Motion of Spinning Egg

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- VSOP-20
- Quy Nhon, August 3-15, 2014

Motion of hard-boiled egg

- **Moffatt & Shimomura**:
  - analyzed the spinning motion of hard-boiled egg
    Nature 416, 385-386 (2002)
  - Discovery of Gyroscopic balance condition (GBC) in rapidly spinning state
  - They derived a first-order differential eq. for Θ
    → a uniform spheroid will rise
- **Sasaki**:
  - analyzed the spinning motion of the body with oval curves
    Am. J. Phys. 72, 775-781 (2004)
  - A real egg has thin and fat ends
  - Which end of the spinning egg will rise?

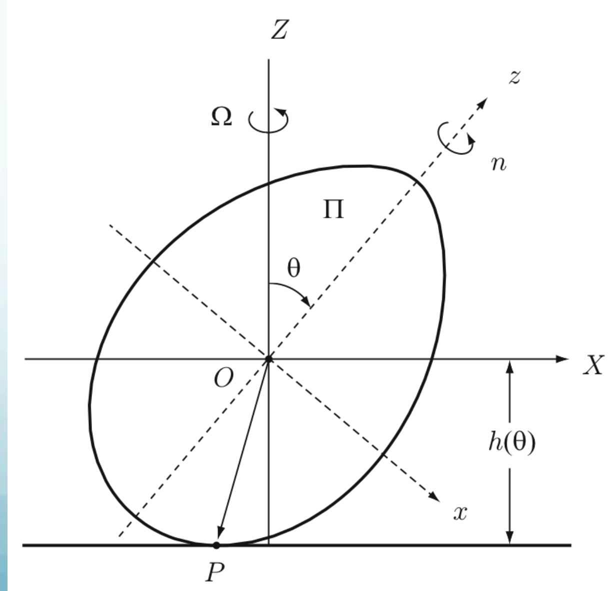
Equation of motion

- In a rotating frame of reference *OXYZ*
- Evolution of angular momentum *L* around CM O (Euler Eq.)

$$\frac{\partial \boldsymbol{L}}{\partial t} + \boldsymbol{\Omega} \times \boldsymbol{L} = \boldsymbol{X}_P \times (\boldsymbol{F} + \boldsymbol{N})$$

$\mathbf{L}$	:	Angular momentum of Egg
$\boldsymbol{\Omega} = (0, 0, \Omega)$	:	Angular velocity of the rotating system
$\mathbf{X}_P = (X_P, 0, Z_P)$	:	Position vector of the contact point P
$\mathbf{F} = (F_X, F_Y, 0)$	:	Frictional force at P
$\mathbf{N} = (0, 0, N)$	:	Normal reaction at P

$$X_P = \frac{dh(\theta)}{d\theta}$$
$$Z_P = -h(\theta)$$



- When Ω is sufficiantly large, GBC holds
  - a first-order differential eq. for *Θ* :

$$J\dot{\theta} = -h^2(\theta) F_Y$$

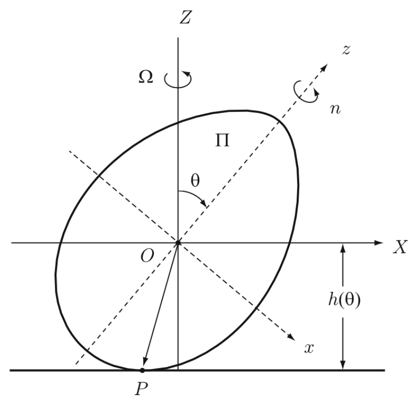
  - Coulomb sliding friction:

$$F_Y = -\mu N \frac{V_{PY}}{|\boldsymbol{V}_P|}$$

  - *Y* component of slipping velocity:

$$V_{PY} = (\boldsymbol{\omega} \times \boldsymbol{X}_P)_Y \equiv \frac{J}{A} \tilde{V}_{PY}(\theta)$$

