in the MSSM

To complete the description of the MSSM, we need to specify the soft susy breaking terms. In section 4.3, we learned how to write down the most general set of such terms. Applying this recipe to the MSSM, we have:

$$
\begin{aligned}
\mathcal{L}_{\rm soft}^{\rm MSSM} \;=\; & -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B} + {\rm c.c.}\right) \\
& -\left(\widetilde{u}_R^c\,\mathbf{a_u}\,\widetilde{Q}H_u - \widetilde{d}_R^c\,\mathbf{a_d}\,\widetilde{Q}H_d - \widetilde{u}_R^e\,\mathbf{a_e}\,\widetilde{L}H_d + {\rm c.c.}\right) \\
& -\widetilde{Q}^\dagger\,\mathbf{m_Q^2}\,\widetilde{Q} - \widetilde{L}^\dagger\,\mathbf{m_L^2}\,\widetilde{L} - \widetilde{u}_R^c\,\mathbf{m_U^2}\,\widetilde{u}_R^{c\dagger} - \widetilde{d}_R^c\,\mathbf{m_D^2}\,\widetilde{d}_R^{c\dagger} - \widetilde{e}_R^c\,\mathbf{m_E^2}\,\widetilde{e}_R^{c\dagger} \\
& -m_{H_u}^2 H_u^\dagger H_u - m_{H_d}^2 H_d^\dagger H_d - (bH_uH_d + {\rm c.c.})\ , \qquad (5.7)
\end{aligned}
$$

where the adjoint representation gauge indices on the wino and gluino fields, as well as the gauge indices on all the chiral supermultiplet fields have been suppressed. In the first line, M_1, M_2, M_3 are the bino, wino, and gluino masses . The second line contains the scalar trilinear couplings of the type a^{ijk} in (4.32). Each of $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$ is a complex 3×3 matrix in family space, with dimensions of [mass]. They are in one-to-one correspondence with the Yukawa couplings of the superpotential. The third line consists of squark and slepton mass terms of the $(m^2)_i^j$ type in (4.32). The 3×3 matrices in family space $\mathbf{m_Q^2}$, $\mathbf{m_U^2}$, $\mathbf{m_D^2}$, $\mathbf{m_L^2}$ and $\mathbf{m_E^2}$ can have complex entries, but they must be hermitian so that the Lagrangian is real. Finally, in the last line we have soft susy breaking contributions to the Higgs potential: $m_{H_u}^2$ and $m_{H_d}^2$ are squared mass terms of the $(m^2)_i^j$ type, while b is the only squared mass term of the type b^{ij} in (4.32) that can occur in the MSSM. As argued in section 4.3, we expect:

$$
\begin{aligned}
& M_1,\, M_2,\, M_3,\, \mathbf{a_u},\, \mathbf{a_d},\, \mathbf{a_e} \sim m_{\rm soft}\ , \\
& \mathbf{m_Q^2},\, \mathbf{m_L^2},\, \mathbf{m_U^2},\, \mathbf{m_D^2},\, \mathbf{m_E^2},\, m_{H_u}^2,\, m_{H_d}^2,\, b \sim m_{\rm soft}^2\ , \qquad (5.8)
\end{aligned}
$$

with $m_{\rm soft}$ not much larger than 1 TeV.

Unlike the susy preserving part of the Lagrangian, the above $\mathcal{L}_{\rm soft}^{\rm MSSM}$ introduces many new parameters that were not present in the SM. A careful count reveals that there are 105 real parameters (including possible phases) in the MSSM Lagrangian that cannot be rotated away by redefining the phases and flavor basis for the quark and lepton supermultiplets, and that have no counterpart in the SM. Thus, in principle, susy *breaking* (as opposed to susy itself) appears to introduce a tremendous arbitrariness in the Lagrangian. Moreover, most of the new parameters in (5.7) imply flavor changing neutral currents (FCNC) or CP violating processes, both severely restricted by experiment.

These potentially dangerous effects can be evaded if one assumes that susy breaking is suitably "flavor blind". Consider a limit in which the squark and slepton squared mass matrices are diagonal in family space and degenerate for the two first families:

$$
\mathbf{m_Q^2} = \begin{pmatrix} m_{Q_1}^2 & 0 & 0 \\ 0 & m_{Q_1}^2 & 0 \\ 0 & 0 & m_{Q_3}^2 \end{pmatrix}\ ,\quad
\mathbf{m_U^2} = \begin{pmatrix} m_{U_1}^2 & 0 & 0 \\ 0 & m_{U_1}^2 & 0 \\ 0 & 0 & m_{U_3}^2 \end{pmatrix}\ ,\quad
\mathbf{m_D^2} = \begin{pmatrix} m_{D_1}^2 & 0 & 0 \\ 0 & m_{D_1}^2 & 0 \\ 0 & 0 & m_{D_3}^2 \end{pmatrix}\ ,
$$

$$
\mathbf{m_L^2} = \begin{pmatrix} m_{L_1}^2 & 0 & 0 \\ 0 & m_{L_1}^2 & 0 \\ 0 & 0 & m_{L_3}^2 \end{pmatrix}\ ,\quad
\mathbf{m_E^2} = \begin{pmatrix} m_{E_1}^2 & 0 & 0 \\ 0 & m_{E_1}^2 & 0 \\ 0 & 0 & m_{E_3}^2 \end{pmatrix}\ . \qquad (5.9)
$$

Then susy contributions to FCNC will be very small, up to flavor mixings induced by $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$. Making the further assumption that the scalar trilinear couplings are each proportional to the corresponding Yukawa coupling matrices:

$$\mathbf{a_u} = A_u\,\mathbf{y_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t A_u \end{pmatrix} \,, \quad \mathbf{a_d} = A_d\,\mathbf{y_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b A_d \end{pmatrix} \,,$$
$$\mathbf{a_e} = A_e\,\mathbf{y_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau A_e \end{pmatrix} \,, \tag{5.10}$$

will ensure that only the squarks and sleptons of the third family can have large trilinear couplings. Finally, one can avoid disastrously large CP violating effects by assuming that the soft parameters do not introduce new complex phases. This is automatic for scalar mass terms if (5.9) is assumed (if the soft masses were not real, the Lagrangian would not be real). One can also fix μ in the superpotential and b in (5.7) to be real, by appropriate phase rotations of fermion and scalar components of the H_u and H_d supermultiplets. If one then assumes that

$$\arg(M_1),\, \arg(M_2),\, \arg(M_3),\, \arg(A_{u0}),\, \arg(A_{d0}),\, \arg(A_{e0}) \;=\; 0 \text{ or } \pi, \tag{5.11}$$

then the only CP violating phase in the theory will be the usual CKM phase found in the ordinary Yukawa couplings. The conditions (5.9)-(5.11) make up what is called the hypothesis of *Minimal Flavor Violation* and the resulting model is referred to as *Phenomenological MSSM*. It has far fewer parameters than the most general MSSM. Besides the usual SM gauge and Yukawa coupling parameters, there are 3 gaugino masses, 10 squark and slepton squared masses, 3 scalar trilinear coupling parameters, and 4 Higgs mass parameters (one of which can be traded for the known EWSB scale), hence a total of 19 independent parameters.