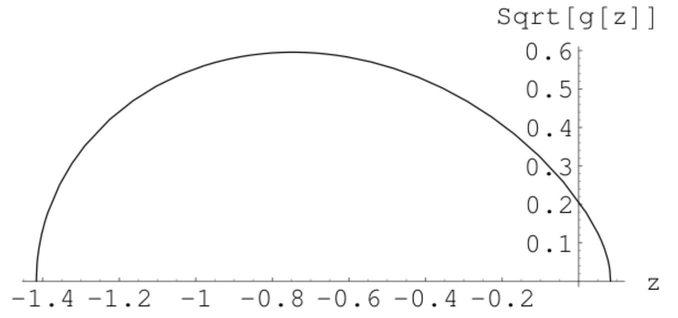
$$\dot{\theta} = \frac{\mu N h^2}{A|\boldsymbol{V}_P|} \tilde{V}_{PY}(\theta)$$



Models of Oval Curve

- Shape of a three-dimensional egg: reconstructed by rotating its two-dimensional cross section around the axis of symmetry

$$x^2 = g(z) \ , \quad g(z) > 0 \quad \text{for} \quad z_{\text{min}} < z < z_{\text{max}}$$
$$g(z_{\text{min}}) = g(z_{\text{max}}) = 0$$



- Assume uniform density

Volume: $V = \pi \int_{z_{\text{min}}}^{z_{\text{max}}} g(z) \ dz$

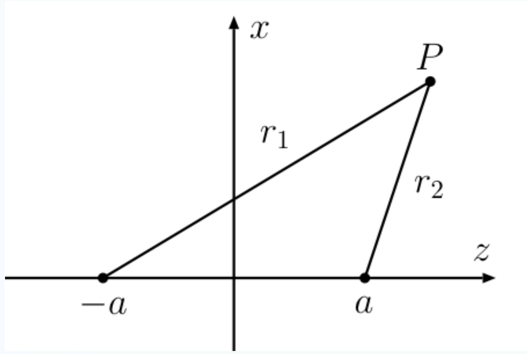
CM: $z_g = \frac{\pi}{V} \int_{z_{\text{min}}}^{z_{\text{max}}} z g(z) \ dz$

Principal moments of inertia at CM:

$$A = M\frac{\pi}{V}\int_{z_{\text{min}}}^{z_{\text{max}}} \left[\frac{1}{4}[g(z)]^2 + g(z)(z - z_g)^2\right] dz$$

$$C = \frac{M}{2}\frac{\pi}{V}\int_{z_{\text{min}}}^{z_{\text{max}}} [g(z)]^2 \ dz$$