One can further decrease the number of parameters of the MSSM by making assumptions on how spontaneous susy breaking is transmitted from the hidden sector where it occurs to the visible sector where we live. Suppose that the spontaneous susy breaking sector connects with the MSSM only (or dominantly) through gravitational-strength interactions. This means that the effective Lagrangian contains non-renormalizable terms that communicate between the two sectors and are suppressed by powers of the Planck mass $M_{\rm Planck}$:

$$\begin{aligned} \mathcal{L}_{\rm NR} \;=\; & -\frac{1}{M_{\rm Planck}} F \left(\frac{1}{2} f_a \lambda^a \lambda^a + \frac{1}{6} y'^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} \mu'^{ij} \phi_i \phi_j \right) + \text{c.c.} \\ & - \frac{1}{M_{\rm Planck}^2} F F^\dagger k^i_j \phi_i \phi^{\dagger j} \,, \end{aligned} \tag{5.12}$$

where F is the auxiliary field of a chiral supermultiplet in the hidden sector which gets a non-trivial vev (therefore breaking susy spontaneously), ϕ_i and λ^a are the scalar and gaugino fields in the MSSM, f^a, y'^{ijk}, k^i_j are dimensionless constants and μ'^{ij} is a constant with dimensions of [mass]. Now if one assumes that $\sqrt{\langle F \rangle} \sim 10^{11}$ GeV, then $\mathcal{L}_{\rm NR}$ will give us a Lagrangian of the form $\mathcal{L}_{\rm soft}^{\rm MSSM}$ in (5.7), with soft terms of order:

$$m_{\rm soft} \sim \frac{\langle F \rangle}{M_{\rm Planck}} \sim 1 \text{ TeV} \,. \tag{5.13}$$

The parameters f_a, k^i_j, y'^{ijk} and μ'^{ij} in $\mathcal{L}_{\rm NR}$ are to be determined by the underlying theory. This is a difficult enterprise in general, but a dramatic simplification occurs if one assumes a "minimal" form for the normalization of kinetic terms and gauge interactions in the full (non-renormalizable) supergravity Lagrangian. In that case, there is a common $f_a = f$ for the three gauginos, $k^i_j = k\delta^i_j$ is the same for all scalars, and the other couplings are proportional to the corresponding superpotential parameters, so that $y'^{ijk} = \alpha y^{ijk}$ and $\mu'^{ij} = \beta\mu^{ij}$ with universal dimensionless constants α and β. Then the soft terms in $\mathcal{L}^{\rm MSSM}_{\rm soft}$ are all determined by just four parameters:

$$M_{1/2} = f\frac{\langle F\rangle}{M_{\rm Planck}}\ , \quad m_0^2 = k\frac{|\langle F\rangle|^2}{M^2_{\rm Planck}}\ , \quad A_0 = \alpha\frac{\langle F\rangle}{M_{\rm Planck}}\ , \quad B = \beta\frac{\langle F\rangle}{M_{\rm Planck}}\ . \tag{5.14}$$

In terms of these quantities, the parameters appearing in (5.7) are:

$$\begin{aligned}
&M_3 = M_2 = M_1 = M_{1/2}\ ,\\
&\mathbf{m^2_Q} = \mathbf{m^2_U} = \mathbf{m^2_D} = \mathbf{m^2_L} = \mathbf{m^2_E} = m_0^2\,\mathbf{1}\ ,\\
&m^2_{H_u} = m^2_{H_d} = m_0^2\ , \qquad b = B\mu\ ,\\
&\mathbf{a_u} = A_0\mathbf{y_u}\ , \quad \mathbf{a_d} = A_0\mathbf{y_d}\ , \quad \mathbf{a_e} = A_0\mathbf{y_e}\ .
\end{aligned} \tag{5.15}$$

This assumption is called *universality* and the resulting model *minimal supergravity* (mSUGRA). It is a matter of some controversy whether universality is well-motivated on theoretical grounds, but from a phenomenological perspective it is clearly very nice. This framework successfully evades the most dangerous types of FCNC processes as (5.15) is just a stronger versions of (5.9) and (5.10). If $M_{1/2}$, A_0 and B all have the same complex phase, then (5.11) will also be satisfied and the susy CP problem solved. Therfore, the parameters of mSUGRA are m_0, $M_{1/2}$, A_0, B and μ (plus the already measured gauge and Yukawa couplings of the SM). Using the EWSB conditions, i.e. the minimization equations for the two neutral Higgs vevs ($\langle H_u^0\rangle$, $\langle H_d^0\rangle$) one can trade B and μ (up to its sign) for the known value of m_Z, proportional to the sum of the Higgs vevs squared, and the ratio of the Higgs vevs, called tanβ (cf. section 6.2). Hence, one is left with 4 free parameters and a sign:

$$M_{1/2}\ , \quad m_0\ , \quad A_0\ , \quad \tan\beta\ , \quad {\rm sign}(\mu)\ . \tag{5.16}$$

Particular models of gravity-mediated susy breaking can be even more predictive, relating some of the parameters $M_{1/2}$, m_0^2, A_0 and B to each other and to the gravitino mass $m_{3/2}$. Three popular kinds of models for the soft terms are:

- Polonyi: $m_0^2 = m^2_{3/2}$, $\quad A_0 = (3-\sqrt{3})m_{3/2}$, $\quad M_{1/2} = m_{3/2}$;
- Dilaton-dominated: $m_0^2 = m^2_{3/2}$, $\quad M_{1/2} = -A_0 = \sqrt{3}m_{3/2}$;
- No-scale: $M_{1/2} \gg m_0, A_0, m_{3/2}$.

Dilaton-dominated and no-scale models arise in a particular limits of superstring theory, while the Polonyi model is the simplest model of susy breaking in the hidden sector.

Of course, universality is usually assumed at a very high scale $Q_0 \approx M_{\rm Planck}$. To compute the low energy MSSM parameters and the sparticle spectrum, one needs to use (5.15) as boundary conditions for the Renormalization Group Equations (RGEs) at the scale $M_{\rm Planck}$ and integrate these coupled differential equations down to the scale $M_{\rm weak}$, as we shall do in the next chapter.

6 The Mass Spectrum of the MSSM

6.1 Renormalization Group Equations

In order to translate a set of predictions for the susy parameters at a high energy scale Q_0 like $M_{\rm Planck}$ into physically meaningful quantities describing physics at a low energy scale $M_{\rm weak}$, it is necessary to evolve the gauge couplings, superpotential parameters, and soft terms using their renormalization group equations (RGEs). The couplings and masses appearing in the Lagrangian are then treated as running parameters depending on the renormalization scale Q and conditions like (5.15) are boundary conditions for the RGEs at the scale Q_0. The RGEs are coupled differential equations for all the parameters as a function of $t \equiv \ln(Q/Q_0)$. This procedure ensures that the loop expansions for calculations of low energy observables will not suffer from very large logarithms $\ln(Q_0/M_{\rm weak})$ which are resummed by the RGEs.

Some care is required in choosing regularization and renormalization schemes in susy. The most popular regularization method for computations of radiative corrections within the SM is dimensional regularization (DREG), in which the number of spacetime dimensions is extended to $d = 4 - 2\epsilon$. Unfortunately, DREG introduces a violation of susy, because it has a mismatch between the gauge boson and the gaugino degrees of freedom off-shell. This mismatch is only 2ϵ, but can be multiplied by factors up to $1/\epsilon^n$ in an n loop calculation. Instead, one uses the slightly different scheme known as regularization by dimensional reduction, or DRED, which does respect susy. In the DRED method, all momentum integrals are still performed in $d = 4 - 2\epsilon$ dimensions, but the 4-vector index μ on the gauge boson fields A^a_μ now runs over all 4 dimensions to maintain the match with the gaugino degrees of freedom. Running parameters are then renormalized using DRED with modified minimal subtraction ($\overline{\rm DR}$) rather than the usual DREG with modified minimal subtraction ($\overline{\rm MS}$). However, at one loop the RGEs are the same in the two schemes.

A general and powerful result in susy theories, known as the *non-renormalization theorem*, states that the logarithmically divergent contributions to any process can always be written in terms of wave-function renormalizations, without any vertex renormalization (it can be proved most easily using superfield techniques). For the parameters appearing in the superpotential (3.44), this implies:

$$\frac{d}{dt}y^{ijk} = \gamma^i_n y^{njk} + \gamma^j_n y^{ink} + \gamma^k_n y^{ijn} \ , \quad \frac{d}{dt}M^{ij} = \gamma^i_n M^{nj} + \gamma^j_n M^{in} \ , \quad \frac{d}{dt}L^i = \gamma^i_n L^n \ , \quad (6.1)$$

where the γ^i_j are anomalous dimension matrices for the supermultiplets i, j which have to be calculated in a perturbative loop expansion. The anomalous dimensions and RGEs for softly broken susy are known up to three loop order, with some partial four loop results. Here we will only use the one loop approximation, for simplicity:

$$\gamma^i_j = \frac{1}{16\pi^2}\left[\frac{1}{2}y^{imn}y^*_{jmn} - 2g_a^2 C_a(i)\delta^i_j\right] \ , \quad (6.2)$$

where $C_a(i)$ are the quadratic Casimir group theory invariants for the supermultiplet $\Phi_i = (\phi_i, \psi_i, F_i)$, defined in terms of the Lie algebra generators T^a by:

$$(T^aT^a)_i{}^j = C_a(i)\delta_i^j \ , \quad (6.3)$$

with gauge couplings g_a. Explicitly, for the MSSM supermultiplets:

$$C_3(i) = \begin{cases} 4/3 & \text{for } \Phi_i = Q, U^c, D^c \\ 0 & \text{for } \Phi_i = L, E^c, H_u, H_d \end{cases} \qquad C_2(i) = \begin{cases} 3/4 & \text{for } \Phi_i = Q, L, H_u, H_d \\ 0 & \text{for } \Phi_i = U^c, D^c, E^c \end{cases}$$
$$C_1(i) = 3Y_i^2/5 \ \text{ for each } \Phi_i \text{ with weak hypercharge } Y_i \ . \tag{6.4}$$

The normalization for the hypercharge is chosen to agree with the canonical covariant derivative for grand unification of the SM gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ into SU(5) or SO(10). Starting with the superpotential (5.1) it is easy to compute the anomalous dimension for all the chiral supermultiplets of the MSSM using (6.2). Putting this into (6.1) one then obtains the RGEs for the Yukawa couplings and the μ term.

For a general susy model, the one loop RGEs for gauge couplings are:

$$\frac{d}{dt} g_a = \frac{1}{16\pi^2} g_a^3 \Big[\sum_i I_a(i) - 3C_a(G) \Big] \ , \tag{6.5}$$

where $C_a(G)$ is the quadratic Casimir invariant of the group (0 for $U(1)$, and N for $SU(N)$), and $I_a(i)$ is the Dynkin index of the chiral supermultiplet Φ_i, normalized to 1/2 for each fundamental representation of SU(N) and to $3Y_i^2/5$ for U(1). In the special case of the MSSM, the one loop RGEs for gauge couplings are:

$$\frac{d}{dt} g_a = \frac{1}{16\pi^2} b_a g_a^3 \ , \quad (b_1, b_2, b_3) = (33/5, \ 1, \ -3) \ . \tag{6.6}$$

Hence, the $\alpha_a = g_a^2/4\pi$ have the nice property that their inverse run linearly with the renormalization scale at one loop order:

$$\frac{d}{dt} \alpha_a^{-1} = -\frac{b_a}{2\pi} \tag{6.7}$$

Figure 6.1 compares the RG evolution of α_a^{-1}, including two loop effects, in the SM and the MSSM. Unlike the SM, the MSSM includes just the right particle content to ensure that the gauge couplings can unify, at a scale $M_U \sim 2 \times 10^{16}$ GeV. While the apparent unification of gauge couplings at M_U might be just an accident, it may also be taken as a strong hint in favor of a grand unified theory (GUT) or superstring models, both of which can naturally accommodate gauge coupling unification below M_{Planck}. Furthermore, if this hint is taken seriously, then we can reasonably expect to be able to apply a similar RGE analysis to the other MSSM couplings and soft masses as well.

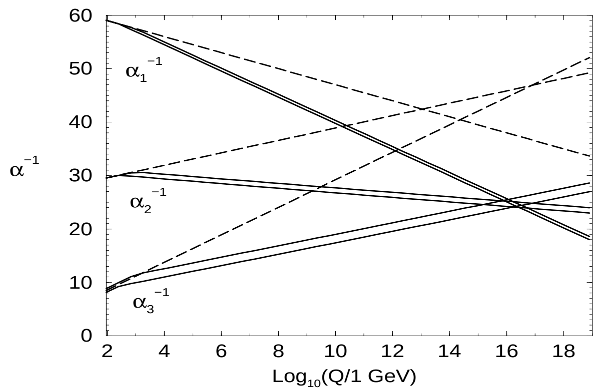


Figure 6.1: Evolution of the inverse gauge couplings $\alpha_a^{-1}(Q)$ in the SM (dashed lines) and the MSSM (solid lines). In the MSSM case, the sparticle mass thresholds are varied between 250 GeV and 1 TeV, and $\alpha_3(m_Z)$ between 0.113 and 0.123. Two loop effects are included.

The one loop RGEs for the general soft susy breaking parameters in (4.32) are:

$$\begin{aligned}
\frac{d}{dt}M_a &= \frac{1}{16\pi^2}g_a^2\Big[2\sum_n I_a(n) - 6C_a(G)\Big]M_a \ , \\
\frac{d}{dt}a^{ijk} &= \frac{1}{16\pi^2}\left[\frac{1}{2}a^{ijp}y^*_{pmn}y^{kmn} + y^{ijp}y^*_{pmn}a^{mnp} + g_a^2C_a(i)(4M_ay^{ijk} - 2a^{ijk})\right] \\
&\quad +(i\leftrightarrow k) + (j\leftrightarrow k) \ , \\
\frac{d}{dt}b^{ij} &= \frac{1}{16\pi^2}\left[\frac{1}{2}b^{ip}y^*_{pmn}y^{jmn} + \frac{1}{2}y^{ijp}y^*_{pmn}b^{mn} + M^{ip}y^*_{pmn}a^{mnj}\right. \\
&\quad \left. +g_a^2C_a(i)(4M_aM^{ij} - 2b^{ij})\right] + (i\leftrightarrow j) \ , \\
\frac{d}{dt}(m^2)_i^j &= \frac{1}{16\pi^2}\left[\frac{1}{2}y^*_{ipq}y^{pqn}(m^2)_n^j + \frac{1}{2}y^{jpq}y^*_{pqn}(m^2)_i^n + 2y^*_{ipq}y^{jpr}(m^2)_r^q\right. \\
&\quad \left. +a^*_{ipq}a^{jpq} - 8g_a^2C_a(i)|M_a|^2\delta_i^j + 2g_a^2(T^a)_i{}^j\mathrm{Tr}(T^am^2)\right] \ . \qquad (6.8)
\end{aligned}$$

Applying the above results to the special case of the MSSM, with soft terms given by (5.7), is straightforward. In particular, the one loop RGEs for the three gaugino mass parameters in the MSSM are determined by the same b_a that appear in the gauge coupling RGEs (6.6):

$$\frac{d}{dt}M_a = \frac{1}{8\pi^2}b_ag_a^2M_a \quad a = 1..3 \ . \qquad (6.9)$$

It follows that the ratios M_a/g_a^2 are scale independent up to small two loop corrections. Since the gauge couplings unify at $Q = M_U$, it is natural to assume that the gaugino masses also unify to a value $M_{1/2}$ near that scale. In GUT models, it is automatic that the gauge couplings and gaugino masses are unified at scales $Q \geq M_U$, because the gauginos all live in the same representation of the unified gauge group. In many superstring models, this can also be a good approximation.

Taking into account only the third family Yukawa couplings, the one loop RGEs for the soft Higgs squared mass parameters $m^2_{H_u}$ and $m^2_{H_d}$ are:

$$\begin{aligned}
\frac{d}{dt}m^2_{H_u} &= \frac{1}{16\pi^2}\left[3X_t - 6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2 + \frac{3}{5}g_1^2S\right] \ , \\
\frac{d}{dt}m^2_{H_d} &= \frac{1}{16\pi^2}\left[3X_b + X_\tau - 6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2 + \frac{3}{5}g_1^2S\right] \ , \\
\text{where} \quad S &= m^2_{H_u} - m^2_{H_d} + \mathrm{Tr}[\mathbf{m^2_Q} - \mathbf{m^2_L} - 2\mathbf{m^2_U} + \mathbf{m^2_D} + \mathbf{m^2_E}] \ . \\
X_t &= 2|y_t|^2(m^2_{H_u} + m^2_{Q_3} + m^2_{U_3}) + 2|a_t|^2 \ , \\
X_b &= 2|y_b|^2(m^2_{H_d} + m^2_{Q_3} + m^2_{D_3}) + 2|a_b|^2, \\
X_\tau &= 2|y_\tau|^2(m^2_{H_d} + m^2_{L_3} + m^2_{E_3}) + 2|a_\tau|^2 \ . \qquad (6.10)
\end{aligned}$$

Note that X_t, X_b, and X_τ are generally positive (at least at a high scale where all the soft scalar squared masses are fixed to be positive), so their effect is to decrease the Higgs masses as one evolves the RGEs down from the high energy input scale to the scale M_{weak}. If y_t is the largest of the Yukawa couplings, as suggested by the experimental fact that the top quark is heavy, then X_t will typically be much larger than X_b and X_τ. This causes $m^2_{H_u}$ to run negative near M_{weak}, helping to destabilize the point $H_u = H_d = 0$ and so inducing EWSB.