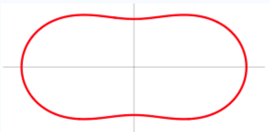
Cassini Oval



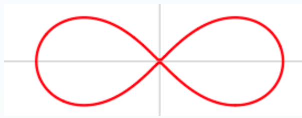
- Def.: $$r_1 \; r_2 = b^2$$

$$\sqrt{(z+a)^2 + x^2}\;\sqrt{(z-a)^2 + x^2} = b^2$$

$a = 1$



$b = 1.2$



$b = 1.0$

$b = 0.98$

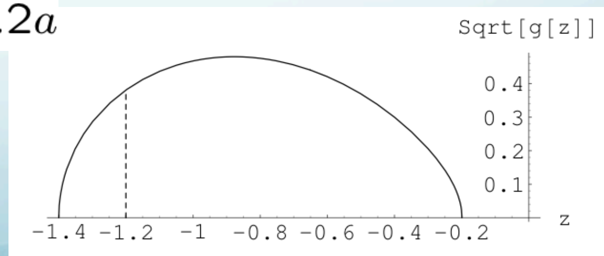
- Without an air chamber

$$x^2 = g(z) = -(z^2 + a^2) + a\sqrt{4z^2 + \lambda^4 a^2}$$

$$\lambda = \frac{b}{a} = 0.98$$

$$z_g = -0.840a \qquad \frac{A}{C} = 1.25$$

- With an air chamber $-1.40a \leq z \leq -1.2a$

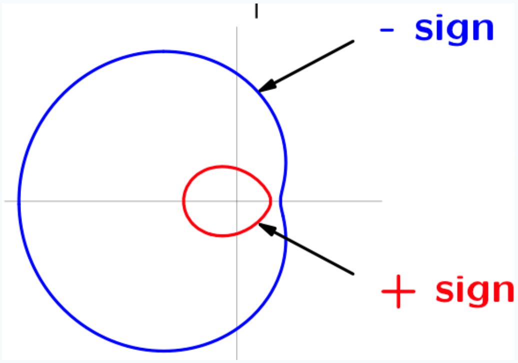


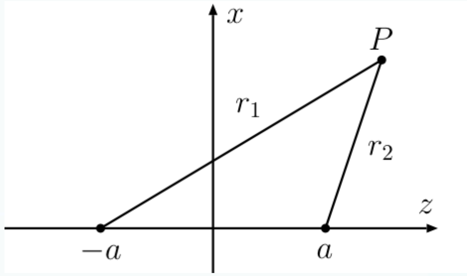
$$z_g = -0.798a \qquad \frac{A}{C} = 1.07$$

Cartesian Oval

- Def.: $m\ r_1 + n\ r_2 = c \quad (m, n\text{: } \textbf{Integer})$

$$m\ \sqrt{(z+a)^2 + x^2}\ \pm n\ \sqrt{(z-a)^2 + x^2} = c$$

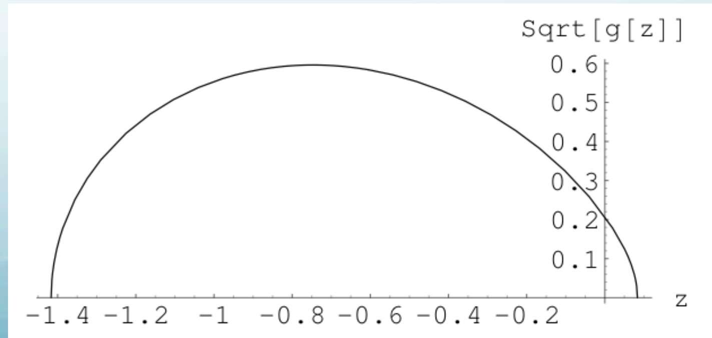




- The inside oval

$$\frac{m}{n} = 2,\ \kappa = \frac{c}{a} = \frac{9}{4}$$

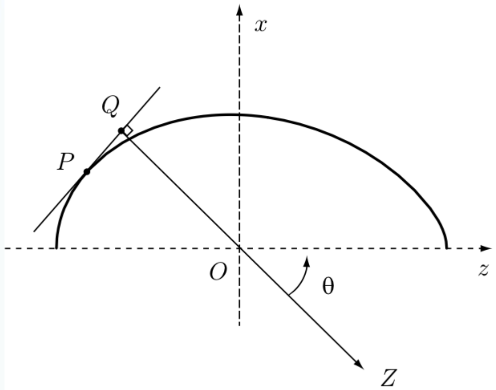
$$z_g = -0.710a \qquad \frac{A}{C} = 1.26$$



Relation between Θ and Oval Curve($x = \sqrt{f(z)}$)

- CM at the origin:

$$x^2 = f(z) = g(z + z_g)$$



- Point *P* on the curve: $P(z, x = \sqrt{f(z)})$
- Slope β of the line tangent to the curve at *P*

$$\beta \equiv \frac{dx}{dz} = \frac{f'(z)}{2\sqrt{f(z)}}$$

- *PQ* is in a plane of table,

  P is a point of contact,

  QO defines the vertical axis *OZ*

$$\tan\theta = \frac{1}{\beta} = \frac{2\sqrt{f(z)}}{f'(z)}$$

Relation between Θ and Oval Curve($x = \sqrt{f(z)}$)

- height:

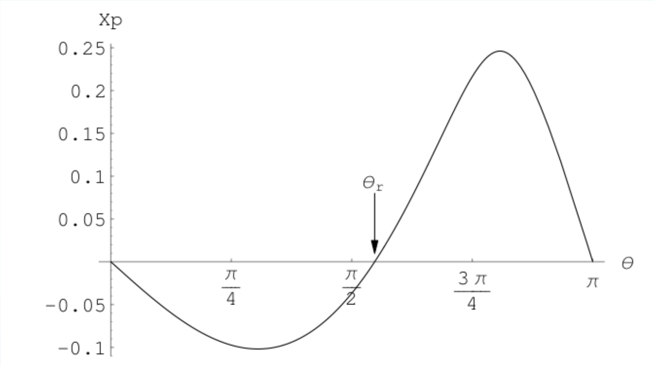
$$h(\theta) = \sqrt{z_Q^2 + x_Q^2} = \frac{1}{\sqrt{\beta^2+1}}(\sqrt{f(z)} - \beta z)$$

$$X_P = \frac{dh(\theta)}{d\theta} = \frac{1}{\sqrt{\beta^2+1}}\left(z + \beta\sqrt{f(z)}\right)$$

- When the body is placed at rest on a table, the CM is just above the contact point *P*

$$X_P = 0$$

- The angle θ_r at which $X_P = 0$



Cassini oval without an air chamber: $\theta_r = 1.77$

Cassini oval with an air chamber: $\theta_r = 1.93$

Cartesian oval : $\theta_r = 1.72$

Which end will rise？

- $$\dot{\theta} = \frac{\mu N h^2}{A|\boldsymbol{V}_P|}\widetilde{V}_{PY}(\theta)$$

  $\widetilde{V}_{PY}(\theta) > 0 \Rightarrow \theta$ Increases with t

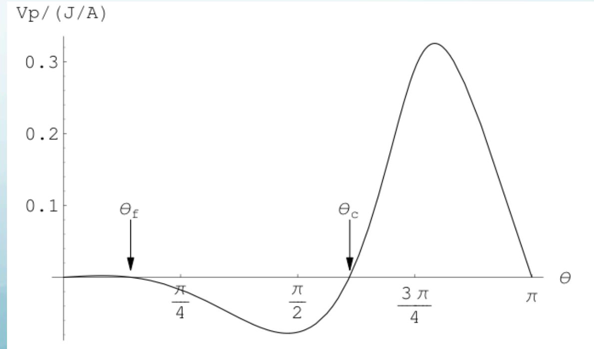
  $\widetilde{V}_{PY}(\theta) < 0 \Rightarrow \theta$ decrease as t increases

  $$\widetilde{V}_{PY}(\theta) = \frac{\beta^2}{\beta^2+1}\left[\left(\frac{1}{\beta^2}+\frac{A}{C}\right)\frac{z+\beta\sqrt{f(z)}}{\sqrt{f(z)}-\beta\ z}+\frac{1}{\beta}\left(\frac{A}{C}-1\right)\right]$$

- Cassini oval without an air chamber
  - $\widetilde{V}_{PY} = 0$ $\theta_f = 0.45$ $\theta_c = 1.92$
  - for $\theta_f < \theta < \theta_c$ $\widetilde{V}_{PY} < 0$

    otherwise $\widetilde{V}_{PY} > 0$
  - $\theta_{\text{initial}} > \theta_c$ $\theta \rightarrow \pi$ the fat end will rise
  - $0 < \theta_{\text{initial}} < \theta_c$ $\theta \rightarrow \theta_f$

  θ_f is the fixed point

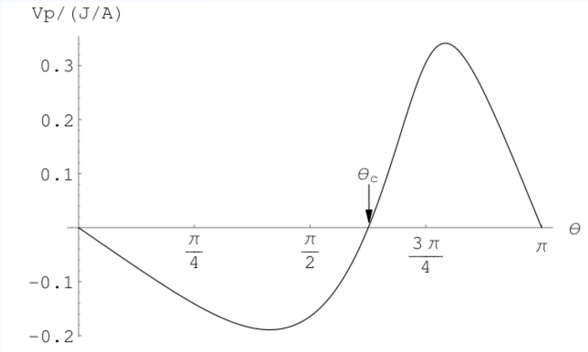


Which end will rise？

- Cassini oval with an air chamber
  - Fixed point θ_f disappears
  - $\tilde{V}_{PY} = 0$ $\theta_c = 1.97$

    for $\theta_c < \theta < \pi$ $\tilde{V}_{PY} > 0$

    for $0 < \theta < \theta_c$ $\tilde{V}_{PY} < 0$



$\theta_r = 1.93$

- $\theta_{\text{initial}} > \theta_c$ $\theta \to \pi$ the fat end will rise
- $\theta_{\text{initial}} < \theta_c$ $\theta \to 0$ the thin end will rise