6.2 EWSB and the Higgs Sector

In the MSSM, the description of EWSB is slightly complicated by the fact that there are two complex Higgs doublets $H_u = (H_u^+,\, H_u^0)$ and $H_d = (H_d^0,\, H_d^-)$ rather than just one as in the SM. The classical potential for the Higgs scalar fields in the MSSM is given by:

$$\begin{aligned} V &= (|\mu|^2 + m_{H_u}^2)(|H_u^0|^2 + |H_u^+|^2) + (|\mu|^2 + m_{H_d}^2)(|H_d^0|^2 + |H_d^-|^2) \\ &\quad + [b\,(H_u^+ H_d^- - H_u^0 H_d^0) + \text{c.c.}] + \frac{1}{2} g^2 |H_u^+ H_d^{0*} + H_u^0 H_d^{-*}|^2 \\ &\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 \ . \end{aligned} \tag{6.11}$$

The terms proportional to $|\mu|^2$ come from F terms. The terms proportional to g^2 and g'^2 are the D term contributions, obtained from the general formula (4.17) after some rearranging. Finally, the terms proportional to $m_{H_u}^2$, $m_{H_d}^2$ and b are just a rewriting of the last three terms of the MSSM soft susy breaking Lagragian (5.7). The full scalar potential of the theory also includes many terms involving the squark and slepton fields. However, we can ignore them as they do not get vevs because they have large positive squared masses.

We now have to check that the minimum of this potential breaks $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ down to $\mathrm{U}(1)_{\mathrm{EM}}$. The freedom to make $\mathrm{SU}(2)_L$ gauge transformations allows us to rotate away a possible vev for one of the scalar fields, so without loss of generality we can take $H_u^+ = 0$ at the minimum of the potential. Then, one can check that a minimum of the potential satisfying $\partial V / \partial H_u^+ = 0$ must also have $H_d^- = 0$. This means that electromagnetism is unbroken. We are then left to consider the scalar potential for the neutral Higgs fields:

$$\begin{aligned} V &= (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 + (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 - (b\, H_u^0 H_d^0 + \text{c.c.}) \\ &\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2 \ . \end{aligned} \tag{6.12}$$

The only term in this potential depending on the phases of the fields is the b term. Therefore, a redefinition of the phase of H_u or H_d can absorb any phase in b, so we can take b to be real and positive. It is then clear that $H_u^0 H_d^0$ is also real and positive at the minimum, so $\langle H_u^0 \rangle$ and $\langle H_d^0 \rangle$ must have opposite phases. We can therefore use a $\mathrm{U}(1)_Y$ gauge transformation to make them both real and positive without loss of generality, since H_u and H_d have opposite weak hypercharges ($\pm 1/2$). It follows that CP cannot be spontaneously broken by the Higgs scalar potential, since the vevs and b can be simultaneously chosen real, as a convention. This means that the Higgs scalar mass eigenstates can be assigned well-defined eigenvalues of CP.

We must now make sure that the potential is bounded from below for arbitrarily large values of the scalar fields, so that V has a true minimum. (Recall that susy scalar potentials are always ≥ 0, therefore bounded from below. But now we have introduced soft susy breaking, so we must be careful.) The scalar quartic interactions in V will stabilize the potential for almost all arbitrarily large values of H_u^0 and H_d^0. However, for the special directions in field space $|H_u^0| = |H_d^0|$, the quartic contributions to V (second line in (6.12)) are identically zero. Such directions in field space are called D-flat directions, because along them the D terms vanish. In order for the potential to be bounded from below, we need the quadratic part of the scalar potential to be positive along the D-flat directions.

This requirement amounts to:

$$0 < 2b < 2|\mu|^2 + m^2_{H_u} + m^2_{H_d} \ . \tag{6.13}$$

Hence $(|\mu|^2 + m^2_{H_u})$ and $(|\mu|^2 + m^2_{H_d})$ cannot both be negative. This implies that the origin $H^0_u = H^0_d = 0$ cannot be a maximum of V. If $(|\mu|^2 + m^2_{H_u})$ and $(|\mu|^2 + m^2_{H_d})$ are both positive then the origin is a minimum (which would lead to an unwanted symmetry preserving solution) unless:

$$b^2 > (|\mu|^2 + m^2_{H_u})(|\mu|^2 + m^2_{H_d}) \ , \tag{6.14}$$

which is the condition for the origin to be a saddle point. The last condition is automatically satisfied if either $(|\mu|^2 + m^2_{H_u})$ or $(|\mu|^2 + m^2_{H_d})$ is negative, but this is not necessary.

The b term favors EWSB but it is not required to be non-zero. On the other hand, if $m^2_{H_u} = m^2_{H_d}$ then the constraints (6.13) and (6.14) cannot both be satisfied. In particular if susy is not broken and $m^2_{H_u} = m^2_{H_d} = 0$ the EWSB does not occur. In models with universal soft terms, $m^2_{H_u} = m^2_{H_d}$ is supposed to hold at the input scale, but the X_t contribution to the RGE (6.10) for $m^2_{H_u}$ naturally pushes it to negative or small values $m^2_{H_u} < m^2_{H_d}$ at the scale M_{weak}. So in these models EWSB is actually driven by quantum corrections. This mechanism is known as *radiative electroweak symmetry breaking.* Note that even if $m^2_{H_u} < 0$, there may be no EWSB if $|\mu|$ is too large or if b is too small. Still, the large negative contribution to $m^2_{H_u}$ from its RGE is an important factor in ensuring that EWSB can occur in models with simple boundary conditions for the soft terms. The fact that this works most naturally with a large top-quark Yukawa coupling provides additional motivation for these models.

Having established the conditions necessary for H^0_u and H^0_d to get non-zero vevs, let us write:

$$v_u = \langle H^0_u \rangle \ , \qquad v_d = \langle H^0_d \rangle \ . \tag{6.15}$$

These vevs are related to the known mass of the Z^0 boson and the electroweak gauge couplings through the kinetic energy terms for the Higgs fields with the proper covariant derivatives:

$$v^2_u + v^2_d = v^2 = 2m^2_Z/(g^2 + g'^2) \approx (174\ \text{GeV})^2 \ . \tag{6.16}$$

The ratio of the vevs is traditionally written as:

$$\tan\beta \equiv v_u/v_d \ . \tag{6.17}$$

The value of $\tan\beta$ is not fixed by present experiments, but it depends on the Lagrangian parameters of the MSSM in a calculable way. Since $v_u = v\sin\beta$ and $v_d = v\cos\beta$ were taken to be real and positive by convention, we have $0 < \beta < \pi/2$, a requirement that will be sharpened below. Now one can write down the conditions $\partial V/\partial H^0_u = \partial V/\partial H^0_d = 0$ under which the potential (6.12) will have a minimum satisfying (6.16) and (6.17):

$$\begin{aligned} m^2_{H_u} + |\mu|^2 - b\cot\beta - (m^2_Z/2)\cos(2\beta) &= 0 \ , \\ m^2_{H_d} + |\mu|^2 - b\tan\beta + (m^2_Z/2)\cos(2\beta) &= 0 \ . \end{aligned} \tag{6.18}$$

It is easy to check that these equations indeed satisfy the necessary conditions (6.13) and (6.14). They allow us to eliminate the two Lagrangian parameters b and $|\mu|$ in favor of $\tan\beta$ and the known value of m_Z, but do not determine the phase of μ (or its sign if μ is assumed to be real).

The Higgs scalar fields in the MSSM consist of two complex $SU(2)_L$ doublets, or eight real scalar degrees of freedom. After EWSB, three of them are the would-be Goldstone bosons G^0, $G^\pm$, which become the longitudinal modes of the Z^0 and $W^\pm$ massive vector bosons. The remaining five Higgs scalar mass eigenstates consist of two CP-even neutral scalars h^0 and H^0 (by convention, h^0 is lighter than H^0), one CP-odd neutral scalar A^0, and a charge +1 scalar H^+ and its conjugate charge −1 scalar H^- ($G^- = G^{+*}$ and $H^- = H^{+*}$). The gauge eigenstates can be expressed in terms of the mass eigenstates as:

$$\begin{pmatrix} H_u^0 \\ H_d^0 \end{pmatrix} = \begin{pmatrix} v_u \\ v_d \end{pmatrix} + \frac{1}{\sqrt{2}} R_\alpha \begin{pmatrix} h^0 \\ H^0 \end{pmatrix} + \frac{i}{\sqrt{2}} R_{\beta_0} \begin{pmatrix} G^0 \\ A^0 \end{pmatrix} , \quad \begin{pmatrix} H_u^+ \\ H_d^{-*} \end{pmatrix} = R_{\beta_\pm} \begin{pmatrix} G^+ \\ H^+ \end{pmatrix} , \tag{6.19}$$

where the orthogonal rotation matrices:

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} , \; R_{\beta_0} = \begin{pmatrix} \cos\beta_0 & \sin\beta_0 \\ -\sin\beta_0 & \cos\beta_0 \end{pmatrix} , \; R_{\beta_\pm} = \begin{pmatrix} \cos\beta_\pm & \sin\beta_\pm \\ -\sin\beta_\pm & \cos\beta_\pm \end{pmatrix} , \tag{6.20}$$

are chosen so that the quadratic part of the potential has diagonal squared masses:

$$\begin{aligned} V &= \frac{1}{2} m_{h^0}^2 (h^0)^2 + \frac{1}{2} m_{H^0}^2 (H^0)^2 + \frac{1}{2} m_{G^0}^2 (G^0)^2 + \frac{1}{2} m_{A^0}^2 (A^0)^2 \\ &\quad + m_{G^\pm}^2 |G^+|^2 + m_{H^\pm}^2 |H^+|^2 + \dots , \end{aligned} \tag{6.21}$$

Then, provided that v_u, v_d minimize the tree level potential, one finds $\beta_0 = \beta_\pm = \beta$ and:

$$\begin{aligned} m_{G^0}^2 &= m_{G^\pm}^2 = 0 , \\ m_{A^0}^2 &= 2b/\sin(2\beta) = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2 , \\ m_{h^0,H^0}^2 &= \frac{1}{2}\Big(m_{A^0}^2 + m_Z^2 \mp \sqrt{(m_{A^0}^2 - m_Z^2)^2 + 4 m_Z^2 m_{A^0}^2 \sin^2(2\beta)} \Big) , \\ m_{H^\pm}^2 &= m_{A^0}^2 + m_W^2 . \end{aligned} \tag{6.22}$$

The mixing angle α is determined, at tree level, by:

$$\frac{\sin(2\alpha)}{\sin(2\beta)} = -\left(\frac{m_{H^0}^2 + m_{h^0}^2}{m_{H^0}^2 - m_{h^0}^2} \right) , \qquad \frac{\tan(2\alpha)}{\tan(2\beta)} = \left(\frac{m_{A^0}^2 + m_Z^2}{m_{A^0}^2 - m_Z^2} \right) , \tag{6.23}$$

and is traditionally chosen to be negative, hence $-\pi/2 < \alpha < 0$ (provided $m_{A^0} > m_Z$). In the so-called decoupling limit where $m_A \gg m_Z$, h^0 has the same couplings as would a SM Higgs boson with the same mass. In this case, $\alpha \simeq \beta - \pi/2$ and $(A^0, H^0, H^\pm)$ form an isospin doublet with a common mass much larger that m_{h^0}.

The masses of A^0, H^0 and $H^\pm$ can in principle be arbitrarily large since they all grow with $b/\sin(2\beta)$. In contrast, the mass of h^0 is bounded above. From (6.22), one finds:

$$m_{h^0} < m_Z |\cos(2\beta)| < m_Z \tag{6.24}$$

This is one of the strongest predictions of low energy susy! Alas, this bound is already ruled out by the current experimental lower bound coming from direct searches at LEP:

$$m_{h^0} \geq 114.4 \text{ GeV } (95\% \text{ c.l.}) . \tag{6.25}$$

Fortunately the tree level mass formulae derived above receive significant radiative corrections, particularly in the case of the h^0, whose mass is shifted upwards by a significant amount. the main contribution comes from the incomplete cancellation of top quark and

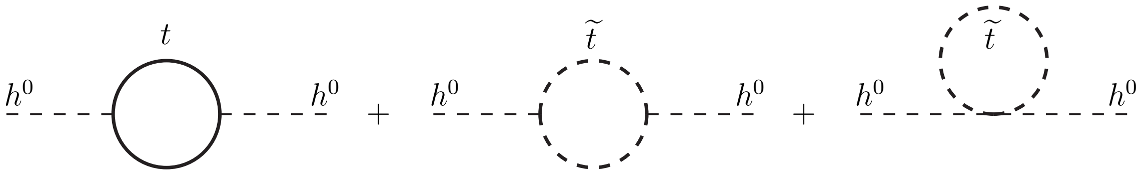


Figure 6.2: Contributions to the MSSM lightest Higgs mass from top/stop one loop diagrams. Incomplete cancellation, due to soft susy breaking, leads to a large positive correction to $m_{h^0}^2$ in the limit of heavy stops.

top squark (also called *stop*) loops of fig. 6.2, which would cancel in the exact susy limit. The magnitude of this contribution depends on the stop masses and mixings, which we shall discuss in the next section. If for simplicity we neglect the stop mixing effects, this contribution modifies (6.24) by a large positive correction:

$$\delta m_{h^0}^2 = \frac{3}{4\pi^2}\, y_t^2 m_t^2 \;\ln\left(\frac{m_{\tilde{t}_1} m_{\tilde{t}_2}}{m_t^2}\right) \;, \qquad (6.26)$$

where $m_{\tilde{t}_1}$, $m_{\tilde{t}_2}$ are the stop masses. The radiative corrections to the h^0 in the MSSM have been studied in detail by many authors and are now known up to two loops with some partial three loops results. If soft susy breaking parameters are ≤ 1 Tev, one generally finds:

$$m_{h^0} \lesssim 140 \text{ GeV} \;. \qquad (6.27)$$

This implies that the lightest CP-even Higgs boson of the MSSM should be discovered at the LHC. If not, then this model will be ruled out, whatever value you may take for its 105 free parameters. However, in extended models, like the NMSSM (*Next-to-Minimal Susy Standard Model*) where a gauge singlet supermultiplet is added in the Higgs sector, the bound (6.27) might be relaxed up to:

$$m_{h^0} \lesssim 200 \text{ GeV} \;. \qquad (6.28)$$

Furthermore, in the NMSSM the lightest CP-even Higgs boson might decay only via $h^0 \to a^0 a^0$, where a^0 is a CP-odd neutral Higgs state which might be extremely light (down to a few GeV). This would seriously complicate the task of the LHC to find the h^0 and in some cases the LHC would see no Higgs at all! However, sparticles should show up at the LHC if susy soft breaking terms are ≤ 1 TeV.