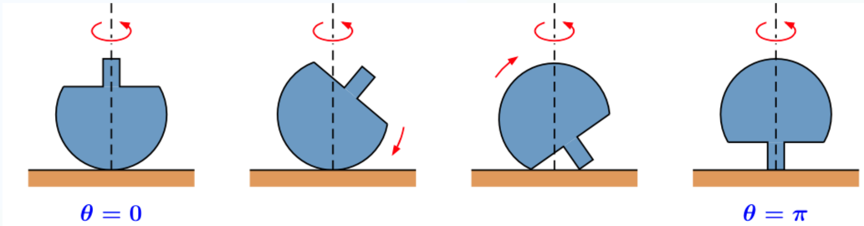
- $\theta_r < \theta_c \Rightarrow$ when the bodies are spun without intention, we expect more chances for $\theta_{\text{initial}} < \theta_c$, so we see spinning states at the fat end more often than at the thin end

Tippe top

T. Ueda, K. Sasaki, S. Watanabe

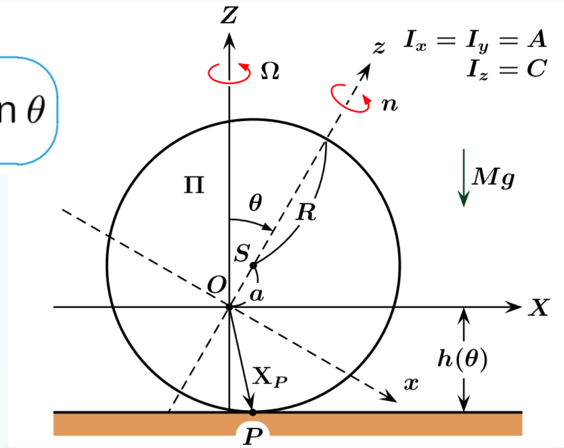
"Motion of the Tippe Top--Gyroscopic Balance Condition and Stability"

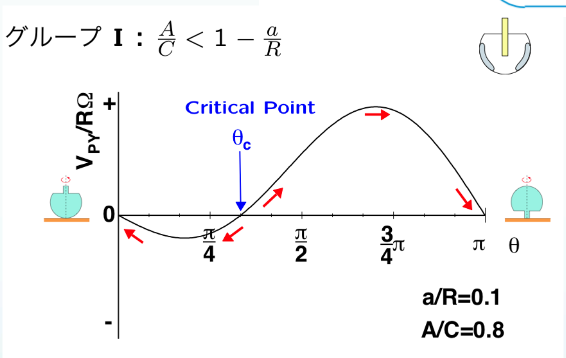
SIAM 4 (2005)

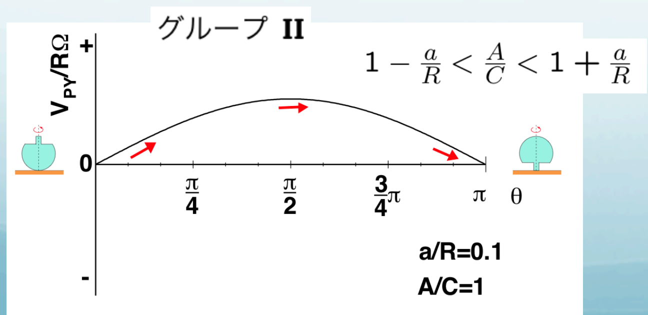


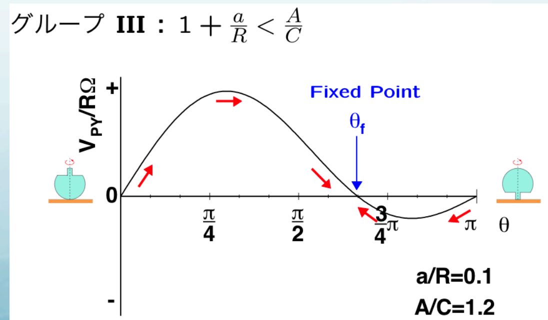
$$A\Omega\dot{\theta} = \mu M g h \,\mathrm{sgn}(V_{PY})$$

$$V_{PY} = R\Omega\left[\frac{a}{R} - \left(1 - \frac{A}{C}\right)\cos\theta\right]\sin\theta$$









When you find a boiled-egg, try spinning it.

But be careful not to drop it from table!!!

- Thank for your attention