In the MSSM, the tree level masses of the quarks and leptons are related to their Yukawa couplings and the Higgs vevs by:

$$m_t = y_t v_u = y_t v \sin\beta \;, \quad m_b = y_b v_d = y_b v \cos\beta \;, \quad m_\tau = y_\tau v_d = y_\tau v \cos\beta \;, \qquad (6.29)$$

It is now clear why we have not neglected y_b and y_τ, even though $m_b, m_\tau \ll m_t$. To a first approximation, $y_b/y_t = (m_b/m_t)\tan\beta$ and $y_\tau/y_t = (m_\tau/m_t)\tan\beta$, so that y_b and y_τ cannot be neglected if $\tan\beta$ is large. For example, GUT models based on the gauge group SO(10) unify the top, bottom and tau Yukawa couplings at the unification scale. This requires $\tan\beta$ to be very roughly of order m_t/m_b. Note that $\tan\beta$ cannot be too large, i.e. $\cos\beta$ cannot be too small, unless y_b and y_τ would become non perturbatively large. Similarly, if one tries to make $\sin\beta$ or $\tan\beta$ too small, y_t will be non perturbatively large. Requiring that y_t, y_b and y_τ do not blow up above M_{weak}, one finds:

$$1 \lesssim \tan\beta \lesssim 60 \;. \qquad (6.30)$$

6.3 The Sparticle Spectrum

The higgsinos and electroweak gauginos mix with each other because of the effects of EWSB. The neutral higgsinos ($\widetilde{H}_u^0$, $\widetilde{H}_d^0$) and the neutral gauginos ($\widetilde{B}$, $\widetilde{W}^0$) combine to form four mass eigenstates called *neutralinos*, $\widetilde{\chi}^0_{i=1..4}$. By convention, these are labeled in ascending order, so that $m_{\widetilde{\chi}^0_1} < m_{\widetilde{\chi}^0_2} < m_{\widetilde{\chi}^0_3} < m_{\widetilde{\chi}^0_4}$. The lightest neutralino, $\widetilde{\chi}^0_1$, is usually assumed to be the LSP (unless there is a lighter gravitino or R_p is not conserved) as it is the only MSSM particle that can make a good dark matter candidate. In the gauge eigenstate basis $\psi^0 = (\widetilde{B}, \widetilde{W}^0, \widetilde{H}_d^0, \widetilde{H}_u^0)$, the Lagrangian for neutralino masses is:

$$\mathcal{L}_{\chi^0} = -\frac{1}{2}(\psi^0)^T \mathbf{M}_{\widetilde{\chi}^0} \psi^0 + \text{c.c.} \; , \tag{6.31}$$

where

$$\mathbf{M}_{\widetilde{\chi}^0} = \begin{pmatrix} M_1 & 0 & -g' v_d/\sqrt{2} & g' v_u/\sqrt{2} \\ 0 & M_2 & g v_d/\sqrt{2} & -g v_u/\sqrt{2} \\ -g' v_d/\sqrt{2} & g v_d/\sqrt{2} & 0 & -\mu \\ g' v_u/\sqrt{2} & -g v_u/\sqrt{2} & -\mu & 0 \end{pmatrix} \; . \tag{6.32}$$

The diagonal terms are just the gaugino soft masses from (5.7), while the $-\mu$ entries can be traced back to the superpotential (5.1). The off-diagonal terms are the result of Higgs-higgsino-gaugino couplings, with the Higgs fields replaced by their vevs, and they are always $\leq m_Z$. The mass matrix $\mathbf{M}_{\widetilde{\chi}^0}$ can be diagonalized by a unitary matrix $\mathbf{N}$ to obtain the mass eigenstates:

$$\widetilde{\chi}^0_i = \mathbf{N}_{ij} \psi^0_j \; , \tag{6.33}$$

so that

$$\mathbf{N}^* \mathbf{M}_{\widetilde{\chi}^0} \mathbf{N}^{-1} = \text{diag}(m_{\widetilde{\chi}^0_1}, m_{\widetilde{\chi}^0_2}, m_{\widetilde{\chi}^0_3}, m_{\widetilde{\chi}^0_4}) \; . \tag{6.34}$$

In general, M_1, M_2, and μ can have arbitrary complex phases. A redefinition of the phases of $\widetilde{B}$ and $\widetilde{W}$ always allows us to choose a convention in which M_1 and M_2 are both real and positive. The phase of μ within this convention is then a physical parameter and cannot be rotated away. We have already used up the freedom to redefine the phases of the Higgs fields, since we have picked b, v_u and v_d to be real and positive. However, if μ is not real, then there can be potentially disastrous CP-violating effects in low-energy physics, including electric dipole moments for both the electron and the neutron. Therefore, it is usual to assume that μ is real in the same set of phase conventions that make M_1, M_2, b, v_u and v_d real and positive. The sign of μ is still undetermined by this constraint. Models that satisfy universality for the gaugino masses as in (5.15) have the nice prediction:

$$M_1 = \frac{5}{3} \tan^2\theta_W \, M_2 \simeq 0.5 M_2 \tag{6.35}$$

at the scale M_{weak}. Furthermore, one also usually has:

$$m_Z \ll |\mu \pm M_1|, \, |\mu \pm M_2|, \tag{6.36}$$

so that ESWB effects can be viewed as small perturbations in the neutralino mass matrix. Finally, models like mSUGRA usually predict $M_1 \simeq M_2/2 \ll |\mu|$. The neutralino mass eigenstates are then a light "bino-like" state $\widetilde{\chi}^0_1 \approx \widetilde{B}$, a "wino-like" state $\widetilde{\chi}^0_2 \approx \widetilde{W}^0$, and two heavier "higgsino-like" states $\widetilde{\chi}^0_3$, $\widetilde{\chi}^0_4 \approx (\widetilde{H}_u^0 \pm \widetilde{H}_d^0)/\sqrt{2}$, with approximative mass eigenvalues M_1, $M_2 \simeq 2M_1$ and $|\mu|$ (resp.).

Similarly, the charged higgsinos ($\widetilde{H}_u^+$, $\widetilde{H}_d^-$) and winos ($\widetilde{W}^+$ and $\widetilde{W}^-$) mix to form two mass eigenstates with charge ± 1 called *charginos*, $\widetilde{\chi}^{\pm}_{i=1,2}$ with masses $m_{\widetilde{\chi}_1^\pm} < m_{\widetilde{\chi}_2^\pm}$. In the gauge eigenstate basis $\psi^\pm = (\widetilde{W}^+, \widetilde{H}_u^+, \widetilde{W}^-, \widetilde{H}_d^-)$, the chargino mass terms in the Lagrangian are:

$$\mathcal{L}_{\widetilde{\chi}^\pm} = -\frac{1}{2}(\psi^\pm)^T \mathbf{M}_{\widetilde{\chi}^\pm} \psi^\pm + \text{c.c.} \,, \tag{6.37}$$

where, in 2×2 block form,

$$\mathbf{M}_{\widetilde{\chi}^\pm} = \begin{pmatrix} \mathbf{0} & \mathbf{X}^T \\ \mathbf{X} & \mathbf{0} \end{pmatrix} , \qquad \text{with} \qquad \mathbf{X} = \begin{pmatrix} M_2 & g v_u \\ g v_d & \mu \end{pmatrix} \tag{6.38}$$

The mass eigenstates are related to the gauge eigenstates by two unitary 2×2 matrices $\mathbf{U}$ and $\mathbf{V}$ according to:

$$\begin{pmatrix} \widetilde{\chi}_1^+ \\ \widetilde{\chi}_2^+ \end{pmatrix} = \mathbf{V} \begin{pmatrix} \widetilde{W}^+ \\ \widetilde{H}_u^+ \end{pmatrix} , \qquad \begin{pmatrix} \widetilde{\chi}_1^- \\ \widetilde{\chi}_2^- \end{pmatrix} = \mathbf{U} \begin{pmatrix} \widetilde{W}^- \\ \widetilde{H}_u^- \end{pmatrix} . \tag{6.39}$$

Note that the mixing matrix for the positively charged left-handed fermions is different from that for the negatively charged left-handed fermions. They are chosen so that

$$\mathbf{U}^* \mathbf{X} \mathbf{V}^{-1} = \begin{pmatrix} m_{\widetilde{\chi}_1^\pm} & 0 \\ 0 & m_{\widetilde{\chi}_2^\pm} \end{pmatrix} , \tag{6.40}$$

with positive real entries $m_{\widetilde{\chi}_i^\pm}$. Because these are only 2×2 matrices, it is not hard to solve for the masses explicitly:

$$\begin{aligned} m^2_{\widetilde{\chi}_1^\pm}, m^2_{\widetilde{\chi}_2^\pm} \;=\; & \frac{1}{2}\Big[|M_2|^2 + |\mu|^2 + 2m_W^2 \\ & \mp \sqrt{(|M_2|^2 + |\mu|^2 + 2m_W^2)^2 - 4|\mu M_2 - m_W^2 \sin(2\beta)|^2}\;\Big] . \end{aligned} \tag{6.41}$$

These are the (doubly degenerate) eigenvalues of the 4×4 matrix $\mathbf{M}^\dagger_{\widetilde{\chi}^\pm} \mathbf{M}_{\widetilde{\chi}^\pm}$, or equivalently the eigenvalues of $\mathbf{X}^\dagger \mathbf{X}$, since

$$\mathbf{V} \mathbf{X}^\dagger \mathbf{X} \mathbf{V}^{-1} = \mathbf{U}^* \mathbf{X} \mathbf{X}^\dagger \mathbf{U}^T = \begin{pmatrix} m^2_{\widetilde{\chi}_1^\pm} & 0 \\ 0 & m^2_{\widetilde{\chi}_2^\pm} \end{pmatrix} . \tag{6.42}$$

But, they are *not* the squares of the eigenvalues of $\mathbf{X}$. In the limit of (6.36) with M_2, μ real and such that $M_2 \ll |\mu|$ the chargino mass eigenstates consist of a wino-like $\widetilde{\chi}_1^\pm$ and a higgsino-like $\widetilde{\chi}_2^\pm$, with approximate masses M_2 and $|\mu|$ (resp.).

The gluino is a color octet fermion, so it cannot mix with any other particle in the MSSM, even if R_p is violated. In this regard, it is unique among all of the MSSM sparticles. In models with universal gaugino masses like mSUGRA, the gluino mass parameter M_3 is related to the bino and wino mass parameters M_1 and M_2:

$$M_3 \;=\; \frac{\alpha_s}{\alpha} \sin^2\theta_W \, M_2 \;=\; \frac{3}{5}\frac{\alpha_s}{\alpha} \cos^2\theta_W \, M_1 \tag{6.43}$$

at any scale, up to small two loop corrections. This implies roughly:

$$M_3 : M_2 : M_1 \;\approx\; 6 : 2 : 1 \tag{6.44}$$

near the TeV scale. It is therefore reasonable to suspect that the gluino is considerably heavier than the lighter neutralinos and charginos.

Let us conclude with the squark and slepton masses. In principle, any scalars with the same electric charge, R_p, and color quantum numbers can mix with each other. This means that with completely arbitrary soft terms, the mass eigenstates of the squarks and sleptons should be obtained by diagonalizing three 6×6 squared mass matrices for up-type squarks ($\widetilde{u}_L$, $\widetilde{c}_L$, $\widetilde{t}_L$, $\widetilde{u}^c_R$, $\widetilde{c}^c_R$, $\widetilde{t}^c_R$), down-type squarks ($\widetilde{d}_L$, $\widetilde{s}_L$, $\widetilde{b}_L$, $\widetilde{d}^c_R$, $\widetilde{s}^c_R$, $\widetilde{b}^c_R$), and charged sleptons ($\widetilde{e}_L$, $\widetilde{\mu}_L$, $\widetilde{\tau}_L$, $\widetilde{e}^c_R$, $\widetilde{\mu}^c_R$, $\widetilde{\tau}^c_R$), and one 3×3 matrix for sneutrinos ($\widetilde{\nu}_e$, $\widetilde{\nu}_\mu$, $\widetilde{\nu}_\tau$). Fortunately, the general hypothesis of flavor blind soft parameters (5.9) and (5.10) predicts that most of these mixing angles are very small. The third family squarks and sleptons can have very different soft masses compared to their two first family counterparts, because of the effects of large Yukawa (y_t, y_b, y_τ) and soft (a_t, a_b, a_τ) couplings in the RGEs. Furthermore, they can have substantial mixing in pairs ($\widetilde{t}_L$, $\widetilde{t}^c_R$), ($\widetilde{b}_L$, $\widetilde{b}^c_R$) and ($\widetilde{\tau}_L$, $\widetilde{\tau}^c_R$). In contrast, the two first family squarks and sleptons have negligible Yukawa couplings, so they end up in 7 very nearly degenerate, unmixed pairs ($\widetilde{e}^c_R, \widetilde{\mu}^c_R$), ($\widetilde{\nu}_e, \widetilde{\nu}_\mu$), ($\widetilde{e}_L, \widetilde{\mu}_L$), ($\widetilde{u}^c_R, \widetilde{c}^c_R$), ($\widetilde{d}^c_R, \widetilde{s}^c_R$), ($\widetilde{u}_L, \widetilde{c}_L$) and ($\widetilde{d}_L, \widetilde{s}_L$). As we have already discussed in section 5.3, this avoids the problem of disastrously large virtual sparticle contributions to FCNC processes.

The masses of the two first family squarks and sleptons are therefore simply given by their soft masses plus a term due to the EWSB. Indeed, when the neutral Higgs scalars H^0_u and H^0_d get vevs, each squark and slepton ϕ will get a contribution Δ_ϕ to its squared mass coming from the $SU(2)_L$ and $U(1)_Y$ D term quartic interactions of the form $\phi^2(\text{Higgs})^2$:

$$\Delta_\phi \;=\; (T_{3\phi}g^2 - Y_\phi g'^2)(v_d^2 - v_u^2) \;=\; (T_{3\phi} - Q_\phi \sin^2\theta_W)\cos(2\beta)\, m_Z^2 \;, \tag{6.45}$$

where $T_{3\phi}$ is the third component of weak isospin, Y_ϕ the weak hypercharge, and Q_ϕ the electric charge of the chiral supermultiplet to which ϕ belongs. Hence:

$$\begin{array}{lllll} m^2_{\widetilde{e}^c_R} = m^2_{\widetilde{\mu}^c_R} = m^2_{E_1} + \Delta_{\widetilde{e}^c_R} & \quad & m^2_{\widetilde{\nu}_e} = m^2_{\widetilde{\nu}_\mu} = m^2_{L_1} + \Delta_{\widetilde{\nu}} \\ m^2_{\widetilde{e}_L} = m^2_{\widetilde{\mu}_L} = m^2_{L_1} + \Delta_{\widetilde{e}_L} & & m^2_{\widetilde{u}^c_R} = m^2_{\widetilde{c}^c_R} = m^2_{U_1} + \Delta_{\widetilde{u}^c_R} \\ m^2_{\widetilde{d}^c_R} = m^2_{\widetilde{s}^c_R} = m^2_{D_1} + \Delta_{\widetilde{d}^c_R} & & m^2_{\widetilde{u}_L} = m^2_{\widetilde{c}_L} = m^2_{Q_1} + \Delta_{\widetilde{u}_L} \\ m^2_{\widetilde{d}_L} = m^2_{\widetilde{s}_L} = m^2_{Q_1} + \Delta_{\widetilde{d}_L} & & \end{array} \tag{6.46}$$

Third generation squarks and sleptons will get additional contributions from F terms. The squared mass matrix for the stops in the gauge eigenstate basis ($\widetilde{t}_L$, $\widetilde{t}_R$) reads:

$$\mathbf{m^2_{\widetilde{t}}} = \begin{pmatrix} m^2_{Q_3} + m_t^2 + \Delta_{\widetilde{u}_L} & a_t^* v_u - \mu y_t v_d \\ a_t v_u - \mu^* y_t v_d & m^2_{U_3} + m_t^2 + \Delta_{\widetilde{u}^c_R} \end{pmatrix} \;. \tag{6.47}$$

This hermitian matrix can be diagonalized by a unitary matrix to give mass eigenstates:

$$\begin{pmatrix} \widetilde{t}_1 \\ \widetilde{t}_2 \end{pmatrix} = \begin{pmatrix} c_{\widetilde{t}} & -s^*_{\widetilde{t}} \\ s_{\widetilde{t}} & c_{\widetilde{t}} \end{pmatrix} \begin{pmatrix} \widetilde{t}_L \\ \widetilde{t}_R \end{pmatrix} \;. \tag{6.48}$$

The eigenvalues of $\mathbf{m^2_{\widetilde{t}}}$ are $m^2_{\widetilde{t}_1} < m^2_{\widetilde{t}_2}$, and $|c_{\widetilde{t}}|^2 + |s_{\widetilde{t}}|^2 = 1$. If the off-diagonal elements of $\mathbf{m^2_{\widetilde{t}}}$ are real, then $c_{\widetilde{t}}$ and $s_{\widetilde{t}}$ are the cosine and sine of a stop mixing angle $\theta_{\widetilde{t}}$, which can be chosen in the range $0 \le \theta_{\widetilde{t}} < \pi$. Because of the large effects proportional to X_t in RGEs for the stop soft masses, at the weak scale one finds that $m^2_{U_3} < m^2_{Q_3}$, and both of these quantities are usually significantly smaller than the soft masses for the two first family squarks. The diagonal terms m_t^2 in (6.47) tend to mitigate this effect somewhat, but the off-diagonal entries will typically induce a significant mixing, which always reduces the lighter stop squared mass eigenvalue. Therefore, models often predict that $\widetilde{t}_1$ is the lightest squark of all, and that it is predominantly $\widetilde{t}_R$.

A similar analysis can be performed for the sbottoms and staus, which in their respective gauge eigenstate bases $(\widetilde{b}_L, \widetilde{b}_R)$ and $(\widetilde{\tau}_L, \widetilde{\tau}_R)$ have squared mass matrices:

$$\begin{aligned} \mathbf{m}^2_{\widetilde{\mathbf{b}}} &= \begin{pmatrix} m^2_{Q_3} + \Delta_{\widetilde{d}_L} & a_b^* v_d - \mu y_b v_u \\ a_b v_d - \mu^* y_b v_u & m^2_{D_3} + \Delta_{\widetilde{d}_R} \end{pmatrix} , \\ \mathbf{m}^2_{\widetilde{\boldsymbol{\tau}}} &= \begin{pmatrix} m^2_{L_3} + \Delta_{\widetilde{e}_L} & a_\tau^* v_d - \mu y_\tau v_u \\ a_\tau v_d - \mu^* y_\tau v_u & m^2_{E_3} + \Delta_{\widetilde{e}_R} \end{pmatrix} , \end{aligned} \tag{6.49}$$

where we have neglected the diagonal contributions from the F terms, equal to the bottom and tau squared masses. These mass matrices can be diagonalized to give mass eigenstates $\widetilde{b}_1, \widetilde{b}_2$ and $\widetilde{\tau}_1, \widetilde{\tau}_2$ in exact analogy with eq. (6.48). For large values of tanβ, the mixing in (6.49) can be quite significant, because y_b, y_τ and a_b, a_τ are non-negligible. Just as in the case of the stops, the lighter sbottom and stau mass eigenstates (denoted $\widetilde{b}_1$ and $\widetilde{\tau}_1$) can then be significantly lighter than their two first family counterparts. Furthermore, $\widetilde{\nu}_\tau$ can also be lighter than the degenerate $\widetilde{\nu}_e$, $\widetilde{\nu}_\mu$ as RGE effects usually yield $m^2_{L_3} < m^2_{L_1}$ at the weak scale.

We are now ready to display the full MSSM sparticle and Higgs spectrum, fig. 6.3. The next task will be for experimentalists at the LHC to hunt all these new states and check that they have all the properties predicted by theorists. Or else you will have read these lecture notes just for the fun of it!

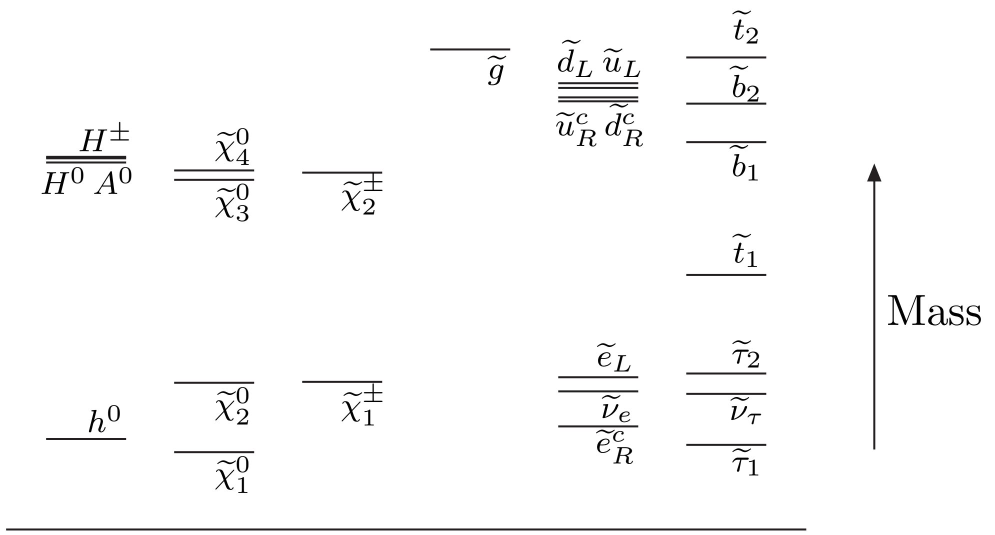


Figure 6.3: Schematic mass spectrum for the undiscovered particles of the MSSM in a typical mSUGRA model.

7 Exercices

A general advice is to try to redemonstrate *all* the formulae of these notes (although this might keep you busy for a substantial amount of time). Here are some selected examples.

7.1 Weyl spinors (beginner)

We take two left-handed (two components) Weyl spinors ψ and χ. Show the following equalities:

$$\psi\chi = \chi\psi \; , \qquad \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi} \; ,$$
$$\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi \; , \quad (\chi\sigma^\mu\bar{\xi})^\dagger = \xi\sigma^\mu\bar{\chi} \; .$$

7.2 O'Raifeartaigh model (medium weight)

We consider three chiral supermultiplets ϕ_i ($i = 1..3$) interacting through the superpotential:

$$W = -k\phi_1 + m\phi_2\phi_3 + \frac{y}{2}\phi_1\phi_3^2 \; , \qquad \text{where } k,\ m,\ y\ \in \mathbb{R}^{+*}.$$

1) Compute the the auxiliary fields $F_i = -\dfrac{\partial W^\dagger}{\partial \phi_i^\dagger}$.

2) Show that it is impossible to have simultaneously $F_i = 0$ for $i = 1..3$.
What does this imply for susy?

3) Write the minimization conditions for the scalar potential $V(\phi_i) = \displaystyle\sum_{i=1}^{3} |F_i|$.
Show that $\dfrac{\partial W}{\partial \phi_3} = 0 \;\Leftrightarrow\; \dfrac{\partial V}{\partial \phi_1} = \dfrac{\partial V}{\partial \phi_2} = 0$. Hence, write V as a function of ϕ_3 only.

4) What are the extrema of $V(\phi_3)$ in the following cases: $m^2 < ky$, $m^2 \geq ky$?

7.3 MSSM γ functions (heavy weight champion)

Applying the general formula (6.2), show that the one loop γ functions of the MSSM chiral supermultiplets are:

$$\gamma_{H_u} = \frac{1}{16\pi^2}\left[3y_t^2 - \frac{3}{2}g_2^2 - \frac{3}{10}g_1^2\right] \; , \qquad \gamma_{H_d} = \frac{1}{16\pi^2}\left[3y_b^2 + y_\tau^2 - \frac{3}{2}g_2^2 - \frac{3}{10}g_1^2\right] \; ,$$
$$\gamma_{Q_3} = \frac{1}{16\pi^2}\left[y_t^2 + y_b^2 - \frac{8}{3}g_3^2 - \frac{3}{2}g_2^2 - \frac{1}{30}g_1^2\right] \; , \quad \gamma_{U_3} = \frac{1}{16\pi^2}\left[2y_t^2 - \frac{8}{3}g_3^2 - \frac{8}{15}g_1^2\right] \; ,$$
$$\gamma_{D_3} = \frac{1}{16\pi^2}\left[2y_b^2 - \frac{8}{3}g_3^2 - \frac{2}{15}g_1^2\right] \; , \qquad \gamma_{L_3} = \frac{1}{16\pi^2}\left[y_\tau^2 - \frac{3}{2}g_2^2 - \frac{3}{10}g_1^2\right] \; ,$$
$$\gamma_{E_3} = \frac{1}{16\pi^2}\left[2y_\tau^2 - \frac{6}{5}g_1^2\right] \; .$$

How can you simply deduce the γ functions for the two first family supermultiplets?

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